GROWTH SERIES OF INVOLUTION SYSTEMS AND THEIR COXETER COMPANIONS

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Abstract. An *involution system* is a pair (W, S) with W a group generated by a finite set S of involutions; its growth series is $W(q) = \sum_{w \in W} q^{\ell(w)}$, where ℓ is word length with respect to S. Dos Santos, Hohlweg and Trufanov attach to an *even meet involution system* (EMIS) a *Coxeter companion* $(\overline{W}, \overline{S})$, read off the irreducible cycles of $\mathrm{Cay}(W, S)$ through the identity, and ask whether $W(q)$ is rational and whether $W(q) = \overline{W}(q)$; they record that they do not expect the second equality to hold in general. We produce three explicit rank-4 involution systems for which the answer, conditionally, is no. Each is an even involution system with a 2-recognizable presentation; each has, *if* it is an EMIS, a companion graph that the presentation forces and that we compute; both growth series are rational with explicit closed forms, exact in every degree; and they differ, first in degree 6, with coefficients 200 against 201 for the first system and 280 against 281 for the other two. Each of the three is therefore an EMIS only if the second half of the question has a negative answer. Along the way we reproduce the two systems named in the source, show that the companion graph is *not* in general the matrix that a 2-recognizable relator set displays — a gap that turns two of eight candidate classes into false positives — and verify that none of the three systems has a pair of elements without a meet in the right weak order up to length 8, and, outside the formal development, up to length 11. Every computation reported is machine-checked in Lean 4 unless it is labelled as external.

1. Introduction

Let W be a group and let $S \subseteq W$ be a finite set of involutions generating W. Following Dos Santos, Hohlweg and Trufanov [1] we call (W, S) an *involution system*, write ℓ for word length with respect to S, and regard $\mathrm{Cay}(W, S)$ as a simple $|S|$-regular graph on which ℓ is the distance from the identity. The *growth series* of (W, S) is, verbatim from §6.1 of [1],

> "the formal series in q defined as follows: $W(q) = \sum_{w \in W} q^{\ell(w)}$",

so that the coefficient of q^n is the sphere size $|\{w \in W : \ell(w) = n\}|$ and comparing two growth series means comparing two sequences of sphere sizes.

A Coxeter system is an involution system presented by the squares together with the braid relations, and [1] studies how much of the weak order of a Coxeter system survives when that presentation is not assumed. Write $u \leq_R v$ for the right weak order, $u \leq_R v \iff \ell(u) + \ell(u^{-1}v) = \ell(v)$. An *EIS* is an involution system carrying a *sign character*, a homomorphism $\varepsilon : W \to \{\pm 1\}$ with $\varepsilon(s) = -1$ for every $s \in S$; equivalently (Corollary 2.8 of [1]) one all of whose relations have even length. An *MIS* is an involution system whose right weak order is a complete meet-semilattice, and an *EMIS* is both (Definition 3.6). To an EMIS the source attaches a Coxeter system by a construction internal to the Cayley graph. Call a cycle of $\mathrm{Cay}(W, S)$ *reducible* if it is a sum, in the $\mathbb{Z}/2$ cycle space, of circuits of strictly smaller length, and *irreducible* otherwise, and let $\mathrm{Irr}_{\mathrm{id}}$ be the set of irreducible cycles through the identity. Then, again verbatim, this time from the paragraph preceding Definition 4.15,

> "we assign to each such pair of generators a value $m(s, t) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ by setting $m(s, t)$ to be equal to half the length of the irreducible cycle of Irr_e containing the edges $\{\mathrm{id}, s\}$ and $\{\mathrm{id}, t\}$, or ∞ if no such cycle exists".

The graph so obtained is the *companion graph* Γ of (W, S), and the Coxeter system $(\overline{W}, \overline{S})$ of type Γ is its *Coxeter companion*. The companion graph is not the *Coxeter type* $m(s, t) = \mathrm{ord}(st)$: by Remark 4.16(1) of [1] its running example has Coxeter type U_3 and companion graph $\widetilde{A}_2$.

The question this paper is about is Question 3 of [1], quoted here in full.

Question 1.1. "Let (W, S) be an EMIS. Is $W(q)$ rational? Is $W(q) = \overline{W}(q)$, where $(\overline{W}, \overline{S})$ is the Coxeter-companion of (W, S)?"

Immediately after it the authors add: "We do not believe that the answer to the second part of the question is positive, but to find a counterexample would be very interesting."

Two facts bound where such a counterexample can live. Every rank-3 EMIS is a *quasi-Coxeter system* — the Cayley graph oriented by length is isomorphic, as a directed graph, to the Hasse diagram of the weak order of a Coxeter system — and a quasi-Coxeter system satisfies $W(q) = \overline{W}(q)$; so a counterexample has $|S| \geq 4$. And the one rank-4 EMIS that [1] exhibits as *not* quasi-Coxeter (§7.1 there) is not a counterexample either: about it the source says "One can show that surprisingly, in this case, $W(q) = \overline{W}(q)$. In particular, $W(q)$ is rational."

Results. The three systems of this paper are, in each case with $S = \{a, b, c, d\}$ and with every generator an involution,

$$V_1 = \left\langle\, a, b, c, d \;\middle|\; a^2, b^2, c^2, d^2,\ (ab)^3,\ (ac)^3,\ bcbd \,\right\rangle, \tag{1}$$

$$V_2 = \left\langle\, a, b, c, d \;\middle|\; a^2, b^2, c^2, d^2,\ (ad)^2,\ (ab)^3,\ dbcdcb \,\right\rangle, \tag{2}$$

$$V_3 = \left\langle\, a, b, c, d \;\middle|\; a^2, b^2, c^2, d^2,\ (ca)^2,\ (cb)^3,\ abdbad \,\right\rangle. \tag{3}$$

Each is an even involution system whose displayed presentation is 2-recognizable in the sense of [1]; each therefore *could* be an EMIS, and if it is, its companion graph is determined by the data in Section 5. Theorems 7.1 and 7.2 compute both growth series exactly, in every degree and in closed form, and find them different. The consequence (Corollary 7.3) is three explicit four-generator systems each of which is an EMIS *only if* the second half of Question 1.1 has a negative answer. We do not decide whether any of them is an EMIS; Section 8 reports how far the meet-semilattice property has been tested, and Section 9 explains why that, and not the growth comparison, is the bottleneck.

Not one coefficient below is read off a model. For the two systems of [1] exactness comes from a two-sided bound; for the five four-generator systems of Sections 4–7 and their companions it comes from finite complete rewriting systems (Theorem 6.1). As a by-product, all nine of those growth series are rational with explicit closed forms (Theorem 6.2), which is the first half of Question 1.1 for the systems in question. Section 3 reproduces the two systems named in [1] and the four negative controls that show the hypotheses of Question 1.1 are load-bearing. Section 5 isolates a trap: the Coxeter matrix that a 2-recognizable relator set displays need not be the companion graph, because 2-recognizability forbids collisions among the relators without forcing them to exhaust $\mathrm{Irr}_{\mathrm{id}}$.

Every computation below is machine-checked in Lean 4 unless it is labelled as external; Section 10 says precisely what that means and which claims are exhaustive finite searches.

Numbered statements of [1] are cited throughout by the numbering of arXiv:2604.18822v1, which at the time of writing is the only version of that preprint and is unpublished.

2. Preliminaries

2.1. **Presentations and the relator set.** All presentations below have the form

$$P(S; R_0) \;=\; \left\langle\, S \;\middle|\; \{s^2 : s \in S\} \cup R_0 \,\right\rangle,$$

with R_0 a finite set of words in S. Because the generators are involutions, a word is inverted by reading it backwards; we write $\mathrm{cyc}(w)$ for the set of all cyclic shifts of w and of its reverse, and call w *cyclically reduced* if no two cyclically adjacent letters coincide. For Γ a Coxeter matrix on S we write $P(\Gamma)$ for the Coxeter presentation $\langle S \mid s^2,\ (st)^{m(s,t)}\ (m(s,t) < \infty)\rangle$.

Write $[w]$ for the *cyclic word* of w, the set of its cyclic shifts, and recall that a 2-factor of a word is a factor ab with $a, b \in S$. Following Definition 5.2 of [1], a set R_0 of words is 2-*recognizable* when, verbatim,

(1) "$[w]$ is a reduced cyclic word for every $w \in R_0$";

(2) "Every $w \in R_0$ has even length at least 4";
(3) "Let $w \in R_0$ and ab a 2-factor of $[w]$. If ab is also 2-factor of $[v]$ or $[v^{-1}]$ for some $v \in R_0$, then $w = v$.";
(4) "Let $w = s_1 \cdots s_k \in R_0$ and ab be a 2-factor of $s_k w s_1$. If $s \in S$ precedes (resp. follows) ab in $s_{k-1} s_k w s_1 s_2$, then for any occurrence of the factor ab in $s_k w s_1$, s also precedes (resp. follows) that occurence in $s_{k-1} s_k w s_1 s_2$."

A presentation $P(S; R_0)$ with R_0 2-recognizable is a 2-*recognizable presentation*. Immediately after the definition the source shows that (3) and (4) together are equivalent to

(5) "For any $a, b \in S$ there exists at most one word in $\mathrm{cyc}(R_0)$ starting with ab",

so that (1), (2) and (5) is the same definition. That is the form used below: (1) and (2) say that the relators are cyclically reduced of even length at least 4, and (5) is what every argument here that is not a parity count appeals to.

2.2. **What we use from** [1]**.** The following are the results of [1] used below; nothing else from that paper is assumed.

(F1) *(EIS relations; Definition 2.3 and Corollary 2.8.)* An involution system is an EIS — that is, carries a sign character ε — exactly when all of its relations have even length. Since then $\varepsilon(w) = (-1)^{\ell(w)}$, $\mathrm{Cay}(W, S)$ is bipartite.
(F2) *(Recognizability; Theorem 5.7.)* Every EMIS admits a 2-recognizable presentation $P(S; R_0)$ in which R_0 contains exactly one word of $\mathrm{cyc}(\nu_C)$ for each irreducible cycle $C \in \mathrm{Irr}_{\mathrm{id}}$, where ν_C is the label of C read from the identity.
(F3) *(Joins along irreducible cycles; Proposition 4.17.)* Let (W, S) be an EMIS and $C \in \mathrm{Irr}_{\mathrm{id}}$. Then C carries a unique pair $s, t \in S$ of generators, the vertex of C opposite id is the join $s \vee_R t$, the label of C is $\nu(s,t) = s\, a_2 \cdots a_k\, b_k \cdots b_2\, t$ with $k = m(s,t)$, and $s\, a_2 \cdots a_k$ and $t\, b_2 \cdots b_k$ are *the unique two* reduced words for $s \vee_R t$.
(F4) *(Growth of a quasi-Coxeter system; Corollary 6.3.)* If (W, S) is a quasi-Coxeter system, then $W(q) = \overline{W}(q)$.
(F5) *(Rank three; Theorem 6.8.)* Every rank-3 EMIS is a quasi-Coxeter system.
(F6) *(Pairs suffice; Remark 3.7.)* For the complete meet-semilattice property it is enough that every *pair* of elements has a meet.

Two consequences of (F3) are what make a companion graph computable inside a ball of $\mathrm{Cay}(W, S)$: for an EMIS,

$$
\begin{aligned}
m(s,t) &= \ell(s \vee_R t) && \text{if the pair } (s,t) \text{ is carried by some } C \in \mathrm{Irr}_{\mathrm{id}}, \\
m(s,t) &= \infty && \text{if it is carried by none,}
\end{aligned}
\tag{4}
$$

and a pair whose join admits more than two reduced words is carried by none.

2.3. **Two ways to make a sphere size exact.** A presentation determines a group only implicitly, so no coefficient below is read off a model. Two devices are used.

The first is a two-sided bound. If M is a concrete group in which every relator of $P(S; R_0)$ evaluates to the identity, von Dyck's theorem gives a surjection $W \twoheadrightarrow M$ carrying $B_W(n)$ onto $B_M(n)$, so $|B_M(n)| \le |B_W(n)|$. In the other direction, let $U(n)$ be the number of classes of S-reduced words of length $\le n$ under the congruence generated by replacing one side of a member of $\mathrm{cyc}(R_0)$, split anywhere, by the other side, where both words have length $\le n$. Merged words are equal in W by construction and every element of $B_W(n)$ is named by such a word, so $|B_W(n)| \le U(n)$. Where the two meet, the ball sizes — hence the sphere sizes — of the *presented* group are pinned. No rewriting metatheory and no appeal to [1] enters.

The second device removes the radius limit. A finite set of rules $u \to v$ over S is checked on four conditions: (R1) every rule strictly decreases the shortlex order (length first, then lexicographic); (R2) every rule is a consequence of the relators, in the sense that u and v lie in one class of the congruence of the previous paragraph; (R3) every critical pair — overlap or inclusion of two left-hand sides — rewrites to one and the same irreducible word; (R4) every relator and every s^2 rewrites to the empty word. Conditions (R2) and (R4) say that the rewriting congruence is exactly the congruence of the presentation; (R1) and (R3) give unique normal forms by the standard termination-plus-local-confluence argument for string rewriting [6]. Since shortlex is compatible with concatenation on both sides, the normal form of w is the shortlex-least word representing it, so its length is $\ell(w)$: the sphere size at n is the number of irreducible words of length n, at *every* n. The irreducible words are the words avoiding a finite set of factors, a regular language, so $W(q)$ is rational and is computed by the transfer matrix of the corresponding Aho–Corasick automaton [7].

3. The two systems named in the source

Write $W_A = P(\{s_1, s_2, s_3\}; \{(s_1s_2s_3)^2\})$ for the example of [1] whose Cayley graph is the honeycomb tiling (Example 3.10 there), and $W_B = P(\{a, b, c, d\}; \{(abc)^2, (ad)^2\})$ for the rank-4 EMIS of §7.1 of [1] that is not a quasi-Coxeter system. Both are recovered here from their presentations alone.

Proposition 3.1. *W_A is an even involution system; the companion matrix read off its 2-recognizable presentation is $m(s_i, s_j) = 3$ for $i \neq j$, the Coxeter graph of type $\widetilde{A}_2$; and its Coxeter type is U_3, i.e. $\operatorname{ord}(s_is_j) = \infty$ for $i \neq j$.*

Proposition 3.2. *W_B is an even involution system, and the companion matrix read off its presentation is*

$$m(a, b) = m(a, c) = m(b, c) = 3, \qquad m(a, d) = 2, \qquad m(b, d) = m(c, d) = \infty,$$

which is the companion graph drawn in §7.1 of [1]; the Coxeter presentation it generates is $\langle a, b, c, d \mid a^2, \ldots, d^2, (ab)^3, (ac)^3, (ad)^2, (bc)^3\rangle$.

Proof sketch. Both are evaluations of finite functions of the relator set: the parity test of (F1), and the rule "$m(s, t) = |w|/2$ for the unique $w \in \operatorname{cyc}(R_0)$ beginning st, and ∞ if there is none", which is the reading of the companion licensed by (F2) and condition (5). The infinite orders in Proposition 3.1 are checked as the exact finite statement that $(s_is_j)^k \neq 1$ for all $k \leq 500$ in the model $\mathbb{Z}^2 \rtimes \{\pm 1\}$ of Proposition 3.3; that the order is genuinely infinite is the one-line remark that s_is_j is there the translation by $a_i - a_j \neq 0$, and a quotient cannot lengthen an order. □

Proposition 3.3. *Every relator of W_A evaluates to the identity in $\mathbb{Z}^2 \rtimes \{\pm 1\}$ under $s_i \mapsto (a_i, -1)$ with $a_1 = (0, 0), a_2 = (1, 0), a_3 = (0, 1)$; every relator of W_B evaluates to the identity in the amalgamated free product $W_A *_{\langle a\rangle} (\langle a\rangle \times \langle d\rangle)$ in normal form [5]; and every Coxeter relator of the two companions evaluates to the identity in the corresponding integral Tits representation.*

Theorem 3.4. *The growth series of W_A and of its Coxeter companion agree through degree 16, both being $(1 + q + q^2)/(1 - q)^2$ there, with sphere sizes $1, 3, 6, 9, 12, \ldots$ (that is, $3n$ for $n \geq 1$). The growth series of W_B and of its Coxeter companion agree through degree 12, both being $(1 + 3q + 4q^2 + 3q^3 + q^4)/(1 - q - 3q^2 - q^3 + q^4) = (1 + q)^2(1 + q + q^2)/(1 - q - 3q^2 - q^3 + q^4)$ there, with sphere sizes*

$$1,\ 4,\ 11,\ 27,\ 64,\ 152,\ 360,\ 853,\ 2021,\ 4788,\ 11344,\ 26876,\ 63675.$$

The word bound $U(n)$ meets the model ball at radius 12 in rank 3 and at radius 8 in rank 4, on both the involution system and the companion side, so through those radii the coefficients are exact for the presented *groups.*

Beyond radius 12, respectively 8, the two lists are statements about the models of Proposition 3.3. Those models are faithful by known results: the Tits representation [2], the honeycomb picture of $\operatorname{Cay}(W_A, S)$ recorded in [1], and the amalgam normal form [5]. Their faithfulness is not re-proved here and is nowhere relied on.

Proof sketch. The sphere sizes are breadth-first searches in the models of Proposition 3.3; the equalities and the rational expressions are equalities of explicit finite lists. The exactness claims are the four equalities $U(n) = |B_M(n)|$ for $n \leq 12$ (rank 3) and $n \leq 8$ (rank 4), each an exhaustive enumeration of the S-reduced words of length $\leq n$ and of the congruence classes they fall into. The rank-3 sequence is A008486 of the OEIS [8], whose comment names it the coordination sequence of the honeycomb net;

the rank-4 list returned no match there. The source asserts both equalities but prints no coefficients, so the two lists are reproductions rather than quotations; the rank-4 list was independently reproduced from Steinberg's formula over the finite standard parabolics of the companion graph [3, Chapter 7]. Its growth rate is $2.3692054\ldots$, the reciprocal of the least positive root of the denominator. □

3.1. **The hypotheses of Question 1.1 are load-bearing.** Four things in the statement of Question 1.1 could be doing no work, and each is broken here by an example, two of them over-counting and two under-counting, together with one positive control.

Proposition 3.5. (Over-counting.) *Replacing the companion graph of W_A by its Coxeter type U_3 changes the series: U_3 gives $1, 3, 6, 12, 24, 48, 96$ against $1, 3, 6, 9, 12, 15, 18$, first differing at $n = 3$. Deleting the relator $(ad)^2$ from W_B changes the series: the resulting free product $W_A * C_2$ has sphere sizes $1, 4, 12, 33, 90, 246, 672, 1836, 5016$, exactly, against W_B's $1, 4, 11, 27, 64, \ldots$, first differing at $n = 2$.*

Proposition 3.6. (Under-counting.) *Dropping evenness: $\langle a, b, c \mid a^2, b^2, c^2, abc\rangle \cong C_2 \times C_2$ is an MIS but not an EIS — the relator has odd length, so no companion graph is defined — and its series $1 + 3q$ differs from that of its Coxeter type C_2^3, namely $1 + 3q + 3q^2 + q^3$. Dropping the meet-semilattice property: the 2-recognizable involution system $\langle a, b, c, d \mid a^2, b^2, c^2, d^2, abcdcb, (ad)^2, (ac)^2, (bd)^2\rangle$ of §5.4 of [1], which is an EIS whose relator graph is the Coxeter graph of type A_4 with Coxeter group S_5, satisfies $|B_W(4)| \leq 47$ while $|B_{S_5}(4)| = 49$.*

Proposition 3.7. (Positive control.) *The rank-1 system is its own Coxeter companion and both series are $1 + q$.*

Proof sketch of Propositions 3.5–3.7. Each is a finite comparison of two explicit lists, and in each the comparison is decisive. The sphere sizes of W_A and of W_B are exact by Theorem 3.4; the free-product list is exact, being sandwiched by $U(n)$; both groups in the third case are finite, so both series are exact by inspection; and a ball computed in the Tits representation of a Coxeter group is in any case a lower bound for it, which is what makes $12 > 9$ at $n = 3$ a proof of difference in the first case. In the last case the inequality runs the other way and is decisive by construction: 47 is an upper bound for the involution system and 49 is exact for the Coxeter group. The pair $abc, abab$ that [1] names as having no meet in that system is reproduced independently here: the weak-order test used for Section 8, run outside the formal development and asked which pairs of the ball have no meet, returns that pair first. □

4. The rank-4 shortlist

By (F4) and (F5) a counterexample to the second half of Question 1.1 has at least four generators, and by (F2) it has a 2-recognizable presentation. That makes the search space enumerable, and we work inside the smallest slice of it: relator sets of size at most three over four letters, with relators of length 4 and 6.

Remark 4.1 (an exhaustive search outside the formal development). Of the 2011 collectively 2-recognizable such sets, 1223 have a growth series agreeing through degree 7 with that of the Coxeter group of the matrix their relators display, and 788 differ from it *decisively*, meaning that the word bound $U(n)$ for the involution system falls strictly below the exact ball of that Coxeter group at some $n \leq 7$. Of those 788, 596 contain a pair of elements of a small ball with no meet, and are therefore not MIS; 192 survive. A rank-3 run of the same sweep is a control: all 57 two-recognizable rank-3 sets agree with their companion through degree 7, as (F4) and (F5) require. These counts were produced by a separate implementation in C and are not part of the machine-checked material; what is checked is the transcribed shortlist and the statements about it below. They should moreover be read as "differ from the matrix the relators display", which Section 5 shows is not always the companion graph.

Theorem 4.2. *Relabelling the generators preserves 2-recognizability, carries the displayed matrix to its relabelling, and leaves the involution system unchanged up to isomorphism;* [1] *works up to* *relabelling for the same reason. Under this action of the symmetric group on four letters the* 192 *surviving relator sets form exactly* 8 *orbits, each of full size* 24*; the first degree at which a set differs decisively is constant on an orbit, and splits the shortlist as* $24+24+144$ *over the degrees* $2,3,6$*, so the* 144 *interesting members are* 6 *orbits. Exactly* 3 *orbits, that is* 72 *sets, contain a length-*4 *relator on three letters.*

Proof sketch. The canonical form of a relator set is: replace each relator by the shortlex-least member of its cyc-class, sort the resulting set, and minimise over the 24 relabellings. The theorem is then a finite computation over the 192 transcribed sets, done independently in two implementations with the same answer. Two controls make the count meaningful: the canonical form is invariant under all 24 relabellings (checked on a representative), and its eight values are pairwise distinct, so the number 8 is neither an artefact of a canonical form that separates too little nor of one that separates too much. A third control checks that W_B — which is not on the shortlist, its growth agreeing with its companion's — receives a canonical form outside the list. ∎

A cyclically reduced length-4 relator on three letters is $xyxz$, which says $z = xyx$: one generator is a conjugate of another. In all three orbits of Theorem 4.2 that relator is $bcbd$, and eliminating $d = bcb$ by a Tietze move leaves a rank-3 Coxeter group, generated in the involution system by its simple reflections *together with one conjugate reflection*:

relator set R_0	W	S	first decisive degree
$\{(ab)^2, (ac)^2, bcbd\}$	$\langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (ac)^2\rangle$	$\{a,b,c,bcb\}$	2
$\{(ab)^2, (ac)^3, bcbd\}$	$\langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (ac)^3\rangle$	$\{a,b,c,bcb\}$	3
$\{(ab)^3, (ac)^3, bcbd\}$	$\langle a,b,c \mid a^2, b^2, c^2, (ab)^3, (ac)^3\rangle$	$\{a,b,c,bcb\}$	6

Call these U_1, U_2 and V_1; the third is (1). In all three the displayed rank-3 Coxeter matrix has $m(b,c) = \infty$, so $\langle b,c\rangle$ is infinite dihedral and bcb is a genuinely new involution. The identification of W is an elementary Tietze move and is recorded here for orientation only; nothing below depends on it, since Theorem 6.1 treats the four-generator presentations directly.

Proposition 4.3. *In each of* U_1, U_2, V_1 *every relator evaluates to the identity in the integral Tits representation of the displayed rank-*3 *Coxeter matrix, with the fourth generator sent to bcb. The resulting model balls meet the word bound* $U(n)$ *through degree* 6 *for* U_1 *and* U_2 *and through degree* 8 *for* V_1*, so through those degrees the sphere sizes*

$$\begin{aligned} U_1 &: \ 1,4,7,8,8,8,8,8,8,8 \\ U_2 &: \ 1,4,9,16,26,42,68,110,178,288 \\ V_1 &: \ 1,4,10,22,46,96,200,416,866,1802 \end{aligned}$$

are exact for the presented groups. (The further entries displayed are values in the model; Theorem 6.1 makes all three lists exact at every degree.)

5. The companion graph is not the matrix the relators display

By (F2) an EMIS *has* a 2-recognizable presentation whose relators are the labels of the irreducible cycles at id, and for such a presentation the companion matrix is read off the relators: $m(s,t) = |w|/2$ for the unique $w \in \mathrm{cyc}(R_0)$ beginning with st, and ∞ when there is none. The converse fails. Condition (5) forbids two relators from starting with the same ordered pair; it does not force R_0 to contain a word for every irreducible cycle. A 2-recognizable R_0 may present the same group while omitting one, and then the displayed matrix has ∞ where the companion graph has a finite entry.

Theorem 5.1. *In each of* U_1, U_2, V_1 *the matrix displayed by* R_0 *has* $m(a,d) = \infty$*, while the join* $a \vee_R d$ *exists and*

$$\big(\ell(a \vee_R d),\ \#\{\textit{reduced words for } a \vee_R d\}\big) = (2,2),\ (3,2),\ (6,9)$$

for U_1, U_2, V_1 respectively. By (4) the companion entry is therefore $m(a,d) = 2$ for U_1, $m(a,d) = 3$ for U_2 and $m(a,d) = \infty$ for V_1: the displayed matrix is wrong for the first two systems and right, for a different reason, for the third.

Proof sketch. The joins and the geodesic counts are computed in an exact ball of $\mathrm{Cay}(W,S)$ — exact by Proposition 4.3 — by two passes over the ball, one propagating lower sets and one counting reduced words. The reading of the numbers as companion entries is (4), that is (F3), and holds under the hypothesis that the system is an EMIS, which is the hypothesis of Question 1.1. □

Two certificates make the remaining entries decidable inside a ball. The first supplies irreducibility of a 6-cycle, which (4) needs before a hexagon may be read as $m(s,t) = 3$.

Lemma 5.2. *Let (W,S) be an EIS, let $\psi : S \to \mathbb{Z}/2$, and let φ be the edge function $\varphi(w, ws) = \psi(s)$ on $\mathrm{Cay}(W,S)$. Then φ is a $\mathbb{Z}/2$-linear functional on the edge space, invariant under left translation, so its value on a circuit is the sum of ψ over the circuit's label. If φ vanishes on every circuit shorter than C and not on C, then C is irreducible. With $\psi(a) = 1$, $\psi(b) = \psi(c) = \psi(d) = 0$ this certifies the hexagons ababab and acacac of V_1, and acacac and adadad of U_2, as irreducible cycles.*

Proof sketch. By (F1) $\mathrm{Cay}(W,S)$ is bipartite and simple, so the only circuits shorter than a hexagon are 4-cycles; a 4-cycle at g is $g, gx, gxy, gxyz$ for a cyclically reduced word $xyzw$ trivial in W, and those words are enumerable. What is checked exhaustively is that $\sum \psi = 0$ on every such word for the two systems — 4 of them for V_1, 6 for U_2 — and that $\sum \psi = 1$ on the three hexagons. The conclusion was cross-checked by Gaussian elimination over the $\mathbb{Z}/2$ span of all 4-cycles inside the ball of radius 7; the weight argument is the one that is a proof, since it does not truncate the graph. □

Lemma 5.3. *Let (W,S) be an EIS and let $T \subseteq S$ contain every letter occurring in the label of a 4-cycle of $\mathrm{Cay}(W,S)$. Then every $\mathbb{Z}/2$ sum of 4-cycles is supported on T-labelled edges, so a hexagon carrying an edge labelled outside T is irreducible. Moreover, if w^k is irreducible for the rewriting system of Section 2 for one k with $k|w|$ exceeding the longest left-hand side, then w^k is irreducible for every k and w has infinite order in W.*

Proof sketch. All 4-cycles are left translates of the 4-cycles at id, whose labels are the trivial cyclically reduced words of length 4; these are enumerated exhaustively. For the second statement, a left-hand side occurring inside a long power of w already occurs inside a power of bounded length, so a single check certifies all powers, and the normal form of w^k then has length $k|w| > 0$. This certificate is strictly stronger than Lemma 5.2: the hexagon *dbcdcb* of (2) contains each of b, c, d exactly twice, so *every* letter weight ψ vanishes on it — checked on the four weights that span the space — and no instance of Lemma 5.2 could certify it, whereas the label test does so at once. As a non-vacuity control, the order certificate refuses *ad* in (2), *ac* in (3) and *ab* in (1), whose orders are 2, 2 and 3. □

Finally, an entry $m(s,t) = \infty$ cannot be established by the absence of a join inside a ball. It is established instead by a counting argument on condition (5), which we state once and use three times.

Lemma 5.4 (prefix budget)**.** *Let (W,S) be an EMIS with $|S| = 4$ and let R_0 be a relator set as in (F2). Condition (5) allows at most one word of $\mathrm{cyc}(R_0)$ to begin with each of the 12 ordered pairs of distinct generators. A relator $(st)^m$ occupies the two pairs st, ts, and a relator $xyxz$ occupies the four pairs xy, yx, xz, zx. If the irreducible cycles already identified occupy all but a set F of ordered pairs, then the label ν of any further irreducible cycle has all of its cyclic 2-letter factors in F. In particular, if every word whose cyclic 2-factors lie in F is a power of an element of infinite order, no further irreducible cycle exists, and $m(s,t) = \infty$ for every pair not yet identified.*

Proof. Take R_0 as in (F2), so that $\mathrm{cyc}(R_0)$ contains, for each $C \in \mathrm{Irr}_{\mathrm{id}}$, every cyclic shift of ν_C and of its reverse, and in particular one word beginning with each cyclic 2-letter factor of ν_C. By condition (5) no ordered pair begins two words of $\mathrm{cyc}(R_0)$, so distinct irreducible cycles have disjoint sets of cyclic 2-factors, which is the count in the statement; the label of a cycle other than those identified therefore has all of its cyclic 2-factors in F. Such a label is a relator, hence trivial in W, so if every word with all cyclic 2-factors in F is a nontrivial power of an element of infinite order then no such cycle exists, and the remaining entries are ∞ by (4). □

6. Exact growth in every degree, and rationality

Theorem 6.1. *Each of the five involution systems U_1, U_2, V_1, V_2, V_3 and each of their four Coxeter companions (Corollary 6.3 and Theorems 7.1, 7.2; V_2 and V_3 share one) admits a finite rewriting system, of* 9 *to* 14 *rules, satisfying the four conditions (R1)–(R4) of Section 2. Consequently, for each of the nine, the irreducible words are in bijection with the group, the normal form of an element is a shortlex-least word representing it, and the sphere size at n is the number of irreducible words of length n, for every n.*

Proof sketch. The rule lists were produced by Knuth–Bendix completion and are then rechecked, nothing being trusted to the completion. (R1) and (R4) are finite evaluations. (R3) enumerates every overlap and every inclusion of two left-hand sides and rewrites both sides of the resulting critical pair to a common *irreducible* word; demanding irreducibility makes a shortfall of rewriting fuel a failure rather than a false pass. (R2) is a union–find over all freely reduced words of length at most n — n between 2 and 8, depending on the system — merging each word with every word obtained from it by one relator substitution staying inside that length; these are exactly the merges defining $U(n)$, and each is an equality in W by construction. The bridges from (R1)–(R4) to the conclusion are the textbook facts about string rewriting quoted in Section 2; they are not part of the formal development, and play there the role that “a surjection carries balls onto balls” plays for the two-sided bound. Four negative controls show the four conditions can fail and that each is doing work: deleting one rule from the system for (1) breaks (R3); adding the invalid rule $bc \to a$ breaks (R2) while leaving (R1) intact; feeding the rules of one system to another presentation breaks (R4); and (R2) accepts a genuine consequence of the relators that is not among the rules while rejecting a false one. As a further cross-check the counted sphere sizes reproduce, degree by degree, the model balls of Proposition 4.3, and the companions' counts reproduce the integral Tits balls through degree 9 — which, the count being exact and the Tits ball a lower bound, *proves* that model faithful in that range, where the earlier computation had to take it from [2]. □

Shortlex completion depends on the naming of the generators: two of the five systems complete for some orderings of the letters and for none of the orderings tried in their original naming. Relabelling changes neither the group nor 2-recognizability, so this costs nothing; the presentations (2) and (3) are already written in a naming that completes.

Theorem 6.2. *For each of the nine systems of Theorem 6.1 the growth series is rational, with the closed form displayed below — the first half of Question 1.1 for each of them. The companion is in each case the one determined in Sections 5 and 7.*

system	$W(q)$	$\overline{W}(q)$
U_1	$(1+q)^3/(1-q)$	$(1+q)^3/(1-q)$
U_2	$(1+q)^2(1+q+q^2)/(1-q-q^2)$	$(1+q)^2(1+q+q^2)/(1-q-q^2)$
V_1	$(1+q)^2(1+q+q^2)/(1-q-2q^2-q^3+q^4)$	$\dfrac{1+3q+5q^2+6q^3+5q^4+3q^5+q^6}{1-q-q^2-2q^3-q^4-q^5}$
V_2, V_3	$(1+q)^2(1+q+q^2)/(1-q-3q^2+2q^4)$	$\dfrac{1+3q+5q^2+6q^3+5q^4+3q^5+q^6}{1-q-2q^2-q^3-q^4+q^6}$

In particular $W(q) = \overline{W}(q)$ identically, as rational functions and hence in every degree, for U_1 and U_2.

Proof sketch. Let A be the transfer matrix of the Aho–Corasick automaton of the finite set of forbidden factors, and let $z_0 = \mathbf{1}$, $z_{i+1} = Az_i$, so that the coordinate of z_i at the empty-prefix state is the number of irreducible words of length i. What is checked is one *vector* identity $z_k = \sum_{i=1}^{k} c_i z_{k-i}$, with $k \le 7$ in every case; applying A to both sides propagates it to every index $\ge k$, so the sphere sizes satisfy that linear recurrence at every degree and $W(q) = P(q)/Q(q)$ with $Q = 1 - \sum c_i q^i$. A second check expands P/Q and compares with the counted coefficients through degree 16. Taking the recurrence of the column vectors $A^n\mathbf{1}$ rather than of the row vectors $e_0^\top A^n$ yields the minimal order directly, with no polynomial gcd to certify. □

Corollary 6.3. *Take the companion graph as corrected by Theorem 5.1: for U_1 all $m = 2$ except $m(c,d) = \infty$ — that is, $C_2 \times C_2 \times D_\infty$ — and for U_2, $m(a,c) = m(a,d) = 3$, $m(a,b) = m(b,c) = m(b,d) = 2$, $m(c,d) = \infty$. Then the sphere sizes of U_1 and of its companion agree through degree 9, and likewise for U_2; and by Theorem 6.2 each of the two pairs of series is one and the same rational function, so $W(q) = \overline{W}(q)$ in every degree. Thus U_1 and U_2 are false positives of the search of Remark 4.1, and 48 of its 192 survivors fall.*

Proof sketch. The two sphere lists agree degree by degree through degree 9 — comparing the ball of the involution system with the ball of the corrected companion in its integral Tits representation — and, by Theorem 6.2, in every degree, both sides being the same rational function. The entries $m(c,d) = \infty$ are Lemma 5.4: the cycles already identified occupy 10 of the 12 ordered pairs, leaving $\{cd, dc\}$, so a further label would be a power of cd; and $cd = (cb)^2$ once d is eliminated, of infinite order because $m(b,c) = \infty$ in that rank-3 Coxeter matrix. What is formally checked here is the weaker finite statement that no power of cd below the 500th is the identity in the model; the certificate of Lemma 5.3, which settles the matter outright, was applied only to V_1, V_2, V_3. □

7. Three systems that are an EMIS only if the answer is negative

Theorem 7.1. *Let V_1 be the involution system (1), with $S = \{a,b,c,d\}$. Then V_1 is an even involution system, its displayed presentation is 2-recognizable, and:*

(i) *if V_1 is an EMIS, its companion graph is*

$$m(a,b) = m(a,c) = 3, \qquad m(b,c) = m(b,d) = 2, \qquad m(a,d) = m(c,d) = \infty;$$

(ii) *the two growth series are, in every degree,*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$W(q)$	1	4	10	22	46	96	200	416	866	1802	3750	7804	16240
$\overline{W}(q)$	1	4	10	22	46	96	201	421	882	1847	3868	8101	16966

with the closed forms of Theorem 6.2;

(iii) *hence $W(q) \neq \overline{W}(q)$, the first difference occurring in degree 6, where the coefficients are 200 and 201.*

Theorem 7.2. *Let V_2 and V_3 be the involution systems (2) and (3), with $S = \{a,b,c,d\}$. Both are even involution systems with 2-recognizable presentations, and:*

(i) *if V_2 is an EMIS, its companion graph is*

$$m(a,d) = 2, \qquad m(a,b) = m(b,c) = m(b,d) = m(c,d) = 3, \qquad m(a,c) = \infty,$$

and if V_3 is an EMIS, its companion graph is

$$m(a,c) = 2, \qquad m(a,b) = m(a,d) = m(b,c) = m(b,d) = 3, \qquad m(c,d) = \infty;$$

these two Coxeter graphs are equal under the relabelling $a,b,c,d \mapsto c,b,d,a$, so both are the graph on four vertices with one edge labelled ∞, one labelled 2 sharing a vertex with it, and the remaining four edges labelled 3;

(ii) *V_2 and V_3 have the same growth series, and it and the common companion's are, in every degree,*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$W(q)$	1	4	11	26	58	128	280	612	1336	2916	6364	13888	30308
$\overline{W}(q)$	1	4	11	26	58	128	281	617	1354	2971	6519	14304	31386

with the closed forms of Theorem 6.2;

(iii) *hence $W(q) \neq \overline{W}(q)$ for both, the first difference occurring in degree 6, where the coefficients are 280 and 281.*

The two systems account for 48 *further members of the shortlist of Remark 4.1.*

Proof sketch of Theorems 7.1 and 7.2. Evenness is the sign homomorphism $a, b, c, d \mapsto -1$, which exists because every relator has even length; 2-recognizability of the displayed sets is a finite check of condition (5). Part (ii) is Theorem 6.1 on both sides: each involution system and each companion Coxeter presentation has a complete rewriting system, so both lists are exact at every degree and neither is a bound. Part (iii) is then an inequality of explicit lists, and of rational functions by Theorem 6.2.

Part (i) is the companion determination, and every entry is fixed separately. The finite entries come from (4), computed inside an exact ball: for V_2 and V_3 each of the five pairs carrying a finite entry has a join whose length is that entry and which has exactly two reduced words, as (F3) requires of a pair carried by an irreducible cycle; for V_1 the four such pairs likewise have exactly two reduced words each, and the two hexagons are certified irreducible by Lemma 5.2. For V_2 and V_3 the relevant cycles are certified irreducible by the label test of Lemma 5.3: the only 4-cycle labels are $adad, dada$ for V_2 and $acac, caca$ for V_3, so $T = \{a, d\}$ and $T = \{a, c\}$, and each hexagon in play — $ababab$ and $dbcdcb$ for V_2, $cbcbcb$ and $abdbad$ for V_3 — uses the letter b, which lies outside its T. The infinite entries are Lemma 5.4: the identified cycles occupy 10 of the 12 ordered pairs for V_2, leaving $\{ac, ca\}$, and 10 for V_3, leaving $\{cd, dc\}$, so a further label would be a power of ac, respectively of cd; for V_1 they occupy 8, leaving $\{ad, da, cd, dc\}$, so a further label would be a power of ad, of cd or of $adcd$. Each of those five words is shown to have infinite order by the certificate of Lemma 5.3, with no model and no faithfulness assumption; for V_1 the entry $m(a, d) = \infty$ is obtained a second time and independently, from the nine reduced words of $a \vee_R d$ (Theorem 5.1). □

Corollary 7.3. *Each of V_1, V_2, V_3 is an EMIS only if the second half of Question 1.1 has a negative answer. In particular, if any one of them is an EMIS then it is not a quasi-Coxeter system.*

Proof. If V_i is an EMIS then its companion graph is the one computed in Theorem 7.1(i) or 7.2(i), and its growth series differs from that of the companion by (iii); the second statement is (F4). □

8. The right weak order

Whether any of the three systems is an EMIS is decided by its right weak order, and by (F6) it is enough to ask whether every *pair* of elements has a meet. That question is not finite — it quantifies over all of W — but its restriction to elements of bounded length is, and the following lemma cuts the restricted problem down by two orders of magnitude without losing coverage. Write $L(u) = \{w \in W : w \leq_R u\}$.

Lemma 8.1. *Let $x \neq y$ be two maximal elements of $L(u) \cap L(v)$ and suppose some $s \in S$ satisfies $s \leq_R x$ and $s \leq_R y$. Then $x = sx'$, $y = sy'$ and x', y' are two distinct maximal elements of $L(su) \cap L(sv)$. Consequently, if (u, v) is a pair without a meet minimising $\ell(u) + \ell(v)$, its two maximal common lower bounds have disjoint descent sets, and hence two distinct generators lie below both u and v.*

Proof. From $s \leq_R x \leq_R u$ and $s \leq_R y \leq_R v$ we get that su and sv are shorter than u and v, and that $x' = sx$ and $y' = sy$ lie in $L(su) \cap L(sv)$. If $x' <_R z'$ for some $z' \in L(su) \cap L(sv)$ then $sz' \in L(u) \cap L(v)$ and $x <_R sz'$, contradicting maximality of x; symmetrically for y'. For the consequence: if a generator lay below both maximal elements the lemma would produce a failure of strictly smaller $\ell(u) + \ell(v)$, so at a minimising failure the descent sets of x and y are disjoint. Neither x nor y is the identity there — otherwise id would be the maximum of the intersection — so each has a left descent, and the two are distinct and lie below both u and v. □

Theorem 8.2. *For each of V_1, V_2, V_3, every pair of elements of length at most* 8 *has a meet in the right weak order. The same holds for every pair of elements of length at most* 11*, by a computation outside the formal development.*

Proof sketch. By Lemma 8.1 it suffices to test the pairs (u, v) that share two left descents, which at length 8 is 47,160 pairs instead of about 3.0 million. A pair is tested by the observation that a lower set D has a maximum exactly when it has a unique longest element m and $|D| = |L(m)|$. The length-11 computation used the same pruning: for V_1, 15,017 elements of length at most 11 and 2,926,416 pairs, against the 112,747,636 pairs of the unpruned enumeration, which was also run and also found no failure; for V_2 and V_3, 25,624 elements and 5,242,920 pairs each. Three controls make the zeros falsifiable. The involution system $(C_2^3, \{x, y, z, xyz\})$ is *not* an MIS — xy and xz have all four generators as maximal common lower bounds — and both the pruned and the unpruned test report the failure; the first implementation of the unpruned test had its maximality condition inverted and returned zero on that control, and every count reported here is post-correction. The example W_A of Section 3 is an EMIS and no pair in its ball fails. And the companion rule (4), run on W_A, returns $\widetilde{A}_2$, which is the companion graph [1] records for it. □

Remark 8.3. A single meet failure at length 12 or beyond would remove V_1, V_2 or V_3 from consideration; no length of testing can establish the semilattice property. The evidence above is therefore evidence of a candidate's survival and nothing more, and no claim is made here that any of the three systems is an EMIS.

9. What is left open

Certifying the meet-semilattice property is the bottleneck for Question 1.1, and it is a statement about all of W that no ball computation settles. A decidable sufficient condition for the right weak order of a 2-recognizable even involution system to be a complete meet-semilattice would, applied to V_1, V_2 or V_3, answer the second half of the Question negatively. By (F6) such a condition need only handle pairs, which is what makes it plausible that a finite criterion exists; the source's own machinery — irreducible cycles that weakly intersect, and its discussion of Garside shadows — is the natural place to look.

Three orbits of the shortlist remain unsettled: those of the relator sets $\{(ab)^2, (ac)^3, X\}$ with X one of *bcbdcd*, *bcdbcd*, *adcbcd*. These are 72 of the 192 survivors of Remark 4.1, and for them neither the growth series nor the companion graph is known: their displayed matrices are unverified in exactly the way Theorem 5.1 shows can fail, so any of them may be a false positive as well. A Knuth–Bendix search over the 48 length-first orderings of the four letters, with left-hand sides allowed up to length 26 and 120,000 completion steps, produced no finite complete system: for the first two sets all 48 orderings were exhausted without one, and for the third the run reached its time cap with no completion among the orderings tried. The signature is divergence rather than exhaustion of the budget, one run reaching a left-hand side of length 25 with only 38 rules. The structural route is visible instead: *bcbdcd* is the commutator relator $[c, bd]$, *bcdbcd* is $(bcd)^2$, which makes $\langle b, c, d\rangle$ the honeycomb system W_A, and *adcbcd* makes a redundant. In the first two the relators split by support — $\{(ab)^2, (ac)^3\}$ on $\{a, b, c\}$ and X on $\{b, c, d\}$, overlapping in the infinite dihedral $\langle b, c\rangle$ — so that, if both embeddings are injective, W is the corresponding amalgamated free product; a normal form for it is a construction rather than a search and would make the growth exact. That was not attempted here, and these completion runs are outside the formal development.

The first half of Question 1.1 — rationality of $W(q)$ for an arbitrary EMIS — is untouched here. Theorem 6.2 settles it for the nine systems considered, by producing a complete rewriting system in each case; whether every EMIS has one, for some ordering of its generators, is open, and the three unsettled orbits above are a reason for caution.

The median case. The source proves that an EIS whose Cayley graph is median is an EMIS (Proposition 3.11 of [1]), and records that the groups generated by involutions having a median Cayley graph are "studied and classified" by Scott [4] under the name of *right-angled mock reflection groups*. Growth series for that sub-class may already be known, and would either supply a counterexample or close off a region.

10. Verification

The computational content of every statement above is machine-checked in Lean 4 (toolchain `v4.32.0`), except where a claim is explicitly labelled as produced outside the formal development; the general lemmas that read those computations — Lemmas 5.2, 5.3, 5.4 and 8.1 — are proved here. The development is free of `sorry`, and the modules for this material import only Lean's standard library, not *Mathlib*, so no result of Coxeter theory is imported as a hypothesis. Each statement is established by compiled evaluation of a finite computation (Lean's `native_decide`) and depends on no axioms beyond `propext`, `Quot.sound` and one axiom per declaration recording that evaluation; `Classical.choice` and `Lean.ofReduceBool` appear nowhere. What is exhaustive computation is exactly this: the four rewriting conditions (R1)–(R4) of Theorem 6.1, including every critical pair and, for (R2), a union–find over all freely reduced words of length at most 8; the counts of irreducible words and the single vector recurrence of Theorem 6.2; the balls, joins, geodesic counts and 4-cycle censuses behind Theorems 5.1, 7.1 and 7.2; the two-sided bounds of Theorem 3.4 and Proposition 4.3, whose upper half enumerates all S-reduced words of length at most 12 (rank 3) or 8 (rank 4); the orbit computation of Theorem 4.2 over the 192 transcribed relator sets; and the meet tests of Theorem 8.2 up to length 8. The two bridges argued rather than formalised are named where they are used: that a surjection carries balls onto balls, and that termination plus local confluence gives unique normal forms. Every number quoted was first produced by an independent implementation in C or Python and the formal statement was then written against it; the deeper meet tests, the completion searches and the counts of Remark 4.1 are those implementations' output and are labelled as such wherever they appear. Repository reference to be added at submission.

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