THE SIGN OF THE COEFFICIENTS IN THE IMPROVED DISCRETE WEIGHTED HARDY INEQUALITY

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Abstract. For a real parameter α put

$$b_k(\alpha) \;=\; \binom{\alpha}{k} - (-1)^k\binom{(1-\alpha)/2}{k} - \binom{(1+\alpha)/2}{k}, \qquad k \ge 0.$$

These are the coefficients of the infinite tail of lower-order terms that upgrades the discrete weighted Hardy inequality with power weight n^α to the improved inequality of Gupta, and the upgrade is available exactly where they are all non-negative. Gupta proved $b_k(\alpha) \ge 0$ for $\alpha \in [1/3, 1) \cup \{0\}$ and conjectured a converse: for every $\alpha \in (0, 1/3) \cup [5, \infty)$ some $b_k(\alpha)$ is negative. Towards it he reached only the odd integers $\alpha = 2m + 1$. We prove the conjecture in full, and more: some $b_k(\alpha)$ is negative for every real $\alpha \in (0, 1/3)$ and for every real $\alpha > 1$, so that the only real parameters at which the whole sequence stays non-negative are those of $[1/3, 1) \cup \{0, 1\}$. The proof is elementary and uniform — one algebraic identity, whose requirement that a certain coefficient vanish identically is what produces the constant $1/3$, together with a decay estimate for $\prod_j (1 - \frac{c}{2j})$ that needs nothing beyond Bernoulli's inequality. We also correct Gupta's partial result at the odd integers: its strict inequality fails at the first index it asserts, where the true value is 0, and we prove the corrected form of that strict half. All results are machine-checked in Lean 4.

1. Introduction

1.1. **The coefficients.** Let u be finitely supported on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ with $u(0) = 0$. The discrete weighted Hardy inequality with power weights reads

$$\sum_{n\ge1} |u(n) - u(n-1)|^2\, n^\alpha \;\ge\; \frac{(\alpha-1)^2}{4} \sum_{n\ge1} \frac{|u(n)|^2}{n^2}\, n^\alpha, \tag{1}$$

and Gupta [1] proved it, with the sharp constant $(\alpha - 1)^2/4$, for every $\alpha \in [0, 1) \cup [5, \infty)$. In the same paper he showed that on a smaller range (1) carries an infinite tail of extra non-negative terms,

$$\sum_{n\ge1} |u(n) - u(n-1)|^2 n^\alpha \;\ge\; \frac{(\alpha-1)^2}{4} \sum_{n\ge1} \frac{|u(n)|^2}{n^2} n^\alpha \;+\; \sum_{k\ge3} b_k(\alpha) \sum_{n\ge2} \frac{|u(n)|^2}{n^k} n^\alpha, \tag{2}$$

where, with $\binom{\gamma}{k} = \gamma(\gamma-1)\cdots(\gamma-k+1)/k!$ the binomial coefficient at a real upper index,

$$b_k(\alpha) \;:=\; \binom{\alpha}{k} - (-1)^k\binom{(1-\alpha)/2}{k} - \binom{(1+\alpha)/2}{k}. \tag{3}$$

The $b_k(\alpha)$ are the Taylor coefficients at $x = 0$ of

$$g(x) \;=\; 1 + (1+x)^\alpha - (1-x)^{(1-\alpha)/2} - (1+x)^{(1+\alpha)/2}, \tag{4}$$

and $b_2(\alpha) = (\alpha-1)^2/4$ is the sharp constant of (1) itself. Here g is $g_{\alpha,\beta}(x) = 1 + (1+x)^\alpha - (1-x)^\beta - (1+x)^{\alpha+\beta}$ at $\beta = (1-\alpha)/2$; both (1) and (2) are obtained in [1, Theorem 2.1] from a two-parameter inequality with weight $w_{\alpha,\beta}(n) = n^\alpha g_{\alpha,\beta}(1/n)$ for $n \ge 2$ and $w_{\alpha,\beta}(1) = 1 + 2^\alpha - 2^{\alpha+\beta}$, by bounding that weight below by $\frac{(\alpha-1)^2}{4} n^{\alpha-2}$. What makes (2) an improvement of (1) is precisely that $b_k(\alpha) \ge 0$ for all $k \ge 3$, and Gupta established this on $[1/3, 1) \cup \{0\}$ [1, Lemma 4.1].

The unweighted case of this circle of ideas is older: an improvement of the classical discrete Hardy inequality, that is of (1) at $\alpha = 0$, is due to Keller, Pinchover and Pogorzelski [4], and to Fischer, Keller and Pogorzelski [5] for general exponent $p > 1$. Gupta notes that (2) at $\alpha = 0$ follows from

[4], and that the inequality of [4] is in fact strictly stronger there; what [1] adds is the whole family of power weights n^α, and with it the question of this paper.

The whole question is therefore one about a sequence of explicit polynomials in α, with no analysis left in it: *for which real α is $b_k(\alpha) \geq 0$ for every $k \geq 3$?*

1.2. **The conjecture.** Gupta conjectured that his range is the whole answer. Verbatim from [1, Remark 5.4], quoted from arXiv:2108.01500v2:

> "We conjecture that constants $b_k(\alpha)$ given by (1.8) are not non-negative for all k when α does not lie in $[1/3, 1) \cup \{0\}$, i.e, for all $\alpha \in (0, 1/3) \cup [5, \infty)$, there exists $i \geq 1$ such that $b_i(\alpha) < 0$."

The same conjecture is stated in [1, Remark 1.2], and again — in nearly identical words, and with the same partial result attached — in the author's doctoral thesis [2], so the question was still open to its proposer at the time of that thesis and of its posting to the arXiv in March 2024. Towards it [1, Lemma 5.5] settles the countable set of odd integers $\alpha = 2m + 1$, $m \geq 2$. No non-integer α is reached anywhere in [1].

Two remarks on how the sentence is to be read, both used below. First, the quantifier "$i \geq 1$" may be replaced by "$k \geq 3$" without changing anything: $b_1(\alpha) = 0$ and $b_2(\alpha) = (\alpha - 1)^2/4 \geq 0$ for every real α, so a negative coefficient of index ≥ 1 automatically has index ≥ 3. (The reading "$i \geq 0$" would instead be vacuous, since $b_0(\alpha) = -1$ always.) Second, the clause after the "i.e." is strictly weaker than the clause before it: the parameters lying outside $[1/3, 1) \cup \{0\}$ include the negative reals and the interval $(1, 5)$, which $(0, 1/3) \cup [5, \infty)$ omits.

1.3. **Results.** We prove the conjecture, in the stronger of its two readings, for every real parameter.

Theorem A (Theorems 5, 7 and 8)**.** *For every real α with $0 < \alpha < 1/3$ there is a $k \geq 3$ with $b_k(\alpha) < 0$, and for every real $\alpha > 1$ there is a $k \geq 3$ with $b_k(\alpha) < 0$. In particular Gupta's conjecture holds: for every $\alpha \in (0, 1/3) \cup [5, \infty)$ there is a $k \geq 3$ with $b_k(\alpha) < 0$.*

The half $\alpha > 1$ is strictly larger than the conjectured $[5, \infty)$; it covers the gap $(1, 5)$ that the "i.e." of Remark 5.4 drops, and it contains no integrality assumption, whereas [1, Lemma 5.5] reaches only $\alpha \in \{5, 7, 9, \dots\}$. Adding the negative axis, which costs one line, closes the real line:

Theorem B (Theorem 9)**.** *For every real $\alpha \notin [1/3, 1) \cup \{0, 1\}$ there is a $k \geq 3$ with $b_k(\alpha) < 0$.*

The excluded point $\alpha = 1$ is a genuine exception rather than a convenience: $b_k(1) = 0$ for every $k \geq 1$ (Proposition 14). So the broad clause of the conjecture — [1, Remark 1.2], verbatim: "the constant $b_k(\alpha)$ is not non-negative for all $k \geq 3$ when α lies outside $[1/3, 1) \cup \{0\}$" — is true at every real α outside $[1/3, 1) \cup \{0\}$ except $\alpha = 1$, where it fails. Combining Theorem B with [1, Lemma 4.1] gives the complete dichotomy stated in Corollary 10.

Finally we correct the partial result:

Theorem C (Theorem 11)**.** *For every integer $m \geq 0$ one has $b_{m+2}(2m + 1) = 0$; in particular $b_4(5) = 0$, $b_5(7) = 0$, $b_6(9) = 0$. Consequently the strict inequality asserted in* [1, Lemma 5.5] *for all $i > m + 1$ fails at $i = m + 2$, the first index it asserts. For $m \geq 2$ the corrected assertion holds: $b_{m+2}(2m + 1) = 0$ and $b_i(2m + 1) < 0$ for every $i \geq m + 3$.*

The lemma's role in [1] — evidence for the conjecture at the odd integers — is untouched by this, since $b_i(2m + 1) < 0$ for $i \geq m + 3$ is still true, and now proved rather than read off a factorisation. Where the slip comes from is explained in Section 7: at $i = m + 2$ two bracketed ranges in the printed proof are empty, so the bracket is $1 - 1 = 0$ and not negative.

1.4. **The mechanism.** The two halves of Theorem A use different arrangements of one elementary mechanism, and no special functions at all. The classical route to the asymptotics of (3) runs through $\binom{\gamma}{k} \sim (-1)^k k^{-\gamma-1}/\Gamma(-\gamma)$; the exponents $-\alpha - 1$ and $-(3 - \alpha)/2$ produced by the three terms of (3) cross exactly at $\alpha = 1/3$, which explains the constant but requires Gamma-function asymptotics. We avoid it. Writing $\alpha_1 = (1-\alpha)/2$ and $\alpha_2 = (1+\alpha)/2$, so that $\alpha_1 + \alpha_2 = 1$, the definition (3) for even k becomes

$$k!\, b_k(\alpha) \;=\; -\alpha\,\Pi(\alpha,k) + \alpha_1\Pi(\alpha_1,k) + \alpha_2\Pi(\alpha_2,k), \qquad \Pi(\gamma,k) = \prod_{j=1}^{k-1}(j-\gamma),$$

and on $(0,1)$ all three products are positive. Everything then rests on the single identity

$$\Big(1-\frac{c}{2j}\Big)(j-\alpha) - (j-\alpha_1) \;=\; \frac{(1-3\alpha-c)\,j + c\alpha}{2j},$$

whose j-coefficient vanishes *identically* exactly for $c = 1-3\alpha$; and the resulting numerator $c\alpha$ is positive exactly when $\alpha < 1/3$. The constant $1/3$ is not imported from anywhere — it is the unique value that makes a linear term disappear. With c so chosen, both shifted products are dominated factor by factor by $\prod_{j=1}^{k-1}\big(1-\frac{c}{2j}\big)$ times $\Pi(\alpha,k)$, and it remains to know that this product tends to 0. That is Lemma 3, proved by dyadic doubling out of Bernoulli's inequality alone; it is the discrete replacement for the divergence of $\sum 1/j$. For $\alpha > 1$ the same estimate is used the other way round: there $\alpha_1 < 0$, the term $(-1)^k\binom{\alpha_1}{k}$ becomes a positive, growing product entering b_k with a minus sign, and the other two terms are shown to be negligible against it.

That the threshold of our argument coincides with the threshold of the positivity lemma of [1] is a reason to think this is the mechanism and not an artefact: for $\alpha > 1/3$ the two comparisons reverse.

1.5. **What is *not* claimed.** Theorems A–C are statements about the sign of the numbers $b_k(\alpha)$, and nothing more. In particular they say nothing about whether the inequality (1) admits an improvement of *some other* shape outside $[1/3,1)\cup\{0\}$. This distinction matters here. Das and Manna [3, §2] prove that Gupta's own weight satisfies

$$w_{\alpha,(1-\alpha)/2}(n) \;>\; \frac{(\alpha-1)^2}{4}\,n^{\alpha-2} \qquad \text{for every } n \geq 1 \text{ and every } \alpha \in [0,1),$$

and thereby extend to all of $[0,1)$ the improvement of (1) that [1] obtains on $\{0\}\cup[1/3,1)$. That is a pointwise statement about the weight — one that [1, Remark 4.3] had already recorded as numerically true, with the comment that proving it "mathematically becomes a bit tricky" — and it does not pass through the Taylor coefficients (3); the improvement it yields is not (2). It does, however, contradict the corollary Gupta draws immediately after his conjecture ("Therefore, we don't have the improvement (2.4) of inequality (2.3) when α does not lie in $[1/3,1)\cup\{0\}$") once "improvement" is read in the general sense. We therefore state everything below in terms of the coefficients themselves.

We also do not reprove [1, Lemma 4.1] ($b_k(\alpha) \geq 0$ on $[1/3,1)\cup\{0\}$); it is used only where explicitly cited, in Corollary 10. The remaining open question of [1, Remark 1.1], whether some exponent β other than $(1-\alpha)/2$ makes $w_{\alpha,\beta}(n) \geq c(\alpha)n^{\alpha-2}$ with $c(\alpha) \geq 0$, is a different, two-parameter problem and is not addressed.

Provenance. All searches reported here were run on 29 August 2026. The conjecture is stated in arXiv:2108.01500v1 (2021) and again in v2 (2022), the version that the arXiv comment records as accepted for publication as [1], and it is restated as open in [2]. Works citing [1] were collected from OpenAlex and from Semantic Scholar (13 and 17 records) and deduplicated across preprint/journal pairs, leaving sixteen distinct works; the twelve carrying an arXiv identifier were searched in full text, and the four appearing only in journals were not obtainable. A full-text search of a local mirror of the arXiv (367 shards, coverage through August 2026) finds the combination (3) itself in exactly two documents, [1] and [2]. Just one citing work touches these coefficients at all: the quantities $C_k(q,\alpha)$ of [3, §3], attached there to a positive weight sequence $\{q_n\}$, reduce to $b_k(\alpha)/n^k$ when $q_n \equiv 1$, and [3] proves them positive for $k \geq 3$ only when $\alpha \in [1/3,1)$ — exactly Gupta's range, so nothing there bears on the conjecture. The other eleven do not contain (3) at all; they cite [1] for its inequalities, at the level of the weight $w_{\alpha,\beta}$. The string `b_i(2k+1)`, in which [1, Lemma 5.5] states its conclusion, occurs in the mirror only in [1] and [2]: no third work restates that lemma. The integer sequences $\big(b_i(2m+1)\big)_i$ for $m = 2, 3$ return no match from the OEIS (a control query for the Fibonacci numbers returns A000045). We found no Hardy inequality, discrete or continuous, in Mathlib (checked against the repository at commit `81a5d257`, Lean toolchain `v4.32.0`); searches of the Archive of Formal Proofs and of the Coq/Rocq libraries also turned up none.

2. Notation and elementary identities

Throughout, α is a real parameter, k and i are non-negative integers, and

$$\mathrm{F}(\gamma, k) \;:=\; \prod_{j=0}^{k-1}(\gamma - j), \qquad \binom{\gamma}{k} \;=\; \frac{\mathrm{F}(\gamma, k)}{k!}, \qquad \Pi(\gamma, k) \;:=\; \prod_{j=1}^{k-1}(j - \gamma),$$

with the usual convention that empty products equal 1. We abbreviate

$$\alpha_1 := \frac{1-\alpha}{2}, \qquad \alpha_2 := \frac{1+\alpha}{2}, \qquad \text{so that} \qquad \alpha_1 + \alpha_2 = 1.$$

The definition (3) is thus $b_k(\alpha) = \binom{\alpha}{k} - (-1)^k\binom{\alpha_1}{k} - \binom{\alpha_2}{k}$, and $k!\,b_k(\alpha) = \mathrm{F}(\alpha, k) - (-1)^k\,\mathrm{F}(\alpha_1, k) - \mathrm{F}(\alpha_2, k)$. Since $k! > 0$, the sign of $b_k(\alpha)$ is the sign of $k!\,b_k(\alpha)$, and we pass between the two without comment.

Directly from (3): $b_0(\alpha) = 1 - 1 - 1 = -1$, and $b_1(\alpha) = \alpha + \alpha_1 - \alpha_2 = 0$, for every real α.

The next proposition records the closed forms printed in [1]; we reproduce them from (3) as polynomial identities over any field of characteristic zero. Their role here is a check that the object we study is the object of [1], and they are used again in Sections 6 and 8.

Proposition 1. *For every α,*

$$\begin{aligned}
b_2(\alpha) &= \frac{(\alpha-1)^2}{4},\\
b_3(\alpha) &= \frac{\alpha(1-\alpha)(3-\alpha)}{8},\\
b_4(\alpha) &= \frac{(5-\alpha)(1-\alpha)\,(7\alpha^2 - 6\alpha + 3)}{192},\\
b_6(\alpha) &= \frac{(1-\alpha)(9-\alpha)\,(31\alpha^4 - 170\alpha^3 + 536\alpha^2 - 310\alpha + 105)}{23040}.
\end{aligned}$$

Proof. Expand $\mathrm{F}(\gamma, k)$ for $k = 2, 3, 4, 6$ at $\gamma \in \{\alpha, \alpha_1, \alpha_2\}$, substitute into (3), clear denominators and compare coefficients. Each of the four is a polynomial identity in one variable of degree at most 6. ∎

The formulas for b_4 and b_6 are equations (4.8) and (4.9) of [1]; the formula for b_3 is equivalent to the computation $E'''(0) = \frac{3}{4}\alpha(1-\alpha)(3-\alpha)$ carried out in the proof of [1, Lemma 4.2] and quoted again in that of [1, Lemma 5.1], since $E(x) = g(x) - \frac{(\alpha-1)^2}{4}x^2 = \sum_{k\ge 3} b_k(\alpha)x^k$ and hence $b_3 = E'''(0)/3!$; and b_2 is the coefficient of x^2 in the expansion (4), i.e. the sharp Hardy constant. All section, lemma, remark and equation numbers of the source quoted in this paper are those of arXiv:2108.01500v2; see the bibliography entry for [1].

The following rearrangement is what the whole of Section 4 runs on.

Lemma 2. *For $k \ge 1$ and every real γ, $\mathrm{F}(\gamma, k) = (-1)^{k-1}\gamma\,\Pi(\gamma, k)$. Consequently, for* even *k,*

$$k!\,b_k(\alpha) \;=\; -\alpha\,\Pi(\alpha, k) + \alpha_1\Pi(\alpha_1, k) + \alpha_2\Pi(\alpha_2, k). \tag{5}$$

Proof. Peel the factor $j = 0$ off $\mathrm{F}(\gamma, k) = \prod_{j<k}(\gamma - j)$ and flip the sign of each of the remaining $k-1$ factors. For even k one has $(-1)^{k-1} = -1$ and $(-1)^k = 1$, so each of the three terms of $k!\,b_k(\alpha)$ contributes $-\gamma\,\Pi(\gamma, k)$ with $\gamma = \alpha$, α_1, α_2 and the middle one changes sign once more. ∎

Note that for $0 < \alpha < 1$ each of $\alpha, \alpha_1, \alpha_2$ lies in $(0,1)$, so every factor $j - \gamma \geq 1 - \gamma > 0$ and all three products in (5) are positive. For even k, therefore,

$$b_k(\alpha) < 0 \iff \alpha_1 \Pi(\alpha_1, k) + \alpha_2 \Pi(\alpha_2, k) \;<\; \alpha\, \Pi(\alpha, k) \qquad (0 < \alpha < 1). \tag{6}$$

3. A decay lemma

Write

$$\tau_c(j) \;:=\; 1 - \frac{c}{2j} \qquad (j \geq 1,\; c > 0).$$

The only analytic input to this paper is that the products of τ_c over long ranges tend to 0. This is of course the divergence of $\sum_j 1/j$ in disguise, but in the following form it needs nothing but Bernoulli's inequality, and in particular no logarithm.

Lemma 3. *Let $c > 0$ and let $n_0 \geq 1$ be an integer with $c \leq 2n_0$. Put $\rho := \left(1 + \frac{c}{4}\right)^{-1} \in (0,1)$. Then for every $M \geq 0$*

$$\prod_{j=n_0}^{n_0 2^M - 1} \tau_c(j) \;\leq\; \rho^M.$$

Consequently, for every $\varepsilon > 0$ and every m there is an $M \geq m$ with $\prod_{j=n_0}^{n_0 2^M - 1} \tau_c(j) < \varepsilon$.

Proof. Each factor is non-negative: for $j \geq n_0$ we have $c \leq 2n_0 \leq 2j$. Fix a dyadic block $j \in [n, 2n)$ with $n \geq n_0$. On it $\tau_c(j) \leq 1 - x$ where $x := c/(4n) \in [0,1]$, so the block product is at most $(1-x)^n$. Now $(1-x)^n(1+x)^n = (1-x^2)^n \leq 1$ and, by Bernoulli, $(1+x)^n \geq 1 + nx$; hence

$$(1-x)^n \;\leq\; \frac{1}{1+nx} \;=\; \frac{1}{1+c/4} \;=\; \rho,$$

because $nx = c/4$ independently of n. Splitting $[n_0, n_0 2^M)$ into the M dyadic blocks $[n_0 2^t, n_0 2^{t+1})$, $0 \leq t < M$, and multiplying gives the bound ρ^M. The last assertion follows since $\rho < 1$. ☐

The point to notice is that the bound ρ per block does not depend on where the block sits: doubling the range buys a fixed factor. This is exactly what a Γ-function asymptotic would give, and it is all that is needed.

4. The interval $(0, 1/3)$

Here is the identity that fixes the constant $1/3$.

Lemma 4. *Let $0 < \alpha$ and set $c := 1 - 3\alpha$. Then for every $j \geq 1$*

$$\tau_c(j)(j - \alpha) - (j - \alpha_1) = \frac{c\alpha}{2j}, \tag{7}$$

$$\tau_c(j)(j - \alpha) - (j - \alpha_2) = \frac{2\alpha j + c\alpha}{2j}. \tag{8}$$

Both right-hand sides are positive for all $j \geq 1$ if and only if $c > 0$, that is, if and only if $\alpha < 1/3$.

Proof. Multiplying the left-hand side of (7) by $2j$ gives $(1 - 3\alpha - c)\, j + c\alpha$, an identity in j; the coefficient of j vanishes identically precisely for $c = 1 - 3\alpha$, which is the definition of c, and what is left is $c\alpha$. Similarly $2j$ times the left-hand side of (8) is $(1 - \alpha - c)j + c\alpha = 2\alpha j + c\alpha$. Since $\alpha > 0$, the numerator $c\alpha$ of (7) is positive exactly when $c > 0$; and $c > 0$ means $\alpha < 1/3$. When $c > 0$ the numerator $2\alpha j + c\alpha = \alpha(2j + c)$ of (8) is positive as well, so positivity of both right-hand sides is equivalent to positivity of the first. ☐

We stress the shape of this computation. One asks for a factor $\tau_c(j)$ that dominates the ratio of the shifted product to the unshifted one; if the coefficient of j in the difference did not vanish, the difference would grow linearly with the wrong sign for one of the two comparisons, and no choice of c would serve. Vanishing of that coefficient determines $c = 1 - 3\alpha$, and then positivity of the remainder determines $\alpha < 1/3$. The interval of [1, Lemma 4.1] and the interval here are the same interval for the same reason.

Theorem 5. *For every real α with $0 < \alpha < 1/3$ there exists $k \geq 3$ with $b_k(\alpha) < 0$.*

Proof. Put $c := 1 - 3\alpha$, which is positive because $\alpha < 1/3$; since $\alpha > 0$ we have $c < 1 \leq 2 \cdot 1$, so Lemma 3 applies with $n_0 = 1$. Choose, by that lemma with $\varepsilon := \alpha$ and $m := 2$, an integer $M \geq 2$ with

$$P \;:=\; \prod_{j=1}^{k-1} \tau_c(j) \;<\; \alpha, \qquad k := 2^M.$$

Then k is even and $k \geq 4$. All the factors $j - \gamma$ occurring below are positive, because $\alpha, \alpha_1, \alpha_2 \in (0, 1)$. By Lemma 4, factor by factor,

$$\Pi(\alpha_1, k) = \prod_{j=1}^{k-1} (j - \alpha_1) \;\leq\; \prod_{j=1}^{k-1} \tau_c(j)(j - \alpha) \;=\; P\,\Pi(\alpha, k),$$

and identically $\Pi(\alpha_2, k) \leq P\,\Pi(\alpha, k)$. Since $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$,

$$\alpha_1 \Pi(\alpha_1, k) + \alpha_2 \Pi(\alpha_2, k) \;\leq\; (\alpha_1 + \alpha_2)\, P\,\Pi(\alpha, k) \;=\; P\,\Pi(\alpha, k) \;<\; \alpha\,\Pi(\alpha, k),$$

the last step because $P < \alpha$ and $\Pi(\alpha, k) > 0$. By (6) — that is, by the identity (5), which is where k even is used — $b_k(\alpha) < 0$. ☐

The witness produced is $k = 2^M$, in particular even. Some restriction of this kind is forced: $b_3(\alpha) = \alpha(1 - \alpha)(3 - \alpha)/8 > 0$ throughout $(0, 1)$ (Proposition 15), so the conclusion cannot be strengthened to "for every $k \geq 3$", and the identity (5) that drives the proof holds only for even k. No bound on M is extracted, and Section 8 explains why none of a useful kind exists on this interval.

5. The half-line $(1, \infty)$

For $\alpha > 1$ the middle term of (3) changes character. Put

$$m := \frac{\alpha - 1}{2} > 0, \qquad \text{so } \alpha_1 = -m < 0,$$

and $\mathrm{F}(-m, k) = \prod_{j<k}(-m - j) = (-1)^k R_k$ with $R_k := \prod_{j=0}^{k-1}(m + j) > 0$. The factor $(-1)^k$ in (3) cancels against this one, and

$$k!\, b_k(\alpha) \;=\; \mathrm{F}(\alpha, k) \;-\; R_k \;-\; \mathrm{F}(\alpha_2, k), \qquad R_k = \prod_{j=0}^{k-1}(m + j) > 0, \tag{9}$$

for every k, even or odd. So a positive, rapidly growing product enters b_k with a minus sign, and it suffices to show that the two remaining products are eventually small compared with R_k. The comparison is again factorwise, with $c := \alpha$ this time, and again rests on two identities.

Lemma 6. *Let $\alpha > 1$, $m = (\alpha - 1)/2$, and let j be an integer with $j \geq 2\alpha$. Then*

$$(j - \alpha) \;\leq\; \tau_\alpha(j)\,(m + j), \qquad \textit{indeed}\;\; \tau_\alpha(j)(m + j) - (j - \alpha) = \frac{m(2j - \alpha) + \alpha j}{2j}, \tag{10}$$

$$(j - \alpha_2) \;\leq\; \tau_\alpha(j)\,(m + j), \qquad \textit{indeed}\;\; \tau_\alpha(j)(m + j) - (j - \alpha_2) = \frac{m(2j - \alpha) + j}{2j}. \tag{11}$$

Moreover $0 \leq j - \alpha$ and $0 \leq j - \alpha_2$ for such j, so (10) and (11) bound $|\alpha - j|$ and $|\alpha_2 - j|$.

Proof. Both displayed equalities are checked by multiplying out and clearing $2j$. Their right-hand sides are positive because $2j \geq 4\alpha > \alpha$, $m > 0$ and $j > 0$. Finally $j \geq 2\alpha > \alpha > \alpha_2$ since $\alpha > 1$. $\square$

Theorem 7. *For every real $\alpha > 1$ there exists $k \geq 3$ with $b_k(\alpha) < 0$. In particular this holds for every $\alpha \geq 5$.*

Proof. Set $n_0 := \lceil 2\alpha \rceil$, so $n_0 \geq 3$ and $\alpha \leq 2n_0$; the hypotheses of Lemma 3 hold with $c = \alpha$. Split each of the three products of (9) at n_0 and abbreviate the heads

$$H_A := \prod_{j<n_0} |\alpha - j|, \qquad H_B := \prod_{j<n_0} |\alpha_2 - j|, \qquad H_R := \prod_{j<n_0} (m + j) > 0.$$

These are fixed non-negative reals depending on α only, with $H_R > 0$ because every factor $m + j$ is positive; they are never evaluated. (H_A or H_B vanishes when α or α_2 is an integer below n_0, which does no harm below.) Apply Lemma 3 with $\varepsilon := H_R / \big(2(H_A + H_B + 1)\big) > 0$ to get an M with

$$T \ := \ \prod_{j=n_0}^{k-1} \tau_\alpha(j) \ < \ \varepsilon, \qquad k := n_0 2^M \ \geq \ n_0 \ \geq \ 3.$$

Write $R^{\mathrm{t}} := \prod_{j=n_0}^{k-1}(m + j)$, so that $R_k = H_R\, R^{\mathrm{t}}$. By Lemma 6, for $\gamma \in \{\alpha, \alpha_2\}$,

$$\big|\mathrm{F}(\gamma, k)\big| \ = \ \prod_{j<n_0} |\gamma - j| \ \cdot \ \prod_{j=n_0}^{k-1} |\gamma - j| \ \leq \ (H_A + H_B) \cdot T\, R^{\mathrm{t}} \ = \ \frac{H_A + H_B}{H_R}\, T\, R_k.$$

By the choice of ε, $2(H_A + H_B)T < H_R$, hence $2 \cdot \frac{H_A + H_B}{H_R} T R_k < R_k$, so the two bounds above sum to strictly less than R_k. Feeding them into (9),

$$k!\, b_k(\alpha) \ \leq \ \big|\mathrm{F}(\alpha, k)\big| + \big|\mathrm{F}(\alpha_2, k)\big| - R_k \ < \ 0. \qquad \square$$

Theorem 8 (Gupta's conjecture)**.** *For every real $\alpha \in (0, 1/3) \cup [5, \infty)$ there exists $k \geq 3$ with $b_k(\alpha) < 0$.*

Proof. Theorem 5 on $(0, 1/3)$; Theorem 7 on $[5, \infty) \subset (1, \infty)$. $\square$

6. The whole real line

Theorem 9. *For every real α with $\alpha \neq 0$, $\alpha \neq 1$ and $\alpha \notin [1/3, 1)$, there exists $k \geq 3$ with $b_k(\alpha) < 0$.*

Proof. If $\alpha < 0$ then $b_3(\alpha) = \alpha(1-\alpha)(3-\alpha)/8 < 0$ by Proposition 1, since $1 - \alpha > 0$ and $3 - \alpha > 0$; take $k = 3$. If $0 < \alpha < 1/3$, apply Theorem 5. Otherwise $\alpha \geq 1/3$ and, as $\alpha \notin [1/3, 1)$ and $\alpha \neq 1$, we get $\alpha > 1$; apply Theorem 7. $\square$

The sign of b_3 for negative α is the sign of $E'''(0) = 3!\, b_3(\alpha)$, whose negativity there is observed in the proof of [1, Lemma 5.1]; from it that paper draws only the conclusion that its own method fails for $\alpha < 0$, and the sign of the coefficient itself is not recorded there as a fact about (3).

Corollary 10. *Assume* [1, Lemma 4.1]*: $b_k(\alpha) \geq 0$ for $\alpha \in [1/3, 1) \cup \{0\}$ and $k \geq 3$. Then for real α,*

$$b_k(\alpha) \geq 0 \text{ for every } k \geq 3 \qquad \Longleftrightarrow \qquad \alpha \in [1/3, 1) \cup \{0, 1\}.$$

Proof. ($\Leftarrow$) On $[1/3, 1) \cup \{0\}$ this is Gupta's lemma; at $\alpha = 1$ every $b_k(1)$ vanishes by Proposition 14. ($\Rightarrow$) is Theorem 9. $\square$

Only the direction ($\Rightarrow$) is proved here; the direction ($\Leftarrow$) on $[1/3, 1) \cup \{0\}$ is quoted from [1] and is the one part of Corollary 10 that is not verified in the formal development described in Section 9.

7. The partial result at the odd integers, corrected

[1, Lemma 5.5] reads, verbatim from arXiv:2108.01500v2: for $\alpha = 2m+1$,

$$b_i(2m+1) = \binom{2m+1}{i} - (-1)^i\binom{-m}{i} - \binom{m+1}{i},$$

and if $m \geq 2$ then

$$b_i(2m+1) \;\geq\; 0 \;\text{ for } 2 \leq i \leq m+1, \qquad b_i(2m+1) \;<\; 0 \;\text{ for } i > m+1.$$

(The paper writes k for our m.) The first display is a correct specialisation of (3). The strict half of the second is false at its first index.

Specialising (3) at $\alpha = 2m+1$, so that $\alpha_1 = -m$ and $\alpha_2 = m+1$, and writing the three falling factorials as products, gives

$$i!\, b_i(2m+1) \;=\; A_i - B_i - C_i, \quad A_i = \prod_{j<i}(2m+1-j), \;\; B_i = \prod_{j<i}(m+j), \;\; C_i = \prod_{j<i}(m+1-j), \tag{12}$$

where $B_i = m(m+1)\cdots(m+i-1)$ is the ascending product coming from $(-1)^i\,\mathrm{F}(-m,i)$.

Theorem 11. *For every integer $m \geq 0$,*

$$b_{m+2}(2m+1) \;=\; 0.$$

In particular $b_4(5) = b_5(7) = b_6(9) = b_7(11) = b_8(13) = 0$, so the assertion $b_i(2m+1) < 0$ for all $i > m+1$ of [1, Lemma 5.5] *fails at $i = m+2$. Moreover, for $m \geq 2$,*

$$b_{m+2}(2m+1) = 0 \qquad \textit{and} \qquad b_i(2m+1) < 0 \;\; \textit{for every } i \geq m+3,$$

which is the corrected form of the strict half of that lemma.

Proof. Take $i \geq m+2$ in (12). The factor of C_i at $j = m+1$ is 0, so $C_i = 0$.

At $i = m+2$ the products $A_{m+2} = (2m+1)(2m)\cdots m$ and $B_{m+2} = m(m+1)\cdots(2m+1)$ run over the same $m+2$ integers in opposite orders, so $A_{m+2} = B_{m+2}$ and $(m+2)!\,b_{m+2}(2m+1) = 0$. This is the whole of the first assertion, and it holds for every $m \geq 0$; the listed values are the cases $m = 2, \ldots, 6$.

For the second assertion let $m \geq 2$, so $A_{m+2} = B_{m+2} = m(m+1)\cdots(2m+1) > 0$. We claim $A_i < B_i$ for all $i \geq m+3$, whence $i!\,b_i(2m+1) = A_i - B_i < 0$ as required. First, $A_i \geq 0$ for every i: as j runs upward, the factor $2m+1-j$ of A reaches the value 0 at $j = 2m+1$ before it can become negative, so either all factors so far are positive or one of them is 0. Next, the base case:

$$A_{m+3} = A_{m+2}\,(m-1) \;<\; B_{m+2}\,(2m+2) = B_{m+3},$$

because $A_{m+2} = B_{m+2} > 0$ and $m-1 < 2m+2$. For the induction step, suppose $A_i < B_i$ with $i \geq m+3$. If $2m+1-i \leq 0$ then $A_{i+1} = A_i(2m+1-i) \leq 0 < B_{i+1}$, since $B_{i+1} > 0$. Otherwise $2m+1-i \leq m+i$, because this says $m+1 \leq 2i$ and $2i \geq 2m+6$; then, using $A_i \geq 0$,

$$A_{i+1} = A_i(2m+1-i) \;\leq\; A_i(m+i) \;<\; B_i(m+i) = B_{i+1}. \qquad \square$$

Note that Theorem 11 does not reprove the non-negative half $b_i(2m+1) \geq 0$ for $2 \leq i \leq m+1$; that half is Gupta's and is used here only as context. The corrected lemma therefore reads: $b_i(2m+1) \geq 0$ *for $2 \leq i \leq m+2$, with equality at $i = m+2$, and $b_i(2m+1) < 0$ for $i \geq m+3$*, the non-negative range having grown by one index and the strict range having shrunk by one.

Remark 12 (where the slip comes from). The proof in [1] treats the range $m+1 < i \leq 2m+1$ by factoring $m(m+1)\cdots(2m+1)$ out of both products and writing

$$b_i(2m+1) = \frac{1}{i!}\,m(m+1)\cdots(2m+1)\Big[(m-1)\cdots\big(2m+1-(i-1)\big) - (2m+2)\cdots(m+i-1)\Big] < 0.$$

At $i = m+2$ the first bracketed range descends from $m-1$ to m and the second ascends from $2m+2$ to $2m+1$: *both are empty.* Each is then the empty product 1, the bracket is $1-1 = 0$, and the inequality is not strict. The strict sign is read off a factorisation that degenerates at exactly the first index it is applied to. For $i \geq m+3$ both ranges are non-empty and the argument of [1] is in force; the proof above reproves that range from scratch.

The slip appears to be unrecorded: we found no work that restates or uses Lemma 5.5, the string in which that lemma states its conclusion occurring nowhere in the arXiv mirror outside [1] and [2] (see the Provenance note in the introduction). The same lemma, with the same strict inequality, is [2, Lemma 2.8].

Remark 13. The values around the vanishing cell, in exact arithmetic, are

α	b_2	b_3	b_4	b_5	b_6	b_7	b_8
5	4	5	**0**	-5	-7	-8	-9
7	9	21	19	**0**	-21	-35	-45
9	16	54	86	69	**0**	-84	-156
11	25	110	245	330	251	**0**	-330

the boldface entry in each row being the first index at which [1, Lemma 5.5] asserts strict negativity. Two of these cells are visible already in Proposition 1: $b_4(5) = 0$ because $b_4(\alpha)$ carries the factor $(5-\alpha)$, and $b_6(9) = 0$ because $b_6(\alpha)$ carries $(9-\alpha)$. The vanishing cells are the case $i = m+2$ of Theorem 11 and $b_5(5) = -5$ is Proposition 16; the remaining entries of the table are exact rational evaluations of (3), reported here as computations.

8. Sharpness, witnesses, and what a finite computation can see

This section collects the elementary facts that fix the hypotheses of Theorems 5 and 7 in place, together with exact data on the size of the witness k. Everything in it is a finite computation in $\mathbb{Q}$; we say in each case exactly what was computed.

Proposition 14. *$b_k(1) = 0$ for every $k \geq 1$.*

Proof. At $\alpha = 1$ we have $\alpha_1 = 0$, so $\mathrm{F}(\alpha_1, k) = 0$ for $k \geq 1$, while $\alpha_2 = 1 = \alpha$, so the first and third terms of $k!\, b_k(1)$ cancel. $\square$

Hence the hypothesis $\alpha > 1$ of Theorem 7 cannot be relaxed to $\alpha \geq 1$: at $\alpha = 1$ no k whatsoever witnesses negativity. The exclusion of $\alpha = 1$ from Corollary 10 is forced, and, as noted in the introduction, the broad clause of [1, Remarks 1.2 and 5.4] is false at that one point.

Proposition 15. *$b_3(\alpha) > 0$ for every $\alpha \in (0, 1)$; for instance $b_3(1/4) = 33/512$.*

Proof. $b_3(\alpha) = \alpha(1-\alpha)(3-\alpha)/8$ by Proposition 1, a product of three positive numbers on $(0, 1)$. $\square$

So on $(0, 1/3)$ no statement of the form "$b_k(\alpha) < 0$ for all $k \geq 3$" can hold: Gupta's conjecture is genuinely existential, and the restriction to even k in the proof of Theorem 5 is load-bearing rather than cosmetic.

Proposition 16. *$b_5(5) = -5, \quad b_5(11/2) = -23661/4096, \quad b_5(13/2) = -16731/4096.$*

Proof. Three evaluations of (3) at $k = 5$, each a product of five rational factors in each of three terms, carried out in exact rational arithmetic. $\square$

The last two are the smallest illustration that Theorem 7 covers ground that [1, Lemma 5.5] does not: $11/2$ and $13/2$ are not odd integers, and [1] reaches no non-integer parameter anywhere.

We turn to the size of the witness. Call $K(\alpha)$ the least $k \geq 3$ with $b_k(\alpha) < 0$, when one exists.

Proposition 17. *$K(1/10) = 94, \quad K(1/5) = 60, \quad K(1/4) = 78, \quad K(3/10) = 228$. In particular $b_{78}(1/4) < 0$.*

Proof. Exhaustive computation: at each of the four rational parameters, $b_k(\alpha)$ was evaluated in exact rational arithmetic for every k with $3 \leq k \leq K$, all values with $k < K$ found non-negative and the value at $k = K$ found negative. No claim is made about any other parameter. $\square$

Proposition 18. $K(11/2) = 5, \quad K(13/2) = 5, \quad K(100) = 52.$

Proof. As in Proposition 17, an exhaustive scan of $3 \le k \le K$ in exact rational arithmetic at each of the three parameters. □

Proposition 19. *For each* $\alpha \in \{0,\ 1/3,\ 1/2,\ 8/25\}$ *and each* k *with* $3 \le k \le 40$ *one has* $b_k(\alpha) \ge 0$.

Proof. An exhaustive sweep of the 4×38 cells in exact rational arithmetic. □

For $\alpha \in \{0, 1/3, 1/2\}$ Proposition 19 is subsumed by [1, Lemma 4.1], which gives non-negativity for *all* k. The informative cell is $\alpha = 8/25 < 1/3$. There Theorem 5 guarantees a witness, and a sweep of the first 38 admissible indices does not find it. It is worth stating what that means: a bounded search cannot decide Gupta's conjecture, however far it is pushed.

Remark 20 (on the shape of K, and on choosing a proof route). The following measurements sit outside the formal development of Section 9 and are reported as computations, not as theorems. On $(0, 1/3)$ the least witness $K(\alpha)$ is U-shaped, with minimum about 60 near $\alpha = 1/5$, and it diverges at *both* endpoints. On the left, $K = 1084$ at $\alpha = 1/50$ and $K = 238$ at $\alpha = 1/20$. On the right, $K = 1030$ at $\alpha = 8/25$, $K = 16\,592$ at $\alpha = 0.33$, $K = 128\,344$ at $\alpha = 0.332$, and no witness at all below $3 \cdot 10^6$ at $\alpha = 0.333$. Two consequences. First, an exhaustive route to Theorem 5 is closed. Concretely: no finite list of pairs $(I_1, k_1), \ldots, (I_N, k_N)$, with I_r an interval on which $b_{k_r} < 0$ throughout — the shape a family of interval positivity certificates would take — can cover $(0, 1/3)$, since near either endpoint every fixed index k eventually fails and only finitely many indices are available. A limit argument is not a convenience here, it is the only shape of proof available. Second, and in contrast, on $[5, \infty)$ the witness is small and regular: $K(\alpha) = \lfloor(\alpha+5)/2\rfloor$ at all sixteen sampled parameters $\alpha = 5,\ 11/2,\ 6,\ 13/2,\ 7,\ 8,\ 9,\ 10,\ 12,\ 15,\ 20,\ 25,\ 30,\ 40,\ 50,\ 100$. That is an observation, not a theorem, and proving $b_{\lfloor(\alpha+5)/2\rfloor}(\alpha) < 0$ for all $\alpha \ge 5$ — replacing the existential of Theorem 7 by a formula — is an obvious next question. The cheap diagnostic suggested by the contrast: before committing to a certificate-based route for a statement of the form "for every α in an interval there is a k", compute the least witness at both ends of the interval.

The values reported in this remark that lie outside the machine-checked development were computed twice, by independent programs. The witnesses 1084, 238 and 1030 are exact rational computations. For $\alpha = 0.33$ and $\alpha = 0.332$ the scan $3 \le k < K$ was carried out in double precision on the ratios $\Pi(\alpha_1, k)/\Pi(\alpha, k)$ and $\Pi(\alpha_2, k)/\Pi(\alpha, k)$, and the two cells that decide the answer were then redone in exact integer arithmetic: at $\alpha = 33/100$, $b_{16591} > 0$ and $b_{16592} < 0$; at $\alpha = 83/250$, $b_{128343} > 0$ and $b_{128344} < 0$. At $\alpha = 0.333$ the same scan reaches $k = 3 \cdot 10^6$ with the quantity that changes sign still as large as $1.3 \cdot 10^{-4}$, some nine orders of magnitude above the accumulated rounding error, and still decreasing.

9. Verification

Every numbered result of this paper has been formally verified in the Lean 4 proof assistant (toolchain `v4.32.0`) against *Mathlib* [6]; the development contains no `sorry` and no added axiom. Three qualifications. First, the hypothesis of Corollary 10 is quoted from [1] and is not formalised, so only the direction (⇒) of that corollary is verified. Second, the numerical measurements of Remarks 13 and 20 lie outside the development and are reported as computations. Third, the three auxiliary Lemmas 2, 4 and 6 are formalised in the form in which they are used — the peeling identity of Lemma 2 and the two factorwise inequalities of each of Lemmas 4 and 6 are separate declarations, whereas the displayed algebraic identities, the "if and only if" of Lemma 4 and the specialisation (5) of Lemma 2 to even k are carried inside the proofs of Theorems 5 and 7 rather than stated on their own. Every numbered theorem and every numbered proposition is formalised exactly as printed here. It is worth being precise about which parts are checked by the kernel alone. The definition (3), the closed forms of Proposition 1, the decay Lemma 3, Theorems 5, 7, 8, 9 and 11, and Propositions 14, 15 and 16 depend on nothing beyond Lean's three standard axioms `propext`, `Classical.choice` and `Quot.sound`: *no step on the path to Gupta's conjecture is delegated to compiled evaluation.* This matters for a result whose content is a limit statement about an unbounded search. Twelve statements of exact rational data — the sweeps of Proposition 19, the least-witness computations of Propositions 17 and 18, and the transfer of $b_{78}(1/4) < 0$ from $\mathbb{Q}$ to $\mathbb{R}$ — are decided by Lean's compiled-evaluation tactic and carry, in addition, the axiom that tactic generates; they are controls and data, and no theorem above depends on them. The formal development also carries the negative controls that this kind of work needs: that the hypothesis $\alpha > 1$ of Theorem 7 cannot be weakened (Proposition 14), that no "for all k" strengthening of Theorem 5 holds (Proposition 15), and the bounded negative of Proposition 19. Repository reference to be added at submission.

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