NONNEGATIVITY OF DIFFERENCES OF TERM-NORMALISED COMPLETE HOMOGENEOUS SYMMETRIC FUNCTIONS: DEGREES 8, 9 AND 10 IN THREE VARIABLES, DECIDED EXACTLY

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ABSTRACT. For a partition λ of d let h_λ be the product of complete homogeneous symmetric polynomials in n variables and $H_{n,\lambda} = h_\lambda / h_\lambda(1^n)$ its term-normalised form. Cuttler, Greene and Skandera proved that dominance of partitions implies $H_{n,\lambda} \leq H_{n,\mu}$ on the nonnegative orthant and conjectured the converse; Heaton and Shankar refuted the converse at $n = 3$, $d = 8$ with a sum of 41 squares, reported that their search found "many other counterexamples, in degrees 8, 9, and 10", and stated that those were left in floating point. We settle, exactly and in full, the three degrees they name, for $n = 3$. Of the 270 ordered questions "is $H_{3,\mu} - H_{3,\lambda} \geq 0$ on $\mathbb{R}^3_{\geq 0}$?" posed by the 135 dominance-incomparable pairs of degree 8, 9 or 10, exactly 58 have the answer yes and 212 have the answer no; none is left undecided. Each positive answer is certified by between 42 and 63 nonnegative integers — the coefficients of the difference after the substitution $x_1 = a$, $x_2 = a + b$, $x_3 = a + b + c$, that is, one step of the classical successive difference substitution method — and each negative answer by one of only five rational points. We also correct one printed coefficient in the expansion displayed by Heaton and Shankar, prove that their counterexample survives in four variables, and observe that the number of counterexamples is not monotone in n. Nonnegativity, not sum-of-squares representability, is what is decided here. All statements are machine-checked in Lean 4.

1. Introduction

The objects. Let $h_d(x_1, \ldots, x_n)$ denote the complete homogeneous symmetric polynomial of degree d, that is, the sum of all $\binom{n+d-1}{d}$ monomials of degree d, with $h_0 = 1$. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of d put $h_\lambda = \prod_i h_{\lambda_i}$, and let

$$H_{n,\lambda}(x) \;=\; \frac{h_\lambda(x)}{h_\lambda(1, \ldots, 1)}$$

be the associated *term-normalised* symmetric function, in which every polynomial is scaled to take the value 1 at the all-ones point; see [12, Ch. 7] for the symmetric function background. Write $\mu \succeq \lambda$ for the dominance (majorization) order: $\mu_1 + \cdots + \mu_k \;\geq\; \lambda_1 + \cdots + \lambda_k$ for all k.

Cuttler, Greene and Skandera [2] classified the Muirhead-type inequalities $G_\lambda \leq G_\mu$ on the nonnegative orthant for the monomial, elementary, power-sum and Schur bases: in each case the inequality holds exactly when the two partitions are comparable in dominance. For the complete homogeneous basis they proved one implication,

$$\mu \succeq \lambda \implies H_{n,\lambda} \leq H_{n,\mu} \text{ on } \mathbb{R}^n_{\geq 0} \text{ for every } n,$$

verified the converse for $d \leq 7$, and conjectured it in general.

Heaton and Shankar [4] disproved the converse. Their counterexample uses the pair $\mu = (4, 4)$ and $\lambda = (5, 2, 1)$, incomparable in dominance, with $n = 3$: they exhibit

$$\big(H_{3,(4,4)} - H_{3,(5,2,1)}\big)(x_1^2, x_2^2, x_3^2)$$

as a sum of 41 squares with rational coefficients, obtained from a symmetry-reduced semidefinite program followed by continued-fraction rounding and an exact LDL^T factorisation. Neither the 45×45 Gram matrix nor the list of 41 squares is printed in the paper: it displays the first row of the matrix and the first of the polynomials to be squared, and refers the reader to the first author's repository [1] for the matrix and the remaining squares and to a repository page at MPI MiS [5] for verification code. The size of the certificate can be read off the part that is printed: of the ten coefficients of the displayed polynomial, nine are rationals whose numerators and denominators have between 70 and 80 decimal digits.

The gap this paper closes. Section 3 of [4] reports, verbatim:

> our search over other pairs of partitions returned many other counterexamples, in degrees 8, 9, and 10. We left these other counterexamples in floating point, though we found an exact, rational SOS certificate in the case of $H_{44} - H_{521}$.

The same section displays a Hasse diagram of the resulting order on partitions of degree 8, 9 and 10 in three variables, whose arrows are, in the authors' words, "certified numerically". Two things are therefore left open by that figure. First, the positive relations it asserts are numerical, not proved. Second — and this is the larger gap — the figure says nothing at all about the pairs it does *not* join: a semidefinite program that fails to find a certificate has refuted nothing, since a nonnegative polynomial need not be a sum of squares, and a numerical search that fails has not even refuted sum-of-squares representability.

We close both, for $n = 3$ and for exactly the three degrees named. Fix $n = 3$ and a degree $d \in \{8, 9, 10\}$. Each pair $\{\mu, \lambda\}$ of partitions of d that is incomparable in dominance poses two ordered questions,

$$Q(\mu, \lambda): \qquad \text{is } \; H_{3,\mu} - H_{3,\lambda} \;\geq\; 0 \; \text{ on } \mathbb{R}^3_{\geq 0}\,?$$

(Dominance-comparable pairs are settled by [2] and pose no question.) There are 15, 35 and 85 incomparable pairs in degrees 8, 9, 10, hence 270 ordered questions in all. Our main results answer every one of them.

- **Degree** 8 (Theorems 3.1–3.4): 6 of the 30 ordered questions have the answer yes and 24 have the answer no. One of the six is the theorem of [4]; one is the $n = 3$ instance of a theorem of Xu and Yao [8]; the remaining four are proved here for the first time.
- **Degree** 9 (Theorem 3.5): 14 yes, 56 no, out of 70.
- **Degree** 10 (Theorem 3.6): 38 yes, 132 no, out of 170.
- No question is left undecided, and nothing is left in floating point.

The instrument is not the one used in [4], and it is not new. Because h_λ is symmetric, nonnegativity on $\mathbb{R}^3_{\geq 0}$ is nonnegativity on the sorted cone $0 \leq x_1 \leq x_2 \leq x_3$, which is the image of $\mathbb{R}^3_{\geq 0}$ under $(a, b, c) \mapsto (a,\, a+b,\, a+b+c)$. A certificate is then simply the statement that the difference, rewritten in the coordinates (a, b, c), has all coefficients nonnegative. This is the first level of the *successive difference substitution* method of Yang [10] (see also [7], [11]), which Xu and Yao [9] apply to precisely these objects; equivalently it is a minimal-degree Handelman representation [3] in the linear forms x_1, $x_2 - x_1$, $x_3 - x_2$ that cut out the cone. (We cite [9] only for its record of which method its authors used and of how far they computed; the arXiv comment field of that preprint states that the article contains uncorrectable errors, and none of our results rely on it.) We claim nothing for the method. What we record is that at $n = 3$ its *first* level — no iteration, no degree elevation, no subdivision — decides every one of the 270 questions, and that the certificates are short vectors of nonnegative integers where [4] needed 41 squares with seventy-digit coefficients. For the Heaton–Shankar pair itself the certificate is 42 nonnegative integers, the largest being 143 910.

Two further results are recorded, and one erratum.

An erratum (Section 4). The 45-term expansion of $\big(H_{(4,4)} - H_{(5,2,1)}\big)(x_1^2, x_2^2, x_3^2)$ displayed in Section 1 of [4] prints -6 as the coefficient of $x_1^{10}x_2^4x_3^2$; the correct value is -48. The slip is typographical and does not touch the theorem of [4], whose certificate is the off-paper Gram matrix rather than this display. We were unable to find it reported.

Four variables (Section 5). The closing remark of [4] expects the counterexamples to thin out as n grows and asks whether the conjecture is true asymptotically. We prove $H_{4,(5,2,1)} \leq H_{4,(4,4)}$ on $\mathbb{R}^4_{\geq 0}$, so the degree-8 counterexample survives the first step; and we observe, outside the formal development, that the count of counterexamples is not monotone in n: at $d = 9$ the positive ordered cells number exactly 14 at $n = 3$, by Theorem 3.5, but at least 15 at $n = 4$, and one cell that provably fails at $n = 3$ holds at $n = 4$.

What is not decided. Everything below concerns *nonnegativity* on the orthant. Nonnegativity does not imply sum-of-squares representability, and no claim about the latter is made or contradicted here. In particular the numerically-certified SOS assertions of [4] are untouched: for the 58 positive cells we prove the inequality and say nothing about whether the corresponding even polynomial is a sum of squares, and for the 212 negative cells we prove that the inequality fails, which does of course rule out SOS-ness there as well. See Section 7.

A caveat on the number of variables. The notation $H_\lambda \le H_\mu$ of [4] means, by its own definition, that the inequality holds for *every* number of variables. As Xu and Yao point out in a footnote of [8], the $n = 3$ certificate of [4] therefore refutes the "for some n" variant of the Cuttler–Greene–Skandera converse rather than the "for all n" variant. Every statement in this paper fixes n explicitly — $n = 3$ throughout, except for Section 5, where $n = 4$ — so the distinction does not arise, but the reader comparing our tables with the figure of [4] should keep it in mind.

2. Preliminaries

2.1. **Normalisation, and the integer polynomial behind each cell.** Throughout $n = 3$ unless stated otherwise, and we abbreviate $H_\lambda = H_{3,\lambda}$. Put

$$A = h_\lambda(1,1,1), \qquad B = h_\mu(1,1,1), \qquad P_{\mu,\lambda} \;=\; A\,h_\mu \;-\; B\,h_\lambda.$$

Then $A, B > 0$ and $P_{\mu,\lambda} = AB\,(H_\mu - H_\lambda)$, so $P_{\mu,\lambda}$ is an integer polynomial with the same sign as $H_\mu - H_\lambda$ everywhere. Every certificate and every witness below is a statement about $P_{\mu,\lambda}$.

Lemma 2.1 (Normalisation)**.** *Let λ, μ be partitions of d, let $A = h_\lambda(1,1,1)$ and $B = h_\mu(1,1,1)$, and suppose $A, B > 0$. If $A\,h_\mu(x) - B\,h_\lambda(x) \ge 0$ for all $x \in \mathbb{R}^3_{\ge 0}$, then $H_\lambda \le H_\mu$ on $\mathbb{R}^3_{\ge 0}$.*

Proof. $H_\mu - H_\lambda = (A\,h_\mu - B\,h_\lambda)/(AB)$ and $AB > 0$. □

2.2. **The sorted cone.** Let $\Delta = \{(x_1, x_2, x_3) : 0 \le x_1 \le x_2 \le x_3\}$ be the sorted cone. The map

$$\sigma(a,b,c) = (a,\; a+b,\; a+b+c)$$

carries $\mathbb{R}^3_{\ge 0}$ onto Δ. The following is the first level of successive difference substitution, in the form we use it.

Lemma 2.2 (Cone reduction)**.** *Let $P\colon \mathbb{R}^3 \to \mathbb{R}$ satisfy $P(x,y,z) = P(y,x,z)$ and $P(x,y,z) = P(x,z,y)$ for all x, y, z. If $P(\sigma(a,b,c)) \ge 0$ for all $a, b, c \ge 0$, then $P \ge 0$ on $\mathbb{R}^3_{\ge 0}$.*

Proof. The two hypotheses generate all of S_3 acting on the arguments. Given $x, y, z \ge 0$, sort them; the sorted triple is $\sigma(a,b,c)$ for the nonnegative successive differences a, b, c, so P is nonnegative there, and it takes the same value at (x,y,z) by symmetry. Formally the proof splits into the six orderings. □

Definition 2.3 (Certificate)**.** A *cone certificate* for the pair (μ, λ) is a finite list of quadruples $(c_t, i_t, j_t, k_t) \in \mathbb{N}^4$ such that

$$P_{\mu,\lambda}\big(\sigma(a,b,c)\big) \;=\; \sum_t c_t\, a^{i_t} b^{j_t} c^{k_t} \qquad \text{identically in } a, b, c.$$

Two features of this shape are worth naming, because they are what makes the classification cheap to verify rather than merely cheap to find. First, the coefficients c_t are natural numbers, so the nonnegativity of the right-hand side on $\mathbb{R}^3_{\ge 0}$ is structural: there is no inequality to check and no rounding to trust, only a polynomial identity. Second, the identity itself is a single expansion. Combining Definition 2.3 with Lemmas 2.2 and 2.1:

Proposition 2.4. *If the pair (μ, λ) admits a cone certificate, then $H_\lambda \le H_\mu$ on $\mathbb{R}^3_{\ge 0}$.*

The dual side needs no machinery at all: a single point $x \in \mathbb{R}^3_{\geq 0} \cap \mathbb{Q}^3$ with $P_{\mu,\lambda}(x) < 0$ refutes $H_\lambda \leq H_\mu$ on $\mathbb{R}^3_{\geq 0}$.

2.3. **Sizes.** A polynomial of degree d in three variables has $\binom{d+2}{2}$ monomials, so a cone certificate in degree d has at most 45, 55, 66 terms for $d = 8, 9, 10$. The certificates occurring below are nearly dense: exactly 42 terms in degree 8, either 51 or 52 in degree 9, either 62 or 63 in degree 10. Their coefficients are small: the largest occurring anywhere is 4 017 600 in degree 8, 63 218 880 in degree 9 and 1 380 784 320 in degree 10. For the Heaton–Shankar pair the certificate has 42 terms and its largest coefficient is 143 910.

3. Degrees 8, 9 and 10 in three variables

3.1. **Degree** 8**.** We begin with the pair of [4], whose statement is not new but whose proof here is of a different size and kind.

Theorem 3.1. *$H_{(5,2,1)} \leq H_{(4,4)}$ on $\mathbb{R}^3_{\geq 0}$, and $(4,4)$ and $(5,2,1)$ are incomparable in dominance order.*

Proof sketch. Here $A = h_{(5,2,1)}(1,1,1) = 378$ and $B = h_{(4,4)}(1,1,1) = 225$. Expanding

$$378\, h_{(4,4)}\big(\sigma(a,b,c)\big) - 225\, h_{(5,2,1)}\big(\sigma(a,b,c)\big)$$

one finds that all 42 nonzero coefficients are positive integers, the largest 143 910; the claim follows from Proposition 2.4. The verification is a polynomial identity between two explicit expressions, not a search. Incomparability is a comparison of partial sums: $4 < 5$ but $4 + 4 > 5 + 2$. □

For contrast, the proof in [4] runs a symmetry-reduced semidefinite program, rounds its floating-point output by continued fractions, factors the resulting rational 45×45 matrix exactly, and obtains 41 squares with seventy-digit coefficients, published outside the paper. Both proofs are correct and they prove different things: [4] proves the stronger statement that the associated even polynomial is a sum of squares, which the argument above does not give.

Theorem 3.2. *The following four inequalities hold on $\mathbb{R}^3_{\geq 0}$, each between a dominance-incomparable pair of partitions of* 8*:*

$$H_{(5,1,1,1)} \leq H_{(4,4)}, \quad H_{(3,3,2)} \leq H_{(5,1,1,1)}, \quad H_{(4,1,1,1,1)} \leq H_{(3,3,2)}, \quad H_{(2,2,2,2)} \leq H_{(4,1,1,1,1)}.$$

Proof sketch. Four cone certificates, of 42 nonnegative integers each, and Proposition 2.4. □

Relations of this kind are asserted by the figure of [4, §3] as arrows "certified numerically", and the accompanying text says that the counterexamples behind them were left in floating point. Theorem 3.2 makes four of them exact. They are not covered by [8], which proves one pair in each degree; that pair, at $d = 8$, is the following.

Theorem 3.3. *$H_{(3,1,1,1,1,1)} \leq H_{(2,2,2,2)}$ on $\mathbb{R}^3_{\geq 0}$.*

Proof sketch. A cone certificate of 42 nonnegative integers, and Proposition 2.4. □

This is the $n = 3$ instance of the main theorem of Xu and Yao [8], who prove $H_{n,(2^{\lfloor d/2 \rfloor},1^{d-2\lfloor d/2 \rfloor})} \geq H_{n,(3,1^{d-3})}$ on $\mathbb{R}^n_{\geq 0}$ for every n and every $d \geq 8$, by an induction on n that passes through a polynomial optimisation problem on the standard simplex. Only the proof, not the statement, is ours.

The negative half is where the classification becomes a classification.

Theorem 3.4. *For each of the remaining* 24 *ordered dominance-incomparable pairs (μ, λ) of partitions of* 8 *there is an explicit point $x \in \mathbb{R}^3_{\geq 0} \cap \mathbb{Q}^3$ with $H_\mu(x) < H_\lambda(x)$; consequently $H_\lambda \leq H_\mu$ fails on $\mathbb{R}^3_{\geq 0}$. Together with Theorems 3.1, 3.2 and 3.3, all* 30 *ordered questions of degree* 8 *at $n = 3$ are answered:* 6 *yes,* 24 *no, none undecided.*

$H_{(5,2,1)} \leq H_{(4^2)}$ *	$H_{(5,1^3)} \leq H_{(4^2)}$
$H_{(3^2,2)} \leq H_{(5,1^3)}$	$H_{(4,1^4)} \leq H_{(3^2,2)}$
$H_{(2^4)} \leq H_{(4,1^4)}$	$H_{(3,1^5)} \leq H_{(2^4)}$ †

Table 1. The six ordered dominance-incomparable pairs of degree 8 for which $H_\lambda \leq H_\mu$ holds on $\mathbb{R}^3_{\geq 0}$. For each of the other 24, it fails. * Heaton–Shankar [4]. † the $n = 3$ instance of Xu–Yao [8].

$H_{(6,1^3)} \leq H_{(5,4)}$	$H_{(5,2^2)} \leq H_{(6,1^3)}$
$H_{(4,3,2)} \leq H_{(6,1^3)}$	$H_{(3^3)} \leq H_{(6,1^3)}$
$H_{(5,2,1^2)} \leq H_{(4^2,1)}$	$H_{(5,1^4)} \leq H_{(4^2,1)}$
$H_{(3^2,2,1)} \leq H_{(5,1^4)}$	$H_{(3,2^3)} \leq H_{(5,1^4)}$
$H_{(4,2^2,1)} \leq H_{(3^3)}$	$H_{(4,2,1^3)} \leq H_{(3^3)}$
$H_{(4,1^5)} \leq H_{(3^3)}$	$H_{(4,1^5)} \leq H_{(3^2,2,1)}$
$H_{(2^4,1)} \leq H_{(4,1^5)}$	$H_{(3,1^6)} \leq H_{(2^4,1)}$ †

Table 2. The 14 ordered dominance-incomparable pairs of degree 9 for which $H_\lambda \leq H_\mu$ holds on $\mathbb{R}^3_{\geq 0}$. For each of the other 56, it fails. † the $n = 3$ instance of Xu–Yao [8].

Proof sketch. Each refutation is the evaluation of $P_{\mu,\lambda}$ at one rational point and the observation that the value is negative; the twenty-four points used are among the five listed in Remark 3.7. Completeness is a finite enumeration: the 22 partitions of 8 were listed, the 15 dominance-incomparable pairs among them extracted by comparing partial sums, and each of the resulting 30 ordered questions assigned to exactly one of the two lists. The incomparability of the 15 pairs is itself verified (Proposition 3.10). □

3.2. **Degree** 9.

Theorem 3.5. *Let* $d = 9$ *and* $n = 3$. *Of the* 70 *ordered questions posed by the* 35 *dominance-incomparable pairs of partitions of* 9*, exactly* 14 *have the answer yes and* 56 *have the answer no; none is undecided. The* 14 *positive ones are those listed in Table 2, each certified by a cone certificate of* 51 *or* 52 *nonnegative integers, and each of the* 56 *negative ones is refuted at an explicit rational point of* $\mathbb{R}^3_{\geq 0}$.

3.3. **Degree** 10.

Theorem 3.6. *Let* $d = 10$ *and* $n = 3$. *Of the* 170 *ordered questions posed by the* 85 *dominance-incomparable pairs of partitions of* 10*, exactly* 38 *have the answer yes and* 132 *have the answer no; none is undecided. The* 38 *positive ones are those listed in Table 3, each certified by a cone certificate of* 62 *or* 63 *nonnegative integers, and each of the* 132 *negative ones is refuted at an explicit rational point of* $\mathbb{R}^3_{\geq 0}$.

Proof sketch of Theorems 3.5 and 3.6. The shape is that of Theorem 3.4. The finite search is the following, and it is the only place where exhaustiveness is used. For $d = 9$ (resp. $d = 10$) the 30 (resp. 42) partitions of d are enumerated, the 35 (resp. 85) dominance-incomparable pairs among the $\binom{30}{2} = 435$ (resp. $\binom{42}{2} = 861$) pairs are selected by comparing partial sums, and for each of the 70 (resp. 170) resulting ordered questions the polynomial $P_{\mu,\lambda}(\sigma(a, b, c))$ is expanded over $\mathbb{Q}$: either all its coefficients are nonnegative, in which case the coefficient list is a cone certificate, or a negative coefficient appears and a rational point of $\mathbb{R}^3_{\geq 0}$ with $P_{\mu,\lambda} < 0$ is exhibited. The search terminated

$H_{(6,4)} \leq H_{(8,1^2)}$	$H_{(5^2)} \leq H_{(8,1^2)}$
$H_{(6,2^2)} \leq H_{(7,1^3)}$	$H_{(5,3,2)} \leq H_{(7,1^3)}$
$H_{(4^2,2)} \leq H_{(7,1^3)}$	$H_{(4,3^2)} \leq H_{(7,1^3)}$
$H_{(6,3,1)} \leq H_{(5^2)}$	$H_{(6,2^2)} \leq H_{(5^2)}$
$H_{(6,2,1^2)} \leq H_{(5^2)}$	$H_{(6,1^4)} \leq H_{(5^2)}$
$H_{(6,1^4)} \leq H_{(5,4,1)}$	$H_{(5,2^2,1)} \leq H_{(6,1^4)}$
$H_{(4,3,2,1)} \leq H_{(6,1^4)}$	$H_{(4,2^3)} \leq H_{(6,1^4)}$
$H_{(3^3,1)} \leq H_{(6,1^4)}$	$H_{(3^2,2^2)} \leq H_{(6,1^4)}$
$H_{(5,2^2,1)} \leq H_{(4^2,2)}$	$H_{(5,2,1^3)} \leq H_{(4^2,2)}$
$H_{(5,2,1^3)} \leq H_{(4^2,1^2)}$	$H_{(5,2,1^3)} \leq H_{(4,3^2)}$
$H_{(3^2,2^2)} \leq H_{(5,2,1^3)}$	$H_{(5,1^5)} \leq H_{(4^2,2)}$
$H_{(5,1^5)} \leq H_{(4^2,1^2)}$	$H_{(5,1^5)} \leq H_{(4,3^2)}$
$H_{(3^2,2,1^2)} \leq H_{(5,1^5)}$	$H_{(3,2^3,1)} \leq H_{(5,1^5)}$
$H_{(2^5)} \leq H_{(5,1^5)}$	$H_{(4,2^2,1^2)} \leq H_{(3^3,1)}$
$H_{(4,2,1^4)} \leq H_{(3^3,1)}$	$H_{(4,2,1^4)} \leq H_{(3^2,2^2)}$
$H_{(2^5)} \leq H_{(4,2,1^4)}$	$H_{(4,1^6)} \leq H_{(3^3,1)}$
$H_{(4,1^6)} \leq H_{(3^2,2^2)}$	$H_{(4,1^6)} \leq H_{(3^2,2,1^2)}$
$H_{(2^4,1^2)} \leq H_{(4,1^6)}$	$H_{(3,2,1^5)} \leq H_{(2^5)}$
$H_{(3,1^7)} \leq H_{(2^5)}$ †	$H_{(3,1^7)} \leq H_{(2^4,1^2)}$

TABLE 3. The 38 ordered dominance-incomparable pairs of degree 10 for which $H_\lambda \leq H_\mu$ holds on $\mathbb{R}^3_{\geq 0}$. For each of the other 132, it fails. † the $n = 3$ instance of Xu–Yao [8].

with every question in one of the two classes. Each individual verdict is then re-proved from scratch: a positive one as a polynomial identity together with Proposition 2.4, a negative one as an arithmetic evaluation at the exhibited rational point. No step uses floating point. □

Remark 3.7 (The refuting points). Across all three degrees the 212 refutations use only five points of $\mathbb{R}^3_{\geq 0}$, namely

$$(0,0,1), \quad (0, \tfrac{1}{2}, 1), \quad (0,1,1), \quad (\tfrac{1}{2}, \tfrac{1}{2}, 1), \quad (\tfrac{2}{3}, \tfrac{2}{3}, 1),$$

with multiplicities 133, 19, 10, 28, 22. The first of these is degenerate and explains the majority: since $h_d(0,0,1) = 1$ for every d, one has $P_{\mu,\lambda}(0,0,1) = h_\lambda(1,1,1) - h_\mu(1,1,1)$, so those 133 refutations reduce to a comparison of the two normalising constants. Every point used has homogeneous representative in $\{0,1,2,3\}^3$.

Corollary 3.8. *For $n = 3$ and $d \in \{8,9,10\}$ the relation $H_\lambda \leq H_\mu$ on $\mathbb{R}^3_{\geq 0}$ is decided for all* 270 *ordered dominance-incomparable pairs: it holds in* 58 *cases and fails in* 212*. In particular no incomparable pair satisfies the inequality in both directions, so at each of these three degrees the relation "$H_\lambda \leq H_\mu$ on $\mathbb{R}^3_{\geq 0}$" strictly contains dominance — by* 6*,* 14 *and* 38 *ordered pairs at $d = 8$, 9, 10 respectively — and is nevertheless far from total:* 9*,* 21 *and* 47 *of the incomparable pairs stay incomparable for H as well.*

Proof. The first sentence is Theorems 3.1–3.6. For the second, the 58 positive ordered cells listed in Tables 1–3 involve 58 distinct unordered pairs, so no pair occurs in both directions; subtracting them from the 15, 35 and 85 incomparable pairs leaves 9, 21 and 47 pairs refuted in both directions. Proposition 3.10(ii) exhibits one of the nine. □

Remark 3.9 (A caveat on priority). The relations of Theorems 3.2, 3.5 and 3.6, and the refutations that complete them, are as far as we can determine not in the literature: they are not in [4], which says in as many words that it left them in floating point, and not in [8], which proves one pair per degree. They may nevertheless have been computed before. The predecessor preprint [9] reports that its authors verified their own, different, conjecture "by explicit computation up through $d = 12$ and $n = 12$", by the same difference substitution method applied to the same objects. It publishes no table of relations, so nothing of what is below can be read off it; but a reader assessing priority should know that a computation of this kind was very likely carried out and not written down.

3.4. **Controls.** Two failure modes would make the above vacuous: the pairs might not really be incomparable, or the orthant hypothesis might be inert. Both are excluded.

Proposition 3.10. (i) *Each of the* 15 *pairs of partitions of* 8 *used above is incomparable in dominance in both directions, and dominance is not vacuous at $d = 8$: $(8) \succeq (1^8)$ and $(1^8) \not\succeq (8)$.*

(ii) *Not every dominance-incomparable pair is comparable for H. For the pair $(5,3)$, $(6,1,1)$ of partitions of* 8*, both $H_{(5,3)} \le H_{(6,1,1)}$ and $H_{(6,1,1)} \le H_{(5,3)}$ fail on $\mathbb{R}^3_{\ge 0}$.*

(iii) *The hypothesis $x \in \mathbb{R}^3_{\ge 0}$ is not decorative. The inequality $H_{(3,3,2)} \le H_{(5,1,1,1)}$ holds on $\mathbb{R}^3_{\ge 0}$ but fails at $(-2,0,1)$, where $600\, h_{(5,1,1,1)} - 567\, h_{(3,3,2)}$ takes the value $-29\,925$.*

4. An erratum to a displayed expansion

Section 1 of [4] displays $\big(H_{(4,4)} - H_{(5,2,1)}\big)(x_1^2, x_2^2, x_3^2)$ as $\frac{1}{9450}$ times an explicit integer polynomial with 45 terms. The normalising constant is right: $h_{(4,4)}(1,1,1) = 15^2 = 225$, $h_{(5,2,1)}(1,1,1) = 21{\cdot}6{\cdot}3 = 378$, and $\operatorname{lcm}(225, 378) = 9450$, so that

$$9450\big(H_{(4,4)} - H_{(5,2,1)}\big) \;=\; 42\, h_{(4,4)} - 25\, h_{(5,2,1)}, \tag{1}$$

whose 45 coefficients are exactly what the display should carry. Forty-four of them do.

Theorem 4.1. *Let $D_{\mathrm{pr}}(x_1, x_2, x_3)$ be the* 45*-term integer polynomial printed inside the parentheses in the display of* [4, §1]*, and let D be the same expression with the coefficient of $x_1^{10}x_2^4x_3^2$ changed from -6 to -48. Then*

$$D(x_1,x_2,x_3) \;=\; 9450\big(H_{(4,4)} - H_{(5,2,1)}\big)(x_1^2,x_2^2,x_3^2), \qquad D_{\mathrm{pr}} - D \;=\; 42\, x_1^{10}x_2^4x_3^2,$$

and D_{pr} is not a symmetric function: $D_{\mathrm{pr}}(1,2,1) \ne D_{\mathrm{pr}}(2,1,1)$.

Proof sketch. The first identity is (1) expanded and compared coefficient by coefficient; the coefficient in question is that of $x_1^5x_2^2x_3$ in $42\, h_{(4,4)} - 25\, h_{(5,2,1)}$. Its value in $h_{(4,4)}$ is the number of ways of splitting the exponent vector $(5,2,1)$ into two vectors of total degree 4, which is 6; its value in $h_{(5,2,1)}$ is the number of ways of splitting it into vectors of total degrees 5, 2 and 1, which is 12; so the coefficient is $42 \cdot 6 - 25 \cdot 12 = -48$. The second identity is subtraction. The third is two evaluations: $D_{\mathrm{pr}}(1,2,1) = 1\,256\,550$ while $D_{\mathrm{pr}}(2,1,1) = 1\,298\,886$. □

There is a second, independent reason to believe -48, and it uses no computation of ours at all. A difference of symmetric functions is symmetric, so its coefficients are constant on S_3-orbits of monomials. The orbit of $x_1^{10}x_2^4x_3^2$ has six members, and [4] itself prints -48 on the other five, namely $x_1^4x_2^{10}x_3^2$, $x_1^{10}x_2^2x_3^4$, $x_1^2x_2^{10}x_3^4$, $x_1^4x_2^2x_3^{10}$ and $x_1^2x_2^4x_3^{10}$. A display with a lone -6 among five -48's cannot be the expansion of a difference of symmetric functions, which is what the last clause of Theorem 4.1 says in the form of a single pair of evaluations.

We stress what this does and does not affect. It is a typographical slip in a display, and the theorem of [4] does not rest on the display: the proof there factors the 45×45 Gram matrix published at [1], and Theorem 3.1 above independently confirms the inequality that the display illustrates. A reader who takes the printed polynomial at face value, however, is looking at a non-symmetric polynomial, and one that is $42\, x_1^{10}x_2^4x_3^2$ too large. We were unable to find the correction recorded anywhere. On 2026-08-29 the works citing [4] numbered one in OpenAlex and six in Semantic Scholar, the former among the latter; all six were read, and none is an erratum or records this correction.

5. Four variables, and the dependence on n

The closing remark of [4] reads, in part: "as n increases, we will likely see fewer counterexamples. An interesting question is whether the conjecture is true asymptotically i.e. does $H_\lambda \le H_\mu$ imply $\mu \succeq \lambda$ for large enough n?" Their own certificate is for $n = 3$ only. The same route works one variable up.

Theorem 5.1. *$H_{4,(5,2,1)} \le H_{4,(4,4)}$ on $\mathbb{R}^4_{\ge 0}$.*

Proof sketch. Lemmas 2.1 and 2.2 have four-variable analogues; the cone reduction now sorts four arguments, which we do in eight cases (choose the slot holding the maximum, then sort the remaining three) rather than the naive twenty-four. Here $h_{(5,2,1)}(1^4) = 2240$ and $h_{(4,4)}(1^4) = 1225$, and the expansion of $2240\, h_{(4,4)} - 1225\, h_{(5,2,1)}$ in the coordinates $(a, b, c, e) \mapsto (a,\ a+b,\ a+b+c,\ a+b+c+e)$ has 161 terms, all with nonnegative integer coefficients. $\square$

We record this without novelty claim: once difference substitution is on the table this is a single expansion, and Xu and Yao report having run computations of exactly this kind up to $n = 12$ [9]. What we have not found is anyone writing it down for this pair, which is the pair the footnote of [8] singles out as having been settled "merely" at $n = 3$.

Remark 5.2 (Outside the formal development)*.* The following two computations were carried out in exact rational arithmetic in a separate program and are *not* part of the machine-checked development; we state them as observations, not theorems.

(a) The same construction certifies $H_{n,(5,2,1)} \le H_{n,(4,4)}$ on $\mathbb{R}^n_{\ge 0}$ for $n = 2, \dots, 8$: the expansion of $h_{(5,2,1)}(1^n)\, h_{(4,4)} - h_{(4,4)}(1^n)\, h_{(5,2,1)}$ in successive-difference coordinates has all coefficients nonnegative, with 7, 42, 161, 490, 1281, 2996 and 6427 terms respectively. So the degree-8 counterexample survives at least to eight variables.

(b) The expectation that counterexamples thin out as n grows fails already at the first step. In degree 9 the number of ordered incomparable cells admitting a first-level cone certificate is 14 at $n = 3$ and 15 at $n = 4$; the extra cell is $H_{(5,4)} \le H_{(7,1,1)}$, which *fails* at $n = 3$. The mechanism is visible. At the point $(0, \dots, 0, 1)$ every h_d equals 1, so $P_{\mu,\lambda}(0, \dots, 0, 1) = h_\lambda(1^n) - h_\mu(1^n)$; for $\lambda = (5, 4)$ and $\mu = (7, 1, 1)$ this is

$$n = 3 :\ 21 \cdot 15 - 36 \cdot 9 = 315 - 324 = -9, \qquad n = 4 :\ 56 \cdot 35 - 120 \cdot 16 = 1960 - 1920 = +40.$$

The $n = 3$ refutation of that cell is exactly this corner value — it is one of the 56 refutations of Theorem 3.5 — and the sign change at $n = 4$ is a single comparison of products of binomial coefficients.

What is claimed here, and what is not. At $n = 3$ the count 14 is exact, because Theorem 3.5 decides all 70 cells. At $n = 4$ we exhibit 15 cells where the inequality holds, so the count strictly increases; that is the non-monotonicity. We do *not* claim that 15 is the exact count at $n = 4$: having a first-level cone certificate is sufficient for the inequality and need not be necessary, and we have not shown that the remaining 55 cells fail at $n = 4$.

Question 5.3. Is the set of ordered incomparable pairs (μ, λ) of a fixed degree d with $H_{n,\mu} \ge H_{n,\lambda}$ on $\mathbb{R}^n_{\ge 0}$ eventually monotone in n? Remark 5.2 shows it is not monotone at $d = 9$ between $n = 3$ and $n = 4$.

6. Why a box-subdivision certificate cannot work here

It is natural to try to certify these inequalities by dehomogenising and running a Bernstein box method: set $x_3 = 1$, restrict to the square $[0, 1]^2$ by symmetry, expand in the tensor Bernstein basis of bidegree (N, N), and subdivide until every coefficient is nonnegative. The following remark explains why that fails for every one of the positive cells, at every subdivision depth and every degree. Like Remark 5.2 it is an observation about the method rather than a machine-checked theorem, and we record it because the failure is structural and easy to mistake for a budget problem.

Remark 6.1 (Outside the formal development). Let μ, λ be partitions of d and $P = P_{\mu,\lambda}$. Since $H_\nu(t,t,t) = t^d$ for every ν, the difference $H_\mu - H_\lambda$ vanishes identically on the diagonal; so P vanishes there, and if $P \geq 0$ on $\mathbb{R}^3_{\geq 0}$ the diagonal is a line of minima. At $(1,1,1)$ the gradient of P therefore vanishes, and its Hessian is an S_3-invariant matrix $\beta I + \gamma(J - I)$ killed by $(1,1,1)$, by Euler's relation for homogeneous functions; hence $\gamma = -\beta/2$ and the Hessian is $\alpha(I - \frac{1}{3}J)$ with $\alpha = \frac{3}{2}\beta$. For the Heaton–Shankar pair, $\alpha = 14\,175$.

Now dehomogenise at $x_3 = 1$ and put $x_1 = 1 - s$, $x_2 = 1 - t$, so that the diagonal point sits at the corner $s = t = 0$. The quadratic part of P there is

$$\tfrac{\alpha}{3}\big(s^2 - st + t^2\big),$$

positive definite, but with a *negative* cross term. That negative cross term is fatal to the box method. On a subbox $[0,h]^2$ anchored at the corner, the Bernstein coefficient of bidegree (N,M) at the multi-index $(1,1)$ picks out only the monomials 1, s, t, st of P: the Bernstein coefficient of s^p at index 1 is proportional to the falling factorial $1 \cdot 0 \cdots (2-p)$, which vanishes as soon as $p \geq 2$. The constant and linear terms are zero, so that Bernstein coefficient equals $-\frac{\alpha}{3} \cdot h^2/(NM)$, which is negative for every side length $h > 0$ and every bidegree. Subdivision is scale-invariant here: whatever the depth, the subbox containing the diagonal corner still has a negative Bernstein coefficient, so no box-subdivision certificate exists at any depth. A direct run confirms the prediction — the check fails at the root and at every depth up to 40.

The sorted-cone substitution repairs exactly this. In the coordinates (a,b,c) the diagonal is the edge $b = c = 0$ and a runs along it, so the transverse quadratic is the one to look at: substituting $s = b + c$ and $t = c$ into the display above turns it into $\frac{\alpha}{3}(b^2 + bc + c^2)$, and the cross term changes sign. The reason is that the cone $\{0 \leq x_1 \leq x_2 \leq x_3\}$ meets the diagonal along one of its edges rather than transversally. Concretely, for the Heaton–Shankar pair the substituted difference $P(\sigma(a,b,c))$ has $4725\,a^6(b^2 + bc + c^2)$ as its part of degree at most 2 in (b,c), and $4725 = \alpha/3$.

7. What is not decided

Sums of squares. Nonnegativity on the orthant is strictly weaker than the property studied in [4], which is that $(H_\mu - H_\lambda)(x_1^2, x_2^2, x_3^2)$ is a sum of squares. We prove the former and say nothing about the latter, except where the inequality fails, in which case the polynomial is not nonnegative and hence not a sum of squares. For the 58 positive cells the SOS question remains exactly where [4] left it: one of them, the Heaton–Shankar cell, has an exact rational SOS certificate; the rest were, in the authors' words, left in floating point. Deciding SOS-ness exactly for the other 57 would need exact rational semidefinite programming, which the present method does not provide and does not replace.

Higher degrees, and other n. Degrees 11 and 12 at $n = 3$ have been classified by the same exact rational computation outside the formal development — 340 and 770 ordered questions, all decided at the first level of difference substitution — but are not formalised here. The obstruction is not time but the size of the compiled artefacts (see Section 8).

Tightness of the first level. Every one of the 270 questions is decided at the *first* level of successive difference substitution: whenever the inequality holds, the substituted polynomial already has all coefficients nonnegative, with no iteration and no degree elevation. We have no explanation for this and no proof that it persists.

Question 7.1. Is the first level of successive difference substitution complete for differences of term-normalised complete homogeneous symmetric functions in a fixed number of variables? Writing $\sigma_n(a_1, \ldots, a_n) = (a_1,\ a_1 + a_2,\ \ldots,\ a_1 + \cdots + a_n)$, does $H_{n,\lambda} \leq H_{n,\mu}$ on $\mathbb{R}^n_{\geq 0}$ imply that $\big(h_\lambda(1^n)h_\mu - h_\mu(1^n)h_\lambda\big) \circ \sigma_n$ has all coefficients nonnegative?

A uniform certificate in n. Remark 5.2(a) certifies $H_{n,(5,2,1)} \leq H_{n,(4,4)}$ for $n \leq 8$, one n at a time. The coefficient patterns are visibly regular, which suggests an induction on n. Such a proof would give for the Heaton–Shankar pair what [8] gives for $\big((2^{\lfloor d/2 \rfloor}, 1^*), (3, 1^{d-3})\big)$, and would settle the "for all n" form of the counterexample at the minimal degree.

8. Verification

Every theorem and proposition in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [6]; the development is free of `sorry` and contains no floating-point arithmetic anywhere. It comprises 353 theorems: the reusable lemmas of Section 2; the 58 positive cells, each as a polynomial identity between the substituted difference and its certificate followed by the structural nonnegativity of a sum of monomials with natural-number coefficients; the 212 refutations, each as an arithmetic evaluation at the exhibited rational point; the three statements of Theorem 4.1; the controls of Proposition 3.10, of which the incomparability of the fifteen degree-8 pairs is decided by kernel evaluation of the partial-sum comparison; and the four-variable machinery behind Theorem 5.1. Nothing is delegated to compiled evaluation: the development uses no `native_decide`, and every headline declaration depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. Making the certificate coefficients natural numbers is what removes the need for any trusted numeric step: their nonnegativity is a property of the type, not a computation.

The finite searches that produced the data are the ones described in the proof sketch of Theorems 3.5 and 3.6, and they were run in a separate implementation over exact rationals before any Lean was written. Their output — which cells are positive, the certificate for each, the refuting point for each of the rest — is data, and the formal development re-derives every individual verdict from the definitions without consulting the search. Both Remark 5.2 and Remark 6.1 lie outside the formal development and are labelled as such in the text.

Cost, measured on a single machine: about 18 CPU-minutes in total, dominated by the degree-10 module (170 cells, 648 CPU-seconds). The binding constraint is not time but artefact size and memory: the degree-10 module peaks at 13.1 GB of resident memory and compiles to a 127 MB object file, because its 38 certificate lists carry 2393 quadruples, some 9600 numerals, which the elaborator turns into 38 very large terms. Extrapolating, degrees 11 and 12 would together cost roughly 1.5 CPU-hours but some 950 MB of object files, which is why they are not included; splitting one module per degree and sign would remove the obstruction.

The formal development is available in the authors' repository; repository reference to be added at submission.

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counter-examples and new conjectures, and on 2026-08-29 the preprint's arXiv comment field read "There are uncorrectable errors in this article".

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