ISOSPECTRAL SPHERICAL SPACE FORMS WITH NON-ISOMORPHIC FUNDAMENTAL GROUPS IN DIMENSIONS 7 AND 11: AN EXHAUSTIVE SEARCH OVER THE GROUPS OF TYPE I

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Abstract. Colantonio and Lauret (arXiv:2602.12939) produced the first pair of isospectral spherical space forms with non-isomorphic fundamental groups, on S^{15}, and asked for the smallest dimension in which such a pair exists; by results of Ikeda the only open dimensions below 15 are 7 and 11, and their own reductions leave, among fundamental groups of Type I, exactly three cells: $\Gamma_4(m,n,r)$ acting irreducibly on S^7, $\Gamma_6(m,n,r)$ acting irreducibly on S^{11}, and $\Gamma_3(m,n,r)$ acting on S^{11} through a sum of two irreducible representations. We search all three cells exhaustively with Ikeda's generating function evaluated in a prime field, which certifies non-isospectrality: there is no isospectral pair with non-isomorphic Type I fundamental groups of order at most 100 000 in dimension 7 (33 682 groups), none of order at most 200 000 in dimension 11 with $d = 6$, and none of order at most 10 000 in dimension 11 with $d = 3$ and two summands (777 groups, every pair of summands) — a cell the source's algorithm, which searches irreducible space forms only, never covered at any order. As controls, the 55 pairs of the source's Table 1 are recomputed by two independent implementations, its S^{15} pair is shown almost conjugate by an exact comparison of eigenvalue multisets, and its Proposition 5.2 and Remark 4.5 are re-derived over every parameter tuple of the cell each concerns ($d = 2$, resp. $d = 4$ with $n = 8$) of order at most 100 000. All statements are finite facts about the groups $\Gamma_d(m,n,r)$, their representations and Ikeda's rational function, verified in Lean 4.

1. Introduction

A *spherical space form* is a quotient S^q/G of the round sphere by a finite subgroup $G \subset O(q+1)$ acting freely; its fundamental group is G. Two Riemannian manifolds are *isospectral* when their Laplace–Beltrami operators have the same spectrum, with multiplicities. Ikeda's examples of isospectral, non-isometric spherical space forms [4] (lens spaces, and quotients by non-cyclic groups) all have isomorphic fundamental groups, and so do the later ones of Gilkey [7] and Wolf [6]: every construction fixes a group and varies its representation. Colantonio and Lauret [1] gave the first pair with *non-isomorphic* fundamental groups; their smallest example (their Remark 4.5, equation (4.5)) is

$$S^{15}/\rho_{1,1}\big(\Gamma_8(85,16,2)\big) \qquad \text{and} \qquad S^{15}/\rho_{1,1}\big(\Gamma_8(85,16,42)\big), \qquad |\Gamma| = 1360, \tag{1}$$

in the notation recalled in Section 2. They then ask ([1], Question 5.1, quoted verbatim):

> *What is the smallest possible dimension for a pair of isospectral spherical space forms with non-isomorphic fundamental groups?*

The table they print under the question records the state of knowledge by dimension $2d-1$: for $d = 2$, no (isospectral 3-dimensional spherical space forms are isometric, [2, Thm. I]); for $d = 3$, no ([3, Thm. 3.9]); for $d = 5$ and $d = 7$, no ([3, Thm. 3.1]); for $d = 8$, yes, by (1); and for $d = 4$ and $d = 6$ — dimensions 7 and 11 — *Unknown.* Ikeda's Theorem 3.1 reads, verbatim: "Let d be an odd prime, and let M, N be $(2d-1)$-dimensional spherical space forms. Suppose M is isospectral to N. Then their fundamental groups are isomorphic." The hypothesis is on d, and 4 and 6 are not prime; dimensions 3 and 5 were settled by hand. A second published closure of this shape, [6, Cor. 7.3] — for $q = 2p-1$ with p prime and G non-cyclic, isospectrality, strong isospectrality and $G \simeq G'$ are equivalent — carries the same primality hypothesis, and it obtains the fundamental-group half by citing Ikeda's Theorem 3.1. This is why 7 and 11 are the open cells.

The source narrows the two open dimensions further. Fundamental groups of spherical space forms fall into Wolf's six types [5, Ch. 7]; Type I is the class of finite groups all of whose Sylow subgroups are cyclic, presented by Burnside as $\Gamma_d(m,n,r)$ (Section 2). Verbatim from [1], after Question 5.1:

> *Note that in dimensions 7 and 11 there are spherical space forms with fundamental groups of other types (see [Wo, §7.5]). One may concentrate in Type I groups, though it is not known either whether there are isospectral spherical space forms of dimension 7 or 11 with non-isomorphic fundamental groups of Type I. In dimension 7 (resp. 11), the possible Type I groups have $d = 2, 4$ (resp. $d = 2, 3, 6$). The following result, combined with Theorem 1.7, excludes the case $d = 2$.*

The "following result" is their Proposition 5.2 (two groups $\Gamma_2(m,n,r_1)$, $\Gamma_2(m,n,r_2)$ are isomorphic), and their Theorem 1.7 says that an isospectral partner of a Type I space form has a Type I fundamental group with the same (d,m,n). Since every irreducible fixed point free representation of $\Gamma_d(m,n,r)$ has degree $2d$ [5, Thm. 5.5.10], a space form with this fundamental group and t irreducible summands has dimension $2dt-1$, and the Type I content of Question 5.1 in dimensions 7 and 11 is exactly three cells:

$$(d,t) = (4,1) \text{ on } S^7; \qquad (d,t) = (6,1) \text{ and } (d,t) = (3,2) \text{ on } S^{11}.$$

What is proved here. We search the three cells exhaustively up to an order bound, using Ikeda's generating function $F_G(z) = \frac{1-z^2}{|G|}\sum_{g\in G}\det(I-gz)^{-1}$, whose equality characterises isospectrality [2], evaluated in a prime field $\mathbb{F}_p$ at a root of unity of exact order $|G|$: because the evaluation is a ring homomorphism, two different values in $\mathbb{F}_p$ are a proof that the two generating functions differ, hence that the two space forms are not isospectral (Lemma 3.1). The results are:

- **Dimension** 7 (Theorem 4.1). No two non-isomorphic groups $\Gamma_4(m,n,r)$ of the same order $mn \le 100\,000$ have isospectral irreducible space forms; the search covers 33 682 groups.
- **Dimension** 11**,** $d = 6$ (Theorem 4.2). The same for $\Gamma_6(m,n,r)$ with $mn \le 200\,000$.
- **Dimension** 11**,** $d = 3$**, two summands** (Theorem 4.3). No two non-isomorphic groups $\Gamma_3(m,n,r)$ of the same order $mn \le 10\,000$ (777 groups) have isospectral space forms $S^{11}/(\rho\oplus\rho')(\Gamma)$, for *any* choice of the two irreducible summands on either side.

Combined with the source's Theorem 1.7 and Proposition 5.2 these give Corollary 4.4: an isospectral pair of spherical space forms of dimension 7 (resp. 11) with non-isomorphic fundamental groups, one of which is of Type I, has both groups of Type I with $d = 4$ and order above 100 000 (resp. $d = 6$ and order above 200 000, or $d = 3$ and order above 10 000).

Two caveats belong in the statement of what is new. First, the source's own search (its Algorithm 4.7, run for all d and $|\Gamma| \le 30\,000$, its Table 1) is restricted verbatim to irreducible space forms $S^{2d-1}/\rho_{1,1}(\Gamma_d(m,n,r))$; it therefore *mechanically* covers the cells (4, 1) and (6, 1) up to order 30 000 — Table 1 has no row with $d = 4$ or $d = 6$ — although the paper never states this as a result (its table says "Unknown"). The new content of Theorems 4.1 and 4.2 is the range of orders above 30 000; the range below is a reproduction. The cell (3, 2) carries no such caveat: an irreducible space form of Γ_3 is 5-dimensional, so Algorithm 4.7 never touched dimension 11 through $d = 3$, and Theorem 4.3 is the first bound of any size on that cell. Second, these are bounded negatives, read as "not found", not as theorems closing a dimension: fundamental groups of Wolf's Types II–VI in dimensions 7 and 11 are untouched here as in [1], and the Type I cells remain open above the bounds.

Controls. The same programs that return the empty list in dimensions 7 and 11 reproduce the source's Table 1 — all 55 pairs of order at most 30 000, in the source's order, no extras and none missing (Proposition 5.1); a second, independent implementation in C had returned the same 55. Below order 1300 the list is empty (the source's example (1) is the smallest), the test separates the other classes of order 1360, and an isomorphic copy of a group is never counted twice. The pair (1) is shown *almost conjugate* by an exact comparison of the multisets of eigenvalues of all 1360 elements — no finite field, no evaluation point — which is the hypothesis of [6, Cor. 2.13] and so re-proves the source's Theorem 4.4 on this pair by direct computation (Proposition 5.2). The source's Proposition 5.2 and Remark 4.5, on which the reduction to three cells rests, are re-derived over every parameter tuple of the cell each concerns — $d = 2$, respectively $d = 4$ with $n = 8$ — of order at most 100 000 (Proposition 5.3), and the two reductions the searches use — Ikeda's theorem that the choice of irreducible representation does not change the spectrum, and the class reduction for the second summand — are checked computationally (Proposition 5.4).

What is not claimed, and the search record. Nothing here is a statement about a manifold, a metric or a Laplacian: every theorem is a finite fact about the groups $\Gamma_d(m,n,r)$, their representations $\rho_{k,l}$ and the rational function F_G, and the geometry enters only through the cited equivalence of Ikeda (Section 2). The literature was searched on 2026-09-07: in a local text corpus of 890 739 arXiv e-prints, the 66 records containing both "isospectral" and "spherical space form" were read, and the arXiv API returned no record for "metacyclic" with "isospectral", for "exhaustive search" with "isospectral", or for "isospectral" with "dimension 7". No search over fixed point free groups with a stated order bound is on record apart from the source's Table 1; the bounded searches in print are over lens spaces, where the fundamental group is cyclic and isomorphism is automatic. The source is a preprint: its arXiv page (v2 of 2026-08-13) carries no journal reference and no DOI beyond the arXiv one, zbMATH and OpenAlex index it as a preprint, and OpenAlex records no citing work. Its co-author's later preprint [8] lists it as "Doc. Math., in press since August 2026"; that status could not be confirmed independently on 2026-09-07 — Crossref holds no record of it under either ISSN of Documenta Mathematica, and the journal publishes no list of accepted papers — so it is reported here as the co-author's claim, and nothing below depends on it. That one citing paper, [8], answers a different question (isospectral but not strongly isospectral, with isomorphic fundamental groups) and contains no search of this kind.

2. Preliminaries

Statement numbers attributed to [1] are those of its compiled e-print (v2; see Section 6). Throughout, $\xi_c = e^{2\pi i/c}$.

2.1. **Type I groups.** A finite group is of *Type I* if all its Sylow subgroups are cyclic. By Burnside ([5, Thm. 5.4.1]; [1, Lemma 2.1]), for positive integers m, n, r with

$$\gcd\big((r-1)n,\, m\big) = 1 \qquad \text{and} \qquad r^n \equiv 1 \pmod{m}, \tag{2}$$

the group $\Gamma = \langle A, B \mid A^m = B^n = 1,\ BAB^{-1} = A^r\rangle$ has order mn and is of Type I, and every Type I group is isomorphic to one of these. Write $\Gamma_d(m,n,r)$ for it, where d is the order of r in $(\mathbb{Z}/m)^\times$; $d \mid n$, every element is uniquely A^aB^b with $0 \le a < m$, $0 \le b < n$, m is odd, and $\Gamma_d(m,n,r)$ is cyclic iff $m = 1$ iff $d = 1$ ([1, Rem. 2.3]). The group is *fixed point free* — it admits a real representation acting freely on a sphere — iff every prime divisor of d divides n/d ([1, Lemma 2.2]); in particular $\operatorname{rad}(n) = \operatorname{rad}(n/d)$. Finally ([1, Prop. 2.5] and the sentence preceding its Theorem 1.7),

(3)

$$\Gamma_{d_1}(m_1,n_1,r_1) \simeq \Gamma_{d_2}(m_2,n_2,r_2) \iff m_1 = m_2,\ n_1 = n_2,\ d_1 = d_2 \text{ and } r_1 \equiv r_2^c \pmod{m} \text{ for some } c,$$

the last condition being $\langle r_1\rangle = \langle r_2\rangle$ in $(\mathbb{Z}/m)^\times$.

Definition 2.1 (admissible tuples)**.** A quadruple $\tau = (m,n,d,r)$ of positive integers is *admissible* if m is odd, $d \ge 2$, $2 \le r < m$, $\gcd(r,m) = \gcd(r-1,m) = \gcd(n,m) = 1$, r has order exactly d in $(\mathbb{Z}/m)^\times$, every prime divisor of d divides n/d, and r is the least element of $\{\, r^c \bmod m : 1 \le c < d,\ \gcd(c,d) = 1 \,\}$. Write $\Gamma(\tau) = \Gamma_d(m,n,r)$, $|\tau| = mn$, and for $N \ge 1$ let $\mathcal{T}_d(N)$ be the set of admissible tuples with third entry d and $|\tau| \le N$, and $\mathcal{T}_d^{=N}$ those with $|\tau| = N$.

These are exactly the conditions of step (2) of [1, Alg. 4.7]: (2), non-cyclic, fixed point free, and r minimal in its class. By (3), every non-cyclic fixed point free Type I group is isomorphic to $\Gamma(\tau)$ for exactly one admissible τ, and distinct admissible tuples give non-isomorphic groups.

2.2. **Fixed point free representations.** By [5, Thm. 5.5.10] ([1, Thm. 4.1]), for $\Gamma = \Gamma_d(m,n,r)$ fixed point free of order at least 3 and integers k, l with $\gcd(k,m) = 1 = \gcd(l,n)$ there is an irreducible fixed point free real representation $\rho_{k,l}$ of degree $2d$, in which A acts by the block-diagonal rotation by the angles $2\pi k r^j/m$, $j = 0, \dots, d-1$, and B by a cyclic shift of the blocks composed with the rotation by $2\pi l/(n/d)$; $\rho_{k,l} \simeq \rho_{k',l'}$ iff

$$k' \equiv \epsilon k r^c \pmod m \quad \text{and} \quad l' \equiv \epsilon l \pmod{n/d} \qquad \text{for some } \epsilon \in \{\pm 1\},\ c \in \{0, \dots, d-1\}; \tag{4}$$

and a real representation of Γ is fixed point free iff it is equivalent to a sum of representations $\rho_{k,l}$. Hence a spherical space form with fundamental group $\Gamma_d(m,n,r)$ and t irreducible summands has dimension $2dt-1$. For a list $\kappa = ((k_1,l_1), \dots, (k_t,l_t))$ write $\rho_\kappa = \rho_{k_1,l_1} \oplus \cdots \oplus \rho_{k_t,l_t}$ and $G(\tau,\kappa) = \rho_\kappa(\Gamma(\tau)) \subset O(2dt)$.

Ikeda's theorem quoted as [1, Thm. 1.2] (from [4]) says that for a non-cyclic fixed point free Type I group Γ and two irreducible real representations ρ_1, ρ_2 acting freely on S^q, $\mathrm{Spec}(S^q/\rho_1(\Gamma)) = \mathrm{Spec}(S^q/\rho_2(\Gamma))$. For Type I groups it is recovered below as Lemma 3.4(a).

2.3. **Ikeda's generating function.** For a finite $G \subset O(q+1)$ acting freely on S^q let $F_G(z) = \sum_{k\ge 0} \dim H^G_{q,k}\, z^k$, where $H_{q,k}$ is the space of harmonic homogeneous polynomials of degree k in $q+1$ variables. Ikeda proved, for odd-dimensional spherical space forms S^q/G_1, S^q/G_2 ($q = 2n-1$), that

$$\mathrm{Spec}(S^q/G_1) = \mathrm{Spec}(S^q/G_2) \iff F_{G_1} = F_{G_2}, \qquad F_G(z) = \frac{1-z^2}{|G|} \sum_{g\in G} \frac{1}{\det(I-gz)}. \tag{5}$$

The left half is his Proposition 2.1 and the right his Theorem 2.2, the latter printed as $F_G(z) = \frac{1}{|G|}\sum_g \frac{1-z^2}{\det(1_{2n}-gz)}$ [2]; [1] quotes both for every q, as its (4.6) and (4.7). Every dimension used below is odd. In particular $F_G \in \mathbb{Q}(z)$, and $\det(I-gz) = \prod_{\lambda\in \mathrm{Eig}(g)}(1-\lambda z)$ depends only on the multiset $\mathrm{Eig}(g)$ of eigenvalues of g.

For $\Gamma = \Gamma_d(m,n,r)$ the eigenvalues are explicit. Following [1] (proof of its Theorem 4.4, equation (4.3), quoting [4, (3.9)]), for $g = A^aB^b$ put $s = \gcd(d,b)$ (with $\gcd(d,0) = d$) and

$$\alpha(b) = \sum_{h=1}^{d/s} r^{sh} \pmod m. \tag{6}$$

The representation $\rho_{k,l}$ is $\pi_{k,l} \oplus \overline{\pi_{k,l}}$ for a complex representation $\pi_{k,l}$ of degree d, and the eigenvalues of $\pi_{k,l}(A^aB^b)$ are the roots of

$$\prod_{j=0}^{s-1}\Big(z^{d/s} - \omega_j\Big), \qquad \omega_j = \xi_m^{k\,a\,\alpha(b)\,r^j}\ \xi_{n/d}^{\,l\,b/s}. \tag{7}$$

Consequently

$$\det\big(I - \rho_{k,l}(A^aB^b)\,z\big) = \prod_{j=0}^{s-1}\big(1-\omega_j z^{d/s}\big)\big(1-\omega_j^{-1} z^{d/s}\big), \tag{8}$$

and for ρ_κ the determinant is the product of (8) over the summands. (The source states (7) for $k = l = 1$; the general case is the same computation with A acting through $\xi_m^{k(\cdot)}$ and the shift through $\xi_{n/d}^l$, and we use it in this form.) Since $\gcd(m,n) = 1$, the number $M = d\cdot \mathrm{lcm}(m, n/d)$ divides $N = mn$, so every eigenvalue is an N-th root of unity and every $\det(I-gz)$ lies in $\mathbb{Z}[\xi_N][z]$.

2.4. **The reduction to three cells.** Two further results of [1] are used as cited inputs and not reproved here.

- **Theorem 1.7 of** [1]**.** Let M_1, M_2 be isospectral spherical space forms with fundamental groups G_1, G_2. If G_1 is of Type I, $G_1 \simeq \Gamma_d(m,n,r_1)$, then G_2 is of Type I with $G_2 \simeq \Gamma_d(m,n,r_2)$, and $\gcd(r_1^c - 1, m) = \gcd(r_2^c-1,m)$ for every positive divisor c of d.
- **Proposition 5.2 of** [1]**.** Two Type I groups of the form $\Gamma_2(m,n,r_i)$, $i = 1,2$, are isomorphic. (Its proof: $m \mid r_i^2 - 1$ and $\gcd(m, r_i - 1) = 1$ force $r_i \equiv -1 \pmod m$.)

Let S^q/G_1, S^q/G_2 be isospectral with $G_1 \not\simeq G_2$ and G_1 of Type I. By Theorem 1.7 both groups are $\Gamma_d(m,n,r_i)$ with the same (d,m,n); $d=1$ is impossible (both would be cyclic of the same order) and $d=2$ is impossible by Proposition 5.2. With $q=7$, $2dt=8$ leaves $(d,t)=(4,1)$; with $q=11$, $2dt=12$ leaves $(6,1)$ and $(3,2)$. These are the three cells searched below.

3. The finite test

3.1. **Specialisation to a prime field.** Fix $N \geq 2$, a prime $p \equiv 1 \pmod N$, an element $\omega \in \mathbb{F}_p^\times$ of exact order N, and $z \in \mathbb{F}_p$. Since $p \nmid N$, the roots of $x^N - 1$ in $\mathbb{F}_p$ are simple and those of exact order N are the roots of the cyclotomic polynomial Φ_N, so $\xi_N \mapsto \omega$ defines a ring homomorphism $\mathbb{Z}[\xi_N] \to \mathbb{F}_p$, which we extend to $\phi_z \colon \mathbb{Z}[\xi_N][z] \to \mathbb{F}_p$ by evaluating at z. For an admissible τ of order N and a list κ, let $D_g = \det(I - \rho_\kappa(g)z) \in \mathbb{Z}[\xi_N][z]$ for $g \in \Gamma(\tau)$ (computed by (8)), and, provided $\phi_z(D_g) \neq 0$ for every g, put

$$v_{\tau,\kappa}(z) = \sum_{g \in \Gamma(\tau)} \phi_z(D_g)^{-1} \in \mathbb{F}_p. \tag{9}$$

If some $\phi_z(D_g)$ vanishes, $v_{\tau,\kappa}(z)$ is undefined (and the programs below report failure rather than a value).

Lemma 3.1 (soundness of the specialisation)**.** *Let τ, τ' be admissible of the same order N and κ, κ' lists of the same length t. If $F_{G(\tau,\kappa)} = F_{G(\tau',\kappa')}$, then $v_{\tau,\kappa}(z) = v_{\tau',\kappa'}(z)$ for every $z \in \mathbb{F}_p$ at which both are defined. Equivalently: if $v_{\tau,\kappa}(z) \neq v_{\tau',\kappa'}(z)$ for one such z, then $S^{2dt-1}/G(\tau,\kappa)$ and $S^{2dt-1}/G(\tau',\kappa')$ are not isospectral.*

Proof. Both groups have order N, so equality of the generating functions is, by (5), the identity $\sum_g D_g^{-1} = \sum_{g'} D_{g'}^{-1}$ in $\mathbb{Q}(\xi_N)(z)$. Clearing denominators, with $Q = \prod_g D_g$, $P = \sum_g \prod_{h \neq g} D_h$ and likewise P', Q', this is the polynomial identity $PQ' = P'Q$ in $\mathbb{Z}[\xi_N][z]$. Applying ϕ_z gives $\phi_z(P)\phi_z(Q') = \phi_z(P')\phi_z(Q)$ in $\mathbb{F}_p$, where $\phi_z(Q) = \prod_g \phi_z(D_g) \neq 0$ and $\phi_z(Q') \neq 0$ by hypothesis, and $\phi_z(P)/\phi_z(Q) = \sum_g \phi_z(D_g)^{-1} = v_{\tau,\kappa}(z)$, similarly for the primed side. The second sentence is the contrapositive combined with (5). ☐

The programs carry v as an unreduced fraction $(\mathrm{num}, \mathrm{den})$ and compare two values by cross-multiplication, which is equality in $\mathbb{F}_p$ because both denominators are nonzero. This finite-field replacement of $\mathbb{C}[[z]]$ is also what the source proposes in its Remark 4.8; the point of Lemma 3.1 is that it is a certificate in the negative direction, which is the only direction the theorems below use.

Definition 3.2 (the modular data)**.** For $N \geq 2$: p_N is the least prime $p \equiv 1 \pmod N$ with $p > 2^{30}$ (searched among $p = kN+1$ for at most 20 000 consecutive values of k); $\omega_N = g^{(p_N-1)/N}$ for the least $g \in \{2, \ldots, 399\}$ such that this element has exact order N, verified as $\omega_N \neq 1$ and $\omega_N^{N/q} \neq 1$ for every prime $q \mid N$, the primes of N being found by trial division by all $q < 2000$ (complete for $N < 4 \cdot 10^6$); and the two evaluation points are

$$z_1(N) = (7919\,N + 13) \bmod p_N, \qquad z_2(N) = (104729\,N + 7) \bmod p_N.$$

Primality of the candidates p is decided by the Miller–Rabin test with bases $2, 3, 5, 7$.

The Miller–Rabin test with these four bases is a correct primality test below 3 215 031 751 [9]. That every modulus used stays below this bound is not part of the formal development: over all $N \leq 200\,000$ (a superset of every order any theorem below touches) the largest p_N is 1 104 935 233, at $N = 185\,641$ — a measurement of the founding session, re-run independently for this paper in Python with the same search rule and a primality test valid far beyond 2^{31}, with the same result. Every product of two residues therefore stays below 2^{62}, inside the machine word range of the implementation. The exact order of ω_N and the non-vanishing of every denominator, by contrast, are checked by the programs themselves, inside the kernel- verified evaluation.

3.2. **The sweeps.**

Definition 3.3. Let $d \geq 2$ and $1 \leq N_0 \leq N_1$.

(a) The *irreducible test at d is negative on* $[N_0, N_1]$ if for every $N \in [N_0, N_1]$ with $|\mathcal{T}_d^{=N}| \geq 2$ the data of Definition 3.2 exist, $v_{\tau,((1,1))}(z_i(N))$ is defined for every $\tau \in \mathcal{T}_d^{=N}$ and $i = 1, 2$, and for every two distinct $\tau, \tau' \in \mathcal{T}_d^{=N}$ there is $i \in \{1, 2\}$ with $v_{\tau,((1,1))}(z_i(N)) \neq v_{\tau',((1,1))}(z_i(N))$.

(b) For admissible $\tau = (m, n, d, r)$ let $K(\tau)$ be the set of pairs (k, l) with $1 \leq k < m$, $\gcd(k, m) = 1$, $1 \leq l < n/d$, $\gcd(l, n) = 1$, that are the lexicographically least element of their class under (4) (the class of (k, l) being $\{(\epsilon k r^c \bmod m,\ \epsilon l \bmod n/d)\}$). The *two-summand test at d is negative on* $[N_0, N_1]$ if for every $N \in [N_0, N_1]$ with $|\mathcal{T}_d^{=N}| \geq 2$ the modular data exist, $v_{\tau,((1,1),(k,l))}(z_i(N))$ is defined for every $\tau \in \mathcal{T}_d^{=N}$, $(k, l) \in K(\tau)$ and $i = 1, 2$, and for every two distinct $\tau, \tau' \in \mathcal{T}_d^{=N}$, every $(k, l) \in K(\tau)$ and $(k', l') \in K(\tau')$ there is i with $v_{\tau,((1,1),(k,l))}(z_i(N)) \neq v_{\tau',((1,1),(k',l'))}(z_i(N))$.

These are precisely the statements `collisionsIrr d N₀ N₁ = some #[]` and `collisionsSum2 d N₀ N₁ = some #[]` of the formal development (Appendix A): the programs enumerate $\mathcal{T}_d(N_1)$ by scanning every odd m and every $r < m$ and applying the conditions of Definition 2.1 literally, group the tuples by order, build the modular data of each order that carries at least two tuples, evaluate (9) at both points, and return the list of pairs whose values agree at both points — or `none` if any modulus or evaluation failed, in which case the run proves nothing. An empty list is the negative of Definition 3.3. Note that the tests compare *all* tuples of a given order, including those with different (m, n); they do not use Theorem 1.7 of [1] to discard such pairs.

3.3. **Why $\rho_{1,1}$ and class representatives suffice.**

Lemma 3.4 (scaling invariance). *Let $\tau = (m, n, d, r)$ be admissible, $\kappa = ((k_i, l_i))_{i \leq t}$, and let σ, u be integers with $\gcd(\sigma, m) = 1 = \gcd(u, n)$. Then $F_{G(\tau,\kappa)} = F_{G(\tau,\kappa')}$ for $\kappa' = ((\sigma k_i,\ u l_i))_{i \leq t}$. Consequently:*

(a) *$F_{\rho_{k,l}(\Gamma)} = F_{\rho_{1,1}(\Gamma)}$ for every admissible (k, l), so all irreducible fixed point free space forms of $\Gamma = \Gamma(\tau)$ are isospectral (Ikeda's theorem,* [1, Thm. 1.2]*, for Type I groups);*

(b) *for $t = 2$, $F_{G(\tau,((k_1,l_1),(k_2,l_2)))} = F_{G(\tau,((1,1),(k,l)))}$ for some $(k, l) \in K(\tau)$.*

Proof. Write $g = A^a B^b$ with $a \in \mathbb{Z}/m$, $b \in \mathbb{Z}/n$; by (8) the factor of $\det(I - \rho_\kappa(g)z)$ coming from the i-th summand is determined by $s = \gcd(d, b)$ and the numbers $\omega_{ij} = \xi_m^{k_i a \alpha(b) r^j} \xi_{n/d}^{l_i b/s}$, which depend on b only modulo n (since $s \mid d$ and n/d divides n/s). The substitution $a \mapsto \sigma a$ is a bijection of $\mathbb{Z}/m$ and turns k_i into σk_i; the substitution $b \mapsto ub$ is a bijection of $\mathbb{Z}/n$ that fixes s (as $\gcd(u, d) = 1$) and $\alpha(b)$ (which depends only on s), and turns l_i into $u l_i$. Hence the sum $\sum_g \det(I - \rho_\kappa(g)z)^{-1}$ is unchanged when κ is replaced by κ', and so is F by (5). For (a) take $\sigma \equiv k^{-1} \pmod{m}$, $u \equiv l^{-1} \pmod{n}$. For (b) take $\sigma \equiv k_1^{-1}$, $u \equiv l_1^{-1}$; the second summand becomes $(k, l) = (k_1^{-1} k_2 \bmod m,\ l_1^{-1} l_2 \bmod n)$. Reducing l modulo n/d to the range $[1, n/d)$ does not change $\rho_{k,l}$ and keeps $\gcd(l, n) = 1$, because l is a unit modulo n/d and $\mathrm{rad}(n) = \mathrm{rad}(n/d)$; finally the class representative of (k, l) under (4) gives an equivalent representation, hence the same eigenvalue multisets and the same F. □

Lemma 3.4 is a paper proof, not part of the formal development; what is formally verified is the computational control of Proposition 5.4: on two test groups, $v_{\tau,((k,l))}(z_1) = v_{\tau,((1,1))}(z_1)$ for every admissible (k, l), and $v_{\tau,((1,1),(k,l))}(z_1)$ equals its value at the class representative for every (k, l).

3.4. **Exact almost conjugacy.** Let τ, τ' be admissible with the same (m, n, d), $L = \mathrm{lcm}(m, n/d)$ and $M = Ld$. By (7), the $2d$ eigenvalues of $\rho_{1,1}(A^a B^b)$ are the d/s-th roots of ω_j and of ω_j^{-1}, $j < s$; writing $\omega_j = \xi_L^{f_j}$ with $f_j = \frac{L}{m}(a\alpha(b)r^j \bmod m) + \frac{L}{n/d}\frac{b}{s} \pmod{L}$, they are $\xi_M^{\pm(s f_j + \ell L s)}$, $0 \leq \ell < d/s$. Let $E(\tau; a, b)$ be the sorted list of these $2d$ exponents modulo M. We say the pair (τ, τ') is *exactly almost conjugate* if $E(\tau; a, b) = E(\tau'; a, b)$ for all $0 \leq a < m$, $0 \leq b < n$: then $A_1^a B_1^b \mapsto A_2^a B_2^b$ is a bijection $\rho_{1,1}(\Gamma(\tau)) \to \rho_{1,1}(\Gamma(\tau'))$ preserving eigenvalue multisets, which is the hypothesis of [1, Prop. 4.3] ([6, Cor. 2.13]): the two space forms are strongly isospectral. This test involves no finite field and no evaluation point.

4. Main results

Theorem 4.1 (dimension 7)**.**

(i) *The irreducible test at $d = 4$ is negative on $[1, 40\,000]$, on $[40\,001, 70\,000]$ and on $[70\,001, 100\,000]$, hence on $[1, 100\,000]$.*

(ii) *$|\mathcal{T}_4(100\,000)| = 33\,682$; the tuples $(65, 8, 4, 8)$ and $(65, 8, 4, 18)$ are admissible, of order 520; and every τ with $d = 4$ has $2 \cdot 4 \cdot 1 - 1 = 7$.*

(iii) *Consequently, for any two distinct $\tau, \tau' \in \mathcal{T}_4(100\,000)$ of the same order, $F_{\rho_{1,1}(\Gamma(\tau))} \neq F_{\rho_{1,1}(\Gamma(\tau'))}$; and no two 7-dimensional spherical space forms $S^7/\rho(\Gamma)$, $S^7/\rho'(\Gamma')$, with $\Gamma \not\simeq \Gamma'$ non-cyclic Type I groups with $d = 4$ of the same order at most $100\,000$ and ρ, ρ' irreducible fixed point free representations, are isospectral.*

Proof. (i) and (ii) are the formally verified statements `dim7_no_isospectral_pair_le_100000`, `dim7_nonvacuous` and `dims` of Appendix A; each range of (i) is one compiled evaluation of the program `collisionsIrr 4` (three evaluations partitioning $[1, 100\,000]$, of a few minutes each, Section 6). For (iii): distinct admissible tuples give non-isomorphic groups by (3), and every non-cyclic Type I group with $d = 4$ of order $\leq 100\,000$ is $\Gamma(\tau)$ for a unique $\tau \in \mathcal{T}_4(100\,000)$, the space form being the same subgroup of $O(8)$ whichever presentation is used. By [1, Thm. 4.1] an irreducible fixed point free ρ is equivalent to some $\rho_{k,l}$, so $F_{\rho(\Gamma)} = F_{\rho_{k,l}(\Gamma)} = F_{\rho_{1,1}(\Gamma)}$ by Lemma 3.4(a) (equivalent representations have the same eigenvalue multisets). If the two space forms were isospectral, (5) would give $F_{\rho_{1,1}(\Gamma(\tau))} = F_{\rho_{1,1}(\Gamma(\tau'))}$ and Lemma 3.1 would force equal values at both $z_1(N)$ and $z_2(N)$, contradicting (i). $\square$

Theorem 4.2 (dimension 11, $d = 6$)**.**

(i) *The irreducible test at $d = 6$ is negative on $[1, 100\,000]$ and on $[100\,001, 200\,000]$, hence on $[1, 200\,000]$.*

(ii) *$|\mathcal{T}_6(100\,000)| = 6\,067$; the tuple $(7, 36, 6, 3)$ is admissible, of order 252; and every τ with $d = 6$ has $2 \cdot 6 \cdot 1 - 1 = 11$.*

(iii) *Consequently, for any two distinct $\tau, \tau' \in \mathcal{T}_6(200\,000)$ of the same order, $F_{\rho_{1,1}(\Gamma(\tau))} \neq F_{\rho_{1,1}(\Gamma(\tau'))}$, and no two 11-dimensional spherical space forms $S^{11}/\rho(\Gamma)$, $S^{11}/\rho'(\Gamma')$ with $\Gamma \not\simeq \Gamma'$ non-cyclic Type I groups with $d = 6$ of the same order at most $200\,000$ and ρ, ρ' irreducible, are isospectral.*

Proof. As for Theorem 4.1, from `dim11_d6_no_isospectral_pair_le_200000`, `dim11_d6_nonvacuous` and `dims` (two compiled evaluations partitioning $[1, 200\,000]$). The count in (ii) is the one recorded in the formal statement, to $100\,000$; the cell is sparse because a fixed point free $\Gamma_6(m, n, r)$ needs $36 \mid n$ and $\gcd(m, 6) = 1$. $\square$

Theorem 4.3 (dimension 11, $d = 3$, two irreducible summands)**.**

(i) *The two-summand test at $d = 3$ is negative on $[1, 6\,000]$ and on $[6\,001, 10\,000]$, hence on $[1, 10\,000]$.*

(ii) *$|\mathcal{T}_3(10\,000)| = 777$; the tuples $(91, 9, 3, 9)$ and $(91, 9, 3, 16)$ are admissible, of order 819; and every τ with $d = 3$ has $2 \cdot 3 \cdot 2 - 1 = 11$.*

(iii) *Consequently no two 11-dimensional spherical space forms $S^{11}/(\rho_1 \oplus \rho_2)(\Gamma)$ and $S^{11}/(\rho'_1 \oplus \rho'_2)(\Gamma')$, with $\Gamma \not\simeq \Gamma'$ non-cyclic Type I groups with $d = 3$ of the same order at most $10\,000$ and ρ_i, ρ'_i irreducible fixed point free representations, are isospectral.*

Proof. (i), (ii) are `dim11_d3_sum2_no_isospectral_pair_le_10000`, `dim11_d3_nonvacuous` and `dims`. For (iii): as before both groups are $\Gamma(\tau)$, $\Gamma(\tau')$ for distinct tuples of the same order, and by [1, Thm. 4.1] the representations are equivalent to $\rho_{k_1,l_1} \oplus \rho_{k_2,l_2}$, resp. $\rho_{k'_1,l'_1} \oplus \rho_{k'_2,l'_2}$. Lemma 3.4(b) replaces these by $\rho_{1,1} \oplus \rho_{k,l}$ with $(k, l) \in K(\tau)$ and $\rho_{1,1} \oplus \rho_{k',l'}$ with $(k', l') \in K(\tau')$ without changing F. Isospectrality would give equal F, hence by Lemma 3.1 equal values at both points, contradicting (i). $\square$

Corollary 4.4. *Let S^q/G_1 and S^q/G_2 be isospectral spherical space forms with $G_1 \not\simeq G_2$ and G_1 of Type I.*

(a) *If $q = 7$, then $G_1 \simeq \Gamma_4(m, n, r_1)$ and $G_2 \simeq \Gamma_4(m, n, r_2)$ with $mn > 100\,000$.*

(b) *If $q = 11$, then $G_i \simeq \Gamma_d(m, n, r_i)$ with either $d = 6$ and $mn > 200\,000$, or $d = 3$ and $mn > 10\,000$.*

Proof. By Theorem 1.7 of [1] both groups are of Type I with the same (d, m, n); $d = 1$ would make both cyclic of the same order and $d = 2$ is excluded by its Proposition 5.2. The degree count $2dt = q + 1$ leaves $d = 4$, $t = 1$ for $q = 7$ and $(d, t) \in \{(6, 1), (3, 2)\}$ for $q = 11$, and Theorems 4.1(iii), 4.2(iii), 4.3(iii) exclude orders up to the stated bounds. □

Remark 4.5 (what is new relative to the source). The source's Algorithm 4.7 restricts itself, verbatim, to "spherical space forms of the form $S^{2d-1}/\rho_{1,1}(\Gamma_d(m, n, r))$" and was run for all d and $N \leq 30\,000$; its Table 1 contains only rows with $d = 8$ or $d = 16$. So the source's own computation covers Theorems 4.1 and 4.2 for orders up to 30 000, without saying so; the new range is $(30\,000, 100\,000]$ at $d = 4$ and $(30\,000, 200\,000]$ at $d = 6$. Theorem 4.3 has no counterpart in the source at any order. In all three cells the smallest order that carries two non-isomorphic groups with the same (m, n) is far below the bound: 520 at $d = 4$ (the pair in Theorem 4.1(ii); two classes of r need m to have two distinct prime factors $\equiv 1 \pmod 4$, the least m being 65), 1 260 at $d = 6$ ($m = 35$, $n = 36$), and 819 at $d = 3$ (two classes need two primes $\equiv 1 \pmod 3$, the least m being 91, and $n = 9$ is forced). At $d = 4$ the enumeration also contains, well below 520, pairs of groups with *different* (m, n) and equal order — the first is order 120, with $\Gamma_4(5, 24, 2)$ and $\Gamma_4(15, 8, 2)$ — and the tests compare those as well, rather than appealing to Theorem 1.7 of [1] to discard them; at $d = 3$ and at $d = 6$ the orders just named are already the smallest that carry two groups of any shape.

Remark 4.6 (outside the formal development: the structure of the $d = 4$ cell). For $d = 4$ write $u = \gcd(r + 1, m)$ and $v = \gcd(r^2 + 1, m)$. Since $m \mid r^4 - 1 = (r - 1)(r + 1)(r^2 + 1)$ with $\gcd(m, r - 1) = 1$, m odd and $\gcd(r + 1, r^2 + 1) \mid 2$, one has $m = uv$ with $\gcd(u, v) = 1$, $r \equiv -1 \pmod u$ and $r^2 \equiv -1 \pmod v$. Splitting the sum in (5) by $s = \gcd(4, b)$: for b odd, $\alpha(b) \equiv 0 \pmod m$ and the term is independent of r and of a; for $b \equiv 2 \pmod 4$, $\alpha(b) \equiv r^2 + 1$ and the term depends on r only through u, which Theorem 1.7 of [1] (with $c = 2$) already forces to agree for an isospectral pair; for $b \equiv 0 \pmod 4$, $\alpha(b) \equiv 1$ and the term depends on the orbit $a\langle r\rangle \subset \mathbb{Z}/m$. Whether two different subgroups $\langle r_1\rangle \neq \langle r_2\rangle$ of order 4 in $(\mathbb{Z}/m)^\times$ can give the same sum over $b \equiv 0 \pmod 4$ is therefore the whole of the dimension-7 question for Type I groups. This reduction is stated as an observation and is not used by any theorem above.

5. Reproductions and controls

Proposition 5.1 (the source's Table 1, and three controls)**.**

- (i) *Over all d, the pairs of distinct admissible tuples of equal order at most* 30 000 *whose values $v_{\tau,((1,1))}$ agree at both evaluation points are exactly the* 55 *pairs of Table 1, sorted by $(|\tau|, m, n, d, r_1, r_2)$; the modular data and all evaluations exist. These are the* 55 *rows of* [1, Table 1]*, in its order.*
- (ii) *Over all d and orders at most* 1300 *the list is empty.*
- (iii) *Among the admissible tuples with $d = 8$ of order* 1360 *exactly one pair agrees at both points, namely $\big((85, 16, 8, 2), (85, 16, 8, 42)\big)$.*
- (iv) 43 *is not the least element of its class modulo* 85 *under powers coprime to* 8 *($43 \equiv 2^7 \pmod{85}$), so* $(85, 16, 8, 43)$ *is not admissible; and* $|\mathcal{T}_8(1400)| = 12$.

Proof. These are `table1_reproduced`, `gate_too_small`, `f_test_separates` and `isomorphic_copies_excluded`. (i) is one compiled evaluation over all 38 458 admissible tuples of order at most 30 000 (about six minutes); the right-hand side is the flat list of 275 integers, five per row. The data of Table 1 is the source's; what is certified is that an independent implementation returns it. A second implementation, a 230-line C program with a different modulus size (p near 2^{40}), batch inversion instead of unreduced fractions, and a different summation order, returned the same 55 rows (38 458 groups, 6 219 orders carrying at least two groups, 299 s); that run is outside the formal development. Note that (i) certifies the *list*: the modular test cannot show that the 55 pairs are isospectral, which is the source's Theorem 4.4. For (iii), the classes of order 1360 with $d = 8$ are $(17, 80, 8, 2)$ and $(85, 16, 8, r)$ for $r \in \{2, 9, 42\}$; of the six pairs only the source's is not separated. For (iv), $2^7 = 128 \equiv 43 \pmod{85}$ and $\gcd(7, 8) = 1$, so $\Gamma_8(85, 16, 43) \simeq \Gamma_8(85, 16, 2)$ by (3). □

Table 1. The 55 pairs of Proposition 5.1(i): order $N = mn$, then (m, n, d) and the two classes r_1, r_2. Forty-four rows have $d = 8$ and eleven $d = 16$.

N	m	n	d	r_1, r_2	N	m	n	d	r_1, r_2	N	m	n	d	r_1, r_2
1360	85	16	8	$2, 42$	15440	965	16	8	$43, 588$	22304	697	32	16	$14, 44$
2720	85	32	16	$3, 12$	15520	485	32	16	$8, 202$	22304	697	32	16	$73, 173$
3280	205	16	8	$3, 68$	15776	493	32	16	$12, 41$	22480	1405	16	8	$192, 473$
3536	221	16	8	$8, 138$	16400	1025	16	8	$68, 232$	23120	1445	16	8	$423, 468$
5840	365	16	8	$22, 168$	16592	1037	16	8	$111, 172$	23504	1469	16	8	$18, 44$
6800	425	16	8	$32, 168$	17680	1105	16	8	$8, 138$	24208	1513	16	8	$144, 212$
7072	221	32	16	$5, 44$	17680	1105	16	8	$83, 213$	24208	1513	16	8	$166, 319$
7120	445	16	8	$12, 37$	18080	565	32	16	$42, 48$	24272	1517	16	8	$68, 191$
7760	485	16	8	$33, 227$	18512	1157	16	8	$190, 749$	25040	1565	16	8	$188, 308$
7888	493	16	8	$70, 104$	18640	1165	16	8	$12, 97$	26384	1649	16	8	$47, 64$
8528	533	16	8	$44, 96$	19024	1189	16	8	$191, 249$	26384	1649	16	8	$172, 366$
9040	565	16	8	$18, 357$	19280	1205	16	8	$8, 693$	26960	1685	16	8	$252, 1122$
10064	629	16	8	$43, 117$	19856	1241	16	8	$100, 246$	27472	1717	16	8	$111, 818$
10960	685	16	8	$127, 452$	19856	1241	16	8	$302, 387$	28240	1765	16	8	$237, 943$
11152	697	16	8	$9, 196$	20128	629	32	16	$6, 80$	28496	1781	16	8	$96, 421$
11152	697	16	8	$38, 55$	20176	1261	16	8	$47, 629$	28832	901	32	16	$23, 182$
13600	425	32	16	$7, 82$	20560	1285	16	8	$193, 253$	29200	1825	16	8	$168, 793$
14416	901	16	8	$76, 348$	22304	697	32	16	$3, 232$	29648	1853	16	8	$76, 185$
15184	949	16	8	$83, 229$										

Proposition 5.2 (the S^{15} pair, exactly)**.**

(i) *The pair* $\big((85, 16, 8, 2), (85, 16, 8, 42)\big)$ *is exactly almost conjugate: with* $M = 1360$*, for every* $0 \le a < 85$ *and* $0 \le b < 16$ *the sorted lists of* 16 *exponents* $E(\tau; a, b)$ *and* $E(\tau'; a, b)$ *coincide. Hence the two space forms* (1) *are strongly isospectral (*[1, Prop. 4.3]*).*

(ii) *Both tuples are admissible, of order* 1360*, and* $2 \cdot 8 \cdot 1 - 1 = 15$*.*

(iii) *The tuple* $(85, 16, 8, 9)$ *is admissible, and it is exactly almost conjugate neither to* $(85, 16, 8, 2)$ *nor to* $(85, 16, 8, 42)$*.*

Proof. `s15_pair_almost_conjugate`, `s15_pair_nonvacuous`, `s15_negative_controls`: compiled evaluations comparing 1360 pairs of sorted 16-element lists of residues modulo 1360. The source proves the isospectrality of (1) by its algebraic Theorem 4.4; (i) re-proves it for this pair by exhausting the hypothesis of its Proposition 4.3 directly. (iii) shows that the exact test does not identify everything in sight, and (i) together with Proposition 5.1(iii) shows that the exact and the modular test agree on the four classes of order 1360. □

Proposition 5.3 (the source's Proposition 5.2 and Remark 4.5, re-derived)**.**

(i) *Every admissible tuple* $(m, n, 2, r)$ *with* $mn \le 100\,000$ *has* $r = m - 1$*.*

(ii) *For every admissible tuple* $(m, 8, 4, r_1)$ *with* $8m \le 100\,000$ *and every* $1 \le r_2 < m$ *with* $r_1 r_2 \equiv -1 \pmod m$*,* $\gcd(r_2, m) = \gcd(r_2 - 1, m) = 1$ *and* r_2 *of order* 4 *modulo* m*, one has* $r_2 = r_1$*.*

Proof. The formal statements are `prop52_d2_forced` and `remark45_d4_forced`, compiled evaluations over $\mathcal{T}_2(100\,000)$ and $\mathcal{T}_4(100\,000)$. (i) is the content of [1, Prop. 5.2]: for $d = 2$ the class of r consists of r alone, so admissibility forces $r = m - 1$, i.e. $r \equiv -1$, and any two groups $\Gamma_2(m, n, r_i)$ coincide. (ii) is the source's Remark 4.5: under the hypotheses of its Theorem 4.4 ($n = 2d$, $r_1 r_2 \equiv -1$) the case $d = 4$ gives $r_1 \equiv r_2$, so that construction cannot produce a dimension-7 pair. Both are published, general arguments; only the machine check over the stated range is new, and it is recorded because Proposition 5.2 is what removes the $d = 2$ cells in Corollary 4.4. □

Proposition 5.4 (the two reductions, checked)**.** *Let* $\tau_1 = (85, 16, 8, 2)$ *and* $\tau_2 = (133, 9, 3, 11)$*, with modular data as in Definition 3.2 and* $z_1 = z_1(|\tau_i|)$*.*

(i) *For* $\tau \in \{\tau_1, \tau_2\}$ *and every* (k,l) *with* $1 \le k < m$, $\gcd(k,m) = 1$, $1 \le l < n/d$, $\gcd(l,n) = 1$*:* $v_{\tau,((k,l))}(z_1) = v_{\tau,((1,1))}(z_1)$.

(ii) *For* $\tau = \tau_2$ *and every such* (k,l)*, with* (k^*, l^*) *its class representative:* $v_{\tau,((1,1),(k,l))}(z_1) = v_{\tau,((1,1),(k^*,l^*))}(z_1)$.

Proof. `rep_invariance_control` and `kl_reduction_control`. (i) is the computational shadow of Lemma 3.4(a), that is of Ikeda's theorem [1, Thm. 1.2] on the two test groups (for τ_1, $n/d = 2$ and only $l = 1$ occurs; for τ_2, $l \in \{1,2\}$ and k runs over the 108 units modulo 133); (ii) is the control for the restriction of the two-summand sweep to $K(\tau)$. Neither replaces the lemma; both would fail if the formula (8) had been implemented wrongly. □

6. Verification

Theorems 4.1(i)–(ii), 4.2(i)–(ii), 4.3(i)–(ii) and Propositions 5.1–5.4 have been formally verified in Lean 4 [10]; their statements are reproduced verbatim in Appendix A. The development lives in a project built against Mathlib [11], but none of its six modules imports Mathlib: the computational core (`Core.lean`, 526 lines, 31 definitions and two structures — modular exponentiation, the Miller–Rabin test, trial division, the modulus of Definition 3.2, the tuples and the predicate of Definition 2.1, the enumeration, the value (9), the two sweeps, the class representative, the exact exponent lists $E(\tau; a, b)$, and the two checks of Proposition 5.3) imports nothing at all, and the five theorem modules (`Table1`, 95 lines, one theorem; `Gate`, 128 lines, eleven theorems; `Dim7`, 74 lines, five; `Dim11`, 47 lines, four; `Dim11Sum2`, 67 lines, four) import only the core and the project's bookkeeping module. Every computational statement is discharged by compiled evaluation (`native_decide`; 21 uses in all); the one non-computational statement, the dimension bookkeeping `dims`, is proved by `simp`. Axiom prints (`#print axioms`, re-run for this paper over every pinned declaration and every chunk theorem, against the compiled modules): `dims` depends on `propext` alone; every other theorem depends on `propext`, `Quot.sound` and the single `native_decide` axiom of its own evaluation; the three combined statements `dim7_no_isospectral_pair_le_100000`, `dim11_d6_no_isospectral_pair_le_200000` and `dim11_d3_sum2_no_isospectral_pair_le_10000` depend on exactly the `native_decide` axioms of their chunks (three, two and two). There is no `sorry`, no `Classical.choice`, and no axiom declared by hand.

The core module is compiled to a shared library and loaded into the elaborator (`--load-dynlib`), so that `native_decide` runs compiled code rather than Lean's interpreter; on this sweep the difference is a factor 21 (80.8 s against 3.8 s for the $d = 4$ test to order 8 000), and the compiled Lean runs level with the C prototype (26.5 s against 24.7 s to order 20 000). As recorded, one Lean process at a time under `nice`, the final checks took: `Table1` 6 min 20 s; `Gate` 1 min 55 s; `Dim7` 14 min 40 s; `Dim11` 10 min 51 s; `Dim11Sum2` 4 min 26 s; the core about 2 s plus 6 s to build the library. The whole founding session, including the C reproduction (299 s for Table 1, 893 s for the $d = 4$ cell), the scaling anchors and two discarded runs, is recorded at 126 processor-minutes (2.1 processor-hours), with peak resident memory under 1 GB. The cost of the irreducible sweep scales as about $B^{2.14}$ in the order bound B (26.5 s at 20 000, 116.6 s at 40 000) and the two-summand sweep as about $B^{3.8}$; pushing $d = 4$ to 400 000 is estimated at 5.4 processor-hours and $d = 3$, $t = 2$ to 30 000 at 7.4, and extending Table 1 itself to order 60 000 at about 22 minutes. These figures are measurements, outside the formal development. Two implementation notes are recorded for reproducibility: a `native_decide` whose evaluation runs for minutes must have the elaborator's heartbeat limit lifted, since the default budget is consumed by the evaluation together with the decidable equality on a 275-element literal, after the sweep has already run; and the trial-division bound of Definition 3.2 is load-bearing — with the bound 400 used in a first version, the cofactor left after trial division is guaranteed prime only for $N < 160\,000$, and the certificate that ω_N has exact order N would have been vacuous precisely on the range $(160\,000, 200\,000]$ of the $d = 6$ sweep; this was caught before any statement was recorded, and the bound is now 2000. The development is currently in a private repository: repository reference to be added at submission.

On the reference list. Statement, equation and table numbers attributed to [1] are those of its compiled e-print (`tectonic` on the e-print source of v2, one file `hearTypeI.tex`; the numbers read off `hearTypeI.aux`):

Theorem 1.2 (p. 2), Question 1.4 (p. 2), Theorems 1.5 and 1.7 (p. 3), Lemmas 2.1 and 2.2 (p. 4), Remark 2.3 and Proposition 2.5 (p. 5), Theorem 4.1, Remark 4.2 and Proposition 4.3 (p. 11), Theorem 4.4 with (4.2) and (4.3) (pp. 12–13), Remark 4.5 and (4.5)–(4.7) (p. 14), Table 1 and Algorithm 4.7 (p. 15), Remark 4.8, Question 5.1 and Proposition 5.2 (p. 16). The arXiv listing showed versions v1 (2026-02-13) and v2 (2026-08-13) on 2026-09-07. The statements attributed to [2], [3] and [6] were read in those papers themselves (the Project Euclid scans of the two Ikeda papers, and the author's own copy of [6]); the numbers attributed to [4] are as cited by [1].

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). A term `⟨m, n, d, r⟩` is the tuple (m, n, d, r); `admissible` is Definition 2.1; `TypeIParam.card` is mn and `TypeIParam.dim g t` is $2dt - 1$; `enumGroups d N` lists $\mathcal{T}_d(N)$ (all d when the first argument is `0`); `collisionsIrr` and `collisionsSum2` are the sweeps of Definition 3.3, returning `some` of the list of agreeing pairs or `none` on any failure; `collisionTable` and `collisionFlat` are the same list over all d, sorted by (mn, m, n, d, r_1, r_2), as 5-tuples (m, n, d, r_1, r_2) and as a flat list of integers; `almostConjugate` is exact almost conjugacy; `canonicalR r m d` says r is least in its class; `propDTwo N` and `remarkDFour N` are Proposition 5.3(i), (ii) over $\mathcal{T}_2(N)$, $\mathcal{T}_4(N)$; `repInvariance` and `klReductionOK` are Proposition 5.4(i), (ii).

Module `Table1` (Proposition 5.1(i)).

```
theorem table1_reproduced : collisionFlat 0 1 30000 = some [
    85, 16, 8, 2, 42,
    85, 32, 16, 3, 12,
    205, 16, 8, 3, 68,
    221, 16, 8, 8, 138,
    365, 16, 8, 22, 168,
    425, 16, 8, 32, 168,
    221, 32, 16, 5, 44,
    445, 16, 8, 12, 37,
    485, 16, 8, 33, 227,
    493, 16, 8, 70, 104,
    533, 16, 8, 44, 96,
    565, 16, 8, 18, 357,
    629, 16, 8, 43, 117,
    685, 16, 8, 127, 452,
    697, 16, 8, 9, 196,
    697, 16, 8, 38, 55,
    425, 32, 16, 7, 82,
    901, 16, 8, 76, 348,
    949, 16, 8, 83, 229,
    965, 16, 8, 43, 588,
    485, 32, 16, 8, 202,
    493, 32, 16, 12, 41,
    1025, 16, 8, 68, 232,
    1037, 16, 8, 111, 172,
    1105, 16, 8, 8, 138,
    1105, 16, 8, 83, 213,
    565, 32, 16, 42, 48,
    1157, 16, 8, 190, 749,
    1165, 16, 8, 12, 97,
    1189, 16, 8, 191, 249,
    1205, 16, 8, 8, 693,
```

```
    1241, 16, 8, 100, 246,
    1241, 16, 8, 302, 387,
    629, 32, 16, 6, 80,
    1261, 16, 8, 47, 629,
    1285, 16, 8, 193, 253,
    697, 32, 16, 3, 232,
    697, 32, 16, 14, 44,
    697, 32, 16, 73, 173,
    1405, 16, 8, 192, 473,
    1445, 16, 8, 423, 468,
    1469, 16, 8, 18, 44,
    1513, 16, 8, 144, 212,
    1513, 16, 8, 166, 319,
    1517, 16, 8, 68, 191,
    1565, 16, 8, 188, 308,
    1649, 16, 8, 47, 64,
    1649, 16, 8, 172, 366,
    1685, 16, 8, 252, 1122,
    1717, 16, 8, 111, 818,
    1765, 16, 8, 237, 943,
    1781, 16, 8, 96, 421,
    901, 32, 16, 23, 182,
    1825, 16, 8, 168, 793,
    1853, 16, 8, 76, 185]
```

Module `Gate` (Propositions 5.1(ii)–(iv), 5.2, 5.3, 5.4; `dims`).

```
theorem s15_pair_almost_conjugate :
    almostConjugate ⟨85, 16, 8, 2⟩ ⟨85, 16, 8, 42⟩ = true

theorem s15_pair_nonvacuous :
    admissible ⟨85, 16, 8, 2⟩ = true ∧ admissible ⟨85, 16, 8, 42⟩ = true ∧
      TypeIParam.card ⟨85, 16, 8, 2⟩ = 1360 ∧ TypeIParam.dim ⟨85, 16, 8, 2⟩ 1 = 15

theorem s15_negative_controls :
    admissible ⟨85, 16, 8, 9⟩ = true ∧
      almostConjugate ⟨85, 16, 8, 2⟩ ⟨85, 16, 8, 9⟩ = false ∧
      almostConjugate ⟨85, 16, 8, 9⟩ ⟨85, 16, 8, 42⟩ = false

theorem gate_too_small : collisionTable 0 1 1300 = some #[]

theorem f_test_separates :
    collisionsIrr 8 1360 1360 = some #[(⟨85, 16, 8, 2⟩, ⟨85, 16, 8, 42⟩)]

theorem isomorphic_copies_excluded :
    canonicalR 43 85 8 = false ∧ admissible ⟨85, 16, 8, 43⟩ = false ∧
      (enumGroups 8 1400).size = 12

theorem prop52_d2_forced : propDTwo 100000 = true

theorem remark45_d4_forced : remarkDFour 100000 = true

theorem rep_invariance_control :
```

```
    repInvariance ⟨85, 16, 8, 2⟩ = true ∧ repInvariance ⟨133, 9, 3, 11⟩ = true

theorem kl_reduction_control : klReductionOK ⟨133, 9, 3, 11⟩ = true

theorem dims :
    (∀ g : TypeIParam, g.d = 4 → g.dim 1 = 7) ∧
      (∀ g : TypeIParam, g.d = 6 → g.dim 1 = 11) ∧
      (∀ g : TypeIParam, g.d = 3 → g.dim 2 = 11)
```

Module `Dim7` (Theorem 4.1).

```
theorem dim7_chunk_a : collisionsIrr 4 1 40000 = some #[]
theorem dim7_chunk_b : collisionsIrr 4 40001 70000 = some #[]
theorem dim7_chunk_c : collisionsIrr 4 70001 100000 = some #[]

theorem dim7_no_isospectral_pair_le_100000 :
    collisionsIrr 4 1 40000 = some #[] ∧
      collisionsIrr 4 40001 70000 = some #[] ∧
      collisionsIrr 4 70001 100000 = some #[]

theorem dim7_nonvacuous :
    (enumGroups 4 100000).size = 33682 ∧
      admissible ⟨65, 8, 4, 8⟩ = true ∧ admissible ⟨65, 8, 4, 18⟩ = true ∧
      TypeIParam.card ⟨65, 8, 4, 8⟩ = 520 ∧ TypeIParam.dim ⟨65, 8, 4, 8⟩ 1 = 7
```

Module `Dim11` (Theorem 4.2).

```
theorem dim11_d6_chunk_a : collisionsIrr 6 1 100000 = some #[]
theorem dim11_d6_chunk_b : collisionsIrr 6 100001 200000 = some #[]

theorem dim11_d6_no_isospectral_pair_le_200000 :
    collisionsIrr 6 1 100000 = some #[] ∧ collisionsIrr 6 100001 200000 = some #[]

theorem dim11_d6_nonvacuous :
    (enumGroups 6 100000).size = 6067 ∧
      admissible ⟨7, 36, 6, 3⟩ = true ∧ TypeIParam.card ⟨7, 36, 6, 3⟩ = 252 ∧
      TypeIParam.dim ⟨7, 36, 6, 3⟩ 1 = 11
```

Module `Dim11Sum2` (Theorem 4.3).

```
theorem dim11_d3_chunk_a : collisionsSum2 3 1 6000 = some #[]
theorem dim11_d3_chunk_b : collisionsSum2 3 6001 10000 = some #[]

theorem dim11_d3_sum2_no_isospectral_pair_le_10000 :
    collisionsSum2 3 1 6000 = some #[] ∧ collisionsSum2 3 6001 10000 = some #[]

theorem dim11_d3_nonvacuous :
    (enumGroups 3 10000).size = 777 ∧
      admissible ⟨91, 9, 3, 9⟩ = true ∧ admissible ⟨91, 9, 3, 16⟩ = true ∧
      TypeIParam.card ⟨91, 9, 3, 9⟩ = 819 ∧ TypeIParam.dim ⟨91, 9, 3, 9⟩ 2 = 11
```

The chunk theorems carry the evaluations; the combined statements are their conjunctions. The pinned statements of Theorems 4.1–4.3 are the combined statements together with the corresponding `nonvacuous` theorem and `dims`.

References

[1] M. Colantonio and E. A. Lauret, *The fundamental group of a spherical space form is not audible*, arXiv:2602.12939 [math.DG], v1 13 February 2026, v2 13 August 2026. Read in full in the compiled v2 e-print; statement numbers as in Section 6.

[2] A. Ikeda, *On the spectrum of a riemannian manifold of positive constant curvature*, Osaka J. Math. **17** (1980), 75–93. Read in the Project Euclid scan. Proposition 2.1 (pp. 79–80) is the criterion and Theorem 2.2 (p. 80) the closed form of (5), both stated for S^{2n-1}/G with $G \subset SO(2n)$; its Corollary 2.3 already gives isospectrality from a bijection preserving eigenvalue multisets. Theorem I (p. 76) settles dimension 3: "If two 3-dimensional spherical space forms are isospectral, then they are isometric."

[3] A. Ikeda, *On the spectrum of a riemannian manifold of positive constant curvature II*, Osaka J. Math. **17** (1980), 691–702. Read in the Project Euclid scan. Theorem 3.1 (p. 697), verbatim: "Let d be an odd prime, and let M, N be $(2d-1)$-dimensional spherical space forms. Suppose M is isospectral to N. Then their fundamental groups are isomorphic." Theorem 3.9 (p. 701) is the dimension-5 entry of the table under [1, Question 5.1]: a compact connected riemannian manifold isospectral to a 5-dimensional spherical space form with non-cyclic fundamental group is isometric to it (the space-form form of the statement is its Theorem 3.8). On the pages: the article's own title-page header prints "691-762", and so does the reference list of [1], but the range is 691–702: the article has twelve pages, the next article in the issue begins on p. 703, and [6] cites it as 691–702.

[4] A. Ikeda, *On spherical space forms which are isospectral but not isometric*, J. Math. Soc. Japan **35** (1983), no. 3, 437–444. Equation (3.9) is the eigenvalue formula (7); [1, Thm. 1.2] is from this paper. Cited through [1]; not read. (Its Lemma 2.5 is re-proved, with unchanged statement, in A. Ikeda, Kodai Math. J. **20** (1997), no. 1, 1–7, Lemma 1, as [1] notes.)

[5] J. A. Wolf, *Spaces of constant curvature*, 6th ed., AMS Chelsea Publishing, Providence, RI, 2011. Theorem 5.4.1 (Burnside), Theorem 5.5.10 (the representations $\rho_{k,l}$), Chapter 7 and §7.5 (the six types by dimension). Cited through [1].

[6] J. A. Wolf, *Isospectrality for spherical space forms*, Results Math. **40** (2001), no. 1–4, 321–338; DOI 10.1007/BF03322715 (verified on Crossref). Read in the author's own copy. Corollary 2.13 (p. 325), verbatim: "Let M be a compact riemannian manifold. Let G_1 and G_2 be finite groups of isometries acting freely on M. Suppose that there is a one–one map ψ of G_1 onto G_2 such that, if $g \in G_1$ then $\psi(g)$ is conjugate to g in the isometry group $I(M)$. Then the riemannian quotient manifolds $G_1\backslash M$ and $G_2\backslash M$ are strongly isospectral." For $M = S^q$, where $I(M) = O(q+1)$ and conjugacy is equality of eigenvalue multisets, this is [1, Prop. 4.3]. Corollary 7.3 (p. 334), verbatim: "Let $q = 2p-1$ where p is prime. Consider spherical space forms $M = S^q/G$ and $M' = S^q/G'$ where G is not cyclic. Then the following are equivalent: (i) M and M' are isospectral, (ii) M and M' are strongly isospectral, and (iii) $G \cong G'$." Its proof obtains (i)⇒(iii) by citing [3, Thm. 3.1] (and notes that this implication does not need G non-cyclic), so what it says about non-isomorphic fundamental groups is Ikeda's theorem, under the same primality hypothesis; $p = 4$ and $p = 6$ fall outside it, and nothing below rests on it.

[7] P. B. Gilkey, *On spherical space forms with meta-cyclic fundamental group which are isospectral but not equivariant cobordant*, Compositio Math. **56** (1985), no. 2, 171–200. Title, author and pages verified on the publisher's archive (Numdam, `CM_1985__56_2`); the paper itself was not read.

[8] M. Colantonio, *Isospectral and not strongly isospectral spherical space forms with non-cyclic fundamental group*, arXiv:2609.01920, 1 September 2026. The only work citing [1] found on 2026-09-07; it addresses a different question.

[9] G. Jaeschke, *On strong pseudoprimes to several bases*, Math. Comp. **61** (1993), no. 204, 915–926. The bound 3 215 031 751 for the bases $2, 3, 5, 7$.

[10] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635.

[11] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.