HARMONIC INDEX TUPLES: COMPLETE CLASSIFICATIONS IN A_5 AND $C_2 \times S_4$, AND THE GROUP ORDERS THAT ADMIT A COUNTEREXAMPLE

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Abstract. A tuple $(a_1, \ldots, a_n)$ of positive integers is *G-harmonic* if the group G has subgroups $U_1, \ldots, U_n$ with $[G : U_i] = a_i$ whose left cosets can be translated so as to be pairwise disjoint, and *$\mathbb{Z}$-harmonic* if there are integers $r_1, \ldots, r_n$ with $r_i \not\equiv r_j \pmod{\gcd(a_i, a_j)}$ for $i \neq j$. Ginosar asked whether every G-harmonic tuple is $\mathbb{Z}$-harmonic; Margolis and Schnabel proved that it is for $n \leq 4$, and a recent preprint of Menon answers the question negatively at $n = 5$ by exhibiting five harmonic 5-tuples that are not $\mathbb{Z}$-harmonic: three in the alternating group A_5 and two in a group of order 48 isomorphic to $C_2 \times S_4$. We settle the question completely in both of those groups: an A_5-harmonic tuple of length at most 5 is $\mathbb{Z}$-harmonic unless its index multiset is $\{6, 6, 6, 10, 15\}$, $\{6, 6, 6, 15, 20\}$ or $\{6, 6, 10, 12, 15\}$, and a $C_2 \times S_4$-harmonic one unless its index multiset is $\{4, 6, 6, 6, 6\}$ or $\{6, 6, 6, 6, 16\}$, so Menon's five families are all of them; being statements about whole subgroup lattices, these also yield twelve index multisets that the two groups do not realise at all, of a kind neither source states. We then leave the two groups and ask how small a counterexample can be. The answer is exactly 12: no group of order less than 12 carries a G-harmonic tuple that fails to be $\mathbb{Z}$-harmonic, at any length, while $C_2 \times S_3$ carries Menon's own multiset $(4, 6, 6, 6, 6)$ — four times smaller than the smallest order he reports. A refined count against a subgroup of index 2 and odd order then shows that no group of order 18 carries one either, by proof rather than by search; the same count, carried out on paper, settles every order $2p^k$ with p an odd prime, and every group order below 48 except 42 is decided, by theorem or by exhaustive computation. We further extend Menon's four remaining families, and the order-12 counterexample, to genuine coset partitions. Every theorem below is machine-checked in Lean 4, apart from two that are marked where they are stated as proved on paper and verified by machine only outside the kernel, and one minimality clause marked where it is proved.

1. Introduction

The two notions. Let G be a group and $n \geq 1$. Following Margolis and Schnabel [2], a tuple $(a_1, \ldots, a_n)$ of positive integers is *G-harmonic* if there are subgroups $U_1, \ldots, U_n \leq G$ and elements $g_1, \ldots, g_n \in G$ with

$$[G : U_i] = a_i \quad (1 \leq i \leq n), \qquad g_i U_i \cap g_j U_j = \varnothing \quad (i \neq j),$$

and *$\mathbb{Z}$-harmonic* if there are integers $r_1, \ldots, r_n$ such that the arithmetic progressions $r_i + a_i\mathbb{Z}$ are pairwise disjoint, equivalently

$$\gcd(a_i, a_j) \nmid r_i - r_j \qquad (i \neq j).$$

The second notion is precisely the first in the case $G = \mathbb{Z}$: the subgroups of $\mathbb{Z}$ of index a are the $a\mathbb{Z}$, their cosets are the progressions $r + a\mathbb{Z}$, and two progressions $r_i + a_i\mathbb{Z}$ and $r_j + a_j\mathbb{Z}$ are disjoint exactly when $\gcd(a_i, a_j) \nmid r_i - r_j$. The question that organises the subject is whether $\mathbb{Z}$ is the hardest group.

Question 1.1 (Ginosar; recorded as Question 1 of [2])**.** Let G be a group and n a positive integer. Is every G-harmonic n-tuple also $\mathbb{Z}$-harmonic?

The interest of Question 1.1 is that a positive answer, for every group and every n, would imply the Herzog–Schönheim conjecture [3] — that a non-trivial partition of a group into cosets cannot have pairwise distinct indices — through the theorem of Davenport, Mirsky, Newman and Rado, which says that if $\mathbb{Z}$ is partitioned into finitely many arithmetic progressions then the largest difference occurs more than once. Margolis and Schnabel record the implication immediately below Question 1.1: "if Question 1 admits a positive answer for any group G and any positive integer n then by the theorem of Davenport, Mirsky, Newman and Rado in particular the Herzog-Schönheim Conjecture holds" [2, §1]. They proved the following.

Theorem 1.2 ([2, Theorem B])**.** *Let G be a group and let $(a_1, \dots, a_n)$ be a G-harmonic tuple with $n \leq 4$. Then $(a_1, \dots, a_n)$ is also $\mathbb{Z}$-harmonic.*

They add that they are not aware of any group carrying the length-5 subgroup configuration that their analysis leaves open — the one described in [2, Lemma 4.6] — and that such a group would answer Question 1.1 negatively [2, §4]. In August 2026 Menon [1] answered the question negatively, in the alternating group A_5; and, separately, proved that no group at all carries the Margolis–Schnabel configuration [1, Theorem 4.2]. So the counterexample does not come from where it was expected.

What is known, and what this paper adds. Menon's Theorem 3.1 exhibits subgroups $D \cong D_{10}$, $T \cong S_3$, $V \cong C_2 \times C_2$ of A_5, of indices 6, 10, 15, and elements $g_1, \dots, g_5$ making g_1D, g_2D, g_3D, g_4T, g_5V pairwise disjoint, so $(6, 6, 6, 10, 15)$ is A_5-harmonic; it is not $\mathbb{Z}$-harmonic by an elementary residue argument. Menon's Corollary 3.2 extends these five cosets to a partition of A_5 into nine cosets of index profile $\{6, 6, 6, 10, 10, 10, 15, 15, 15\}$, and his §5 lists four further counterexamples: $(6, 6, 6, 15, 20)$ and $(6, 6, 10, 12, 15)$ in A_5, and $(4, 6, 6, 6, 6)$ and $(6, 6, 6, 6, 16)$ in a group of order 48 isomorphic to $C_2 \times S_4$. Nowhere does he claim that these are all of them; his phrasing is that the same phenomenon produces further counterexamples. About the size of a counterexample he says only that "the smallest counterexamples we have found therefore have order 48" [1, §5], and neither he nor [2] gives any bound on the order of a group in which Question 1.1 can fail.

This paper answers both questions. In each of Menon's two groups his families are all of them:

Theorem 1.3 (Theorem 7.2 and Corollary 7.3)**.** *Let $(a_1, \dots, a_n)$ be an A_5-harmonic tuple with $n \leq 5$. Then $(a_1, \dots, a_n)$ is $\mathbb{Z}$-harmonic, unless $n = 5$ and the multiset $\{a_1, \dots, a_5\}$ is one of*

$$\{6, 6, 6, 10, 15\}, \qquad \{6, 6, 6, 15, 20\}, \qquad \{6, 6, 10, 12, 15\}.$$

Theorem 1.4 (Theorem 8.2 and Corollary 8.3)**.** *Let $(a_1, \dots, a_n)$ be a $C_2 \times S_4$-harmonic tuple with $n \leq 5$. Then $(a_1, \dots, a_n)$ is $\mathbb{Z}$-harmonic, unless $n = 5$ and the multiset $\{a_1, \dots, a_5\}$ is $\{4, 6, 6, 6, 6\}$ or $\{6, 6, 6, 6, 16\}$.*

The five exceptions really are exceptions — each of them is harmonic in the relevant group and not $\mathbb{Z}$-harmonic — so neither list can be shortened, and in both groups the two bounds in the statement are attained: the conclusion fails at $n = 5$ without the exceptions, and the list of exceptions is not complete at $n = 6$ (Propositions 7.4 and 8.5). Together the two theorems say that the answer to Question 1.1 is completely known in both of the groups where the source finds it to be negative (Corollary 8.4).

A classification is a statement about the entire subgroup lattice, and so it yields non-realisability results, which an exhibition of examples cannot. Neither [1] nor [2] states any.

Theorem 1.5 (Theorems 7.5 and 8.6)**.** *None of the tuples*

$$(5, 6), \quad (5, 12), \quad (6, 6, 6, 6, 10), \quad (6, 6, 6, 6, 20), \quad (6, 6, 6, 10, 12)$$

is A_5-harmonic, and none of

$$(6, 6, 6, 6, 8),\ (4, 4, 4, 8, 12),\ (4, 4, 4, 12, 16),\ (2, 8, 8, 8, 12),\ (2, 8, 8, 12, 16),\ (4, 6, 8, 8, 8),\ (4, 6, 8, 8, 16)$$

is $C_2 \times S_4$-harmonic.

The first two say that a coset of a subgroup of A_5 of index 5 (an A_4) always meets a coset of a subgroup of index 6 (a D_{10}), and likewise of index 12 (a C_5). The next three are the minimal shapes of the residue obstruction that [1, Lemma 2.1(ii)] uses on the $\mathbb{Z}$ side; the source proves only that they are not $\mathbb{Z}$-harmonic, and says nothing about whether A_5 realises them. In the order-48 list, $(6, 6, 6, 6, 8)$ is the sharp case: it has four entries 6 and a fifth entry with $\gcd(6, \cdot) = 2$, exactly as the source's own $(6, 6, 6, 6, 16)$ does, and yet one of the two is realised in $C_2 \times S_4$ and the other is not. No counting argument reaches any of the twelve: each satisfies $\sum_i 1/a_i \leq 1$, so its cosets do fit inside the group.

The second half of the paper leaves those two groups. Question 1.1 is a question about all groups, and the invariant that organises the counterexamples turns out to be the order of G. Three necessary conditions on the index tuple of a G-harmonic configuration (Theorem 3.1), together with a greedy residue assignment for tuples whose entries form a divisibility chain (Lemma 3.2), settle every order below 12 using no information about G beyond Lagrange's theorem; and Menon's own multiset $(4, 6, 6, 6, 6)$ turns out to be realised one symmetric group down.

Theorem 1.6 (Theorem 4.1, Proposition 4.2 and Corollary 4.3)**.** *Let G be a finite group with $|G| < 12$. Then every G-harmonic tuple, of any length, is $\mathbb{Z}$-harmonic. The tuple $(4, 6, 6, 6, 6)$ is $C_2 \times S_3$-harmonic and not $\mathbb{Z}$-harmonic. Hence the smallest order of a group carrying a G-harmonic tuple that is not $\mathbb{Z}$-harmonic is exactly 12.*

The lower half of Theorem 1.6 uses no group data at all — no multiplication tables and no classification of small groups — and it also gives, for free, that no group of prime-power order carries a counterexample at any length (Corollary 3.4). The next order that resists is $18 = 2 \cdot 3^2$, whose divisors 6 and 9 are neither coprime nor comparable. It is settled by sharpening the counting: if $P \leq G$ has index 2 and *odd* order, then a coset of a subgroup of even order meets P in exactly half of its elements while a coset of a subgroup of odd order lies wholly inside P or wholly outside, so the single inequality $\sum_i |G|/a_i \leq |G|$ splits into two (Lemma 5.1).

Theorem 1.7 (Theorem 5.2)**.** *Let G be a group of order 18. Then every G-harmonic tuple over G, of any length, is $\mathbb{Z}$-harmonic.*

This is a proof, not a search: the only group-theoretic input is that a Sylow 3-subgroup of G has order 9, hence index 2 and odd order, and no normality, uniqueness or classification of the five groups of order 18 is used. The same argument, carried out on paper rather than formally, settles every order $2p^k$ with p an odd prime (Theorem 5.4); and those theorems together with an exhaustive computation outside the formal development decide every group order below 48 except 42 (Remark 5.5). Counterexamples are, in addition, not sporadic: for every odd $k \geq 3$ the group $C_2 \times D_{2k}$ of order $4k$ carries one (Theorem 4.5, also proved on paper).

Finally, the coset-partition phenomenon that [1] exhibits once is general.

Theorem 1.8 (Theorems 9.2, 9.3 and 9.4)**.** *Each of the four families of* [1, §5] *extends to a coset partition:* $(6, 6, 6, 15, 20)$ *to a partition of A_5 into eleven cosets with profile* $\{6^3, 15^6, 20^2\}$*,* $(6, 6, 10, 12, 15)$ *to one into ten cosets with profile* $\{6^2, 10^3, 12^2, 15^3\}$*,* $(4, 6, 6, 6, 6)$ *to a partition of $C_2 \times S_4$ into six cosets with profile* $\{4, 6^4, 12\}$*, and* $(6, 6, 6, 6, 16)$ *to one into ten cosets with profile* $\{6^4, 12, 16^2, 24^3\}$*. The order-12 counterexample of Theorem 1.6 extends to a partition of $C_2 \times S_3$ with profile* $\{4, 6^4, 12\}$*. None of the five profiles is $\mathbb{Z}$-harmonic.*

As the engine of Theorems 1.3 and 1.4, the two subgroup lattices are settled formally: A_5 has exactly 59 subgroups and $C_2 \times S_4$ exactly 98 (Propositions 6.2 and 6.3). The numbers are textbook; what is used is not the number but the certified list, since every non-realisability statement quantifies over it. The same enumerations confirm Theorem 1.2 independently inside each of the two groups (Theorems 7.1 and 8.1), by a route that has nothing in common with the proof of [2]: they classify G-harmonic n-tuples abstractly for all groups, whereas the argument here exhausts one lattice at a time.

Scope. Sections 3–5 are about arbitrary finite groups. Sections 6–8 are about the two groups in which [1] finds Question 1.1 to fail, but the argument is about neither in particular: §6 states it once for a finite group presented by a multiplication table together with four finite checks (Proposition 6.1), and the two groups enter only as instances, so a third group would cost its table and its checks and nothing else. What we do not determine is which orders admit a counterexample in general; below 48 the answer is complete except for order 42 (Remark 5.5), and the general question is open.

Related work. The nearest recent work on the Herzog–Schönheim circle of questions for A_5 is that of Garonzi and Margolis [4], who prove the Herzog–Schönheim conjecture for simple groups and for symmetric groups. Their method turns on the quantity $\mathcal{J}(G) = \sum_m 1/m$, summed over the *distinct* indices m of subgroups of G, which they show to be less than 2 for every finite simple group; it therefore reads only the set of indices G realises and not which multisets it realises, and does not overlap with what follows. Order-bounded results do occur in this circle of questions — Theorem A of [2] is that any group of order less than 1440 satisfies the Herzog–Schönheim conjecture — but they concern that conjecture and not Question 1.1, for which we have found no statement in the literature bounding the order of a counterexample. A neighbouring line of work bounds the arithmetic of coset *partitions*: Ginosar and Schnabel [5] give conditions on the prime factorisation of $|G|$ that force a multiplicity in the index multiset of a coset partition of G. That is the Herzog–Schönheim setting, in which the cosets are required to cover G; Question 1.1 is about pairwise disjoint cosets that need not cover, and the two-sided count of §5 is a statement about that weaker configuration. We note that [1] is a single-author preprint, at version 1 and uncited at the time of writing, whose own footnotes attribute part of its construction to machine assistance. That is one reason the reproduction of §10 was worth doing independently; every numerical assertion of [1] that we checked is correct, and the gaps filled below are questions the source does not raise.

Changes in this version. Versions 1 and 2 of this paper treated only the two groups of [1]: the classifications of §§6–8, the non-realisability theorems, and the coset partitions of [1, §5]. The present version adds §§3–5 — the necessary conditions and the chain lemma, the exact smallest order 12, the order-18 theorem with the refined count that proves it, the two unformalised theorems (the infinite dihedral family and the orders $2p^k$), and the external map of the orders below 48 — together with Theorem 9.4, the coset partition extending the order-12 counterexample; the title, the abstract and the related-work paragraph are revised to match. Nothing stated in version 2 is withdrawn; its scope paragraph, which recorded the smallest order of a counterexample group as undetermined and priced settling it at a sweep over the multiplication tables of all groups of order below 48, is superseded by Theorem 1.6, which needs no such sweep.

2. Preliminaries

Throughout, G is a group, written multiplicatively, and gU denotes the left coset $\{gu : u \in U\}$ of a subgroup $U \leq G$. Tuples are indexed by $\{1, \ldots, n\}$ and identified with functions on that set, so "the multiset $\{a_1, \ldots, a_n\}$" means the tuple up to reordering.

Definition 2.1. Let $a = (a_1, \ldots, a_n)$ be a tuple of positive integers.

(1) a is *G-harmonic* if there are subgroups $U_i \leq G$ and elements $g_i \in G$ with $[G : U_i] = a_i$ for all i and $g_i U_i \cap g_j U_j = \varnothing$ for all $i \neq j$.
(2) a is *$\mathbb{Z}$-harmonic* if there are integers $r_1, \ldots, r_n$ with $\gcd(a_i, a_j) \nmid r_i - r_j$ for all $i \neq j$.

These are the definitions of [2, §1]; the gcd form of (2) is the one stated in [1], and it is equivalent to pairwise disjointness of the progressions $r_i + a_i\mathbb{Z}$ because two such progressions meet if and only if $\gcd(a_i, a_j) \mid r_i - r_j$.

Lemma 2.2. *Both properties pass to sub-tuples: if a is G-harmonic (resp. $\mathbb{Z}$-harmonic) and $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is injective, then $a \circ \sigma$ is G-harmonic (resp. $\mathbb{Z}$-harmonic).*

Proof. Restrict the data along σ; injectivity is what preserves the pairwise conditions. $\square$

Consequently a tuple fails to be G-harmonic as soon as one of its sub-tuples does, which is the monotonicity that makes a search over multisets finite and cheap. Dually, a tuple fails to be $\mathbb{Z}$-harmonic as soon as one of its sub-tuples does, so a group carries a counterexample to Question 1.1 if and only if it realises one of the *minimal* index multisets that are not $\mathbb{Z}$-harmonic.

The residue obstruction. All the $\mathbb{Z}$-side non-existence statements used below come from a handful of finite residue facts. The first two are [1, Lemma 2.1] stated with no tuple in sight.

Lemma 2.3. *Let $r_0, r_1, r_2, r_3, x, y \in \mathbb{Z}$.*

(1) *If r_0, r_1, r_2 are pairwise incongruent modulo 6, none of them is congruent to x modulo 2, and none of them is congruent to y modulo 3, then a contradiction follows.*
(2) *If r_0, r_1, r_2, r_3 are pairwise incongruent modulo 6 and none of them is congruent to x modulo 2, then a contradiction follows.*

Proof. Both are statements about residues alone. The formal proof pushes the integers into $\mathbb{Z}/6$, $\mathbb{Z}/2$ and $\mathbb{Z}/3$ along the reduction homomorphisms and checks every pattern exhaustively: $6^3 \cdot 2 \cdot 3 = 1296$ of them for (1) and $6^4 \cdot 2 = 2592$ for (2). The reason is visible through $\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$. In (1), avoiding one class in each factor leaves $1 \cdot 2 = 2$ admissible classes modulo 6, which cannot hold three pairwise incongruent residues; in (2), avoiding one class modulo 2 leaves the three classes modulo 6 of the other parity, which cannot hold four. □

Applied to a putative $\mathbb{Z}$-harmonicity witness, (1) shows that a tuple with three entries 6, an entry x with $\gcd(6, x) = 2$ and an entry y with $\gcd(6, y) = 3$ is not $\mathbb{Z}$-harmonic, and (2) shows the same for a tuple with four entries 6 and an entry x with $\gcd(6, x) = 2$. An entry 12 can play the role of a 6, since $\gcd(6, 12) = 6$; that is how the multiset $\{6, 6, 10, 12, 15\}$ falls under (1) and $\{6, 6, 6, 10, 12\}$ under (2).

Remark 2.4 (non-vacuity). Both extra hypotheses in Lemma 2.3(1) are needed: $(6, 6, 6, 10)$ and $(6, 6, 6, 15)$ are both $\mathbb{Z}$-harmonic, with witnesses $(0, 2, 4, 1)$ and $(0, 1, 3, 2)$; and $(6, 6, 6, 6)$ is $\mathbb{Z}$-harmonic with witness $(0, 1, 2, 3)$, so the entry x in part (2) is needed. Also $(6, 10, 15)$ is both A_5-harmonic and $\mathbb{Z}$-harmonic, so A_5-harmonicity by itself obstructs nothing.

The order-48 statements of §8 need two more residue facts of the same elementary kind; the first is a pigeonhole, the second the analogue of Lemma 2.3(1) with 8 in place of 6.

Lemma 2.5. (1) *Let $m \geq 1$ and let $(a_1, \dots, a_n)$ be a tuple with $n > m$ all of whose pairwise greatest common divisors divide m. Then $(a_1, \dots, a_n)$ is not $\mathbb{Z}$-harmonic.*
(2) *Let $r_0, r_1, r_2, x, y \in \mathbb{Z}$. If r_0, r_1, r_2 are pairwise incongruent modulo 8, none of them is congruent to x modulo 4, none of them is congruent to y modulo 2, and $x \not\equiv y \pmod 2$, then a contradiction follows.*

Proof. (1) If $(r_1, \dots, r_n)$ were a witness then for $i \neq j$ we would have $\gcd(a_i, a_j) \nmid r_i - r_j$, hence $m \nmid r_i - r_j$ since $\gcd(a_i, a_j) \mid m$; so $i \mapsto r_i \bmod m$ is injective from an n-element set into $\mathbb{Z}/m$, and $n \leq m$. This is an argument about cardinalities, not a finite check.

(2) The parity class of y is excluded for r_0, r_1, r_2 and also for x, so x and r_0, r_1, r_2 all lie in the other parity class; inside it, the class of x modulo 4 is excluded for the three r_i, leaving a single class modulo 4, which contains only two residues modulo 8 — too few for three pairwise incongruent ones. The formal proof is one exhaustive check over the $8^3 \cdot 4 \cdot 2 = 4096$ residue patterns. □

Applied to a witness, (2) shows that a tuple with three entries whose pairwise greatest common divisors are 8, an entry x with $\gcd(8, x) = 4$, and an entry y with $\gcd(8, y) = \gcd(x, y) = 2$ is not $\mathbb{Z}$-harmonic; an entry 16 can play the role of an 8, since $\gcd(8, 16) = 8$.

3. What a harmonic tuple forces on the order of G

Everything in this section is about an arbitrary finite group. Three conditions are necessary for G-harmonicity; a fourth condition, on the divisibility relations among the entries, is sufficient for $\mathbb{Z}$-harmonicity; and the two meet.

Theorem 3.1. *Let G be a finite group and let $(a_1, \ldots, a_n)$ be G-harmonic. Then*

(1) *a_i is positive and divides $|G|$, for every i;*
(2) *$\sum_i |G|/a_i \le |G|$;*
(3) *$\gcd(a_i, a_j) \ne 1$ for all $i \ne j$.*

Since each a_i divides $|G|$ by (1), the division in (2) is exact and (2) says $\sum_i 1/a_i \le 1$.

Proof. (1) is Lagrange's theorem. For (2), the cosets g_iU_i are pairwise disjoint subsets of G of size $|U_i| = |G|/a_i$; formally, $(i, u) \mapsto g_iu$ is injective on the disjoint union $\coprod_i U_i$, and the bound is the resulting cardinality inequality.

(3) is Poincaré's inequality made strict by the missing coset pair. The map $x(U_i \cap U_j) \mapsto (xU_i, xU_j)$ is an injection $G/(U_i \cap U_j) \hookrightarrow (G/U_i) \times (G/U_j)$, and disjointness of g_iU_i and g_jU_j says exactly that the pair (g_iU_i, g_jU_j) is not in its image. So the injection is not surjective and $[G : U_i \cap U_j] < a_ia_j$, while a_i and a_j both divide $[G : U_i \cap U_j]$. Were a_i and a_j coprime their product would divide $[G : U_i \cap U_j]$, forcing $a_ia_j \le [G : U_i \cap U_j]$; contradiction. □

Only the non-strict inequality $[G : U \cap V] \le [G : U][G : V]$ is standard; the strict form used here needs the two cosets, and is built from the injection above.

Lemma 3.2 (the chain lemma)**.** *Let $N \ge 1$ and let $a_1, \ldots, a_n$ be positive divisors of N that are pairwise comparable under divisibility, with $\sum_i N/a_i \le N$. Then $(a_1, \ldots, a_n)$ is $\mathbb{Z}$-harmonic.*

Proof. Induction on n. Reorder so that $a_n = M$ is a maximal entry — both properties are invariant under reordering — and take a witness $r_1, \ldots, r_{n-1}$ for the first $n - 1$ entries. Each earlier entry a_j divides M, so the residues modulo M forbidden to r_n by a_j form a single class modulo a_j and number at most M/a_j. Now $\sum_{j \le n} N/a_j \le N$ and $N/a_n = N/M$, so scaling by M/N gives $\sum_{j<n} M/a_j \le M - 1$: at most $M - 1$ of the M residues modulo M are forbidden, and one is free. The counting is a union bound together with the observation that a class modulo d meets $\{0, \ldots, M - 1\}$ in at most M/d points when $d \mid M$. □

Theorem 3.1 and Lemma 3.2 fit together whenever the divisor lattice of $|G|$ is thin enough that "non-coprime" already implies "comparable".

Corollary 3.3. *Let G be a finite group in which any two divisors d, e of $|G|$ with $\gcd(d, e) \ne 1$ satisfy $d \mid e$ or $e \mid d$. Then every G-harmonic tuple, of any length, is $\mathbb{Z}$-harmonic.*

Proof. By Theorem 3.1 the entries are positive divisors of $|G|$ with $\sum_i |G|/a_i \le |G|$, and no two of them are coprime, hence any two are comparable; apply Lemma 3.2 with $N = |G|$. □

Corollary 3.4. *No group of prime-power order carries a counterexample to Question 1.1, at any length: if $|G| = p^k$ with p prime, every G-harmonic tuple is $\mathbb{Z}$-harmonic.*

Proof. The divisors of p^k form a chain, so Corollary 3.3 applies. □

Corollary 3.4 is the group-theoretic form of Kraft's inequality, and is stated in neither [1] nor [2].

Remark 3.5 (elementary; not part of the formal development). The orders to which Corollary 3.3 applies are exactly the prime powers and the products of two distinct primes. Indeed if N is divisible by p^2q with $p \ne q$ prime then p^2 and pq are non-coprime and incomparable, and if N is divisible by three distinct primes p, q, r then pq and pr are; conversely the divisors of p^k form a chain and those of pq are $1, p, q, pq$, whose only non-coprime pairs are comparable. So $12 = 2^2 \cdot 3$ is the smallest order Corollary 3.3 does not reach, and the offending pair of divisors is $(4, 6)$ — which is the shape of the counterexample found in §4.

Remark 3.6 (elementary; not part of the formal development). For the three exceptional multisets of Theorem 1.3 the sums $\sum_i 1/a_i$ of Theorem 3.1(2) are $2/3$, $37/60$ and $7/12$, and for the two of Theorem 1.4 they are $11/12$ and $35/48$; for the order-12 counterexample of §4 it is $11/12$. Equality holds exactly when the cosets partition G, so none of the six configurations is itself a partition — which is what makes the extensions of §9 a separate question.

4. The smallest order of a counterexample group

Theorem 4.1. *Let G be a finite group with $|G| < 12$. Then every G-harmonic tuple over G, of any length, is $\mathbb{Z}$-harmonic. Equivalently, a counterexample to Question 1.1 forces $|G| \geq 12$.*

Proof. By Corollary 3.3 it suffices to know that for every $N < 12$ any two divisors of N with a common divisor ≥ 2 are comparable. That is an exhaustive check over the quadruples (N, d, e, k) with $N, d, e, k < 12$, where k is a common divisor of d and e with $k \geq 2$; carrying k explicitly rather than invoking gcd inside the check keeps it independent of how gcd is evaluated. No information about G beyond $|G|$ enters. □

The bound is attained, and by the source's own index multiset. Write $C_2 \times S_3$ for $\mathbb{Z}/2 \times S_3$, the dihedral group of order 12, and its elements as pairs (b, π) with $b \in \mathbb{Z}/2$ and π a permutation of $\{0, 1, 2\}$.

Proposition 4.2. *Let $U_1 = \{0\} \times A_3 \leq C_2 \times S_3$, of order 3 and index 4, and let*

$$U_2 = \langle (0, (1\,2)) \rangle, \qquad U_3 = \langle (1, e) \rangle, \qquad U_4 = \langle (1, (0\,1)) \rangle$$

be three subgroups of order 2, hence of index 6, generated by a transposition in the S_3 factor, by the central involution, and by a twisted involution respectively. Then the five cosets

$$(0, e)\, U_1, \quad (1, e)\, U_2, \quad (1, (0\,1))\, U_2, \quad (0, (0\,2))\, U_3, \quad (0, (1\,2))\, U_4$$

are pairwise disjoint, of sizes 3, 2, 2, 2, 2, and cover 11 of the 12 elements. Hence $(4, 6, 6, 6, 6)$ — the index multiset that [1, §5] realises in a group of order 48 — is $C_2 \times S_3$-harmonic; it is not $\mathbb{Z}$-harmonic by Lemma 2.3(2), since $\gcd(6, 4) = 2$.

Proof sketch. Each subgroup is given by its element list, from which its order, and hence by Lagrange its index against $|C_2 \times S_3| = 12$, follows; nothing about the group is assumed. Disjointness of $g_i U_i$ and $g_j U_j$ reduces to $g_i u \neq g_j v$ for u in the element list of U_i and v in that of U_j, a finite check over at most 3×3 pairs of lists, carried out exhaustively. The element missed by the five cosets is $(0, (0\,1))$. □

Corollary 4.3. *The smallest order of a group carrying a G-harmonic tuple that is not $\mathbb{Z}$-harmonic is exactly 12.*

Proof. Theorem 4.1 and Proposition 4.2. □

Menon reports only that the smallest counterexamples he found have order 48, and neither source bounds the order from below. Note that the lower half of Corollary 4.3 needs no group data whatever: it is not a sweep over the groups of order < 12 but a statement about their orders.

Proposition 4.4 (controls)**.**
(1) *Both 4-element sub-multisets of $(4, 6, 6, 6, 6)$, namely $(6, 6, 6, 6)$ and $(4, 6, 6, 6)$, are $\mathbb{Z}$-harmonic with explicit witnesses. So the order-12 counterexample really needs length 5, and the bound of Theorem 1.2 is attained here too, not beaten.*
(2) *The comparability hypothesis of Lemma 3.2 cannot be weakened to "pairwise non-coprime": $(4, 6, 6, 6, 6)$ satisfies every other hypothesis at $N = 12$ — its entries divide 12, no two are coprime, and $\sum_i 12/a_i = 11 \leq 12$ — and is not $\mathbb{Z}$-harmonic. In particular the bound 12 of Theorem 4.1 cannot be raised by this argument.*
(3) *The counting hypothesis cannot be dropped: $(2, 2, 2)$ is a chain of divisors of 12 that is not $\mathbb{Z}$-harmonic, and $\sum_i 12/a_i = 18 > 12$.*

(4) *Lemma 3.2 does prove things:* $(2,4,4)$ *is a chain of divisors of* 12 *with* $\sum_i 12/a_i = 12$*, and it is* $\mathbb{Z}$*-harmonic by the lemma. The four subgroups of Proposition 4.2 are pairwise distinct,* U_1 *is not the whole group and* U_2 *is not trivial, so the configuration is not a disguised repetition.*

Counterexamples of order 12 are not isolated. The following theorem is proved on paper and machine-checked outside the kernel; it is stated here because it explains Corollary 4.3, but it is *not* part of the formal development.

Theorem 4.5 (not part of the formal development)**.** *Let* $k \geq 3$ *be odd and let* $G_k = C_2 \times D_{2k}$*, where* D_{2k} *is the dihedral group of order* $2k$*, so* $|G_k| = 4k$*. Then the* $(k+2)$*-tuple* $(4, 2k, \ldots, 2k)$ *with* $k+1$ *entries equal to* $2k$ *is* G_k*-harmonic and not* $\mathbb{Z}$*-harmonic. In particular Question 1.1 fails in infinitely many groups, beginning at the smallest order that admits any failure at all.*

Proof. Not $\mathbb{Z}$-harmonic: since k is odd, $\gcd(4, 2k) = 2$, so in a witness the $k+1$ residues attached to the entries $2k$ would have to be pairwise incongruent modulo $2k$ and all incongruent to r_0 modulo 2, hence all in the parity class opposite to r_0; that class contains exactly k residues modulo $2k$, and $k+1 > k$.

G_k-harmonic: write the elements of G_k as (b, ρ^i) and $(b, \sigma\rho^i)$ with $b \in \mathbb{Z}/2$, ρ a rotation of order k and σ a reflection. The subgroup $\{0\} \times \langle\rho\rangle$ has order k and index 4; take it as the first coset. The remaining $3k$ elements split into $A = \{1\} \times \{\text{rotations}\}$, $B = \{1\} \times \{\text{reflections}\}$ and $C = \{0\} \times \{\text{reflections}\}$, each of size k. Three shapes of coset of an order-2 subgroup, hence of index $2k$, are available: $\{(0, f), (1, f)\}$ with f a reflection is a coset of $\langle(1, e)\rangle$ and uses one point of B and one of C; $\{(1, \rho^i), (1, \rho^i\sigma\rho^j)\}$ is a coset of $\langle(0, \sigma\rho^j)\rangle$ and uses one point of A and one of B; and $\{(1, \rho^i), (0, \rho^i\sigma\rho^j)\}$ is a coset of $\langle(1, \sigma\rho^j)\rangle$ and uses one point of A and one of C. Take one coset of the first shape, $(k+1)/2$ of the second and $(k-1)/2$ of the third. They use all k points of A, $1 + (k+1)/2 \leq k$ of B and $1 + (k-1)/2 \leq k$ of C (both because $k \geq 3$), so they are $k+1$ pairwise disjoint cosets of index $2k$, disjoint also from the first coset. □

The configuration was in addition produced by exhaustive machine search at the seven odd values $k = 3, 5, \ldots, 15$, that is at the orders 12, 20, 28, 36, 44, 52, 60, covering $3k+2$ of the $4k$ elements in every case as the construction predicts; at $k = 3$ the search returns exactly the configuration of Proposition 4.2. Formalising the induction over k is not attempted here.

5. Order 18, and a refined count against an index-2 subgroup

By Corollary 3.3 and Remark 3.5, the orders not settled by §3 are those divisible by p^2q or by three distinct primes. The smallest of them are 12, which does carry a counterexample, and $18 = 2 \cdot 3^2$, whose divisors 6 and 9 are neither coprime nor comparable. Order 18 is settled here, and in the negative. The tool is a sharpening of Theorem 3.1(2).

Lemma 5.1 (refined counting)**.** *Let* G *be a finite group and let* $P \leq G$ *have index* 2 *and odd order. Let* $U \leq G$ *and* $g \in G$*.*

(1) *If* $|U|$ *is odd then* $U \leq P$*, and* $gU \subseteq P$ *or* $gU \cap P = \varnothing$ *according as* $g \in P$ *or not.*
(2) *If* $|U|$ *is even then* $U \not\leq P$ *and* $|gU \cap P| = |U|/2$*.*

Consequently, if $g_1U_1, \ldots, g_nU_n$ *are pairwise disjoint cosets in* G *then*

$$\sum_i |g_iU_i \cap P| \leq |P| \qquad \text{and} \qquad \sum_i |g_iU_i \setminus P| \leq |P|.$$

Proof. For any subset $S \subseteq G$ the sets $g_iU_i \cap S$ are pairwise disjoint subsets of S, so $\sum_i |g_iU_i \cap S| \leq |S|$; the two displayed inequalities are this with $S = P$ and $S = G \setminus P$, together with $|G \setminus P| = |G| - |P| = |P|$. For (1), the relative index $[U : U \cap P]$ divides $[G : P] = 2$ — if $U \not\leq P$, pick $v \in U \setminus P$; then every $b \in U$ has $b \in P$ or $bv \in P$, since P has index 2 — and it also divides $|U|$, so for $|U|$ odd it cannot be 2; hence $U \leq P$, and then gU lies inside P or misses it according to whether $g \in P$. For (2), a subgroup of even order cannot lie in a subgroup of odd order, so $[U : U \cap P] = 2$ and

$|U \cap P| = |U|/2$; and the fibre $\{u \in U : gu \in P\}$ is non-empty — take $u = 1$ if $g \in P$, and otherwise any $u \in U \setminus P$, which is non-empty and for which $gu \in P$ because P has index 2 — and it is a left coset of $U \cap P$ inside U, hence has $|U \cap P| = |U|/2$ elements. □

Summing whole cosets gives Theorem 3.1(2); summing the pieces inside P and the pieces outside gives two inequalities whose sum is that one but which are strictly more informative as soon as some coset is not split evenly. The odd-order hypothesis is what makes the dichotomy work: without it a subgroup of even order can still lie inside P. The formal development carries the ingredients of Lemma 5.1 as separate statements — the disjointness bound for an arbitrary subset S, the divisibility $[U : U \cap P] \mid 2$, the two containment lemmas, the half-size computation, and the two additivity facts — and the two displayed inequalities are their immediate combination. Mathematically each ingredient is standard; what is not available off the shelf is the assembly into a counting inequality for a family of disjoint cosets.

Theorem 5.2. *Let G be a group of order 18. Then every G-harmonic tuple over G, of any length, is $\mathbb{Z}$-harmonic. Equivalently, no counterexample to Question 1.1 lives in a group of order 18.*

Proof sketch. Let $(a_1, \dots, a_n)$ be G-harmonic, with cosets $C_i = g_i U_i$. By Theorem 3.1 each a_i divides 18, so lies in $\{1, 2, 3, 6, 9, 18\}$, and no two entries are coprime. If some $a_i = 1$ then $\gcd(a_i, a_j) = 1$ for every $j \neq i$, so $n \leq 1$ and there is nothing to prove. If some $a_i = 2$ then every entry is even, hence lies in the divisibility chain $\{2, 6, 18\}$, and Lemma 3.2 applies with $N = 18$. So we may assume every entry lies in $\{3, 6, 9, 18\}$.

A Sylow 3-subgroup $P \leq G$ has order 9, hence index 2 and odd order; this is the only group-theoretic input, and it uses no normality, no uniqueness, and no classification of the groups of order 18. Since $|U_i| = 18/a_i$, the order $|U_i|$ is even exactly when a_i is odd, and Lemma 5.1 gives:

a_i	$\lvert U_i\rvert$	position of C_i	$\lvert C_i \cap P\rvert$	$\lvert C_i \setminus P\rvert$
3	6	split evenly	3	3
6	3	wholly on one side ε	3 or 0	0 or 3
9	2	split evenly	1	1
18	1	wholly on one side ε	1 or 0	0 or 1

Write x for the number of entries 3, z for the number of entries 9, and y_ε, w_ε for the numbers of entries 6, respectively 18, whose coset lies on side $\varepsilon \in \{0, 1\}$ (side 0 being P). The two inequalities of Lemma 5.1 read

$$3x + 3y_0 + z + w_0 \leq 9, \qquad 3x + 3y_1 + z + w_1 \leq 9.$$

The content of the proof is the converse: those two inequalities already force $\mathbb{Z}$-harmonicity. Identify $\mathbb{Z}/18$ with $\mathbb{Z}/2 \times \mathbb{Z}/9$, a grid of two rows (parities) and nine columns, and read the columns of a row in the order $p \mapsto 3(p \bmod 3) + \lfloor p/3 \rfloor$ — reversal of base-3 digits — so that three consecutive positions $[3c, 3c+3)$ form exactly the residue class c modulo 3. Then assign residues by rank within kind:

entry	occupies
$a = 3$, rank k	positions $[3k, 3k+3)$ of *both* rows
$a = 6$, side ε, rank k	positions $[3c, 3c+3)$ of row ε, where $c = 3 - y_\varepsilon + k$
$a = 9$, rank k	position $3x + k$ of *both* rows
$a = 18$, side ε, rank k	position $3x + z + k$ of row ε

The blocks of the second line fill $[9 - 3y_\varepsilon, 9)$, so row ε occupies $[0, 3x + z + w_\varepsilon) \cup [9 - 3y_\varepsilon, 9)$, and those two blocks are disjoint precisely when the inequality for ε holds. The reason the two rows do not interfere is that the $a = 3$ blocks sit at the bottom of *both* rows and the $a = 6$ blocks at the top of *each* row; that alignment leaves the middle free in both rows, which is where the $a = 9$ entries — which also need both rows — go, followed inside row ε by its own single cells of index 18. Each entry is sent to the slot (kind, side, rank), a map that is injective and lands in the admissible range, so the pairwise condition $r_i \not\equiv r_j \pmod{\gcd(a_i, a_j)}$ becomes a statement about slots alone. That statement is verified exhaustively: over the 640 parameter tuples (x, y_0, y_1, z) in range, of which 90 satisfy both inequalities, and their 9610 ordered pairs of slots — about 40 seconds of kernel reduction. The counts w_0, w_1 do not occur in the residue formulas, only in the range of the ranks of the entries of index 18, so they are eliminated in favour of their own upper bounds $9-(3x+3y_\varepsilon+z)$; the greatest common divisors of the four indices are tabulated and the table proved correct by a separate 16-case check, so the main check never evaluates gcd. $\square$

The two inequalities are exactly sharp. There are six minimal index multisets of divisors of 18 that are not $\mathbb{Z}$-harmonic and satisfy the conditions of Theorem 3.1:

$$(3,3,6,9),\ (3,6,6,6,9),\ (3,6,9,9,9,9),\ (6,6,6,6,6,9),\ (6,6,6,9,9,9,9),\ (6,9,9,9,9,9,9,9),$$

and each of them has $\sum_i 18/a_i = 17 \le 18$ — so Theorem 3.1(2) misses all six by exactly one unit of slack — while each violates one of the two refined inequalities by exactly 1, the worst row costing $10 > 9$.

Proposition 5.3 (controls)**.** (1) $(3,3,6,9)$ *is not* $\mathbb{Z}$*-harmonic: its six pairwise greatest common divisors all equal* 3*, so a witness would need four integers pairwise incongruent modulo* 3*, which is impossible. Yet each entry divides* 18*,* $\sum_i 18/a_i = 17 \le 18$*, and no two entries are coprime, so Theorem 3.1 alone — and likewise Corollary 3.3, since* 6 *and* 9 *are non-coprime and incomparable — cannot settle order* 18*. It is a minimal obstruction: each of its three sub-multisets of length* 3*, namely* $(3,3,6)$*,* $(3,3,9)$ *and* $(3,6,9)$*, is* $\mathbb{Z}$*-harmonic with witness* $(0,1,2)$*.*

(2) *The odd-order hypothesis of Lemma 5.1 is essential and cannot be dropped: a group of order* 12 *also has subgroups of index* 2*, but their order* 6 *is even, so the lemma cannot be applied there — and must not be, since Proposition 4.2 exhibits a counterexample of order* 12*.*

(3) *The hypothesis of Lemma 5.1 is met at order* 18*: a group of order* 18 *really does have a subgroup of index* 2 *and odd order, so Theorem 5.2 is not vacuously true for want of a* P*.*

(4) *Theorem 5.2 has to construct residues and not merely derive contradictions:* $(6,9,9,9,9,9,9)$ *is a length-*7 *tuple of divisors of* 18 *that* is $\mathbb{Z}$*-harmonic, with witness* $(0,1,2,4,5,7,8)$*, and adjoining one further entry* 9 *produces the length-*8 *minimal obstruction above.*

Nothing in the argument is special to 9 beyond the fact that the odd divisors of $|G|$ form a chain. The following theorem is proved on paper and machine-checked outside the kernel; it is *not* part of the formal development, and the case $p=3$, $k=2$ that is formalised is exactly Theorem 5.2.

Theorem 5.4 (not part of the formal development)**.** *Let* p *be an odd prime,* $k \ge 1$*, and let* G *be a group of order* $2p^k$*. Then every* G*-harmonic tuple over* G *is* $\mathbb{Z}$*-harmonic. Question 1.1 therefore has a positive answer for every group of order* $2p^k$*.*

Proof. G has a subgroup P of order p^k by Sylow's theorem, necessarily of index 2 and of odd order. Write $a_i = 2^{\delta_i} p^{j_i}$, so $|U_i| = 2^{1-\delta_i} p^{k-j_i}$ and $|U_i|$ is even exactly when $\delta_i = 0$. By Lemma 5.1, $|C_i \cap P| = p^{k-j_i}$ on both sides when $\delta_i = 0$, while for $\delta_i = 1$ the coset C_i lies wholly on one side ε_i and contributes p^{k-j_i} there and 0 on the other. Hence

$$(\star) \qquad \sum_{\delta_i=0} p^{k-j_i} \;+ \sum_{\delta_i=1,\ \varepsilon_i=r} p^{k-j_i} \;\le\; p^k \qquad (r=0,1).$$

Conversely $(\star)$ forces $\mathbb{Z}$-harmonicity. Since every a_i divides $2p^k$, the tuple is $\mathbb{Z}$-harmonic exactly when residues r_i can be chosen so that the classes r_i mod a_i, read as subsets of $\mathbb{Z}/2p^k$, are pairwise disjoint. Identify $\mathbb{Z}/2p^k$ with $\mathbb{Z}/2 \times \mathbb{Z}/p^k$ and read $\mathbb{Z}/p^k$ in base-p digit-reversed order, so that the class of ρ modulo p^j becomes the interval $[c\,p^{k-j}, (c+1)p^{k-j})$ of positions; a class modulo an odd $a_i = p^{j_i}$ is that interval in both rows, and a class modulo an even $a_i = 2p^{j_i}$ is that interval in one row. Place the entries with $\delta_i = 0$, which need both rows, in a block anchored at one end of both rows, sorted by decreasing block length; place the entries with $\delta_i = 1$ and $\varepsilon_i = r$ in a block anchored at the other end of row r, again sorted by decreasing length. Every block length divides every larger one, so consecutive placement from either end is automatically aligned with the class structure, and the two blocks of row r are disjoint exactly when $(\star)$ holds for r. $\square$

Since the orders $2p$ are already covered by Corollary 3.3, the smallest order that Theorem 5.4 adds is $18 = 2 \cdot 3^2$, and the next are 50, 54, 98 and 162. The criterion ($\star$) was tested by machine at the orders $2, 6, 10, 14, 18, 22, 26, 34, 38, 46, 50, 54$, up to tuple length 14, against an independently written decision procedure for $\mathbb{Z}$-harmonicity, and it holds at each. The hypothesis that $|G|/2$ be a prime power cannot be weakened to "$|G|/2$ odd": at $30 = 2 \cdot 15$ and $42 = 2 \cdot 21$ the criterion ($\star$) admits $(2, 6, 10)$ and $(2, 6, 14)$ respectively, neither of which is $\mathbb{Z}$-harmonic — consistent with the fact, recorded in Remark 5.5, that order 30 does carry a counterexample.

Remark 5.5 (computation outside the formal development). Combining the theorems above with an exhaustive search decides every group order below 48 except one. The search takes multiplication tables from a computer-algebra library of small groups, re-verifies the group axioms from each table independently, and backtracks over cosets; a positive verdict is an explicit family of pairwise disjoint cosets and is exact, whereas a negative verdict is only as good as the exhaustiveness of the search that produced it.

no counterexample	1–11, 13–17, 19, 21–23, 25–27, 29, 31–35, 37–39, 41, 43, 46, 47	
		(each is p^a or pq: Corollary 3.3)
	18	(Theorem 5.2)
	45	(exhaustive, uncapped)
counterexample	12, 20, 24, 28, 30, 36, 40, 44	
undetermined	42	

Orders 44 and 45 were decided by exhaustive uncapped searches: at order 44 only the dihedral group carries a counterexample, with $(4, 22, \ldots, 22)$ of length 13, and the other three groups of that order carry none; at order 45 neither of the two groups does. Orders 36 and 40 are settled positively, which needs only one exact family: $(4, 18, \ldots, 18)$ of length 11 in each of four groups of order 36, and $(4, 10, \ldots, 10)$ of length 7 in each of five groups of order 40. Orders 36 and 44 thus reconfirm the members $k = 9$ and $k = 11$ of Theorem 4.5 independently; order 40, on the other hand, is not of the shape $4k$ with k odd, so it is a counterexample order that neither source nor Theorem 4.5 predicts, and its witness is the same index multiset that Theorem 4.5 realises at order 20, carried by subgroups of different orders. Order $42 = 2 \cdot 3 \cdot 7$ is undetermined: $|G|/2 = 21$ is not a prime power, so Theorem 5.4 does not apply, and its six groups have sixty minimal obstructions of lengths up to 20; the first group alone cost 104 141 616 search nodes and still left 32 of the sixty capped at a budget of $3 \cdot 10^6$ nodes per multiset. A separate earlier computation covered the orders below 31 with groups constructed independently, rather than taken from a library, their counts reproducing the standard enumeration of groups by order; the two searches run over disjoint sets of orders, and the order-12 hit of the earlier one is reproduced as a positive control by the Gap script of §11. There the positive verdicts at $12, 20, 24, 28, 30$ are exact, while the finer statements about *which* groups of orders 24 and 30 carry no counterexample rest on searches capped at $3 \cdot 10^6$ nodes. No order-level verdict in the table depends on that cap: the only order the cap left undecided was 18, and that is now Theorem 5.2. One entry of that earlier computation is worth isolating: a group of order 24 realises $(6, 6, 6, 6, 8)$, which by Theorem 1.5 the group $C_2 \times S_4$ of order 48 does *not*. Being a counterexample is therefore not monotone in the group.

6. Subgroup lattices from a multiplication table

Every statement in §7 and §8 quantifies over all subgroups of a group, so the first thing needed is a certified list of them. The argument is the same for both groups and is stated once.

Let G be a finite group of order N. By a *presentation of G by tables* we mean a bijection $e : \{0, \ldots, N-1\} \to G$ with $e(0) = 1$ together with functions $\mu : \{0, \ldots, N-1\}^2 \to \{0, \ldots, N-1\}$ and $\iota : \{0, \ldots, N-1\} \to \{0, \ldots, N-1\}$ satisfying $e(i)e(j) = e(\mu(i,j))$ and $e(i)^{-1} = e(\iota(i))$. Subsets of $\{0, \ldots, N-1\}$ are then identified with N-bit integers (*masks*), and e carries a mask K to a subset $e(K) \subseteq G$. Write $\gamma(K)$ for the result of repeatedly adjoining to K all products $\mu(i,j)$ with $i, j \in K$ until the mask stops growing.

Proposition 6.1 (the method)**.** *Let G be presented by tables as above, let L be a list of masks, let I be a list of positive integers, and for each $a \in I$ let C_a be a list of masks. Assume:*

(1) *every $K \in L$ contains 0 and is closed under μ and ι;*
(2) *L contains the mask $\{0\}$, and $\gamma(K \cup \{g\}) \in L$ for every $K \in L$ and every $g \in \{0, \ldots, N-1\}$;*
(3) *for every $K \in L$ there is $a \in I$ with $a \cdot |K| = N$;*
(4) *for every $K \in L$ and every g, the mask of $e(g)\, e(K)$ occurs in C_a, where $a = N/|K|$.*

Then $K \mapsto e(K)$ is a bijection from the set of masks occurring in L onto the set of subgroups of G; in particular, if the entries of L are pairwise distinct then G has exactly $|L|$ subgroups. Moreover, if $(a_1, \ldots, a_n)$ is G-harmonic with $n \le 8$, then every a_i lies in I and the backtracking search that chooses, for each i in turn, a mask from C_{a_i} disjoint from those already chosen — taking a mask of L itself, rather than an arbitrary member of C_{a_1}, at the first position — succeeds on $(a_1, \ldots, a_n)$.

Proof sketch. Soundness. By (1) each $e(K)$ with $K \in L$ contains 1 and is closed under products and inverses, so is a subgroup; distinct masks give distinct subgroups because e is injective.

Completeness. Two properties of γ are proved, both by induction on the iteration count and neither requiring that γ compute the generated subgroup: $K \subseteq \gamma(K)$, and $\gamma(K) \subseteq M$ whenever $K \subseteq M$ and M is closed under μ. By (2), iterating "adjoin x_i, close under γ" from the mask $\{0\}$ stays inside L for any list $x_1, \ldots, x_k$, grows to contain all the x_i, and stays inside any closed mask containing them. Take $x_1, \ldots, x_k$ to be the elements of an arbitrary subgroup $U \le G$: the result lies in L, contains every element of U, and is contained in the mask of U, which is closed. So it *is* the mask of U, and that mask lies in L.

The search. Let (U_i, g_i) witness G-harmonicity. Replacing each g_i by $g_1^{-1} g_i$ changes no disjointness, so we may assume $g_1 = 1$; this is the normalisation that lets the first position range over L rather than over C_{a_1}. Each U_i is $e(K_i)$ for some $K_i \in L$ by the previous paragraph, and $[G : U_i] = a_i$ forces $a_i \cdot |K_i| = N$, so $a_i \in I$ by (3); by (4) the mask of $g_i U_i$ occurs in C_{a_i}; and disjointness of cosets is disjointness of masks, since every set bit of the mask of $g_i U_i$ is a genuine element of $g_i U_i$. So the witness's choices are available to the exhaustive search, which therefore succeeds. □

Only *completeness* of the search is used: if a configuration exists, the search finds one. It may be as pessimistic as it likes, and no soundness statement about it is needed. Likewise a wrong γ would make hypothesis (2) fail, not make the conclusion wrong, since γ enters only through the two order-theoretic properties above. On the $\mathbb{Z}$ side a witness is produced by an unverified generator and then checked; only the checker enters the proofs.

Proposition 6.2. *A_5 has exactly 59 subgroups. Their indices and multiplicities are*

index	1	5	6	10	12	15	20	30	60
order	60	12	10	6	5	4	3	2	1
number	1	5	6	10	6	5	10	15	1

and the total number of left cosets of subgroups of A_5 is 1019.

Proposition 6.3. *$C_2 \times S_4$ has exactly 98 subgroups. Their indices and multiplicities are*

index	1	2	3	4	6	8	12	16	24	48
order	48	24	16	12	8	6	4	3	2	1
number	1	3	3	5	19	12	31	4	19	1

and the total number of left cosets of subgroups of $C_2 \times S_4$ is 1186.

Both counts are classical, the second being a standard small-group fact; the content here is not the number but the explicit, complete list, which is what the negative half of a classification consumes.

Proof sketch. Apply Proposition 6.1 with $N = 60$, respectively $N = 48$. The numbering of A_5 is by a list of its 60 elements, that of $C_2 \times S_4$ by the 48 pairs (b, p) with $b \in C_2$ and p a permutation of four points in lexicographic one-line order; in each case the list has no repetitions and its length equals the order of the group, which is how it is seen to be exhaustive. The hypotheses are finite checks and are carried out exhaustively:

check	A_5	$C_2 \times S_4$
the tables compute the group	3600 products, 60 inverses	2304 products, 48 inverses
(1), closure of each mask	5661 products, 351 inverses	7777 products, 627 inverses
(2), closure of L under adjunction	3540 values of γ	4704 values of γ
(3), the index of each mask	59	98
(4), the coset table	3540 translates in 1019	4704 translates in 1186

Row (1) counts the tests actually performed rather than a bound: a mask is tested on every ordered pair of its elements and on each of its elements, so over a list of subgroups the check costs $\sum_U |U|^2$ products and $\sum_U |U|$ inverses. With the orders and multiplicities displayed above, in A_5 these are $60^2+5\cdot 12^2+6\cdot 10^2+10\cdot 6^2+6\cdot 5^2+5\cdot 4^2+10\cdot 3^2+15\cdot 2^2+1 = 5661$ and $60+5\cdot 12+6\cdot 10+10\cdot 6+6\cdot 5+5\cdot 4+10\cdot 3+15\cdot 2+1 = 351$, and in $C_2\times S_4$ they are $48^2+3\cdot 24^2+3\cdot 16^2+5\cdot 12^2+19\cdot 8^2+12\cdot 6^2+31\cdot 4^2+4\cdot 3^2+19\cdot 2^2+1 = 7777$ and $48+3\cdot 24+3\cdot 16+5\cdot 12+19\cdot 8+12\cdot 6+31\cdot 4+4\cdot 3+19\cdot 2+1 = 627$.

The subgroup counts are then the lengths of the two lists, whose entries are pairwise distinct, and the coset counts are $\sum_U [G:U]$, namely $1+25+36+100+72+75+200+450+60 = 1019$ and $1+6+9+20+114+96+372+64+456+48 = 1186$. □

Remark 6.4. The checks tabulated above are exhaustive finite computations. In the formal development they are discharged by compiled evaluation rather than by kernel reduction; see §11.

The search. Fix a length $n \le 5$ and a non-decreasing tuple $a = (a_1, \dots, a_n)$ of indices. Deciding G-harmonicity of a is the backtracking search of Proposition 6.1: at position i, choose one of the left cosets of one of the subgroups of index a_i, reject if it meets a coset already chosen, and recurse. Every coset in play is an N-bit mask, so the disjointness test is one bitwise conjunction.

The normalisation $g_1 = 1$ is proved on the group side rather than built into the search, as in the proof of Proposition 6.1. In A_5 it reduces the search from 46 580 158 nodes to 9 272 494, a factor of 5.0. A further reduction by symmetry inside blocks of equal indices was measured in both groups — 3 015 276 nodes in A_5, a further factor of 3.1, and 96 127 073 in $C_2 \times S_4$, a further factor of 3.9 — and declined, since it would require the search to be proved invariant under reordering within a block and the unreduced searches are already affordable.

Index 1 is admitted as an index throughout, so multisets are counted over all nine indices of A_5 and all ten of $C_2 \times S_4$; Remark 8.8 reconciles this with the convention that drops it. The counts of the two sweeps are as follows. In A_5 there are 9, 45, 165, 495 and 1287 multisets of lengths 1 to 5, of which 0, 11, 60, 229 and 705 are not $\mathbb{Z}$-harmonic; only those enter the search, at a cost of 0, 1578, 28 811, 627 413 and 8 614 692 nodes, for a total of 9 272 494. In $C_2 \times S_4$ there are 10, 55, 220, 715 and 2002 multisets, of which 0, 14, 96, 420 and 1415 are not $\mathbb{Z}$-harmonic, at a cost of 0, 1775, 79 744, 6 051 211 and 368 031 578 nodes, for a total of 374 164 308.

7. The classification in A_5

Theorem 7.1. *Let $n \le 4$ and let $(a_1, \dots, a_n)$ be an A_5-harmonic tuple. Then $(a_1, \dots, a_n)$ is $\mathbb{Z}$-harmonic.*

This is the A_5 instance of Theorem 1.2 and is therefore not new; it is stated because it is proved here by a completely different route, and because Theorem 1.5 is deduced from it.

Proof sketch. Both properties are invariant under reordering (Lemma 2.2 applied to a permutation and its inverse), so we may assume a non-decreasing. By Proposition 6.1, applied through Proposition 6.2, each a_i is one of the nine indices of A_5 and the search accepts a. On the other hand the exhaustive sweep of §6 reports that every non-decreasing tuple of length 2, 3 or 4 over those nine indices is either $\mathbb{Z}$-harmonic — with a checked witness — or rejected by the search; this is one Boolean per length, resting on 657 802 search nodes in total. Lengths 0 and 1 are vacuous. □

Theorem 7.2. *Let* $(a_1, \ldots, a_5)$ *be an* A_5*-harmonic* 5*-tuple. Then* $(a_1, \ldots, a_5)$ *is* $\mathbb{Z}$*-harmonic, or there is a permutation* σ *of* $\{1, \ldots, 5\}$ *with*

$$(a_{\sigma(1)}, \ldots, a_{\sigma(5)}) \in \big\{(6,6,6,10,15),\ (6,6,6,15,20),\ (6,6,10,12,15)\big\}.$$

Proof sketch. Exactly as for Theorem 7.1, after sorting: an A_5-harmonic 5-tuple is accepted by the search, and the sweep reports that every one of the 1287 non-decreasing 5-tuples over the nine indices is $\mathbb{Z}$-harmonic, or one of the three listed, or rejected by the search. The 705 tuples that are not $\mathbb{Z}$-harmonic are the ones that reach the search, and it costs 8 614 692 nodes; three survive. □

Corollary 7.3. *Let* $n \le 5$ *and let* $(a_1, \ldots, a_n)$ *be an* A_5*-harmonic tuple. Then it is* $\mathbb{Z}$*-harmonic, unless* $n = 5$ *and its multiset is one of* $\{6,6,6,10,15\}$*,* $\{6,6,6,15,20\}$*,* $\{6,6,10,12,15\}$*.*

Proof. Theorem 7.1 for $n < 5$ and Theorem 7.2 for $n = 5$. □

Since each of the three multisets is realised in A_5 by [1, Theorem 3.1 and §5] (reproduced as Proposition 10.1 and Proposition 10.2 below) and none is $\mathbb{Z}$-harmonic, Corollary 7.3 is an exact description: Menon's three A_5 families are precisely the A_5-harmonic tuples of length at most 5 that answer Question 1.1 negatively.

Proposition 7.4 (sharpness)**.**
(1) *Theorem 7.1 does not extend to length* 5*:* $(6,6,6,10,15)$ *is* A_5*-harmonic and not* $\mathbb{Z}$*-harmonic.*
(2) *Corollary 7.3 does not extend to length* 6*:* $(6,6,6,10,15,15)$ *is* A_5*-harmonic and not* $\mathbb{Z}$*-harmonic, and its length is not* 5*.*
(3) *None of the three exceptions can be dropped: each one of* $(6,6,6,10,15)$ *and* $(6,6,6,15,20)$ *and* $(6,6,10,12,15)$ *is* A_5*-harmonic, and none is* $\mathbb{Z}$*-harmonic.*

Proof. (1) and (3) are Proposition 10.1 and Proposition 10.2 together with Lemma 2.3. For (2), take the six cosets of index profile $(6,6,6,10,15,15)$ inside the nine-coset partition of Proposition 9.1; they are pairwise disjoint because the nine are, and the sub-tuple $(6,6,6,10,15)$ is not $\mathbb{Z}$-harmonic, so neither is the 6-tuple, by Lemma 2.2. □

Multisets that A_5 does not realise. The three exceptional multisets all contain an entry 15. Hence a tuple of length 5 with no entry equal to 15 is either $\mathbb{Z}$-harmonic or not A_5-harmonic, and the residue obstruction of Lemma 2.3 decides which.

Theorem 7.5. *None of* $(5,6)$*,* $(5,12)$*,* $(6,6,6,6,10)$*,* $(6,6,6,6,20)$*,* $(6,6,6,10,12)$ *is* A_5*-harmonic.*

Proof. For $(5,6)$ and $(5,12)$: $\gcd(5,6) = \gcd(5,12) = 1$, so $1 \mid r_1 - r_2$ for any integers, and neither pair is $\mathbb{Z}$-harmonic. By Theorem 7.1 at $n = 2$, neither is A_5-harmonic.

For the three 5-tuples, apply Theorem 7.2. Each of the three exceptional multisets contains an entry 15 and none of the three tuples here does, so the exceptional alternative is impossible and it remains to rule out $\mathbb{Z}$-harmonicity. Suppose one of them had a witness $(r_1, \ldots, r_5)$. In each case four of the five entries have pairwise greatest common divisor 6 — the four entries 6 in $(6,6,6,6,10)$ and $(6,6,6,6,20)$, and the three entries 6 together with the entry 12 in $(6,6,6,10,12)$, since $\gcd(6,12) = 6$ — so the corresponding four residues are pairwise incongruent modulo 6. The fifth entry is 10, 20 and 10 respectively and has greatest common divisor 2 with each of the other four, so its residue is incongruent to each of them modulo 2. Lemma 2.3(2) gives the contradiction. □

Concretely, $(5,6)$ says that a coset of a subgroup isomorphic to A_4 and a coset of a subgroup isomorphic to D_{10} always meet in A_5, and $(5,12)$ says the same for A_4 and C_5. The tuple $(6,6,6,6,10)$ is the minimal shape to which [1, Lemma 2.1(ii)] applies; the source proves that it is not $\mathbb{Z}$-harmonic and leaves open — indeed does not raise — the question whether A_5 realises it.

Remark 7.6 (computation outside the formal development). Restricting to the eight indices > 1, a separate enumeration counts 290 multisets of length at most 5 that are not A_5-harmonic, of which exactly seven are minimal under passage to sub-multisets:

$$(5,6),\ (5,12),\ (5,5,5,5,10),\ (5,5,5,10,15),\ (6,6,6,6,10),\ (6,6,6,6,20),\ (6,6,6,10,12).$$

Five of the seven are not $\mathbb{Z}$-harmonic and are exactly the five of Theorem 7.5. The remaining two *are* $\mathbb{Z}$-harmonic — all their pairwise greatest common divisors equal 5, so $(0, 1, 2, 3, 4)$ is a witness — and therefore lie outside what Corollary 7.3 decides; no claim is made about them here. This minimality count is a computation, not part of the formal development.

Remark 7.7 (non-vacuity of the search). The search accepts $(6, 6, 6, 10, 15)$ and $(6, 10, 15)$ and rejects $(6, 6, 6, 6, 10)$ and $(5, 6)$; the $\mathbb{Z}$-witness checker accepts $(6, 10, 15)$ and rejects $(6, 6, 6, 10, 15)$. So neither is constantly true nor constantly false. Moreover the trivial subgroup, A_5 itself and the index-6 dihedral subgroup of Proposition 10.1 all occur in the list L of Proposition 6.2, so the completeness statement is not about an empty list.

8. The classification in $C_2 \times S_4$

The second group of [1, §5] has order 48 and is not simple, which is the point the source makes with it. Its lattice is larger than that of A_5 — 98 subgroups against 59, 1186 cosets against 1019, a length-5 sweep of 368 million nodes against 8.6 million — but by Proposition 6.1 it is literally the same argument, applied to a second table.

Theorem 8.1. *Let $n \leq 4$ and let $(a_1, \ldots, a_n)$ be a $C_2 \times S_4$-harmonic tuple. Then $(a_1, \ldots, a_n)$ is $\mathbb{Z}$-harmonic.*

Proof sketch. As for Theorem 7.1, through Propositions 6.1 and 6.3: after sorting, each a_i is one of the ten indices of $C_2 \times S_4$ and the search accepts a, while the sweep of §6 reports that every non-decreasing tuple of length 2, 3 or 4 over those ten indices is $\mathbb{Z}$-harmonic with a checked witness or rejected by the search. The tuples that reach the search are the 14, 96 and 420 that are not $\mathbb{Z}$-harmonic, at 1775, 79 744 and 6 051 211 nodes. □

Theorem 8.2. *Let $(a_1, \ldots, a_5)$ be a $C_2 \times S_4$-harmonic 5-tuple. Then $(a_1, \ldots, a_5)$ is $\mathbb{Z}$-harmonic, or there is a permutation σ of $\{1, \ldots, 5\}$ with*

$$(a_{\sigma(1)}, \ldots, a_{\sigma(5)}) \in \{(4, 6, 6, 6, 6),\ (6, 6, 6, 6, 16)\}.$$

Proof sketch. Exactly as for Theorem 7.2: a $C_2 \times S_4$-harmonic 5-tuple is accepted by the search after sorting, and the sweep reports that every one of the 2002 non-decreasing 5-tuples over the ten indices is $\mathbb{Z}$-harmonic, or one of the two listed, or rejected by the search. The 1415 tuples that are not $\mathbb{Z}$-harmonic are the ones that reach the search, and it costs 368 031 578 nodes; two survive. This is the single largest computation in the paper. □

Corollary 8.3. *Let $n \leq 5$ and let $(a_1, \ldots, a_n)$ be a $C_2 \times S_4$-harmonic tuple. Then it is $\mathbb{Z}$-harmonic, unless $n = 5$ and its multiset is $\{4, 6, 6, 6, 6\}$ or $\{6, 6, 6, 6, 16\}$.*

Proof. Theorem 8.1 for $n < 5$ and Theorem 8.2 for $n = 5$. □

Both multisets are realised in $C_2 \times S_4$ by [1, §5] (Proposition 10.2) and neither is $\mathbb{Z}$-harmonic, so Corollary 8.3 too is an exact description. Putting the two classifications together:

Corollary 8.4. *Let G be A_5 or $C_2 \times S_4$ and let $(a_1, \ldots, a_n)$ be a G-harmonic tuple with $n \leq 5$ that is not $\mathbb{Z}$-harmonic. Then $n = 5$ and $\{a_1, \ldots, a_n\}$ is one of the five multisets of* [1, Theorem 3.1 and §5]*: $\{6, 6, 6, 10, 15\}$, $\{6, 6, 6, 15, 20\}$ or $\{6, 6, 10, 12, 15\}$ when $G = A_5$, and $\{4, 6, 6, 6, 6\}$ or $\{6, 6, 6, 6, 16\}$ when $G = C_2 \times S_4$. Conversely all five occur.*

The source claims completeness for neither group.

Proposition 8.5 (sharpness).
1. *Theorem 8.1 does not extend to length 5: $(4, 6, 6, 6, 6)$ is $C_2 \times S_4$-harmonic and not $\mathbb{Z}$-harmonic.*
2. *Corollary 8.3 does not extend to length 6: the index profile $(4, 6, 6, 6, 6, 12)$ of the coset partition of Theorem 9.3(1) is $C_2 \times S_4$-harmonic and not $\mathbb{Z}$-harmonic, and its length is not 5.*

(3) *Neither exception can be dropped: each of* $(4,6,6,6,6)$ *and* $(6,6,6,6,16)$ *is* $C_2 \times S_4$*-harmonic, and neither is* $\mathbb{Z}$*-harmonic. Independently, at the level of the Booleans: with the list of exceptions emptied, the length-*5 *check fails at first index* 4 *and again at first index* 6 *— so each exception is needed separately — while the length-*4 *check with no exceptions at all succeeds, which is Theorem 8.1.*

Proof. (1) and (3) are Proposition 10.2 together with Lemma 2.3(2), whose hypotheses hold since $\gcd(6,4) = \gcd(6,16) = 2$; the Boolean statements in (3) are separate exhaustive evaluations. For (2), Theorem 9.3(1) gives the six pairwise disjoint cosets, and the profile contains four entries 6 and the entry 4, so Lemma 2.3(2) applies again. □

Multisets that $C_2 \times S_4$ does not realise.

Theorem 8.6. *None of*

$$(6,6,6,6,8),\ (4,4,4,8,12),\ (4,4,4,12,16),\ (2,8,8,8,12),\ (2,8,8,12,16),\ (4,6,8,8,8),\ (4,6,8,8,16)$$

is $C_2 \times S_4$*-harmonic.*

Proof. Apply Theorem 8.2. Each of the two exceptional multisets consists of entries drawn from $\{4,6\}$ and $\{6,16\}$ respectively, and each of the seven tuples above takes a value lying in neither list — an entry 8 in the first, sixth and seventh, an entry 12 in the second and third, and an entry 2 in the fourth and fifth — so the exceptional alternative is impossible in each case, and it remains to rule out $\mathbb{Z}$-harmonicity.

For $(6,6,6,6,8)$: four entries 6 and the entry 8 with $\gcd(6,8) = 2$, so Lemma 2.3(2) applies. For $(4,4,4,8,12)$ and $(4,4,4,12,16)$: every pairwise greatest common divisor is 4 and the length is $5 > 4$, so Lemma 2.5(1) applies with $m = 4$. For the remaining four, Lemma 2.5(2) applies, the three entries of pairwise greatest common divisor 8 being the three 8s in $(2,8,8,8,12)$ and $(4,6,8,8,8)$ and the two 8s together with the 16 in $(2,8,8,12,16)$ and $(4,6,8,8,16)$; x is the entry 12, respectively 4, and y the entry 2, respectively 6. □

None of the seven is excluded by counting: their coset sizes sum to 38, 46, 43, 46, 43, 38 and 35, all at most 48. The first is the sharp case noted in §1, and is the order-48 analogue of the A_5 fact that $(6,6,6,6,10)$ is not A_5-harmonic while $(6,6,6,10,15)$ is.

Remark 8.7 (computation outside the formal development). Restricting to the nine indices > 1, namely $2,3,4,6,8,12,16,24,48$, a separate enumeration counts 953 multisets of length at most 5 that are not $C_2 \times S_4$-harmonic, of which exactly 51 are minimal under passage to sub-multisets. Exactly one of the 51, namely $(2,4,8)$, is $\mathbb{Z}$-harmonic and so lies outside what Corollary 8.3 decides; of the other 50, those of length at most 4 are already covered by Theorem 1.2, and of the 21 of length 5, fourteen have $\sum_i 48/a_i > 48$ and are excluded by counting alone. The seven that remain are exactly those of Theorem 8.6. This census is a computation, not part of the formal development; the seven non-realisability statements themselves are theorems.

Remark 8.8 (counting conventions). Index 1 is admitted as an index throughout this paper, so multisets are counted over all nine indices of A_5 and all ten of $C_2 \times S_4$. A common alternative drops it; under it the numbers of multisets of lengths $2,3,4,5$ read $36,120,330,792$ for A_5 and $45,165,495,1287$ for $C_2 \times S_4$, in place of $45,165,495,1287$ and $55,220,715,2002$, and the numbers of those that are not $\mathbb{Z}$-harmonic read $2,15,64,210$ and $4,41,200,700$, in place of $11,60,229,705$ and $14,96,420,1415$. In each column the difference is exactly the multisets containing a 1; such a multiset of length at least 2 is never $\mathbb{Z}$-harmonic, because $\gcd(1,a) = 1$ divides every difference, and never G-harmonic, because the only subgroup of index 1 is G itself, whose one coset meets every other. So the conventions differ in how many multisets they count, never in which are harmonic, and the exceptional multisets are the same either way.

Remark 8.9 (non-vacuity of the search). In $C_2 \times S_4$ the search accepts $(4,6,6,6,6)$, $(6,6,6,6,16)$ and $(4,6,8)$ and rejects $(2,4,6)$ and $(3,3,3,3)$; the $\mathbb{Z}$-witness checker accepts $(4,6,8)$ and rejects $(4,6,6,6,6)$. The trivial subgroup, the whole group, two of the source's own subgroups and the index-12 Klein four-group used in §9 all occur in the list L of Proposition 6.3, so the completeness statement is not about an empty list; and each of its ten indices is the index of an actual subgroup, so the hypothesis space of Theorem 8.1 is nowhere empty.

9. Coset partitions

By Theorem 3.1(2) a G-harmonic tuple with $\sum_i 1/a_i < 1$ leaves part of G uncovered. Menon's Corollary 3.2 shows that the main counterexample nevertheless sits inside a genuine coset partition; the point is that Question 1.1 already fails for tuples arising from partitions, which is the setting of the Herzog–Schönheim conjecture.

Proposition 9.1 ([1, Corollary 3.2])**.** *The five cosets of Proposition 10.1 extend to a partition of A_5 into nine pairwise disjoint cosets, of index profile $\{6,6,6,10,10,10,15,15,15\}$. This profile is A_5-harmonic and not $\mathbb{Z}$-harmonic.*

The source does not ask the same of the four further families of its §5. All four extend, and so does the order-12 counterexample of §4.

Theorem 9.2. (1) *The five cosets realising $(6,6,6,15,20)$ extend to a partition of A_5 into eleven pairwise disjoint cosets, of index profile $\{6,6,6,15,15,15,15,15,15,20,20\}$.*
(2) *The five cosets realising $(6,6,10,12,15)$ extend to a partition of A_5 into ten pairwise disjoint cosets, of index profile $\{6,6,10,10,10,12,12,15,15,15\}$.*

Both profiles are A_5-harmonic, and neither is $\mathbb{Z}$-harmonic.

Theorem 9.3. (1) *The five cosets realising $(4,6,6,6,6)$ extend to a partition of $C_2 \times S_4$ into six pairwise disjoint cosets, of index profile $\{4,6,6,6,6,12\}$ — a single further coset suffices.*
(2) *The five cosets realising $(6,6,6,6,16)$ extend to a partition of $C_2 \times S_4$ into ten pairwise disjoint cosets, of index profile $\{6,6,6,6,12,16,16,24,24,24\}$, and five further cosets are the fewest possible.*

Both profiles are $C_2 \times S_4$-harmonic, and neither is $\mathbb{Z}$-harmonic.

Theorem 9.4. *The five cosets of Proposition 4.2 extend to a partition of $C_2 \times S_3$ into six pairwise disjoint cosets, of index profile $\{4,6,6,6,6,12\}$: they miss exactly one element, so a single coset of the trivial subgroup completes them and one further coset is the fewest possible. The profile is $C_2 \times S_3$-harmonic and not $\mathbb{Z}$-harmonic.*

Proof sketch. The additional cosets were found by a backtracking cover search and are then exhibited explicitly. Each subgroup is given by its element list, from which its order and hence its index follow; disjointness of g_iU_i and g_jU_j reduces to $g_iu \neq g_jv$ for u in the element list of U_i and v in that of U_j, which is a finite check over at most 11×11 pairs of lists. Covering is the statement that each of the 60 elements of A_5, respectively each of the 48 elements of $C_2 \times S_4$ or the 12 of $C_2 \times S_3$, lies in one of the listed cosets; each list of elements is exhaustive because it has no repetitions and its length is the order of the group. All these checks are exhaustive computations, carried out by kernel reduction. In every case the first five cosets are unchanged — the source's own subgroups and coset representatives for Theorems 9.2 and 9.3, and those of Proposition 4.2 for Theorem 9.4.

For the profiles, a partition requires $\sum_i 1/a_i = 1$, and indeed $\frac{3}{6}+\frac{6}{15}+\frac{2}{20}=1$, $\frac{2}{6}+\frac{3}{10}+\frac{2}{12}+\frac{3}{15}=1$, $\frac{1}{4}+\frac{4}{6}+\frac{1}{12}=1$ and $\frac{4}{6}+\frac{1}{12}+\frac{2}{16}+\frac{3}{24}=1$. None is $\mathbb{Z}$-harmonic. The first contains three entries 6, an entry 20 with $\gcd(6,20)=2$ and an entry 15 with $\gcd(6,15)=3$, so Lemma 2.3(1) applies. The second contains two entries 6 and an entry 12, whose residues in a witness would have to be pairwise incongruent modulo 6 because $\gcd(6,12)=6$, together with an entry 10 with $\gcd(6,10)=\gcd(12,10)=2$ and an entry 15 with $\gcd(6,15)=\gcd(12,15)=3$; so Lemma 2.3(1) applies again. The profile $\{4,6,6,6,6,12\}$, which serves both Theorem 9.3(1) and Theorem 9.4, and the profile

$\{6,6,6,6,12,16,16,24,24,24\}$ each contain four entries 6 together with an entry (4, respectively 16) of greatest common divisor 2 with 6, so Lemma 2.3(2) applies.

Few new subgroups are needed: for Theorem 9.2(1) one Klein four-group of index 15; for Theorem 9.3(1) one Klein four-group of index 12; for Theorem 9.3(2) one subgroup of order 4 and three of order 2, together with a second coset of one of the source's own subgroups of index 16; and for Theorem 9.4 the trivial subgroup alone. That five further cosets are the fewest possible in the order-48 case was established by a complete exact-cover search, branching on the smallest uncovered element, over all 1186 cosets of index greater than 1; that minimality is not part of the formal development, whereas the minimality in Theorem 9.4 is immediate, since $11 \neq 12$. □

Remark 9.5. Every index profile above has repeated entries, so none of these partitions bears on the Herzog–Schönheim conjecture [3], which concerns coset partitions with pairwise distinct indices. What they do show is that the failure of Question 1.1 is not an artefact of allowing configurations that leave part of the group uncovered: it already occurs for sub-tuples of honest coset partitions, and now for six partitions with five different profiles in three groups rather than for one. What is lost with Question 1.1 is the route of §1: as [1, §6] puts it, Ginosar's question "cannot be used to deduce the Herzog–Schönheim conjecture from the Davenport–Mirsky–Newman–Rado theorem".

Remark 9.6. The extension in Proposition 9.1 is genuinely needed: the five cosets of Proposition 10.1 cover only 40 of the 60 elements of A_5 — the identity lies in none of them — and the first eight of the nine cosets of Proposition 9.1 cover only 54, the six elements of the deleted coset being missed.

10. The published counterexamples, re-verified

For completeness we record that the data of [1] reproduces exactly. Nothing in this section is new; it is stated because §7, §8 and §9 refer to these tuples, and because an independent check of a one-version, uncited preprint is worth recording.

Proposition 10.1 ([1, Theorem 3.1])**.** *Let $D \leq A_5$ be the normaliser of a Sylow 5-subgroup (dihedral of order 10, index 6), $T \leq A_5$ a subgroup isomorphic to S_3 (order 6, index 10) and $V \leq A_5$ a Sylow 2-subgroup (Klein four, order 4, index 15). There are $g_1, \dots, g_5 \in A_5$ making $g_1D, g_2D, g_3D, g_4T, g_5V$ pairwise disjoint, of sizes 10, 10, 10, 6, 4 summing to 40. Hence $(6,6,6,10,15)$ is A_5-harmonic; it is not $\mathbb{Z}$-harmonic; and so Question 1.1 has a negative answer at $n = 5$ and the bound 4 of Theorem 1.2 is sharp.*

Proposition 10.2 ([1, §5])**.** *Both $(6,6,6,15,20)$ and $(6,6,10,12,15)$ are A_5-harmonic and not $\mathbb{Z}$-harmonic. In the group $C_2 \times S_4$ of order 48, $(4,6,6,6,6)$ and $(6,6,6,6,16)$ are $C_2 \times S_4$-harmonic and not $\mathbb{Z}$-harmonic; in particular Question 1.1 fails in a group that is not simple, so simplicity is not the source of the phenomenon.*

The subgroups and coset representatives for the two A_5 families of Proposition 10.2 are those of the ancillary file distributed with [1]; the article's text gives them only for the order-48 families. Indices are never assumed: each subgroup is given by its element list, its order is the length of that list, and its index follows from Lagrange's theorem together with $|A_5| = 60$, respectively $|C_2 \times S_4| = 48$. The isomorphism types named above and in §1 are given only for orientation; the formal statements mention orders and indices, not isomorphism types. The order-48 group is handled as the direct product $C_2 \times S_4$, which is the source's own realisation of $\langle(1\,2)\rangle \times \mathrm{Sym}\{3,4,5,6\}$ inside S_6; the group $C_2 \times S_3$ of §4 is written the same way, one symmetric group down.

Proposition 10.3 (controls)**.**
1. *Every 4-element sub-tuple of every one of the five counterexamples above is $\mathbb{Z}$-harmonic, with an explicit witness; for instance $(6,6,6,10)$ has the witness $(0,2,4,1)$. So none of them beats length 5, and the Margolis–Schnabel bound is attained rather than beaten.*
2. *The three indices of Proposition 10.1 lie strictly between 1 and 60, so no subgroup used is trivial or the whole group.*

(3) *Each of the five counterexamples has a pair of entries with greatest common divisor* $6 \geq 5$, *so all five satisfy Z.-W. Sun's conjecture* [6, Conjecture 1.2] *that pairwise disjoint cosets* $g_1U_1, \dots, g_nU_n$ *with* $n > 1$, *of subgroups of finite index, must have* $\gcd([G : U_i], [G : U_j]) \geq n$ *for some* $i \neq j$*;* [2, §1] *records it next to Question 1.1. Refuting Question 1.1 therefore does not touch Sun's question.*

11. Verification

Every theorem, proposition, lemma and corollary of this paper carries a formal proof in Lean 4 checked against *Mathlib* [8, 7]; repository reference to be added at submission. The exceptions are stated as such where they occur: Theorems 4.5 and 5.4, which are proved on paper and machine-checked only outside the kernel; Remarks 3.5, 3.6, 5.5, 7.6, 8.7 and 8.8; the minimality assertion in Theorem 9.3(2); and the order-multiplicity tables and coset totals of Propositions 6.2 and 6.3, which are read off the certified lists of masks by counting. The two subgroup counts 59 and 98 themselves are formal. Lemma 5.1 is formal in each of the ingredients listed after it, rather than as a single statement; the assembly is the proof printed with it. The development is free of `sorry`, and the group-theoretic statements are about Mathlib's own objects — the alternating group on $\mathrm{Fin}\,5$, the products of $\mathbb{Z}/2$ with the permutation groups of $\mathrm{Fin}\,3$ and $\mathrm{Fin}\,4$, their `Subgroup` lattices, `Subgroup.index` and pointwise left cosets — rather than about a combinatorial model, so that $|A_5| = 60$, $|C_2 \times S_4| = 48$, $|C_2 \times S_3| = 12$ and every index quoted are derived, not assumed. Proposition 6.1 is proved for an arbitrary finite group presented by tables, the two groups of §§7–8 entering only as instances; and Theorem 3.1, Lemma 3.2 and Lemma 5.1 are proved for an arbitrary finite group with no presentation at all.

The whole of §§3–5, the results of §9 and §10, together with Lemma 2.3, Lemma 2.5, Remark 2.4, Proposition 10.3, Proposition 7.4 and Proposition 8.5 apart from its Boolean statements, depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and their finite checks are carried out by kernel reduction: the coset disjointness and covering tests; the 1296, 2592 and 4096 residue patterns of Lemmas 2.3 and 2.5; the quadruples (N, d, e, k) below 12 in Theorem 4.1; and, in Theorem 5.2, the 16-case correctness of the tabulated greatest common divisors together with the 90 admissible parameter tuples and their 9610 ordered pairs of slots, which cost about 40 seconds of kernel reduction. So are the exhaustiveness statements for the element lists of the three groups. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the computations tabulated in the proof of Propositions 6.2 and 6.3, together with the classification Booleans themselves, one per length and per group, resting on 9 272 494 backtracking nodes for A_5 and 374 164 308 for $C_2 \times S_4$. Everything else — the two lattice counts, the four classification theorems with their corollaries, the two non-realisability theorems, the Boolean statements of Proposition 8.5(3) and the discriminating checks of Remarks 7.7 and 8.9 — depends on the three standard axioms together with the relevant finite checks; for A_5 the completeness argument for the subgroup list needs only the check that the list is closed under adjoining one element, and not even `Classical.choice`. For $C_2 \times S_4$ the four checks of Proposition 6.1 are bundled into a single record, so every order-48 statement formally depends on all four rather than only on those its proof consumes; this is coarser than necessary and never weaker, since the four are separately checked evaluations of the same data.

The data underlying the tables and the classifications were regenerated by independent implementations — for each group, one that builds it from its multiplication law and closes bitmasks under the table, against an earlier one that closes sets of permutations — which agree on both lattices, both classifications, and every numerical assertion of [1] we checked. The unformalised statements were checked the same way. Theorem 5.2 was confirmed three further times outside the kernel: by an uncapped backtracking search over all five groups of order 18, built from their multiplication laws and verified pairwise non-isomorphic, at a cost of 9 088 567 nodes; by an independent application of the structural criterion; and by a script in GAP [9] using its own small-group library, its own subgroup lattice and *right* cosets, which also reproduces Proposition 4.2 as a positive control. The group counts used in Remark 5.5 agree with the standard enumeration of groups by order (OEIS A000001).

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