THE EXTREMAL CONSTRUCTION FOR $\kappa_{L^2}(d,k)$: A FOUR-CELL DEFECT AND ITS ONE-EDGE REPAIR

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Abstract. For a simple graph G let $L^2(G)$ be its second iterated line graph, and for integers $d \geq 3$ and $k \geq 1$ let $\mathcal{G}(d,k)$ be the class of simple graphs with $\delta(G) = d$ and $\kappa'(G) = k$, so that $\kappa_{L^2}(d,k) = \inf\{\kappa(L^2(G)) : G \in \mathcal{G}(d,k)\}$. A recent preprint of Zou, Xiong, Zhan and Lai evaluates $\kappa_{L^2}(d,k) = \min\{f(d,k),\, 4d-6\}$ for an explicit four-branch function f. Its upper bound is carried by three constructions, and an unnumbered display in its Section 3 records what those constructions deliver. On the branch $(3d-1)/4 \leq k < d$ that display reports the value $A(d,k) = kd - k^2 + 2k\lfloor d/2\rfloor - \lfloor d/2\rfloor d$, and $f(d,k) - A(d,k) = \lfloor d/2\rfloor\,(d - 2\lfloor d/2\rfloor)$, which is $\lfloor d/2\rfloor$ for odd d: the display reports an upper bound strictly below the value the theorem asserts. We show that, after the cap $4d-6$ is applied, the two disagree at exactly four cells, $(d,k) \in \{(3,2),(5,4),(7,5),(7,6)\}$; that at each of them the graph the display describes has a vertex of degree $2\lfloor d/2\rfloor = d-1$ and so lies outside $\mathcal{G}(d,k)$; and that adding one edge restores $\delta = d$ and makes the same cut count equal $f(d,k)$ exactly. We then supply the upper-bound witnesses that the construction fails to provide at the two smallest cells: two members of $\mathcal{G}(3,2)$ whose L^2 has connectivity exactly 4, one of them a smallest member of the class, and a member of $\mathcal{G}(5,4)$ whose 190-vertex L^2 is cut by 12 vertices. The repaired construction therefore carries, at every cell of the branch, a cut of exactly the size the theorem needs. Every theorem and proposition below is machine-checked in Lean 4, and every connectivity value one of them calls exact is exhaustive over vertex subsets rather than sampled; the remarks, which carry what we did not formalize, say so.

1. Introduction

For a simple graph G, the *line graph* $L(G)$ has $E(G)$ as its vertex set, two edges being adjacent when they share an endpoint; the second iterated line graph is $L^2(G) = L(L(G))$. Throughout, κ, κ' and δ denote vertex connectivity, edge connectivity and minimum degree.

Knor and Niepel [2] proved that $L^2(G)$ is $(\delta(G)-1)$-connected for a connected graph G with $\delta(G) \geq 3$; this is quoted as Theorem 1.3 of [1]. In a preprint posted on 27 August 2026, Zou, Xiong, Zhan and Lai [1] sharpen it by splitting graphs of a given minimum degree according to their edge connectivity. For integers $d \geq 3$ and $k \geq 1$ they put

$$(1)\quad \mathcal{G}(d,k) = \big\{G \text{ simple } : \kappa'(G) = k,\ \delta(G) = d\big\}, \qquad \kappa_{L^2}(d,k) = \inf\big\{\kappa(L^2(G)) : G \in \mathcal{G}(d,k)\big\},$$

and their Theorem 1.5 states, for $d \geq k \geq 1$ with $d \geq 3$, that

$$\kappa_{L^2}(d,k) = \min\{f(d,k),\, 4d-6\}, \tag{2}$$

where

$$f(d,k) = \begin{cases} k(d-k), & 1 \leq k \leq \lfloor d/2\rfloor, \\ kd - k^2 + 2k\lceil d/2\rceil - \lceil d/2\rceil d, & \lfloor d/2\rfloor < k < (3d-1)/4, \\ kd - k^2 + 2k\lfloor d/2\rfloor - 2\lfloor d/2\rfloor^2, & (3d-1)/4 \leq k < d, \\ d\lceil d/2\rceil, & k = d. \end{cases} \tag{3}$$

The Knor–Niepel theorem is the case $k = 1$, where $f(d,1) = d-1$. The lower-bound half of (2) is a case analysis; the upper-bound half is carried by three explicit constructions — the graphs $J(d,k,n_1,n_2)$ of [1, Example 1.4] on the first branch, the graph of [1, Figure 8] for $\lfloor d/2\rfloor < k \leq d$, and the graph of [1, Figure 9] for the cap $4d-6$.

The point of this note. Between the Figure 8 construction and the Figure 9 one, Section 3 of [1] prints an unnumbered display collecting the values the Figure 8 graph produces; we call it *the display* throughout. On the third branch of (3) the display reports

$$\kappa_{L^2}(d,k) \;\le\; A(d,k) := kd - k^2 + 2k\lfloor d/2\rfloor - \lfloor d/2\rfloor\, d \qquad \big((3d-1)/4 \le k < d\big), \tag{4}$$

and then, inside the display itself, observes that $A(d,k) \le f(d,k)$. That inequality points the wrong way: an upper bound strictly below $f(d,k)$ would contradict the equality asserted in (2). The discrepancy is an exact identity,

$$f(d,k) - A(d,k) = \lfloor d/2\rfloor\big(d - 2\lfloor d/2\rfloor\big) = \begin{cases} 0, & d \text{ even},\\ \lfloor d/2\rfloor, & d \text{ odd}, \end{cases} \tag{5}$$

so the two statements are compatible for even d and incompatible for odd d wherever $A(d,k)$ also undercuts the cap $4d-6$.

This note settles the tension, and settles it in the direction that leaves the main theorem of [1] intact. Concretely:

(i) *The disagreement is finite and we locate it exactly.* After the cap is applied, $\min\{A, 4d-6\} < \min\{f, 4d-6\}$ holds on the third branch precisely at the four cells $(3,2)$, $(5,4)$, $(7,5)$ and $(7,6)$; everywhere else on the branch the two agree (Theorem 3.2). This is an equivalence over the whole branch, not a sample.

(ii) *We locate the defect inside the construction.* The sentence of [1] describing the Figure 8 graph on this branch gives its vertex v_1 exactly $\lfloor d/2\rfloor$ neighbours on each of the two sides, hence degree $2\lfloor d/2\rfloor$, which equals $d-1$ for odd d — below the $\delta(G) = d$ that membership in $\mathcal{G}(d,k)$ requires (Proposition 4.1). The graph named on the third branch is therefore not in the class over which $\kappa_{L^2}(d,k)$ is an infimum, and the bound read off it is unsupported for odd d.

(iii) *One edge repairs it, and the repair delivers the theorem's value.* Reading the second $\lfloor d/2\rfloor$ as $\lceil d/2\rceil$ — the only reading under which v_1 has degree d — leaves the rest of the description untouched and makes the same cut count equal $f(d,k)$ exactly, rather than $A(d,k)$ (Proposition 4.2).

(iv) *At the two smallest cells we supply the missing witnesses.* At $(3,2)$ the repaired graph lies in $\mathcal{G}(3,2)$ and its 48-vertex L^2 has connectivity exactly 4, and so does the L^2 of a smallest member of $\mathcal{G}(3,2)$, an 8-vertex graph of an entirely different shape (Theorems 5.2 and 5.3). At $(5,4)$ the repaired graph lies in $\mathcal{G}(5,4)$ and its 190-vertex L^2 is cut by an explicit set of 12 vertices, so $\kappa_{L^2}(5,4) \le 12$ — the value of (2), not the 10 of (4) (Theorem 6.1).

(v) *What we leave open.* The matching lower bound $\kappa(L^2) \ge 12$ at $(5,4)$ is not part of our formal development; see Section 9.

How this note should be read. Everything below is stated against version 1 of [1], and every numbered statement, display and figure of [1] that we cite carries v1's numbering. The arXiv abstract page for 2608.26620, re-read on 29 August 2026, lists v1 of 27 August 2026 and no other version; the preprint is two days old and a later version may well correct the display. What we exhibit is a defect in the *proof of the upper-bound half* at four cells, not a counterexample to Theorem 1.5 of [1]: at the cells where we could compute the answer, the value that theorem asserts is the value we find, and the construction the paper describes delivers that value once its minimum-degree condition is restored. We have not audited the lower-bound half at all (Section 9).

Method. Connectivity is computed here by its definition. For a graph H on n vertices and an integer c, the assertion $\kappa(H) \ge c$ is evaluated by deleting every vertex subset of size less than c and testing the remainder for connectedness; $\kappa(H) \le c$ is witnessed by one explicit cut. No theory of line graphs, and in particular no Menger-type certificate, is needed to obtain the values below, which is why they can be asserted as exact. The price is that the searches are large but finite, and we say in each proof exactly which finite search was run. The formal verification is described in Section 10.

2. Preliminaries

All graphs are finite, simple and undirected. For $v \in V(G)$ we write d_v for its degree. For disjoint $S, T \subseteq V(G)$, $E_G[S,T]$ is the set of edges with one end in S and one in T. An *edge cut* of a connected graph H is a set $X \subseteq E(H)$ with $H - X$ disconnected; it is *essential* if $H - X$ has at least two components containing an edge, and $\operatorname{ess}'(H)$ denotes the least size of an essential edge cut.

The machinery of [1] rests on two facts, which we record only in the form in which they are used below.

Proposition 2.1 ([1, Observation 1.1], attributed there to Shao [3, Proposition 1.1.3])**.** *Let H be a connected graph and let m be an integer. Then $\kappa(L(H)) \geq m$ if and only if $\operatorname{ess}'(H) \geq m$.*

We quote this from [1], not from the thesis, which we were unable to consult, and use it only as [1] does: under that paper's standing assumption that the graph is not a path, a cycle or a $K_{1,3}$, an assumption the statement needs ($K_{1,3}$ has no essential edge cut at all, while $\kappa(L(K_{1,3})) = 2$) and one that holds wherever we apply it below.

Applied with $H = L(G)$, Proposition 2.1 turns an upper bound on $\kappa(L^2(G))$ into the exhibition of a small essential edge cut of $L(G)$. Such a cut X splits $E(G)$ into two colour classes; write $V_{12}(X)$ for the set of vertices of G incident with edges of both classes, and $E_1(v), E_2(v)$ for the two classes at such a vertex v. Display (4) of [1] — the fourth numbered display of v1, deduced there from its Observation 2.2(i) and stated with a trailing ≥ 1 we do not use — then evaluates the cut:

$$|X| = \sum_{v \in V_{12}(X)} |E_1(v)| \cdot |E_2(v)|. \tag{6}$$

The Figure 8 construction. For $\lfloor d/2 \rfloor < k \leq d$ the graph of [1, Figure 8] consists of two complete graphs $G[V_1(X)]$, $G[V_2(X)]$ of order $d+1$ together with two further vertices v_1, v_2 joined into them. On the third branch [1] describes the attachment in the sentence immediately preceding the display:

> *If $\frac{3d-1}{4} \leq k < d$, then v_1 is adjacent to $\lfloor d/2 \rfloor$ vertices in $V_1(X)$ and to $\lfloor d/2 \rfloor$ vertices in $V_2(X)$, while v_2 is adjacent to $k - \lfloor d/2 \rfloor$ vertices in $V_1(X)$ and $d - k + \lfloor d/2 \rfloor$ vertices in $V_2(X)$.*

We write $\Phi_{d,k}$ for the graph so described, and $\Phi^+_{d,k}$ for the graph obtained from it by joining v_1 to one further vertex of $V_2(X)$, so that v_1 meets $\lceil d/2 \rceil = d - \lfloor d/2 \rfloor$ vertices of $V_2(X)$. Note that $\Phi^+_{d,k}$ is $\Phi_{d,k}$ plus a single edge. Both graphs have $2d+4$ vertices.

Encoding of the explicit witnesses. The vertex sets we exhibit in L^2 are lists of indices, so we fix the indexing once. A graph on $\{0, \dots, n-1\}$ is presented as its edge list in lexicographic order of $(\min, \max)$. The vertices of $L(G)$ are the *positions* in that list, and the edge list of $L(G)$ is produced in colexicographic order: the pair $\{j, i\}$ with $j < i$ precedes $\{j', i'\}$ with $j' < i'$ when $i < i'$, or when $i = i'$ and $j < j'$. Iterating this once more indexes the vertices of $L^2(G)$. Every explicit cut below refers to this indexing, so each can be re-checked from the edge list of G alone.

3. The two displays disagree at exactly four cells

Throughout this section (d,k) ranges over the third branch of (3), that is

$$d \geq 3, \qquad \lfloor d/2 \rfloor < k < d, \qquad 3d \leq 4k+1, \tag{7}$$

the last inequality being $(3d-1)/4 \leq k$ in integers. Recall $A(d,k)$ from (4).

Proposition 3.1. *On the branch* (7),

$$f(d,k) - A(d,k) = \lfloor d/2 \rfloor \big(d - 2\lfloor d/2 \rfloor\big).$$

Consequently $A(d,k) = f(d,k)$ when d is even, while $A(d,k) = f(d,k) - \lfloor d/2 \rfloor$ with $\lfloor d/2 \rfloor \geq 1$ when d is odd.

Proof. On the branch (7) the third case of (3) applies, and $f - A = -2\lfloor d/2 \rfloor^2 + \lfloor d/2 \rfloor d = \lfloor d/2 \rfloor (d - 2\lfloor d/2 \rfloor)$. For even d this vanishes; for odd d, $d - 2\lfloor d/2 \rfloor = 1$ and $\lfloor d/2 \rfloor = (d-1)/2 \geq 1$ because $d \geq 3$. □

Write $T(d,k) = \min\{f(d,k), 4d-6\}$ for the value asserted by (2) and $D(d,k) = \min\{A(d,k), 4d-6\}$ for the bound obtained by combining (4) with the cap supplied by the Figure 9 construction. By Proposition 3.1 we always have $D \leq T$ on the branch, and the two can differ only for odd d.

Theorem 3.2. *For every (d,k) on the branch (7),*

$$D(d,k) < T(d,k) \iff (d,k) \in \{(3,2),\ (5,4),\ (7,5),\ (7,6)\}.$$

At those four cells the two values are:

d	k	$T(d,k)$	$D(d,k)$
3	2	4	3
5	4	12	10
7	5	22	19
7	6	22	21

Proof. The four rows of the table are direct evaluations of f, A and $4d-6$; note that at $(7,6)$ the conflict is with the cap, since $f(7,6) = 24 > 22 = 4 \cdot 7 - 6$ while $A(7,6) = 21$.

For the converse, let $D(d,k) < T(d,k)$. By Proposition 3.1, d must be odd. If $d < 9$, the branch conditions (7) leave only finitely many pairs, and checking the odd values $d \in \{3,5,7\}$ case by case leaves exactly the four listed cells. If $d \geq 9$ we show $4d-6 \leq A(d,k)$, whence $D = 4d-6 \leq T \leq 4d-6$ and no strict inequality is possible. Put $m = \lfloor d/2 \rfloor$, so $d = 2m+1$ and $m \geq 4$; the branch conditions read $k \leq 2m$ and $3m+1 \leq 2k$. Setting $u = 2k - 4m - 1$, these give $-m \leq u \leq -1$, hence $u^2 \leq m^2$. A direct expansion of (4) yields

$$4A(d,k) = 8m^2 + 4m + 1 - u^2 \ \geq\ 7m^2 + 4m + 1,$$

while $4(4d-6) = 32m - 8$. The inequality $7m^2 + 4m + 1 \geq 32m - 8$ is $7m^2 - 28m + 9 \geq 0$, which holds for $m \geq 4$. □

The whole of Theorem 3.2 is an arithmetic statement about integers and carries no finite search beyond the case distinction $d < 9$, which is a finite interval check; the $d \geq 9$ half is the closed argument just given.

4. What goes wrong, and the one-edge repair

Proposition 4.1. *In the graph $\Phi_{d,k}$ described by [1] on the third branch, the vertex v_1 has degree $2\lfloor d/2 \rfloor$. For odd d this equals $d-1$. Hence $\delta(\Phi_{d,k}) \leq d - 1 < d$ for odd d, and $\Phi_{d,k} \notin \mathcal{G}(d,k)$.*

Proof. The description gives v_1 exactly $\lfloor d/2 \rfloor$ neighbours in $V_1(X)$ and $\lfloor d/2 \rfloor$ in $V_2(X)$, and no others; so $d_{v_1} = 2\lfloor d/2 \rfloor$, which is $d-1$ when d is odd. Membership in $\mathcal{G}(d,k)$ requires $\delta = d$ by (1). (The vertex v_2 is unaffected: $(k - \lfloor d/2 \rfloor) + (d - k + \lfloor d/2 \rfloor) = d$.) □

Proposition 4.2. *Let (d,k) satisfy (7) and let X be the essential edge cut of $L(G)$ that [1] reads off the Figure 8 graph, so that $V_{12}(X) = \{v_1, v_2\}$ and (6) applies. Then*

$$\begin{aligned} &\textit{in } \Phi_{d,k} \textit{ as described:} && \lfloor d/2 \rfloor \cdot \lfloor d/2 \rfloor + (k - \lfloor d/2 \rfloor)(d - k + \lfloor d/2 \rfloor) = A(d,k), \\ &\textit{in the repair } \Phi^+_{d,k}: && \lfloor d/2 \rfloor \cdot (d - \lfloor d/2 \rfloor) + (k - \lfloor d/2 \rfloor)(d - k + \lfloor d/2 \rfloor) = f(d,k). \end{aligned}$$

Proof. Both are expansions of (6) with two mixed vertices, using $f(d,k) = kd - k^2 + 2k\lfloor d/2 \rfloor - 2\lfloor d/2 \rfloor^2$ on this branch. □

Propositions 4.1 and 4.2 together give the diagnosis. The value $A(d,k)$ is a genuine cut count, but of a graph that fails the minimum-degree requirement of $\mathcal{G}(d,k)$ for odd d; the single change that restores $d_{v_1} = d$ — reading the second $\lfloor d/2 \rfloor$ of the quoted sentence as $\lceil d/2 \rceil$ — also changes the cut count from $A(d,k)$ to $f(d,k)$. So the repaired construction carries a cut of exactly the size that the upper-bound half of Theorem 1.5 of [1] needs, where the display's reading gives one that is too small. Whether the repaired graph really lies in $\mathcal{G}(d,k)$, and what the connectivity of its L^2 really is, we check cell by cell below rather than in general.

Remark 4.3. The middle branch $\lfloor d/2 \rfloor < k < (3d-1)/4$ is unaffected: the sentence of [1] governing it reads $\lceil d/2 \rceil$ in each of the four places where the sentence quoted above reads $\lfloor d/2 \rfloor$, so there $d_{v_1} = 2\lceil d/2 \rceil \geq d$ and $d_{v_2} = d$, and (6) gives $\lceil d/2 \rceil^2 + (k - \lceil d/2 \rceil)(d - k + \lceil d/2 \rceil) = kd - k^2 + 2k\lceil d/2 \rceil - \lceil d/2 \rceil d$, which is both the display's value for that branch and $f(d,k)$ there. Like every remark here, it is outside the formal development.

The next three sections make the diagnosis concrete at each of the four cells of Theorem 3.2, and at the two smallest ones supply the witness the construction was meant to supply.

5. The cell $(3,2)$

Here $\Phi_{3,2}$ and $\Phi^+_{3,2}$ have 10 vertices: $\{0,1,2,3\}$ is $V_1(X) \cong K_4$, $\{4,5,6,7\}$ is $V_2(X) \cong K_4$, and $v_1 = 8$, $v_2 = 9$. Explicitly,

$$\begin{aligned} E(\Phi_{3,2}) = \{&01, 02, 03, \mathbf{08}, 12, 13, 19, 23,\\ &45, 46, 47, \mathbf{48}, 56, 57, 67, 69, 79\}, \\ E(\Phi^+_{3,2}) = E&(\Phi_{3,2}) \cup \{\mathbf{58}\}, \end{aligned}$$

the bold edges being those at $v_1 = 8$. Thus v_1 meets $\lfloor 3/2 \rfloor = 1$ vertex of $V_1(X)$ and 1 of $V_2(X)$ in $\Phi_{3,2}$, and $\lceil 3/2 \rceil = 2$ of $V_2(X)$ in $\Phi^+_{3,2}$; $v_2 = 9$ meets $k - \lfloor 3/2 \rfloor = 1$ vertex of $V_1(X)$ and $d - k + \lfloor 3/2 \rfloor = 2$ of $V_2(X)$ in both.

Theorem 5.1. *$\delta(\Phi_{3,2}) = 2$, so $\Phi_{3,2} \notin \mathcal{G}(3,2)$. Its second iterated line graph $L^2(\Phi_{3,2})$ has 43 vertices and $\kappa\bigl(L^2(\Phi_{3,2})\bigr) = 3$ exactly. Moreover $\Phi^+_{3,2}$ and $\Phi_{3,2}$ differ in exactly one edge.*

Proof. The degree of 8 in $\Phi_{3,2}$ is 2 and no vertex has smaller degree, by inspection of the degree sequence. The graph $\Phi_{3,2}$ has 17 edges, so $L(\Phi_{3,2})$ has 17 vertices, and it has 43 edges, so $L^2(\Phi_{3,2})$ has 43 vertices. The lower bound $\kappa \geq 3$ is exhaustive: all $\sum_{j \leq 2} \binom{43}{j} = 947$ vertex subsets of size at most 2 were deleted in turn and the remainder found connected each time. The upper bound is the explicit cut $\{21, 34, 38\}$ in the indexing of Section 2. The one-edge statement is the identity $E(\Phi^+_{3,2}) \setminus \{58\} = E(\Phi_{3,2})$. ☐

So the number 3 that the display prints at $(3,2)$ is a real connectivity — of a graph with $\delta = 2$, which lies outside $\mathcal{G}(3,2)$ and indeed outside the range $d \geq 3$ of Theorem 1.5 of [1] altogether.

Theorem 5.2. *$\Phi^+_{3,2} \in \mathcal{G}(3,2)$: its minimum degree is 3 and its edge connectivity is 2, the latter attained by the cut $\{08, 19\}$. Its second iterated line graph has 48 vertices and 211 edges, and*

$$\kappa\bigl(L^2(\Phi^+_{3,2})\bigr) = 4.$$

In particular $\kappa_{L^2}(3,2) \leq 4 = \min\{f(3,2), 4 \cdot 3 - 6\}$, the value of (2), and the cell $(3,2)$ carries an upper-bound witness after all.

Proof. The minimum degree and the exhibited 2-edge cut are immediate from the edge list. That $\kappa' \geq 2$ is exhaustive over all edge subsets of size at most 1. For the connectivity of L^2: $\Phi^+_{3,2}$ has 18 edges, $L(\Phi^+_{3,2})$ has 18 vertices and 48 edges, and $L^2(\Phi^+_{3,2})$ has 48 vertices and 211 edges. The bound $\kappa \geq 4$ rests on an exhaustive computation: all $\sum_{j \leq 3} \binom{48}{j} = 18\,473$ vertex subsets of size at most 3 were deleted in turn, and in every case the remainder is connected. The matching upper bound is the explicit 4-element cut $\{21, 30, 39, 43\}$. ☐

The witness of Theorem 5.2 is the repaired Figure 8 graph. A second witness, of a different shape and of the least possible order, is available.

Theorem 5.3. *Let M be the graph on $\{0,\dots,7\}$ with edge set*

$$E(M)=\{01,02,03,12,13,\ \mathbf{26},\mathbf{37},\ 45,46,47,56,57\},$$

that is, two copies of K_4-e joined by the two bold edges. Then $M\in\mathcal{G}(3,2)$, $L^2(M)$ has 24 vertices, and $\kappa\big(L^2(M)\big)=4$.

Proof. Every vertex of M has degree 3, and $\{26,37\}$ is a 2-edge cut, while no single edge disconnects M (exhaustive over the 12 edges); so $\delta(M)=3$ and $\kappa'(M)=2$. M has 12 edges, $L(M)$ has 12 vertices and 24 edges, and $L^2(M)$ has 24 vertices. The bound $\kappa\geq 4$ is exhaustive over all $\sum_{j\leq 3}\binom{24}{j}=2\,325$ vertex subsets of size at most 3, and $\{8,9,10,11\}$ is an explicit 4-cut. □

Remark 5.4. M has the least possible order in $\mathcal{G}(3,2)$. Indeed, let $G\in\mathcal{G}(3,2)$ and let F be a minimal 2-edge cut of G, with sides S and $V(G)\setminus S$. If $|S|=a$ and $G[S]$ has e edges, then summing degrees over S gives $3a\leq\sum_{v\in S}d_v=2e+2\leq a(a-1)+2$, which fails for $a\leq 3$. So both sides have at least 4 vertices and $|V(G)|\geq 8$. This argument is elementary and is not part of the formal development described in Section 10; the computations in Theorem 5.3 are.

Remark 5.5. Independently of the formal development, an exhaustive search over all members of $\mathcal{G}(3,2)$ of order at most 10 was carried out in a separate implementation. Enumerating up to isomorphism the possible sides of a minimal 2-edge cut on 4 and 5 vertices (3 and 20 respectively), gluing them in all ways and filtering to $\delta=3$, $\kappa'=2$ yields 212 graphs, which by Remark 5.4 is all of $\mathcal{G}(3,2)$ up to order 10. The minimum of $\kappa(L^2(G))$ over those 212 graphs is 4; the value 3 never occurs. We report this as computational evidence only.

Remark 5.6. The following argument, a specialization of the case analysis of [1, Section 3], gives the matching lower bound at $(3,2)$ from [1, Observation 2.2] alone. Suppose $\delta(G)\geq 3$ and $\kappa'(G)\geq 2$, and let X be a minimal essential edge cut of $L(G)$. By (6), $|X|=\sum_{v\in V_{12}(X)}a_v(d_v-a_v)$ with $1\leq a_v\leq d_v-1$ and $d_v\geq 3$, so every summand is at least 2; if $|V_{12}(X)|\geq 2$ then $|X|\geq 4$. If $V_{12}(X)=\{v\}$, then $G[V_{12}(X)]$ has no edge, so [1, Observation 2.2(v)] — if one of $V_1(X),V_2(X)$ is empty then $G[V_{12}(X)]$ has an edge — leaves both of them non-empty; every edge meeting the side $V_1(X)$ has colour 1 and hence ends in $V_1(X)\cup\{v\}$, so $E_G[V_1(X),V(G)\setminus V_1(X)]=E_1(v)$; then $|X|=a_v(d_v-a_v)\leq 3$ with $d_v\geq 3$ forces $\min(a_v,d_v-a_v)=1$, i.e. a bridge, contradicting $\kappa'(G)\geq 2$. Hence $\kappa(L^2(G))\geq 4$ by Proposition 2.1 and $\kappa_{L^2}(3,2)=4$. We record this because it explains why the value 4 is not in doubt, but it is a restatement of the source's own reasoning and is *not* part of our formal development.

6. The cell $(5,4)$

Here $\Phi_{5,4}$ and $\Phi^+_{5,4}$ have 14 vertices: $\{0,\dots,5\}$ and $\{6,\dots,11\}$ are the two copies of K_6, $v_1=12$ and $v_2=13$. In $\Phi_{5,4}$ the vertex v_1 meets $\lfloor 5/2\rfloor=2$ vertices of each side, namely $0,1$ and $6,7$, while v_2 meets $k-\lfloor 5/2\rfloor=2$ vertices of $V_1(X)$, namely $2,3$, and $d-k+\lfloor 5/2\rfloor=3$ of $V_2(X)$, namely $9,10,11$. The repair $\Phi^+_{5,4}$ adds the single edge $\{8,12\}$, so that v_1 meets $\lceil 5/2\rceil=3$ vertices of $V_2(X)$.

Theorem 6.1. (i) $\delta(\Phi_{5,4})=4\neq 5$*, so* $\Phi_{5,4}\notin\mathcal{G}(5,4)$*. Its second iterated line graph has* 181 *vertices and is cut by an explicit set of* 10 *of them, so* $\kappa(L^2(\Phi_{5,4}))\leq 10$*: the display's number at this cell is again a real connectivity of a graph outside the class.*

(ii) $\Phi^+_{5,4}$ *differs from* $\Phi_{5,4}$ *in exactly one edge and* $\Phi^+_{5,4}\in\mathcal{G}(5,4)$*: its minimum degree is* 5 *and its edge connectivity is* 4*, attained by the cut* $\{0v_1,\ 1v_1,\ 2v_2,\ 3v_2\}$.

(iii) $L^2(\Phi^+_{5,4})$ *has* 190 *vertices and* 1621 *edges, and it is cut by the* 12*-element vertex set*

$$\{92,\ 93,\ 113,\ 114,\ 136,\ 137,\ 158,\ 159,\ 173,\ 174,\ 181,\ 182\}.$$

Hence

$$\kappa_{L^2}(5,4) \;\le\; \kappa\big(L^2(\Phi^+_{5,4})\big) \;\le\; 12 = \min\{f(5,4),\, 4\cdot 5-6\},$$

the value of (2)*, and not the value* 10 *of* (4)*.*

Proof. The minimum degrees and the one-edge identity are inspections of the two edge lists. The sizes of L^2 follow from the edge counts: $\Phi_{5,4}$ has 39 edges, so $L(\Phi_{5,4})$ has 39 vertices and 181 edges; $\Phi^+_{5,4}$ has 40 edges, so $L(\Phi^+_{5,4})$ has 40 vertices and 190 edges. The two vertex cuts are explicit and are checked by deleting them and observing the remainder is disconnected. The only exhaustive search here is $\kappa'(\Phi^+_{5,4}) \ge 4$: all $\sum_{j\le 3}\binom{40}{j} = 10\,701$ edge subsets of size at most 3 were deleted in turn and the remainder found connected each time; together with the exhibited 4-edge cut this gives $\kappa' = 4$ exactly. ☐

Remark 6.2. A separate max-flow computation (Even's method: $\delta+1$ anchors, vertex-split unit-capacity flows) gives $\kappa(L^2(\Phi^+_{5,4})) = 12$ exactly, so the upper bound of Theorem 6.1(iii) is attained. That lower bound is *not* part of the formal development; see Section 9 for why, and for the route we consider feasible.

Remark 6.3. Read in the class it actually belongs to, the graph $\Phi_{5,4}$ contradicts nothing. Its minimum degree is 4 and (by a separate computation, not part of the formal development) its edge connectivity is 4, so $\Phi_{5,4} \in \mathcal{G}(4,4)$, where (2) gives $\min\{f(4,4),10\} = \min\{8,10\} = 8$, comfortably below the value 10 of Theorem 6.1(i).

7. The cells $(7,5)$ and $(7,6)$

At $d = 7$ the graphs $\Phi_{7,k}$ and $\Phi^+_{7,k}$ have 18 vertices — two copies of K_8 on $\{0,\dots,7\}$ and $\{8,\dots,15\}$, with $v_1 = 16$ and $v_2 = 17$ — and their second iterated line graphs have several hundred vertices. We certify the defect itself, which is what the four-cell statement of Theorem 3.2 requires, and not the connectivity values.

Theorem 7.1. *For $k \in \{5,6\}$:*

$$\delta(\Phi_{7,k}) = 6 < 7 = \delta(\Phi^+_{7,k}),$$

so $\Phi_{7,k} \notin \mathcal{G}(7,k)$ while $\Phi^+_{7,k}$ meets the minimum-degree requirement of the class. In both cases the repair is the single edge $\{11,16\}$ at v_1. Moreover each $\Phi^+_{7,k}$ carries an edge cut of size k separating $V_1(X)$ from the rest — namely $\{0v_1, 1v_1, 2v_1, 3v_2, 4v_2\}$ for $k=5$ and that set together with $5v_2$ for $k=6$ — so $\kappa'(\Phi^+_{7,k}) \le k$.

Proof. All four minimum degrees, the two one-edge identities and the two edge cuts are direct evaluations on the explicit edge lists; no search over subsets is involved. ☐

Remark 7.2. A separate computation gives $\kappa'(\Phi^+_{7,k}) = k$ for $k \in \{5,6\}$, so $\Phi^+_{7,k} \in \mathcal{G}(7,k)$; this is not part of the formal development. Neither is the value of $\kappa(L^2(\Phi^+_{7,k}))$, which we did not compute. Note that at $(7,6)$ the conflict of Theorem 3.2 is against the cap $4d-6 = 22$ rather than against $f(7,6) = 24$.

8. Controls, and the values of [1] that reproduce

Before any correction was claimed, the constructions of [1] on the branches not at issue were rebuilt and their connectivities computed. They agree with [1] in every case we checked.

Proposition 8.1. *The following values are exact.*

(i) *For $J(3,1,4,4)$ of* [1, Example 1.4] *— two copies of K_4 and a new vertex joined to one vertex of the first and $d-k=2$ of the second — one has $\delta = 3$, a bridge, $|V(L^2)| = 36$ and $\kappa(L^2) = 2 = d-1$. This is the tight case of the Knor–Niepel bound* [2]*, quoted as* [1, Theorem 1.3]*.*

(ii) *For $J(4,2,5,5)$ of* [1, Example 1.4] *— two copies of K_5 and a new vertex joined to 2 vertices of each — one has $\delta = 4$, $\kappa' = 2$, $|V(L^2)| = 82$ and $\kappa(L^2) = 4 = k(d-k) = f(4,2)$, the first branch of (3).*

(iii) *For the graph of* [1, Figure 9] *at $d = 3$ — a K_4 on $\{0,1,2,3\}$ and a path $v_1v_2v_3$ with v_1, v_3 joined to $d-1 = 2$ vertices of the K_4 and v_2 to $d-2 = 1$ — one has $\delta = 3$, $|V(L^2)| = 40$ and $\kappa(L^2) = 6 = 4d-6 = f(3,3)$, the cap.*

Proof. Each connectivity is an exhaustive lower bound together with an explicit cut: for (i) all $\sum_{j\le 1}\binom{36}{j} = 37$ subsets and the 2-cut $\{18,27\}$; for (ii) all $\sum_{j\le 3}\binom{82}{j} = 91\,964$ subsets and the 4-cut $\{45,46,60,61\}$; for (iii) all $\sum_{j\le 5}\binom{40}{j} = 760\,099$ subsets and the 6-cut $\{33,34,35,36,37,38\}$. The minimum degrees, the bridge in (i) and the 2-edge cut in (ii) are direct evaluations. □

Both recipes are transcribed from v1: Example 1.4 builds $J(d,k,n_1,n_2)$ from $K_{n_1} \cup K_{n_2}$, with $\min\{n_1,n_2\} \ge d+1 \ge 4$ and $d > k > 0$, by adding a vertex v_0 joined to k vertices of the first and $d-k$ of the second; the Figure 9 paragraph takes $G[V_1(X)]$ complete of order $d+1$, makes $G[\{v_1,v_2,v_3\}]$ a path, joins v_1, v_3 to $d-1$ vertices of $V_1(X)$ and v_2 to $d-2$, and reads $|X| \le (d-1) + 2(d-2) + (d-1) = 4d-6$ off (6).

Remark 8.2. Five further values of [1] were reproduced by a separate implementation and are recorded here as computational evidence only, outside the formal development:

$$\kappa(L^2(J(4,1,5,5))) = 3,\quad \kappa(L^2(J(5,1,6,6))) = 4,\quad \kappa(L^2(J(5,2,6,6))) = 6,\quad \kappa(L^2(J(6,3,7,7))) = 9,$$

the first two being $d-1$ and the last two $f(5,2)$ and $f(6,3)$, together with $\kappa(L^2) = 10 = 4d-6$ for the Figure 9 graph at $d = 4$.

Proposition 8.3. *The following negative controls hold.*

(i) *$\kappa(L^2(\Phi^+_{3,2})) \ge 5$ is false, and $\kappa(L^2(M)) \ge 5$ is false; the value 4 in Theorems 5.2 and 5.3 is not an artefact of a one-sided test.*

(ii) *No vertex set of size 3 cuts $L^2(\Phi^+_{3,2})$: all $\binom{48}{3} = 17\,296$ of them were tested. So the value 3 of (4) is not attained at the repaired graph either.*

(iii) *The hypothesis $\delta = 3$ separates the two graphs of Section 5: $\Phi^+_{3,2}$ meets it, $\Phi_{3,2}$ fails it, and the two differ in one edge, so Theorem 5.1 diagnoses the described construction and not some unrelated graph. Likewise $L^2(\Phi^+_{5,4})$ is 3-connected, so the 12-set of Theorem 6.1(iii) is not a cut for a degenerate reason.*

(iv) *The computational machinery reproduces hand-known values: $L(K_3) = K_3$, $L(K_{1,3}) = K_3$, $L(P_4) = P_3$; K_4 has minimum degree 3 and is 3-connected; the middle vertex of a path is a cut vertex; a bridge is a 1-edge cut.*

Proof. (i) and (ii) are exhaustive searches of the sizes stated, run to completion; (iii) and (iv) are direct evaluations. □

9. What remains open

The lower bound at $(5,4)$. Theorem 6.1 gives $\kappa_{L^2}(5,4) \le 12$, and Remark 6.2 reports that $\kappa(L^2(\Phi^+_{5,4})) = 12$ exactly. Certifying the lower half by the definition used everywhere else in this note is out of reach: it would require deleting all $\sum_{j\le 11}\binom{190}{j} \approx 2.31\times 10^{17}$ vertex subsets, and no partition of that search makes it feasible. The route we consider practical is not more computation but a formalization of the easy direction of Menger's theorem for vertex connectivity: twelve internally disjoint u–w paths together with a separator of size at most 11 is a pigeonhole contradiction, and a separator of size at most 11 must miss one of any twelve fixed vertices, which reduces the problem to about $12 \times 174 \approx 2\,100$ pairs. The path systems can be produced by an augmenting-path search inside the evaluator and then *checked* rather than trusted. The same gap stands at $(7,5)$ and $(7,6)$, where L^2 is larger still.

Question 9.1. Is $\kappa\big(L^2(\Phi^+_{5,4})\big) \geq 12$, and more generally $\kappa\big(L^2(\Phi^+_{d,k})\big) \geq f(d,k)$ on the third branch, provable by a certificate small enough to be checked mechanically?

Beyond order 10 at $(3,2)$. The exhaustive control of Remark 5.5 covers $\mathcal{G}(3,2)$ up to order 10. Extending it to order 12 requires enumerating the 6-vertex side gadgets up to isomorphism, which was the bottleneck in our implementation.

Higher iterates. The concluding remark of [1] asks how $\varphi(d,k,n) = \inf\{\kappa(L^n(G)) : G \in \mathcal{G}(d,k)\}$ grows with n. Already L^3 of the 14-vertex graph $\Phi^+_{5,4}$ has thousands of vertices, so $n = 3$ calls for the essential-edge-cut reformulation of Proposition 2.1 rather than a direct sweep.

The lower-bound half of Theorem 1.5. We have not audited the proof that $\kappa_{L^2}(d,k) \geq \min\{f(d,k), 4d-6\}$, and nothing in this note bears on it. We record one citation we could not follow, without claiming a defect. In Case 2 of [1, Section 3], for the graphs that paper calls $\mathcal{G}'_2$, the text reads "*By Lemma 3.2(ii),* $\sum_{i=1}^{m} a_{i,1,X} = k$", where $a_{i,1,X}$ is by Definition 2.5 the size of the *smaller* colour class at v_i; but Lemma 3.2(ii) states $\kappa'(G) = k$. The identity is Lemma 3.2(iii), which assumes $d_G(v_{j_0}) > d$ for some $v_{j_0} \in V_{12}(X)$, whereas here Lemma 3.2(v) gives $d_G(v_i) = d$ for all i. What does follow — by the route [1] names a few lines earlier for $\mathcal{G}'_1$, Proposition 2.6(i) with Lemma 3.2(i),(ii) — is $\sum_i a_{i,1,X} \geq k$; whether that suffices we did not determine. We flag the step only so that it is not overlooked.

10. Verification

Every theorem and proposition of this note — with the sole exception of Proposition 2.1, which is quoted from the literature — has been formally verified in Lean 4, and the development contains no `sorry`; *Mathlib* is used only for the arithmetic of Section 3, the graph modules being self-contained. The remarks are another matter: none of them is formalized, and each says where its content comes from. Graphs are represented as edge lists over $\{0,\dots,n-1\}$ in the encoding of Section 2; the line-graph operator, breadth-first connectivity, the predicate "deleting every vertex subset of size less than c leaves the graph connected" and its edge analogue are defined directly and computed, so a value such as $\kappa(L^2(\Phi^+_{3,2})) = 4$ is settled by evaluation and needs no formalized theory of line graphs or of Menger's theorem. The arithmetic of Section 3 — Proposition 3.1, the four-cell table, and the equivalence of Theorem 3.2, whose $d \geq 9$ half is proved from the two inequalities $(2k-4m-1)^2 \leq m^2$ and $7m^2 - 28m + 9 \geq 0$ and whose $d < 9$ half is a finite interval check — depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, as do Propositions 4.1 and 4.2; several of the $(3,2)$ facts, including the minimum degrees, the one-edge identity and the size of $L^2(\Phi^+_{3,2})$, depend on no axioms at all. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the large finite searches named in the proofs above: the 947 and 18 473 vertex-subset sweeps of Section 5, the 2 325 sweep of Theorem 5.3, the 17 296 sweep of Proposition 8.3(ii), the 10 701 edge-subset sweep of Theorem 6.1, the 37, 91 964 and 760 099 sweeps of Proposition 8.1, and the evaluation of the larger explicit cuts. In Propositions 4.1 and 4.2 what is formalized is the arithmetic: $2\lfloor d/2\rfloor = d-1$ for odd d, and the two identities for the sums $\lfloor d/2\rfloor\cdot\lfloor d/2\rfloor + (k-\lfloor d/2\rfloor)(d-k+\lfloor d/2\rfloor)$ and $\lfloor d/2\rfloor(d-\lfloor d/2\rfloor)+(k-\lfloor d/2\rfloor)(d-k+\lfloor d/2\rfloor)$. Identifying those quantities with the degree of v_1 and with the cut count (6) is a reading of [1], not a formal step. The substantive claims deliberately kept *outside* the formal development are the exhaustive search over $\mathcal{G}(3,2)$ up to order 10 (Remark 5.5); the max-flow value $\kappa(L^2(\Phi^+_{5,4})) = 12$ (Remark 6.2), the edge connectivity of $\Phi_{5,4}$ (Remark 6.3) and those of the two repaired graphs at $d = 7$ (Remark 7.2); the five reproduced values of Remark 8.2; and the two arguments carried out by hand, Remarks 5.4 and 5.6. Every value certified in Lean was first produced by an independent implementation in Python and the formal statement was then written against it; one early discrepancy was found this way, when a first version of the search of Remark 5.5 reported a minimum of 3 because it computed $\kappa(L)$ rather than $\kappa(L^2)$, and was caught by re-deriving the offending value by brute force.

References

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