TWELVE-ELEMENT COUNTERMODELS TO WILKIE'S IDENTITY: RIGIDITY, TWO ERRATA, AND THE 2026 CLASSIFICATION

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Abstract. A *Gurevič–Burris algebra* is a finite algebra satisfying the eleven high school identities in which Wilkie's identity fails; twelve is the smallest size at which one exists, as two independent preprints of August 2026 establish. Only two twelve-element examples have been exhibited: Figure 4 of Burris and Yeats (2005) and the algebra of Hajdari and Niederhauser (2026). We ask how rigid such an algebra is. Our main result is the complete profile of the single-entry perturbations of the exponentiation table of the Hajdari–Niederhauser algebra: of the 1,584 ways to change one of its 144 exponentiation entries to another element of the carrier, exactly eighteen again satisfy all eleven identities, and they occupy exactly nine positions, two values at each. Thirteen of the eighteen are again countermodels; the other five are twelve-element models of the eleven identities in which Wilkie's identity holds everywhere. Transported into the classification of all 8,957,952 twelve-element countermodels of Subercaseaux and Przybocki, the nine positions are exactly the nine cells their exponentiation template leaves free one at a time, and the five perturbations that lose the countermodel are exactly the five single-cell violations of that template's ε-condition. As a corollary, the exponentiation table printed by Hajdari and Niederhauser — which differs in thirteen entries from the model their own solver assignment encodes, and fails two of the eleven identities — is repairable in exactly one way at each of those thirteen cells. We also record what the printed tables and the shipped verification program contain, verify Burris and Yeats' Figure 4 and the thirteen-element countermodel to Gurevič's univariate identity, and locate both twelve-element algebras inside the classification. Every statement is machine-checked.

1. Introduction

Tarski's *high school algebra problem* asks whether every identity true of the positive integers under $+$, $\cdot$, exponentiation and the constant 1 is derivable from the eleven *high school identities*

(H1) $x + y = y + x$	(H2) $(x + y) + z = x + (y + z)$	(H3) $x \cdot y = y \cdot x$
(H4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	(H5) $x \cdot (y + z) = x \cdot y + x \cdot z$	(H6) $x \cdot 1 = x$
(H7) $x^{y+z} = x^y \cdot x^z$	(H8) $(x \cdot y)^z = x^z \cdot y^z$	(H9) $(x^y)^z = x^{y \cdot z}$
(H10) $1^x = 1$	(H11) $x^1 = x$.	

Wilkie answered no, by exhibiting an identity valid over the positive integers and not derivable from (H1)–(H11) [9]. Writing

$$P = 1 + x, \qquad Q = 1 + x + x^2, \qquad R = 1 + x^3, \qquad S = 1 + x^2 + x^4,$$

Wilkie's identity is

$$W(x, y): \qquad (P^x + Q^x)^y \cdot (R^y + S^y)^x \; = \; (P^y + Q^y)^x \cdot (R^x + S^x)^y. \tag{1}$$

A finite algebra in the signature $(+, \cdot, \uparrow, 1)$ satisfying (H1)–(H11) in which W fails at some pair of carrier elements refutes derivability semantically; following the literature we call such an algebra a *Gurevič–Burris algebra*. The smallest known size came down from 59, Gurevič's [5], to 12, the twelve-element example being Figure 4 of Burris and Yeats' survey *The Saga of the High School Identities* [4], where it is conjectured to be smallest possible (Conjecture 3 there).

Two preprints of August 2026 settle that conjecture, eight days apart and without citing one another. Subercaseaux and Przybocki [8] prove by a SAT encoding that there is no Gurevič–Burris algebra of size at most 11; they also classify *all* twelve-element Gurevič–Burris algebras up to isomorphism — there are $768 \cdot 11{,}664 = 8{,}957{,}952$ of them — by one addition template with thirteen free parameters, one fixed multiplication table, and one exponentiation template with fourteen free parameters, and they exhibit a thirteen-element countermodel to Gurevič's univariate identity. Hajdari and Niederhauser [7] prove the same non-existence result by a different encoding, and exhibit a twelve-element Gurevič–Burris algebra which they report to be non-isomorphic to the one previously known.

Neither preprint settles the question Burris and Yeats pose as their Problem 10: is there an algebra with fewer than twelve elements that satisfies (H1)–(H11) but is not in the variety generated by the positive integers? A countermodel to W is such an algebra, so twelve is an upper bound; the best published lower bound is three, since all identities of the positive integers hold in every two-element algebra satisfying (H1)–(H11) [2]. Subercaseaux and Przybocki restate the question as their principal open problem and report that Alsulami and Jackson announce a lower bound of four in future work ([8, §8], citing [2] and [1]).

What is settled here. This paper is about how rigid a twelve-element Gurevič–Burris algebra is, and what its rigidity has to say about the classification of [8]. Write $\mathbf{A}$ for the Hajdari–Niederhauser algebra and $\mathbf{B}$ for Burris and Yeats' Figure 4.

Section 5 contains the main computation. Of the $144 \cdot 11 = 1{,}584$ ways to change a single entry of the exponentiation table of $\mathbf{A}$ to a different element of the carrier, exactly eighteen leave an algebra satisfying (H1)–(H11), and these eighteen sit at exactly nine of the 144 positions, two admissible values at each (Theorem 14). So $\mathbf{A}$ is rigid at 135 of its 144 exponentiation cells.

Section 6 reads that profile through the classification of [8]. Under an explicit relabelling carrying $\mathbf{A}$ onto its place in the classification, the nine free positions land exactly on the five δ-cells and four ε-cells of the exponentiation template — the nine cells the template leaves free one parameter at a time — and on nothing else; the eleven cells carrying the template's γ-parameters, which the classification allows to vary only in three joint packages, are rigid to any single-entry change (Theorem 17). Sharper: thirteen of the eighteen perturbations are still Gurevič–Burris algebras, failing W at the same single pair, and the remaining five satisfy W everywhere; those five are exactly the five single-cell moves forbidden by the ε-condition of [8]. The classification and the perturbation computation were produced independently of one another; the condition is stated in [8] as part of a template, and what the five cases here add is that each of its single-cell violations really does destroy the countermodel at this instance.

Section 4 records two discrepancies in the published material of [7], both machine-checked against the sources as they stand. The exponentiation table printed in Table 1 of [7] (v1) differs in thirteen entries from the table that the solver assignment in the authors' own artifact encodes, and as printed it fails two of the eleven identities, at 179 triples in all (Propositions 10 and 11). The non-isomorphism program in the same artifact compares that printed table against a structure whose multiplication and exponentiation tables are not those of Burris and Yeats' Figure 4, and which fails six of the eleven identities (Proposition 12). The conclusion [7] draws is nevertheless correct, and Theorem 8 proves it by a short argument rather than by a 12! search. A consequence of the rigidity profile is that the correction of Table 1 requires no artifact at all: none of the thirteen cells is free, so each printed entry admits exactly one repair (Corollary 15).

Sections 3 and 7 carry out the verifications on which the rest rests: that **A**, **B** and the thirteen-element algebra **G** of [8] really are what they are claimed to be, where W fails in each of them, and that both twelve-element algebras are instances of the template of [8], with their relabellings and parameters given explicitly. As those two algebras predate the classification — one of them by twenty-one years — and neither arose from the search that produced it, this is an outside test of the classification, which it passes.

All finite statements below are proved by exhaustive evaluation inside the Lean 4 kernel; Section 9 says what was run and what it rests on.

2. Preliminaries

Definition 1. For $n \geq 1$, an *n-element structure* $\mathbf{M}$ consists of a carrier $\{1, \dots, n\}$ together with three two-place operations $+$, $\cdot$ and $\uparrow$ on it, the element 1 serving as the constant of the signature. We write x^y for $\uparrow(x, y)$, and present each operation as an $n \times n$ table whose rows and columns are indexed by the carrier in the order $1, \dots, n$. We call $\mathbf{M}$ *closed* if all three tables take values in the carrier, and an *HSI-algebra* if it satisfies (H1)–(H11) at every triple of carrier elements.

In evaluating (1) in a structure whose operations need not be associative or commutative — which happens below — the parse matters, so we fix it. With $s := x \cdot x$ put

$$P := 1 + x, \quad Q := (1 + x) + s, \quad R := 1 + (s \cdot x), \quad S := (1 + s) + (s \cdot s),$$

and set

$$W_L(x, y) := \big((P^x + Q^x)^y\big) \cdot \big((R^y + S^y)^x\big), \qquad W_R(x, y) := \big((P^y + Q^y)^x\big) \cdot \big((R^x + S^x)^y\big).$$

We say W *fails* in $\mathbf{M}$ at (x, y) when $W_L(x, y) \neq W_R(x, y)$, and write $\mathrm{Fail}_W(\mathbf{M})$ for the set of such pairs, listed in lexicographic order. An HSI-algebra $\mathbf{M}$ with $\mathrm{Fail}_W(\mathbf{M}) \neq \emptyset$ is a *Gurevič–Burris algebra.* In an HSI-algebra the bracketing above is immaterial, by (H1)–(H4).

Two further pieces of vocabulary. The *integer chain* of $\mathbf{M}$ is the sequence $c_0 = 1$, $c_{k+1} = c_k + 1$; its members are the *integers* of $\mathbf{M}$. For a structure $\mathbf{M}$ on $\{1, \dots, n\}$, a position (i, j) and a value v in the carrier with $v \neq i^j$, we write

$$\mathbf{M}[(i, j) \mapsto v]$$

for the structure obtained from $\mathbf{M}$ by replacing the single entry of the exponentiation table in row i and column j by v, leaving $+$, $\cdot$ and the other $n^2 - 1$ exponentiation entries untouched. There are $n^2(n-1)$ such *single-entry perturbations*; for $n = 12$ that is $144 \cdot 11 = 1{,}584$.

Finally, a homomorphism convention. By an *isomorphism* $\mathbf{M} \to \mathbf{N}$ of n-element structures we mean a pair of maps p, q carrying the carriers into one another and mutually inverse there, with p commuting with all three operations on carrier elements. Nothing is assumed about the constant: that an isomorphism of HSI-algebras must fix 1 is derived in Theorem 8. We call p a *multiplicative isomorphism* when only the condition on $\cdot$ is imposed.

3. The two twelve-element algebras

3.1. **The Hajdari–Niederhauser algebra.** Let **A** be the twelve-element structure whose addition and multiplication tables are those printed in Table 1 of [7], and whose exponentiation table is the one in Table 1 below. Its three tables are exactly what the solver assignment `12sat.txt` in the artifact accompanying [7] decodes to under the decoding performed by the authors' own checking script in the same artifact. As Proposition 10 records, the addition and multiplication tables of **A** agree entry for entry with the printed ones, and the exponentiation table does not; the thirteen entries in which it differs are set in bold.

$+$	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	11	2	3	9	11	11	3	3	11	11
2	3	11	11	3	11	3	11	11	11	11	11	11
3	11	11	11	11	11	11	11	11	11	11	11	11
4	2	3	11	10	12	10	11	11	3	11	11	11
5	3	11	11	12	11	11	3	12	11	11	11	11
6	9	3	11	10	11	10	11	11	3	11	11	11
7	11	11	11	11	3	11	11	3	11	11	11	11
8	11	11	11	11	12	11	3	11	11	11	11	11
9	3	11	11	3	11	3	11	11	11	11	11	11
10	3	11	11	11	11	11	11	11	11	11	11	11
11	11	11	11	11	11	11	11	11	11	11	11	11
12	11	11	11	11	11	11	11	11	11	11	11	11

$\cdot$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	11	11	10	11	10	11	11	11	11	11	11
3	3	11	11	11	11	11	11	11	11	11	11	11
4	4	10	11	6	12	6	11	11	10	10	11	11
5	5	11	11	12	11	11	11	12	11	11	11	11
6	6	10	11	6	11	6	11	11	10	10	11	11
7	7	11	11	11	11	11	11	11	11	11	11	11
8	8	11	11	11	12	11	11	11	11	11	11	11
9	9	11	11	10	11	10	11	11	11	11	11	11
10	10	11	11	10	11	10	11	11	11	11	11	11
11	11	11	11	11	11	11	11	11	11	11	11	11
12	12	11	11	11	11	11	11	11	11	11	11	11

$\uparrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	11	11	7	7	11	11	11	11	11	11	11
3	3	11	11	5	8	11	11	5	11	11	11	12
4	4	6	6	6	6	6	6	6	6	6	6	6
5	5	11	11	11	12	11	11	11	11	11	11	11
6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**
8	8	11	11	**12**	11	11	11	**12**	11	11	11	11
9	9	11	11	7	11	11	11	11	11	11	11	11
10	10	11	11	11	11	11	11	11	11	11	11	11
11	11	11	11	11	11	11	11	11	11	11	11	11
12	12	11	11	11	11	11	11	11	11	11	11	11

Table 1. The three tables of the Hajdari–Niederhauser algebra **A**. The thirteen bold entries of the exponentiation table are the cells where Table 1 of [7] (v1) prints a different value: 7 in place of each bold 11, and 11 in place of each bold 12.

Proposition 2. **A** *is a closed HSI-algebra.*

Proof. Exhaustive evaluation: $11 \cdot 12^3 = 19{,}008$ axiom instances for (H1)–(H11), and $3 \cdot 144$ table entries for closure. $\square$

Proposition 3. $\mathrm{Fail}_W(\mathbf{A}) = \{(4,5)\}$*, and there* $W_L(4,5) = 11$ *while* $W_R(4,5) = 12$*. In particular* $\mathbf{A}$ *is a Gurevič–Burris algebra.*

Proof. Evaluation of W_L and W_R at all 144 pairs. The failing pair itself is reported in [7]; that it is the only one is not stated there, nor in [8] or [4]. □

Proposition 4. *The integer chain of* $\mathbf{A}$ *begins* $1, 2, 3, 11, 11, 11$*, and* $11 + 1 = 11$*, so it is constant from its fourth term on. Neither* 4 *nor* 5 *is an integer of* $\mathbf{A}$*, and* $4 \cdot x \neq 5$ *for every carrier element* x*.*

Proof. Six additions for the chain, one table lookup for $11 + 1$, and twelve products for the divisibility claim. The three statements reproduce inside $\mathbf{A}$ the constraints that results of Burris and Lee [3] and of Zhang [10] place on any Gurevič–Burris algebra: at least three distinct integers, and a witness pair consisting of non-integers with the first not dividing the second. We have not consulted [3] or [10] directly; those attributions, and the numbered results they rest on, are the ones [7] and [8] give when they recite the constraints ([7, §2], [8, §4]). □

3.2. **Burris and Yeats' Figure 4.** Let $\mathbf{B}$ be the twelve-element structure of Figure 4 of [4], whose elements $1, 2, 3, 4, a, b, c, d, e, f, g, h$ we number $1, \ldots, 12$ in that order.

Proposition 5. $\mathbf{B}$ *is a closed HSI-algebra. Its integer chain begins* $1, 2, 3, 4, 4, 4$*, and* $4 + 1 = 4$*, so it is constant from its fifth term on; neither* a *nor* e *is an integer of* $\mathbf{B}$*.*

Proof. As for Proposition 2. The tables were transcribed by hand from Figure 4 of the authors' preprint of [4]; the journal version is behind a paywall and was not consulted. The transcription is self-checking, in that a wrong transcription would be very unlikely to satisfy all 19,008 axiom instances and to fail W at exactly one pair, as Proposition 6 shows it does. A second, independent check on the addition table alone: it agrees entry for entry with the table the artifact accompanying [7] hard-codes for the same algebra (Proposition 12) — a transcription made by other hands, from a source cited there as the journal version. □

Proposition 6. $\mathrm{Fail}_W(\mathbf{B}) = \{(a, e)\}$*, and there* $W_L = 4$ *while* $W_R = h$*.*

Proof. Evaluation at all 144 pairs. That Figure 4 refutes W is the point of printing it in [4]; where it fails is not stated there, and we have found it nowhere else. □

Remark 7. The failing pairs of $\mathbf{A}$ and $\mathbf{B}$ carry different labels for a reason that is not mathematical: both 2026 preprints normalise a countermodel so that $2 = 1 + 1$, $3 = 2 + 1$ and the witnesses are the first two free labels $4, 5$, whereas [4] was written before that convention.

3.3. **Non-isomorphism.**

Theorem 8. *There is no isomorphism* $\mathbf{B} \to \mathbf{A}$*.*

Proof. Let (p, q) be one. Since q is defined on the whole carrier of $\mathbf{A}$ and $p \circ q$ is the identity there, p is onto; so for every b in the carrier of $\mathbf{A}$ we have $b \cdot p(1) = p(q(b)) \cdot p(1) = p(q(b) \cdot 1) = p(q(b)) = b$, using $x \cdot 1 = x$ in $\mathbf{B}$. Thus $p(1)$ is a multiplicative identity of $\mathbf{A}$; taking $b = 1$ and using commutativity of $\cdot$ in $\mathbf{A}$ gives $p(1) = 1$. Consequently $p(2) = p(1+1) = p(1) + p(1) = 2$ and $p(3) = p(2+1) = 3$. Now count the solutions of $x + 1 = 3$: in $\mathbf{B}$ there are six, namely $2, b, d, e, f, g$, and in $\mathbf{A}$ there are four, namely $2, 5, 9, 10$. Since p carries each solution in $\mathbf{B}$ to a solution in $\mathbf{A}$ and is injective on the carrier, we get an injection of a six-element set into a four-element set, which is absurd.

The two solution counts, the two additions used, and $x \cdot 1 = x$ in $\mathbf{B}$ together with commutativity of $\cdot$ in $\mathbf{A}$ are the only finite checks; the rest is the argument above. In particular no search over the 12! bijections is performed. $\square$

The claim of Theorem 8 is made in [7], where the algebra is reported as "not isomorphic to the previously known 12-element model"; the proof appears to be new, for the reason given in Proposition 12. The next proposition explains why the argument above has to use addition.

Proposition 9. *The multiplicative reducts of* $\mathbf{B}$ *and* $\mathbf{A}$ *are isomorphic, by the permutation*

$$\sigma. : \quad 1, 2, 3, 4, a, b, c, d, e, f, g, h \longmapsto 1, 2, 3, 11, 4, 10, 6, 9, 5, 7, 8, 12.$$

That same permutation does not carry $+$ *across.*

Proof. 144 identities $\sigma.(x \cdot y) = \sigma.(x) \cdot \sigma.(y)$ together with injectivity on the carrier, and one counterexample for $+$. The proposition is a corollary of the classification of [8], whose 8,957,952 models share a single multiplication table (Section 6); neither preprint draws it. It says that no invariant of $\cdot$ alone can separate the two algebras. $\square$

4. What is printed and what is shipped

The propositions of this section are machine-checked comparisons against the sources as they stood on 30 August 2026: the v1 e-print of [7] (its abstract page listing no later version), the files of the artifact accompanying it, and Figure 4 of the authors' preprint of [4]. They are stated as findings about those documents.

Proposition 10. *The addition and multiplication tables printed in Table 1 of* [7] *(v1) agree entry for entry with the ones the solver assignment in the accompanying artifact decodes to. The printed exponentiation table differs from that assignment's in exactly thirteen entries: the eleven entries* $(7, j)$ *for* $2 \leq j \leq 12$*, where the printed value is* 7 *and the assignment's is* 11*; and the two entries* $(8, 4)$ *and* $(8, 8)$*, where the printed value is* 11 *and the assignment's is* 12*.*

Proof. Entrywise comparison of $3 \cdot 144$ pairs of values. Both sides were transcribed mechanically: the printed tables from the LaTeX source of the v1 e-print, the assignment's tables from the artifact file through the decoding the authors' own script performs. $\square$

Write $\mathbf{A}^{\mathrm{pr}}$ for the twelve-element structure whose three tables are those printed in Table 1 of [7] (v1).

Proposition 11. $\mathbf{A}^{\mathrm{pr}}$ *is not an HSI-algebra. Its vector of violation counts for (H1)–(H11) is*

$$(0, 0, 0, 0, 0, 0, 144, 0, 35, 0, 0),$$

so (H7) fails at 144 *triples and (H9) at* 35*,* 179 *in all, while the other nine identities hold throughout. The smallest witness is* $7^{1+1} = 7 \neq 11 = 7 \cdot 7$*. Wilkie's identity fails in* $\mathbf{A}^{\mathrm{pr}}$ *at* $(4, 5)$ *and at no other pair.*

Proof. Exhaustive evaluation of the $11 \cdot 12^3$ axiom instances, counting rather than stopping at the first failure, and of W at all 144 pairs. Nine identities holding and two failing is also the non-vacuity check on the axiom conjunction: it genuinely constrains. □

The artifact accompanying [7] contains a program whose stated purpose is the verification, "by exhaustive permutation search, checking whether any permutation simultaneously preserves" the three operations, of the non-isomorphism claim proved above as Theorem 8. Let $\mathbf{K}$ be the twelve-element structure the program uses as the "previously known" algebra, that is, the structure given by its three hard-coded tables for that side of the comparison.

Proposition 12. *The addition table of* $\mathbf{K}$ *is the addition table of Burris and Yeats' Figure 4. Its multiplication table differs from Figure 4's in eleven entries and its exponentiation table in twenty-four.* $\mathbf{K}$ *is not an HSI-algebra: its vector of violation counts for (H1)–(H11) is*

$$(0, 0, 216, 322, 305, 0, 467, 109, 282, 0, 0),$$

so six of the eleven identities — (H3), (H4), (H5), (H7), (H8), (H9) — fail there, at 1,701 *triples in all; in particular its multiplication is not commutative. Wilkie's identity fails in* $\mathbf{K}$ *at no pair. On the other side of the comparison, the exponentiation table the program uses for the new algebra is entrywise the printed table of Proposition 10, not the assignment's.*

Proof. Entrywise comparison against Figure 4 and against the two exponentiation tables of Proposition 10; exhaustive evaluation of the $11 \cdot 12^3$ axiom instances and of W at all 144 pairs. The program's tables were read from the artifact file and shifted from its 0-based to our 1-based labelling. □

Remark 13. One further observation about that program is a property of its source text rather than of a table, and is therefore not part of the formal development: its main routine invokes the permutation search three times, once per operation, rather than once for the three operations simultaneously. Proposition 9 is what makes the difference visible — on the tables of $\mathbf{A}$ and $\mathbf{B}$ the three separate searches would answer "no", "yes", "no".

5. Rigidity of the exponentiation table

We now fix $\mathbf{A}$ and perturb one entry of its exponentiation table at a time.

Theorem 14. *Of the* 1,584 *single-entry perturbations* $\mathbf{A}[(i, j) \mapsto v]$*, exactly eighteen are HSI-algebras. They occupy exactly nine of the* 144 *positions, two values at each:*

$$\begin{array}{llll}(2,4) \mapsto 11, 12 & (2,5) \mapsto 11, 12 & (2,7) \mapsto 7, 12 & (2,8) \mapsto 7, 12 \\ (3,7) \mapsto 7, 12 & (9,4) \mapsto 11, 12 & (9,5) \mapsto 7, 12 & (9,7) \mapsto 7, 12 \\ (9,8) \mapsto 7, 12. & & & \end{array}$$

At each of the other 135 *positions, every one of the eleven alternative values destroys at least one of (H1)–(H11).*

Proof. The claim is an exhaustive statement about a search space of size 1,584, and both halves of it are verified by computation.

The positive half. For each of the eighteen listed perturbations, a full sweep of the $11 \cdot 12^3 = 19{,}008$ axiom instances confirms that all eleven identities hold.

The negative half. Sweeping the remaining 1,566 perturbations in the same way is possible but expensive, so it is replaced by a certificate. For each rejected perturbation the certificate carries one witness — the index of an identity among (H1)–(H11) and a triple of carrier elements at which that identity fails in the perturbed structure — packed with the perturbation's own (i, j, v) into a single integer, seven base-16 fields to an entry. Checking the certificate is 1,566 evaluations of a single axiom instance rather than 1,566 sweeps. Two further finite checks close the argument: that the list of cells the certificate refers to is exactly the list of the 1,584 perturbations with the eighteen removed, in order; and that every certificate entry is well formed, in that its identity index is one of the eleven and its three witnesses lie in the carrier. Together with the elementary fact that a structure with one failing axiom instance is not an HSI-algebra, this gives the rejection of all 1,566.

The two halves combine into an equivalence: for every (i, j, v) in the search space, $\mathbf{A}[(i, j) \mapsto v]$ is an HSI-algebra if and only if (i, j, v) is one of the eighteen. $\square$

Corollary 15. *None of the thirteen cells of Proposition 10 is among the nine positions of Theorem 14. Hence, at each of those thirteen cells, the value carried by* $\mathbf{A}$ *is the only element of the carrier that keeps all eleven identities when the other* 143 *exponentiation entries and the two other tables are held at their values in* $\mathbf{A}$*: the printed table of* [7] *is repairable in exactly one way at each cell, and the repair reproduces the model without reference to the solver assignment.*

Proof. A containment check on thirteen pairs against the nine positions, together with Theorem 14. The uniqueness asserted is cell by cell: each of the other 143 exponentiation entries is held at its value in $\mathbf{A}$. Whether some simultaneous change of several of the thirteen cells also restores the eleven identities is a different question, not claimed here and not computed. $\square$

6. The rigidity profile inside the classification

Subercaseaux and Przybocki [8, §7] classify the twelve-element Gurevič–Burris algebras up to isomorphism: each is an instance of one addition template with thirteen free parameters, five parameters $\alpha_1, \ldots, \alpha_5$ ranging over three joint options and eight parameters $\beta_1, \ldots, \beta_8$ each in $\{8, 12\}$, giving $3 \cdot 2^8 = 768$ addition tables; of one multiplication table with no free cells at all; and of one exponentiation template with fourteen free parameters, five parameters $\gamma_1, \ldots, \gamma_5$ ranging over three joint options, five parameters $\delta_1, \ldots, \delta_5$ each in $\{6, 7, 12\}$, and four parameters $\varepsilon_1, \ldots, \varepsilon_4$ subject to the condition recalled below, giving $3 \cdot 3^5 \cdot 16 = 11{,}664$ exponentiation tables and $768 \cdot 11{,}664 = 8{,}957{,}952$ algebras in all. The exponentiation template is

$\uparrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	γ_1	γ_1	2	2	2	2	2	2	2	2	2
4	4	12	γ_2	γ_3	γ_2	12	12	12	12	12	12	12
5	5	12	γ_3	γ_2	γ_3	12	12	12	12	12	12	12
6	6	12	12	12	12	12	12	12	12	12	12	12
7	7	12	12	12	12	12	12	12	12	12	12	12
8	8	12	γ_4	γ_5	γ_4	δ_1	7	12	12	12	12	12
9	9	12	ε_1	ε_2	δ_2	δ_3	12	12	12	12	12	12
10	10	12	ε_3	ε_4	δ_4	δ_5	12	12	12	12	12	12
11	11	12	12	12	12	12	12	12	12	12	12	12
12	12	12	12	12	12	12	12	12	12	12	12	12

with the three γ-options $(3,7,12,5,4)$, $(2,12,7,4,5)$, $(2,7,12,5,4)$. The condition on the ε-parameters, which we call the *ε-condition*, is that the pairs $(\varepsilon_1, \varepsilon_3)$ and $(\varepsilon_2, \varepsilon_4)$ each lie in one of

$$\mathcal{A} = \{(6,6)\}, \qquad \mathcal{B} = \{(6,7),(6,12)\}, \qquad \mathcal{C} = \{(7,6),(12,6)\},$$

and not in the same one; there are sixteen such choices. Note the shape of the template: 124 of its 144 cells carry constants, eleven carry the γ-parameters — which the classification allows to move only in three joint packages — and exactly nine, the five δ-cells $(8,6)$, $(9,5)$, $(9,6)$, $(10,5)$, $(10,6)$ and the four ε-cells $(9,3)$, $(9,4)$, $(10,3)$, $(10,4)$, carry a parameter of their own.

Proposition 16. *Both twelve-element algebras are instances of this classification. Explicitly, the relabellings*

$$\sigma_{\mathbf{A}} : \ 1,2,3,4,5,6,7,8,9,10,11,12 \ \longmapsto\ 1,9,8,3,4,2,6,5,10,11,12,7,$$
$$\sigma_{\mathbf{B}} : \ 1,2,3,4,a,b,c,d,e,f,g,h \ \longmapsto\ 1,9,8,12,3,11,2,10,4,6,5,7$$

are isomorphisms of $\mathbf{A}$ *and of* $\mathbf{B}$ *onto the template instances with parameters*

$$\begin{aligned}
\mathbf{A}: \quad & \alpha = (9,10,9,8,8), \quad \beta = (8,12,12,12,12,12,12,12),\\
& \gamma = (2,12,7,4,5), \quad \delta = (12,12,12,12,12), \quad \varepsilon = (6,6,6,12);\\
\mathbf{B}: \quad & \alpha = (9,10,9,8,8), \quad \beta = (8,8,8,8,8,8,8,8),\\
& \gamma = (2,12,7,4,5), \quad \delta = (12,12,12,12,12), \quad \varepsilon = (12,6,6,12).
\end{aligned}$$

Both parameter tuples satisfy the constraints of the classification, both instances are Gurevič–Burris algebras with W *failing at* $(3,4)$ *and nowhere else, and the two instances are distinct: they differ in seven* β*-cells and in* ε_1*.*

Proof. For each algebra: 12 range checks, 12 injectivity checks and $3 \cdot 144$ homomorphism conditions verify the relabelling; a full sweep of $11 \cdot 12^3$ axiom instances and 144 evaluations of W verify that the instance is a Gurevič–Burris algebra with the stated failing pair; membership of the parameter tuple in the classification's ranges is read off the three α-options, the three γ-options, the ranges of β and δ, and the ε-condition. Distinctness is an entrywise comparison of the two instantiated templates. That every twelve-element Gurevič–Burris algebra is an instance of the template is the classification of [8, §7] — whose Theorem 1.2 counts the isomorphism classes — and is not reproved here; what is checked is the two instances. Neither algebra arose from the search that produced the classification — $\mathbf{B}$ predates it by twenty-one years and $\mathbf{A}$ came from a different solver and a different encoding — so the check is an outside test. $\square$

The relabelling $\sigma_{\mathbf{A}}$ now lets us read the profile of Theorem 14 in the classification's own coordinates.

Theorem 17. *Under $\sigma_{\mathbf{A}}$, the nine free positions of Theorem 14 correspond exactly to the five δ-cells and the four ε-cells of the exponentiation template — no more and no fewer:*

$$\begin{array}{lllll} (2,4)\mapsto\varepsilon_1 & (2,5)\mapsto\varepsilon_2 & (2,7)\mapsto\delta_3 & (2,8)\mapsto\delta_2 & (3,7)\mapsto\delta_1 \\ (9,4)\mapsto\varepsilon_3 & (9,5)\mapsto\varepsilon_4 & (9,7)\mapsto\delta_5 & (9,8)\mapsto\delta_4. & \end{array}$$

In particular none of the eleven γ-cells is free: they are rigid to single-entry change. Moreover, of the eighteen perturbations of Theorem 14, thirteen are again Gurevič–Burris algebras — each of them failing W at $(4,5)$ and nowhere else — and five are twelve-element HSI-algebras in which Wilkie's identity holds at every pair. In the classification's coordinates the five are

$$\varepsilon_2\mapsto 12,\qquad \varepsilon_2\mapsto 7,\qquad \varepsilon_3\mapsto 12,\qquad \varepsilon_3\mapsto 7,\qquad \varepsilon_4\mapsto 6,$$

and these are exactly the single-cell moves that the ε-condition forbids at this instance.

Proof. The correspondence of positions is the image of the nine pairs of Theorem 14 under $\sigma_{\mathbf{A}}$, compared against the two lists of cells; the statement about the γ-cells is the corresponding containment check on eleven pairs. The split 13/5 and the identification of the five are 18 evaluations of Fail_W, each over all 144 pairs, followed by relabelling.

The last clause is an eight-case verification against the ε-condition, done by hand and stated here in full since it is short. At this instance $\varepsilon=(6,6,6,12)$, so $(\varepsilon_1,\varepsilon_3)=(6,6)\in\mathcal{A}$ and $(\varepsilon_2,\varepsilon_4)=(6,12)\in\mathcal{B}$, distinct classes as required. The four ε-cells admit two alternative values each, by Theorem 14 read through the correspondence above, and: $\varepsilon_1\mapsto 12$ and $\varepsilon_1\mapsto 7$ put $(\varepsilon_1,\varepsilon_3)$ in $\mathcal{C}$, still distinct from $\mathcal{B}$, and are allowed; $\varepsilon_2\mapsto 12$ and $\varepsilon_2\mapsto 7$ put $(\varepsilon_2,\varepsilon_4)$ at $(12,12)$ and $(7,12)$, in none of $\mathcal{A},\mathcal{B},\mathcal{C}$, and are forbidden; $\varepsilon_3\mapsto 12$ and $\varepsilon_3\mapsto 7$ put $(\varepsilon_1,\varepsilon_3)$ in $\mathcal{B}$, the class of $(\varepsilon_2,\varepsilon_4)$, and are forbidden; $\varepsilon_4\mapsto 6$ puts $(\varepsilon_2,\varepsilon_4)$ in $\mathcal{A}$, the class of $(\varepsilon_1,\varepsilon_3)$, and is forbidden, while $\varepsilon_4\mapsto 7$ keeps it in $\mathcal{B}$ and is allowed. Three allowed and five forbidden; the five forbidden ones are the five listed, and the ten δ-moves together with the three allowed ε-moves are the thirteen that remain countermodels. $\square$

Remark 18. The correspondence extends to the values and not only to the positions. Reading the eighteen perturbations of Theorem 14 through $\sigma_{\mathbf{A}}$, each of the five δ-cells sits at 12 in this instance and its two admissible values are 6 and 7, which are exactly the other two values the template allows a δ to take; and each of the four ε-cells likewise moves only within $\{6,7,12\}$, the three values the ε-condition permits an ε to carry. So at every free cell, the perturbations that survive are precisely the moves into the three-element value range the template gives that cell taken by itself; the ε-condition, which couples the four ε-cells, is what then decides which of them keep the countermodel. This reading is done by hand, off the explicit list in Theorem 14 and the relabelling of Proposition 16; unlike the correspondence of positions in Theorem 17, no single statement of the development asserts it.

Remark 19. The ε-condition is stated in [8] as part of the description of the template. Theorem 17 adds that at the instance $\mathbf{A}$ each of its five single-cell violations really does destroy the countermodel — the resulting structure is still a twelve-element HSI-algebra, but W holds in it everywhere — and that each of the thirteen single-cell moves the condition allows really does preserve one. The classification and the computation of Theorem 14 were carried out independently of each other. Whether the same holds at other instances is open; the cheapest test is **B**, which sits in a different ε-class, $(\varepsilon_1, \varepsilon_3) = (12, 6) \in \mathcal{C}$, and for which the same arithmetic — run on the assumption, not verified here, that the nine free cells and their value ranges are the same at **B** — predicts a split of 14/4 rather than 13/5. An unformalised computation agrees with that prediction; no statement of this paper asserts it, and the perturbation profile of **B** has not been put in the kernel.

7. The thirteen-element algebra and Gurevič's univariate identity

Gurevič singled out among the univariate identities valid over the positive integers but not derivable from (H1)–(H11) the substitution instance $y := 2^x$ of Wilkie's identity, conjecturing it smallest by growth rate and expecting it to have a competitively small countermodel [6, pp. 29–30]. We take both statements from the recitations in [4, Conjecture 4] and [8, §7.4]; [6] itself was not consulted. Appendix A of [8] exhibits one of size 13; write **G** for it.

Proposition 20. **G** *is a closed thirteen-element HSI-algebra with* $1 + 1 = 2$ *and* $2^4 = 5$, *and* $\mathrm{Fail}_W(\mathbf{G}) = \{(4, 5)\}$.

Proof. Exhaustive evaluation: $11 \cdot 13^3 = 24{,}167$ axiom instances, $3 \cdot 169$ table entries for closure, 169 evaluations of W, two table lookups. The tables were parsed from the LaTeX source of the v1 e-print of [8]. □

Proposition 21. *In* **G**, *the univariate identity obtained from* W *by substituting* $y := 2^x$ *fails at exactly one point, namely* $x = 4$, *where* $2^4 = 5$. *In the same algebra* $2 + 1 = 3$ *and the sibling identity obtained by substituting* $y := 3^x$ *holds at every point.*

Proof. Thirteen evaluations of W_L against W_R at the pairs $(x, 2^x)$, and thirteen more at the pairs $(x, 3^x)$, as x runs over the carrier. That **G** refutes the 2^x identity is the point of Appendix A of [8]. Two facts here go beyond it: that the failure is at exactly one point, and that the 3^x sibling holds throughout the same algebra, so that **G** separates the two substitutions. □

8. Controls

Every claim above is an assertion about finitely many evaluations of transcribed expressions, so the transcriptions themselves need checking. The following are the controls that were run.

Proposition 22. *Over the natural numbers, both sides of* (1) *agree at all* 25 *pairs of* $\{1, \ldots, 5\}^2$, *while the expression obtained by adding* 1 *to its right-hand side disagrees at all* 25. *The one-element algebra satisfies (H1)–(H11) and satisfies Wilkie's identity. In* **A** *we have* $3 + 1 = 11$ *and* $11 \notin \{1, 2, 3\}$.

Proof. Direct evaluation. The first statement is the positive control on the transcription of W itself, and it was run before W was used to judge any algebra; the second is its non-vacuity. The third refutes the too-large claim that every finite HSI-algebra refutes W. The fourth refutes the too-small claim that the integers of

A carry a sub-countermodel: they are not even closed under addition, which is the shape the published lower bounds take. Proposition 11, where nine of the eleven identities hold and two fail, is the non-vacuity control on the axiom conjunction. □

9. Verification

Every statement above other than those explicitly marked as read by hand carries a machine-checked proof in Lean 4. The four algebras, the structures $\mathbf{A}^{\mathrm{pr}}$ and $\mathbf{K}$, and the classification template are carried as explicit tables over $\{1, \ldots, n\}$; the eleven identities, W, closure, the isomorphism predicates and the single-entry perturbation are computable predicates on them; and every finite claim is discharged by the Lean kernel's `decide`, that is, by evaluation inside the kernel itself. No use is made of `native_decide` or of any native-evaluation axiom, so no compiled evaluator appears in any proof. Lean's `#print axioms` was run on the fifty-nine theorems of the development that carry the statements above and the facts those rest on; the only axioms occurring are `propext`, `Quot.sound` — which enters through list and finite-set decidability instances — and `Classical.choice`, the last in Theorem 8 alone, which is also the only statement in the development that uses Mathlib at all, for a pigeonhole lemma on finite sets. Seven of the fifty-nine, including the positive control of Proposition 22, depend on no axioms whatever, and no proof uses `sorry`. The one place where an unstructured kernel search would have been too expensive — the 1,566 rejected perturbations of Theorem 14 — is handled by the certificate described in that proof, whose soundness and coverage are themselves machine-checked; the eighteen surviving sweeps are run in six groups of three, for reasons of kernel memory alone. No table in the development is hand-typed except Figure 4 of [4]: a script regenerates every table literal and the certificate from the primary files — the solver assignment and the program of the artifact accompanying [7], the v1 e-prints of [7] and [8], and the authors' preprint of [4] — and re-derives every number quoted above. Repository reference to be added at submission.

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