WHICH GAMMA LAWS ARE FREELY INFINITELY DIVISIBLE, AND STRICT ENLARGEMENT OF THE HANKEL HIERARCHY OF FREE CUMULANTS

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Abstract. For a probability measure with all moments finite, free infinite divisibility forces every shifted free-cumulant Hankel matrix $H_N = (\kappa_{i+j+2})_{i,j=0}^{N}$ to be positive semidefinite. A single vector on which one of those matrices is negative therefore certifies that the measure is *not* freely infinitely divisible, and the point of this paper is that one such vector can be held fixed while a parameter of the law moves. For the classical gamma law $\boldsymbol{\gamma}_p$ the moments $m_n = (p)_n$ are integer polynomials in the shape p, hence so are the free cumulants, so the quadratic form $Q(p) = u^{\mathsf{T}} H_N(\boldsymbol{\gamma}_p) u$ of a fixed integer vector u is one polynomial of degree at most $N+1$, whose sign on a closed rational interval is decided by a Bernstein certificate. One vector at order 20 gives $H_{20}(\boldsymbol{\gamma}_p) \not\succeq 0$ for every real $p \in [\frac{2}{3}, \frac{4}{3}]$; since Hasebe's set $\mathcal{I}$ covers $(\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2})$, this settles part (3) of his 2014 conjecture — $\boldsymbol{\gamma}_p$ is not freely infinitely divisible for any $p \in (\frac{1}{2}, \frac{3}{2})$ — and, with the positive half of his Corollary 1.3(1), completes the classification: $\boldsymbol{\gamma}_p$ is freely infinitely divisible exactly for $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$. A second vector, at order 13 on $[\frac{49}{50}, \frac{6}{5}]$, gives the same conclusion with no appeal to compiled evaluation, and already covers infinitely many of the intervals $\mathcal{I}$ omits, where the exponential law was the only shape previously known.

The rest of the paper studies the hierarchy $\{H_N \succeq 0\}_{N \geq 0}$ itself, which is where the witness vectors come from. Yu exhibited, for the beta laws $\boldsymbol{\beta}_{p,q}$, exact rational parameters at which the $(N+1) \times (N+1)$ test is the first to fire, for $2 \leq N \leq 11$, and asked whether the hierarchy keeps enlarging at higher orders. It does, at infinitely many orders, by a completeness statement — for a measure whose moment sequence is determinate, positive semidefiniteness of *every* H_N is not merely necessary for free infinite divisibility but sufficient — together with the fact that no test fires at $p = \frac{1}{2}$. That argument assembles published theorems and is the one part of this paper that is not machine-checked; it is labelled as such throughout, as is the observation that the second half of Yu's open sentence — that every non-freely-infinitely-divisible beta law is eventually separated by a finite Hankel matrix — is the contrapositive of the criterion Yu's paper itself cites. The computational content: the witness ladder continued from $N = 11$ to $N = 20$ with exact rational parameters and bracketed thresholds; the *gamma ceiling* $\nu(p)$, the first order at which the hierarchy of $\boldsymbol{\gamma}_p$ fails, computed at 36 rational shapes and shown to take every value from 13 to 25; a Schur reduction of the whole hierarchy to the sign of one scalar Φ_N, non-increasing in N wherever the trailing Hankel matrices are positive definite, so that the exclusion never un-fires there; and the first negative Hankel determinant of the free cumulants of the exponential law, of size 14×14, sharper than the 16×16 one on record and certified together with it. Everything computational is verified in Lean 4 by exact integer arithmetic — 159 theorems, no floating point anywhere — and all of it but the order-20 certificate is evaluated by the proof kernel itself.

1. Introduction

A probability measure μ on $\mathbb{R}$ is *freely infinitely divisible* (FID) if for every $n \geq 1$ it is the n-fold free additive convolution power of some probability measure. When all moments of μ are finite, its free cumulants $\kappa_1, \kappa_2, \ldots$ are determined by the moment–cumulant relation, and free infinite divisibility forces a conditional positive definiteness: the shifted cumulant Hankel matrices

$$H_N(\mu) \;=\; \big(\kappa_{i+j+2}(\mu)\big)_{i,j=0}^{N}, \qquad D_N(\mu) \;=\; \det H_N(\mu), \tag{1}$$

are positive semidefinite for every $N \geq 0$ (Lemma 2.1, and Remark 2.2 for the exact form of the criterion each family of laws needs). A single negative D_N is therefore a certificate that μ is *not* FID. Because H_M is a leading principal submatrix of H_N for $M \leq N$, the tests are nested: if the

$(M+1)\times(M+1)$ test fires then so does every larger one. The family $\{H_N \succeq 0\}_{N\geq 0}$ is thus a hierarchy of obstructions, and the natural question is how much each new order adds.

Yu [10] answered this for the beta laws. Write $\boldsymbol{\beta}_{p,q}$ for the beta law on $[0,1]$ with parameters $p,q>0$, whose moments are $m_n=(p)_n/(p+q)_n$ with $(x)_n=x(x+1)\cdots(x+n-1)$. Call a pair (p,q) a *witness at order N* when

$$D_0,\ \ldots,\ D_{N-1}\ >\ 0\ >\ D_N \qquad \text{for } \boldsymbol{\beta}_{p,q}, \tag{2}$$

that is, when the $(N+1)\times(N+1)$ leading Hankel test excludes the law and no smaller leading test does. A witness at order N is exactly a proof that the N-th test strictly enlarges the region excluded by all the preceding ones. Yu's Theorem 5.1 reads, verbatim,

> "Fix $p=7/10$. For $2\leq N\leq 11$, let q_N be given in Table 1. At (p,q_N), $D_0,D_1,\ldots,D_{N-1}>0$, $D_N<0$. Consequently, for every $N=2,\ldots,11$, the $(N+1)\times(N+1)$ leading Hankel test excludes a nonempty open two-dimensional set excluded by none of the preceding leading Hankel tests."

A closing sentence, not quoted here, adds that those open sets can be chosen outside the previously known non-FID regions and outside $0<p,q\leq 1$. The ten witnesses are $q_N=2$, 5/2, 3, 4, 5, 7, 10, 12, 15, 20. The paper's Discussion then states what its table does not do:

> "In particular, the witness table proves neither that the hierarchy is strictly decreasing at every order nor that every non-FID beta law is eventually separated by a finite Hankel matrix."

Both halves of that sentence are settled below, in opposite ways. The first is answered affirmatively — the hierarchy does keep enlarging, at infinitely many orders (Theorem 11.6). The second is not open at all: it is the contrapositive of the very criterion [10] cites in its Introduction and uses for its own Lemma 2.1 (Remark 11.10). Neither statement comes from a computation; both come from reading the hierarchy as a whole, which is the content of Section 11.

Negative Hankel determinants of high order are not themselves new. Hasebe [6, §5.3] reports that the laws $\boldsymbol{\beta}'_{1,q}$ "[are] not FID for $q=60,61,62,70,90,100,150$ because their 26th, 25th, 24th, 21th, 18th, 18th, 16th Hankel determinants are negative respectively", and that $\boldsymbol{\beta}_{1,q}$ has a negative Hankel determinant for $q=1,2,\ldots,15$. Two conventions have to be read off before those numbers can be compared with (1). First, $\boldsymbol{\beta}'_{p,q}$ is the beta law of the second kind, with density $B(p,q)^{-1}x^{p-1}(1+x)^{-p-q}$ on $[0,\infty)$ [6, §1.1] — a different family from the $\boldsymbol{\beta}_{p,q}$ studied here, and not compactly supported. Second, the "nth Hankel determinant of $(r_n)_{n\geq 2}$" is the $n\times n$ minor displayed in [6, §5.3], with first row $r_2,\ldots,r_{n+1}$ and last row $r_{n+1},\ldots,r_{2n}$, which in the indexing (1) is D_{n-1}. So the first list reaches D_{25}, well above the range of [10, Table 1]. But an exclusion is not an enlargement: no claim is made there that the *preceding* leading minors are positive, so those determinants do not say that any particular test is the first to fire. Isolating the first failing order — the content of (2) — is exactly the contribution of [10] that the present paper extends.

The second thread is older. Hasebe [6] determined the free infinite divisibility of the classical gamma law $\boldsymbol{\gamma}_p$ of shape p and unit scale (moments $m_n=(p)_n$) outside an interval, and conjectured the rest. His Corollary 1.3(1) reads

> "The gamma distribution $\boldsymbol{\gamma}_p$ is FID if $p\in(0,\frac{1}{2}]\cup[\frac{3}{2},\infty)$, and is not FID if $p\in\mathcal{I}$,"

where

$$\mathcal{I}\ :=\ \Big(\bigcup_{n\geq 1}\Big(\tfrac{2n-1}{2n},\tfrac{2n}{2n+1}\Big)\Big)\ \cup\ \Big(\bigcup_{n\geq 1}\Big(\tfrac{2n+2}{2n+1},\tfrac{2n+1}{2n}\Big)\Big)\ \subset\ \Big(\tfrac{1}{2},\tfrac{3}{2}\Big). \tag{3}$$

The set $\mathcal{I}$ is a countable union of *open* intervals accumulating at 1 from both sides, so it does not exhaust $(\frac{1}{2},\frac{3}{2})$: it omits the closed gaps $[\frac{2}{3},\frac{3}{4}]$, $[\frac{4}{5},\frac{5}{6}]$, $[\frac{6}{7},\frac{7}{8}]$, … below 1, the point 1 itself, and $[\frac{5}{4},\frac{4}{3}]$, $[\frac{7}{6},\frac{6}{5}]$, $[\frac{9}{8},\frac{8}{7}]$, … above 1. Part (3) of the same paper's Conjecture 5.6 is

> "$\boldsymbol{\gamma}_p$ is not FID for $p\in(\frac{1}{2},\frac{3}{2})$."

The only shape in a gap for which anything was known is $p = 1$, the exponential law: Hasebe records in [6, §5.3] that its 16th Hankel determinant of free cumulants is negative, and a footnote in [6, §1.3] credits the computation to F. Lehner. To our knowledge no shape parameter has been added to that list since 2014; Section 14 says what was searched.

That conjecture is proved here (Corollary 12.8), and with the positive half of [6, Corollary 1.3(1)] it completes the classification of the freely infinitely divisible gamma laws:

$$\boldsymbol{\gamma}_p \text{ is FID} \qquad \Longleftrightarrow \qquad p \in \big(0, \tfrac{1}{2}\big] \cup \big[\tfrac{3}{2}, \infty\big). \tag{4}$$

What makes a whole interval of shapes reachable, where a determinant computation reaches one rational shape at a time, is a change of instrument. A negative determinant is not the only certificate that $H_N \not\succeq 0$: a single vector u with $u^{\mathsf{T}} H_N u < 0$ is one too, it costs one contraction rather than an elimination, and — this is the point — it can be held *fixed* while p moves. Since $m_n(\boldsymbol{\gamma}_p) = (p)_n$ lies in $\mathbb{Z}[p]$, so does every free cumulant of $\boldsymbol{\gamma}_p$, so for a fixed integer vector u the quadratic form

$$Q(p) \;:=\; u^{\mathsf{T}} H_N(\boldsymbol{\gamma}_p)\, u \;=\; \sum_{i,j=0}^{N} u_i\, \kappa_{i+j+2}(\boldsymbol{\gamma}_p)\, u_j \tag{5}$$

is a *single* polynomial with integer coefficients, of degree at most $N+1$, and its sign on a closed rational interval is decided by a Bernstein certificate. Section 12 carries this out at $N = 20$ on $[\frac{2}{3}, \frac{4}{3}]$, which with $\mathcal{I} \supset (\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2})$ covers $(\frac{1}{2}, \frac{3}{2})$, and at $N = 13$ on $[\frac{49}{50}, \frac{6}{5}]$.

These two threads are the same question seen from two sides, and the link is already printed in [10]. Its Appendix B, Lemma B.1, states

$$\lim_{q\to\infty} q^{(N+1)(N+2)} D_N(\boldsymbol{\beta}_{p,q}) \;=\; D_N(\boldsymbol{\gamma}_p), \tag{6}$$

and observes that a negative $D_N(\boldsymbol{\gamma}_p)$ therefore makes $\boldsymbol{\beta}_{p,q}$ non-FID for all sufficiently large q. (The exponent is forced: scaling a law by $c > 0$ multiplies κ_n by c^n, hence multiplies D_N by $c^{(N+1)(N+2)}$.) But [10] evaluates no gamma determinant anywhere, and [6] evaluates exactly one, at $p = 1$.

Results. Except in Section 12, where the parameter runs over a real interval, and in Section 11, which is classical analysis and is marked as such, everything below is stated for the leading Hankel tests (1) at rational parameters, where every determinant sign is an exact rational computation.

An interval of shapes at once (Section 12). Let p_0 be a shape at which the hierarchy of $\boldsymbol{\gamma}_{p_0}$ first fails at order N, and let u be the last column of the adjugate of $H_N(\boldsymbol{\gamma}_{p_0})$. Then $H_N u = D_N e_N$ and $u^{\mathsf{T}} H_N u = D_N D_{N-1} < 0$, so the polynomial (5) attached to that u is negative at p_0 — and u has integer entries once the law is rescaled to have integer moments. A Bernstein certificate then propagates the sign to a closed interval (Proposition 12.3). At $N = 13$, $p_0 = 1$ this gives $H_{13}(\boldsymbol{\gamma}_p) \not\succeq 0$ for every real $p \in [\frac{49}{50}, \frac{6}{5}]$ (Theorem 12.5), and at $N = 20$, $p_0 = \frac{2}{3}$ it gives $H_{20}(\boldsymbol{\gamma}_p) \not\succeq 0$ for every real $p \in [\frac{2}{3}, \frac{4}{3}]$ (Theorem 12.7), hence [6, Conjecture 5.6(3)] and the classification (4) (Corollary 12.8). The order 20 is forced and the interval is essentially the largest available: Remark 12.10 says why.

The ladder to order 20 (Sections 3 and 4). At the same shape $p = 7/10$ used by [10] there are witnesses at orders $12, \dots, 18$, namely $q = 27, 36, 49, 71, 109, 194, 541$ (Theorem 3.1); each is accompanied by its exact reduced final pivot D_N/D_{N-1}, a fraction of up to 3925 digits, and by a bracket that confines the threshold in q to an interval between two consecutive integers (Theorem 3.2). Orders 19 and 20 need a different shape, and $p = 2/3$ supplies them, at $q = 355$ and $q = 3639$ (Theorem 4.1). Together with [10, Table 1] this certifies a strict enlargement of the hierarchy at every order $2 \le N \le 20$ (Corollary 4.4).

The gamma ceiling (Section 4). Why the shape has to move is a theorem, not an artefact of the search. Write $\nu(p)$ for the first order at which the leading Hankel hierarchy of $\boldsymbol{\gamma}_p$ fails. Then $\nu(7/10) = 18$, so by (6) the first failing order of $\boldsymbol{\beta}_{7/10,q}$ is *exactly* 18 for all sufficiently large q: the line $p = 7/10$, on which [10] built its table, is capped in its own variable (Theorem 4.1). A single shape can witness only finitely many orders, and how many is $\nu(p)$.

ν takes every value from 13 to 25 (Section 5). Thirteen explicit rational shapes realise the thirteen values (Theorem 5.1), which turns the question "does the hierarchy strictly enlarge at every order?" into "is ν unbounded?" and extends the certified range of strict enlargement to every order $2 \le N \le 25$ (Corollary 5.2).

Non-FID gamma shapes, one at a time (Section 6). The shapes at which ν was computed include 28 rationals lying in the gaps of $\mathcal{I}$, and at each of them $\boldsymbol{\gamma}_p$ has a negative Hankel determinant of free cumulants, hence is not FID (Theorem 6.1). Every one of those shapes is covered, as a set, by Theorem 12.7; what the pointwise statement adds is the exact order at which the hierarchy fails there, which the interval certificates do not give (Remark 6.4).

The exponential law (Section 7). Its first negative Hankel determinant has size 14×14: consistent with the negative 16×16 determinant recorded in [6, §5.3], and sharper than it (Proposition 7.1). The recorded 16×16 determinant is itself certified here as well, as part of Theorem 10.4.

Controls (Section 8). The obstruction does not fire on the known-FID range; the witnesses are not already excluded one order down; and no single q witnesses two orders (Proposition 8.1).

The hierarchy is one scalar, and it fires monotonically (Section 10). Deleting the first row and column of H_N leaves the Hankel matrix of the twice-shifted cumulant sequence, with determinant E_{N-1}; the Schur complement of that block gives $D_N = E_{N-1}\Phi_N$ for a single scalar Φ_N (Theorem 10.1), and Φ_N is non-increasing in N wherever those trailing matrices are positive definite (Theorem 10.3), so that there the exclusion, once it fires, never un-fires. The certified data match exactly: at $p = 1$ and $p = 6/5$ the pivot word of $H_{18}(\boldsymbol{\gamma}_p)$ is $+^{13} -+^{5}$ with $E_0, \dots, E_{18} > 0$, and at $p = 11/8$ the pivot word of H_{22} is $+^{15} -+^{7}$ while the trailing determinants are positive exactly through E_{21} and negative at E_{22} (Theorem 10.4) — so each of these Hankel matrices has exactly one negative square, and the hypothesis of Theorem 10.3 is neither vacuous nor automatic.

Completeness, and the answer to the open question (Section 11). For a measure with all moments whose moment sequence is determinate, "$H_N \succeq 0$ for every N" is equivalent to free infinite divisibility (Theorem 11.1); for compactly supported measures that is the criterion of [8, Theorem 13.16] which [10] itself cites, and the extension assembles published theorems. The gamma law is determinate, so its entire hierarchy is equivalent to its free infinite divisibility; with [6, Corollary 1.3(1)] on $(\frac{1}{2}, \frac{2}{3}) \subset \mathcal{I}$ and strict positivity of every $D_N(\boldsymbol{\gamma}_{1/2})$ (Lemma 11.3) this gives ν unbounded (Theorem 11.6), hence, through (6), a strict enlargement of the hierarchy at infinitely many orders. It also identifies [6, Conjecture 5.6(3)] with finiteness of the first failing order at every $p \in (\frac{1}{2}, \frac{3}{2})$ (Remark 11.8): the hierarchy is not one instrument among several for that conjecture but exactly equivalent to it. Nothing in that section is machine-checked, and every statement in it says so — and nothing in Section 12 depends on it, the proof of Corollary 12.8 using only the elementary direction of the criterion.

Method, and what the reader is asked to trust. Every claim about a specific parameter is a finite exact computation over $\mathbb{Q}$: the first $2N+2$ moments, the cumulant recursion, the Hankel matrix and an unpivoted symmetric elimination whose pivots are the ratios D_j/D_{j-1}. These computations are carried out by the Lean 4 kernel on arbitrary-precision integers; there is no floating point in any statement or proof, and decimals appear in this paper only to *locate* parameters, never to decide a sign (Section 13). The searches that produced the parameters — a grid in q, bisections, a scan in p — are described where they are used and are not part of any certified statement. The two structural results of Section 10 are formalised as well, the Schur identity over an arbitrary commutative ring and the monotonicity of Φ_N over $\mathbb{R}$, for an arbitrary sequence of cumulants. The statements of Section 12 are of a different shape — they quantify over a real p — and what is computed there is likewise different: a polynomial identity in $\mathbb{Z}[X]$ and a list of integer signs, from which the real statement follows by an argument that is itself formalised. Two integer vectors and two lists of Bernstein coefficients enter as data; how they were found is a search, and nothing certified depends on it.

One disclosure belongs here rather than in Section 13. Every computation in this paper is evaluated by the proof kernel itself, on arbitrary-precision integers, with a single exception: the order-20 certificate behind Theorem 12.7, and hence behind Corollary 12.8, is checked by compiled evaluation instead, because running the cumulant recursion over $\mathbb{Z}[p]$ to the depth it needs costs an estimated 50 GB inside the kernel. That theorem therefore rests on the correctness of the compiler as well as of the kernel, and its statement is recorded with the corresponding axiom. Theorem 12.5 is the same method at order 13, on the smaller interval that the kernel can afford, and carries no such axiom; the two are stated separately for exactly that reason.

Two steps are cited rather than reproved. The passage from "$D_N < 0$", or more generally from "$H_N \not\succeq 0$", to "not FID" is the conditional positive definiteness criterion, applied exactly as [6] applies it to the exponential law; Remark 2.2 says which form of it each family of laws needs and what that form rests on. The passage from a negative gamma determinant to a statement about beta laws with large q is (6), quoted from [10, Lemma B.1].

Section 11 is different in kind and is kept apart for that reason. Its proofs are ordinary analysis — the Hamburger moment problem, the Nevanlinna representation behind the free Lévy–Khintchine formula, and a continuity argument — and they are written out here, but they are not machine-checked and could not cheaply be: the library used for the rest of the paper carries no moment-problem theory and no free probability. Every theorem, lemma and corollary in that section is marked *not machine-checked* in its own heading. What is new there is the answer, not the technique; Remark 11.2 says precisely which published theorem each step is.

2. Preliminaries

Moments and free cumulants. For $p, q > 0$ let $\boldsymbol{\beta}_{p,q}$ be the beta law on $[0,1]$ with density proportional to $x^{p-1}(1-x)^{q-1}$, and let $\boldsymbol{\gamma}_p$ be the classical gamma law of shape p and unit scale. Their moment sequences are

$$m_n(\boldsymbol{\beta}_{p,q}) = \frac{(p)_n}{(p+q)_n}, \qquad m_n(\boldsymbol{\gamma}_p) = (p)_n, \qquad (x)_n = \prod_{i=0}^{n-1}(x+i), \tag{7}$$

so at rational parameters every moment is an explicit rational number. The free cumulants κ_n are obtained from the moment–cumulant relation in the formal-power-series form used in [10, eq. (21)]: with $M(z) = \sum_{j\ge 0} m_j z^j$,

$$\kappa_n \;=\; m_n \;-\; \sum_{k=1}^{n-1} \kappa_k \,[z^{\,n-k}]\, M(z)^k, \qquad n \ge 1. \tag{8}$$

Only $\kappa_2, \dots, \kappa_{2N+2}$ enter H_N, so H_N is a rational function of the parameters, with denominators that are products of Pochhammer factors and hence nonvanishing for positive parameters. In particular $q \mapsto D_N(\boldsymbol{\beta}_{p,q})$ and $p \mapsto D_N(\boldsymbol{\gamma}_p)$ are continuous on $(0,\infty)$.

The test, the hierarchy, and pivots. We use (1) throughout; note the index convention, H_N is $(N+1)\times(N+1)$ and starts at κ_2. The criterion we use is:

Lemma 2.1 (conditional positive definiteness)**.** *Let μ be a probability measure on $\mathbb{R}$ with moments of all orders. If μ is FID then $H_N(\mu) \succeq 0$ for every $N \ge 0$. Consequently $D_N(\mu) < 0$ for a single N implies that μ is not FID.*

Remark 2.2 (which form of the criterion is used where). For *compactly supported* μ this is the standard conditional-positive-definiteness criterion for free cumulants: [10] cites [8, Chapter 13] for it and states it, with the compactness hypothesis, as its own Lemma 2.1. That is all the beta-side statements of this paper need, since $\boldsymbol{\beta}_{p,q}$ lives on $[0,1]$. The gamma law is not compactly supported, and there the form used is the finite-moment one stated above — the form [6, §5.3] applies to the exponential law, likewise without a compactness hypothesis. It is what the free Lévy–Khintchine representation gives, and that representation carries no support assumption: by [4] (see Section 11, step (BV), for how that

is read here), a $\boxplus$-infinitely divisible μ has Voiculescu transform $\varphi_\mu(z) = \gamma + \int (1+tz)(z-t)^{-1}\, d\sigma(t)$ with σ a finite positive measure, so that $\kappa_{n+2}(\mu) = \int t^n(1+t^2)\, d\sigma(t)$ and $H_N(\mu)$ is the moment matrix of the positive measure $(1+t^2)\, d\sigma$, hence positive semidefinite. Identifying the free cumulants of a measure with unbounded support with the coefficients of the expansion of φ_μ at infinity is the step that needs care, and is [3, Theorem 1.3]; we cite it rather than reprove it, and it is not formalised here. Every "not FID" conclusion about a *gamma* law below — Theorem 6.1, and the sentences that quote it — rests on this form of the criterion, exactly as Hasebe's conclusion at $p = 1$ does; neither [10] nor [6] states that form with its hypotheses.

If H_N has all leading principal minors $D_0, \dots, D_{N-1}$ nonzero, unpivoted symmetric Gaussian elimination on H_N runs to completion and produces the factorisation $H_N = LDL^{\mathsf{T}}$ with L unit lower triangular and

$$D = \operatorname{diag}(\ell_0, \dots, \ell_N), \qquad \ell_j = \frac{D_j}{D_{j-1}} \quad (D_{-1} := 1). \tag{9}$$

The list $(\operatorname{sgn}\ell_0, \dots, \operatorname{sgn}\ell_N)$, which we call the *pivot sign word*, therefore determines the signs of $D_0, \dots, D_N$ and conversely. One elimination at size $N+1$ delivers the whole word, hence the sign of every leading minor at once; this is what makes a single computation decide the order at which a given law first fails.

Definition 2.3. A law μ (with finite moments) is a *witness at order* N if $D_0(\mu), \dots, D_{N-1}(\mu) > 0 > D_N(\mu)$; equivalently, if its pivot sign word at size $N+1$ is $+^N-$. A witness at order N is excluded by the $(N+1) \times (N+1)$ leading Hankel test and by no smaller leading test, so that test strictly enlarges the region excluded by its predecessors.

Definition 2.4. The *gamma ceiling* $\nu(p)$ is the unique N for which $\boldsymbol{\gamma}_p$ is a witness at order N, and $\nu(p) = \infty$ if no such N exists.

Finally we record the consequence of (6) that is used repeatedly.

Lemma 2.5. *Let $p > 0$ with $\nu(p) = N < \infty$. Then there is $q_0 = q_0(p)$ such that for every $q \geq q_0$ the law $\boldsymbol{\beta}_{p,q}$ is a witness at order N; in particular $\boldsymbol{\beta}_{p,q}$ is not FID and its first failing leading Hankel test is exactly the $(N+1) \times (N+1)$ one.*

Proof. By (6), $q^{(M+1)(M+2)} D_M(\boldsymbol{\beta}_{p,q}) \to D_M(\boldsymbol{\gamma}_p)$ for each fixed M. For $M = 0, \dots, N$ the limits are nonzero — positive for $M < N$ and negative for $M = N$, by $\nu(p) = N$ — and $q^{(M+1)(M+2)} > 0$, so each of these finitely many sign conditions holds for all large q; take q_0 beyond all $N+1$ thresholds. □

Lemma 2.5 is not effective: (6) is quoted as an asymptotic statement and yields no explicit q_0. Every beta-side claim in this paper is therefore stated either with an explicit exact witness or with "for all sufficiently large q", never both.

3. The witness ladder at $p = 7/10$

Theorem 3.1 (new rungs of the witness table)**.** *Let $p = 7/10$ and*

$$(q_{12}, \dots, q_{18}) = (27,\ 36,\ 49,\ 71,\ 109,\ 194,\ 541).$$

For each $N \in \{12, \dots, 18\}$ the law $\boldsymbol{\beta}_{p,q_N}$ is a witness at order N:

$$D_0(\boldsymbol{\beta}_{p,q_N}),\ \dots,\ D_{N-1}(\boldsymbol{\beta}_{p,q_N}) \ >\ 0 \ >\ D_N(\boldsymbol{\beta}_{p,q_N}).$$

In particular $\boldsymbol{\beta}_{7/10,q_N}$ is not FID, and no leading Hankel test of size $N \times N$ or smaller excludes it. Moreover, in each case the reduced fraction $\ell_N = D_N/D_{N-1}$ is certified exactly, with numerator and denominator of the sizes listed in Table 1.

TABLE 1. The witness ladder. Rows 2–11 are [10, Table 1]; rows 12–20 are Theorems 3.1 and 4.1. The last column is the number of decimal digits of the numerator and denominator of the exact reduced final pivot $\ell_N = D_N/D_{N-1}$, which is certified for every row $12 \le N \le 20$ and, for rows $2 \le N \le 11$, agrees with the author's ancillary certificate (Section 9).

N	p	q_N	source	digits of ℓ_N
2–11	7/10	2, $\frac{5}{2}$, 3, 4, 5, 7, 10, 12, 15, 20	[10]	—
12	7/10	27	new	960/984
13	7/10	36	new	1209/1236
14	7/10	49	new	1407/1437
15	7/10	71	new	1974/2010
16	7/10	109	new	2361/2403
17	7/10	194	new	2814/2865
18	7/10	541	new	3858/3925
19	2/3	355	new	3102/3167
20	2/3	3639	new	4764/4870

Proof sketch. Each of the seven statements is a single exact rational computation, not a search: from (7) form $m_0, \dots, m_{2N+2}$ as reduced fractions, run (8) to get $\kappa_1, \dots, \kappa_{2N+2}$, assemble H_N and run the unpivoted elimination (9) to size $N+1$. The resulting sign word is $+^N-$ in each case, and the final pivot is the exact reduced fraction certified alongside it. All of this is performed by the Lean 4 kernel in arbitrary-precision integer arithmetic (Section 13); no step uses floating point and no step is a decision procedure over a search space.

What *was* a search is the choice of the q_N. At fixed $p = 7/10$ let $\nu_\beta(q)$ denote the index of the first non-positive pivot of the elimination, so that q is a witness at order N exactly when $\nu_\beta(q) = N$. A grid of 617 points over $q \in [1, 1200]$, evaluated exactly at rational (decimal) abscissae, followed by 28-step bisections, located the sign changes of D_{N-1}; q_N was then taken to be the simplest rational strictly inside the located interval. That recipe reproduces the choice of [10, Table 1] at nine of its ten rows (it differs only at $N = 8$, where [10] takes $q = 10$ and the simplest rational is 9), which is evidence that the intervals are correctly located. None of the certified statements depends on the search: each was re-derived exactly at the landed parameter. □

Theorem 3.2 (strictness and the bracketed threshold)**.** *Keep $p = 7/10$ and $q_{12}, \dots, q_{18}$ as in Theorem 3.1. For each $N \in \{12, \dots, 18\}$ the pivot sign word of $\boldsymbol{\beta}_{p,\, q_N - 1}$ at size $N+1$ is*

$$\underbrace{+\cdots+}_{N-1} - +,$$

that is, $D_0, \dots, D_{N-2} > 0 > D_{N-1}$ at $q = q_N - 1$, whereas $D_{N-1} > 0$ at $q = q_N$. Consequently:

1. *$D_{N-1}(\boldsymbol{\beta}_{7/10,\,\cdot})$ vanishes at some q strictly between $q_N - 1$ and q_N;*
2. *$\boldsymbol{\beta}_{7/10,\, q_N - 1}$ is already excluded by the $N \times N$ leading test, so it is not a witness at order N;*
3. *at $(7/10, 27)$ all of $D_0, \dots, D_{11}$ are strictly positive, so the 13×13 test of Theorem 3.1 is genuinely the first one that excludes $\boldsymbol{\beta}_{7/10,27}$.*

Proof sketch. The two sign words at $q = q_N - 1$ and $q = q_N$, and the all-positive word at $(7/10, 27)$ up to order 11, are exact computations of the kind described in the proof of Theorem 3.1. Item (1) is then the intermediate value theorem applied to $q \mapsto D_{N-1}(\boldsymbol{\beta}_{7/10,q})$, which is continuous on $(0, \infty)$ by the remark following (8). Item (2) is immediate from the sign word at $q_N - 1$, and item (3) is the all-positive computation at order 11, run independently of the order-12 one. □

Remark 3.3. No source records a threshold, an interval, or any minimality statement for a witness; [10, Table 1] prints one q per order. Bisection refines the brackets of Theorem 3.2 to

$$q_{12}^* \approx 26.0564, \quad q_{13}^* \approx 35.1978, \quad q_{14}^* \approx 48.6906, \quad q_{15}^* \approx 70.1002,$$
$$q_{16}^* \approx 108.3509, \quad q_{17}^* \approx 193.8533, \quad q_{18}^* \approx 540.9497,$$

where q_N^* is the zero of D_{N-1} located in item (1). These decimals are the output of the search and are stated here for orientation only; they are not part of any certified statement. The brackets themselves — the integer pairs $(q_N - 1, q_N)$ — are.

Remark 3.4. Each point witness upgrades to a two-dimensional statement exactly as in [10, Theorem 5.1]: the conditions (2) are finitely many strict inequalities between continuous functions of (p, q), so they persist on a neighbourhood of the witness. We state our results at points, which is what is computed.

4. The gamma ceiling, and why one shape is not enough

The ladder of Section 3 stops at $N = 18$, and the reason is visible only from the gamma side — which [10] sets up, in (6), and never evaluates.

Theorem 4.1 (the ceiling at $p = 7/10$, and the continuation at $p = 2/3$)**.**

(1) $\nu(7/10) = 18$*; that is, the gamma law* $\boldsymbol{\gamma}_{7/10}$ *satisfies* $D_0, \dots, D_{17} > 0 > D_{18}$.
(2) *Consequently, by Lemma 2.5, there is* q_0 *such that for all* $q \geq q_0$ *the law* $\boldsymbol{\beta}_{7/10,q}$ *is a witness at order* 18*, so its first failing leading Hankel test is the* 19×19 *one. The line* $p = 7/10$ *therefore supplies, in its tail, witnesses at order* 18 *and at no higher order: the table of* [10] *is capped in its own variable.*
(3) $\nu(2/3) = 20$*: for* $\boldsymbol{\gamma}_{2/3}$ *one has* $D_0, \dots, D_{19} > 0 > D_{20}$*. Moving to that shape continues the ladder by two rungs:* $\boldsymbol{\beta}_{2/3,355}$ *is a witness at order* 19 *and* $\boldsymbol{\beta}_{2/3,3639}$ *is a witness at order* 20*, with the same brackets as in Theorem 3.2 — at* $q = 354$ *and* $q = 3638$ *the pivot sign words are* $+^{18} - +$ *and* $+^{19} - +$ *respectively — and with exact final pivots of* 3102/3167 *and* 4764/4870 *digits.*

Proof sketch. Parts (1) and (3) are exact rational computations on the moment sequence $m_n = (p)_n$ of (7), run exactly as in Theorem 3.1: eight of them here (two gamma eliminations, two beta witnesses, two brackets, two exact final pivots). Part (2) is Lemma 2.5 applied to $p = 7/10$ with $N = 18$, using the quoted (6). The two beta parameters 355 and 3639 were located by bisection at $p = 2/3$ in the manner described in Theorem 3.1 and then re-derived exactly. □

Remark 4.2. Part (2) is a statement about the tail of the line $p = 7/10$, because (6) is asymptotic in q. Whether some *small* q on that line witnesses an order above 18 is not settled by it. Outside the certified statements, the 617-point grid of Theorem 3.1 over $q \in [1, 1200]$, together with spot checks up to $q = 10^4$, found no first-failing index above 18 anywhere on the line.

Remark 4.3. The scaling limit behind Lemma 2.5 is present in both sources, but only at $p = 1$: [6, §5.3] records that "the exponential distribution is the limit of $D_q \boldsymbol{\beta}_{1,q}$ as $q \to \infty$" and uses that one instance to move a Hankel computation from the beta side to the gamma side, while [10, Lemma B.1] proves the limit for every p and every order and then evaluates no gamma determinant. Theorem 4.1 is what one gets by evaluating it.

Corollary 4.4. *For every* N *with* $2 \leq N \leq 20$ *there is an explicit pair of rational parameters* (p, q) *such that the* $(N+1) \times (N+1)$ *leading Hankel test excludes* $\boldsymbol{\beta}_{p,q}$ *and no smaller leading test does. The parameters are those of Table 1. Equivalently, the leading Hankel hierarchy strictly enlarges the excluded region at every order from* 2 *to* 20*.*

Proof. Orders 2–11 are [10, Theorem 5.1], recomputed here as Proposition 9.1; orders 12–18 are Theorem 3.1; orders 19 and 20 are Theorem 4.1(3). □

5. The gamma ceiling takes every value from 13 to 25

Theorem 4.1 converts the question left open in [10] into a question about one function of one variable. If ν is unbounded on $(\frac{1}{2}, \frac{3}{2})$ then, by Lemma 2.5, the hierarchy strictly enlarges at every order in the range of ν, hence at infinitely many orders; if ν is bounded, the beta family of [10] cannot by itself settle the question above that bound. Section 11 proves that ν is unbounded. Here we compute it.

Theorem 5.1 (the values of the gamma ceiling)**.** *For each N with $13 \le N \le 25$ the shape p_N listed in Table 2 satisfies $\nu(p_N) = N$; that is, $\boldsymbol{\gamma}_{p_N}$ is a witness at order N: $D_0(\boldsymbol{\gamma}_{p_N}), \dots, D_{N-1}(\boldsymbol{\gamma}_{p_N}) > 0 > D_N(\boldsymbol{\gamma}_{p_N})$. In particular ν takes every value in $\{13, 14, \dots, 25\}$.*

Table 2. Shapes realising each value of the gamma ceiling ν. Seven of the thirteen shapes lie in gaps of $\mathcal{I}$ and are therefore also covered by Theorem 6.1; the remaining six (7/5, 13/20, 63/100, 31/50, 3/5, 149/100) lie inside $\mathcal{I}$, where non-FID was already known.

ν	13	14	15	16	17	18	19	20	21	22	23	24	25
p	$\frac{6}{5}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{7}{5}$	$\frac{3}{4}$	$\frac{7}{10}$	$\frac{17}{25}$	$\frac{2}{3}$	$\frac{13}{20}$	$\frac{63}{100}$	$\frac{31}{50}$	$\frac{3}{5}$	$\frac{149}{100}$

Proof sketch. Thirteen exact computations, one per column, of the kind described in the proof of Theorem 3.1, applied to the moment sequence $m_n = (p)_n$; each returns the pivot sign word $+^N-$ at size $N+1$. The shapes were found by a scan of ν over rationals in $(\frac{1}{2}, \frac{3}{2})$, again evaluated exactly at decimal abscissae, and then re-derived at the landed shape. The cost grows sharply with N: the five largest columns are one computation per file in the formal development, the largest of them measuring between 5.4 and 6.5 GB of kernel memory depending on machine load (Section 13). □

Corollary 5.2. *The leading Hankel hierarchy strictly enlarges at every order $2 \le N \le 25$. Precisely: for $2 \le N \le 20$ there is an explicit beta law that is a witness at order N (Corollary 4.4); and for $13 \le N \le 25$ the gamma law $\boldsymbol{\gamma}_{p_N}$ of Table 2 is a witness at order N, so the $(N+1) \times (N+1)$ leading test excludes a law that no smaller leading test excludes. For $21 \le N \le 25$ one also gets beta witnesses at order N, at the shape p_N and all sufficiently large q, by Lemma 2.5.*

Remark 5.3. Outside the certified statements, the scan describes ν as follows. On $(\frac{1}{2}, \frac{3}{2})$ it is a U-shaped function of p with minimum value 13 attained near $p \approx 1.1$, rising to 18 at $p = 0.7$, to 20 at $p = 2/3$ and to 24 at $p = 3/5$ on the left, and to 25 at $p = 149/100$ on the right; the scan also gives $\nu(299/200) = 28$, which was not certified here because its cost exceeds that of the $\nu = 25$ computation, itself the largest in the development. Outside $(\frac{1}{2}, \frac{3}{2})$ no negative determinant appeared at any tested shape through the orders checked (Proposition 8.1). The evidence is thus that $\nu \to \infty$ at both endpoints of the interval where [6] conjectures non-FID. That ν is indeed unbounded is proved in Section 11, by an argument that uses only the behaviour of the hierarchy near $p = \frac{1}{2}$; the *rate* at which ν diverges is measured in Remark 14.1 and is not explained here.

6. New shapes at which the gamma law is not FID

The computations of Section 5 were run at shapes chosen for a second reason: to land inside the gaps of Hasebe's set $\mathcal{I}$ of (3), where Corollary 1.3(1) of [6] gives no information and its Conjecture 5.6(3) predicts non-FID.

Theorem 6.1 (28 new non-FID gamma shapes)**.** *Let p be any of the 28 rationals of Table 3. Each lies in $(\frac{1}{2}, \frac{3}{2}) \setminus \mathcal{I}$, and the free cumulants of $\boldsymbol{\gamma}_p$ satisfy*

$$D_0(\boldsymbol{\gamma}_p),\ \dots,\ D_{\nu(p)-1}(\boldsymbol{\gamma}_p)\ >\ 0\ >\ D_{\nu(p)}(\boldsymbol{\gamma}_p)$$

with $\nu(p)$ as listed. Hence, by Lemma 2.1 — in the finite-moment form of Remark 2.2, since $\boldsymbol{\gamma}_p$ is not compactly supported — $\boldsymbol{\gamma}_p$ is not freely infinitely divisible at any of these 28 *shapes. These are instances of* [6, Conjecture 5.6(3)] *at shapes not covered by* [6, Corollary 1.3(1)].

TABLE 3. Shapes in the gaps of $\mathcal{I}$ at which $\boldsymbol{\gamma}_p$ is certified non-FID, grouped by gap, with the order $\nu(p)$ at which the leading Hankel hierarchy first fails. The point $p = 1$ is omitted by $\mathcal{I}$ as well; there non-FID was already known (Section 7).

gap of $\mathcal{I}$	shapes, with ν
$[\frac{2}{3}, \frac{3}{4}]$	$\frac{2}{3}$ (20), $\frac{17}{25}$ (19), $\frac{7}{10}$ (18), $\frac{3}{4}$ (17)
$[\frac{4}{5}, \frac{5}{6}]$	$\frac{4}{5}$ (15), $\frac{9}{11}$ (15), $\frac{5}{6}$ (15)
$[\frac{6}{7}, \frac{7}{8}]$	$\frac{6}{7}$ (15), $\frac{13}{15}$ (14), $\frac{7}{8}$ (14)
$[\frac{8}{9}, \frac{9}{10}]$	$\frac{8}{9}$ (14), $\frac{17}{19}$ (14), $\frac{9}{10}$ (14)
$[\frac{10}{11}, \frac{11}{12}]$	$\frac{10}{11}$ (14), $\frac{21}{23}$ (14), $\frac{11}{12}$ (14)
$[\frac{11}{10}, \frac{10}{9}]$	$\frac{11}{10}$ (13), $\frac{21}{19}$ (13), $\frac{10}{9}$ (13)
$[\frac{9}{8}, \frac{8}{7}]$	$\frac{9}{8}$ (13), $\frac{17}{15}$ (13), $\frac{8}{7}$ (13)
$[\frac{7}{6}, \frac{6}{5}]$	$\frac{7}{6}$ (13), $\frac{13}{11}$ (13), $\frac{6}{5}$ (13)
$[\frac{5}{4}, \frac{4}{3}]$	$\frac{5}{4}$ (13), $\frac{13}{10}$ (14), $\frac{4}{3}$ (14)

Proof sketch. That each listed p lies outside $\mathcal{I}$ is a finite check against (3): the intervals of $\mathcal{I}$ below 1 are $(\frac{2n-1}{2n}, \frac{2n}{2n+1})$ and those above 1 are $(\frac{2n+2}{2n+1}, \frac{2n+1}{2n})$, and both endpoints of every gap listed in Table 3 are of that form, so a rational p lies in a gap as soon as it lies between the two listed endpoints inclusive. The sign statements are 28 exact rational computations of the kind described in the proof of Theorem 3.1, on the moment sequence $m_n = (p)_n$, at sizes between 14 and 21. The final step, from a negative determinant to "not FID", is Lemma 2.1; it is cited, not reproved, and it is applied exactly as [6] applies it at $p = 1$. □

Remark 6.2. By Lemma 2.5, each shape of Table 3 also yields non-FID beta laws $\boldsymbol{\beta}_{p,q}$ for all sufficiently large q. Explicit q are given here only for $p = 7/10$ (Theorem 3.1) and $p = 2/3$ (Theorem 4.1); for the other 26 shapes the statement is the ineffective one.

Remark 6.3. The two beta witnesses of Theorem 4.1(3) sit at the shape $p = 2/3$ and at $q = 355$, $q = 3639$, and they are covered by neither part of [6, Theorem 1.2(1),(2)], the two parts of that theorem that concern $\boldsymbol{\beta}_{p,q}$. Its positive half asks for $p, q \geq \frac{3}{2}$, or $0 < p \leq \frac{1}{2}$ with $p + q \geq 2$, or $0 < q \leq \frac{1}{2}$ with $p + q \geq 2$; its negative half asks for $0 < p, q \leq 1$, or $p \in \mathcal{I}$, or $q \in \mathcal{I}$. Here $p = 2/3$ is an endpoint of the gap $[\frac{2}{3}, \frac{3}{4}]$, hence outside the open set $\mathcal{I}$; and $q = 355$, $q = 3639$ lie outside $\mathcal{I} \subset (\frac{1}{2}, \frac{3}{2})$ and outside $(0, 1]$. The same holds for every witness of Theorem 3.1 and of Proposition 9.1, where $p = 7/10$ lies in the same gap and every q is at least 2; [10, Theorem 5.1] records that for its own rows. All of these points are instances of [6, Conjecture 5.6(1)], that $\boldsymbol{\beta}_{p,q}$ is not FID whenever $p \in (\frac{1}{2}, \frac{3}{2})$ and $q > 0$.

Remark 6.4 (what Section 12 does to this theorem). All 28 shapes of Table 3 lie in $[\frac{2}{3}, \frac{4}{3}]$ — the smallest is $\frac{2}{3}$ and the largest is $\frac{4}{3}$ — so Theorem 12.7 covers every one of them, and covers the gaps they sit in entirely rather than at finitely many points; nine of them, the nine lying between $\frac{11}{10}$ and $\frac{6}{5}$, are covered even by the smaller interval of Theorem 12.5. The pointwise theorem is kept for two reasons. It says more at each shape: not only that the hierarchy fails, but at which order, since it exhibits the sign word $+^{\nu(p)}-$, whereas a witness vector at order N shows only that the hierarchy has failed *by* order N. And it is the source of the vectors: both certificates of Section 12 are built at a shape whose exact first failing order was computed here, and Remark 12.10 shows that the order-20 certificate could not have been built without knowing $\nu(\frac{2}{3}) = 20$.

7. The exponential law

The exponential law is $\boldsymbol{\gamma}_1$, and 1 is the one point of $(\frac{1}{2}, \frac{3}{2})$ in a gap of $\mathcal{I}$ where non-FID was already known: [6, §5.3] states that "the 16th Hankel determinant … of $(r_n)_{n\geq 2}$ is negative"; a footnote in [6, §1.3] credits the computation to F. Lehner. In the indexing (1) that determinant is D_{15}.

Proposition 7.1. *For the exponential law $\boldsymbol{\gamma}_1$,*

$$D_0,\ \ldots,\ D_{12}\ >\ 0\ >\ D_{13},$$

i.e. $\nu(1) = 13$: the first negative Hankel determinant of the free cumulants of the exponential law is the 14×14 one.

Proof sketch. One exact rational elimination at size 14 on the moment sequence $m_n = n!$, as in the proof of Theorem 3.1. □

Remark 7.2. This is consistent with the published statement and sharper than it, and it is worth saying how the two fit together. Because H_{13} is a leading principal submatrix of H_{15}, Proposition 7.1 already implies that the 16×16 test excludes $\boldsymbol{\gamma}_1$ — which is what [6, §5.3] reports — and it locates the first leading test that does so two orders earlier. The sign of D_{15} itself is a separate computation, certified in Theorem 10.4(1): the pivot word of $H_{18}(\boldsymbol{\gamma}_1)$ is $+^{13} - +^5$, so each of $D_{13}, \ldots, D_{18}$ is negative, D_{15} among them. No minimality was claimed in [6], so nothing here contradicts it; the point is that the exponential law fails the hierarchy two orders earlier than the record shows, and that $\nu(1) = 13$ is the minimum of ν observed over $(\frac{1}{2}, \frac{3}{2})$ (Remark 5.3).

8. Controls

A hierarchy of tests that fired on everything would make every statement above true and empty. The following computations rule that out, and rule out the opposite degeneracy as well.

Proposition 8.1.

(1) *(The obstruction does not fire on the known-FID range.) For $\boldsymbol{\gamma}_{1/2}$ and $\boldsymbol{\gamma}_{3/2}$ — the two endpoints of the window of* [6, Corollary 1.3(1)] *— every leading Hankel determinant $D_0, \ldots, D_{20}$ is strictly positive; the same holds through D_{15} at the shapes 1/4, 2/5, 2, 5/2 and 3. Moreover $\boldsymbol{\gamma}_{3/2}$ is not a witness at order 13, the order at which shapes just inside the window fire.*

(2) *(No single q witnesses two orders.) At $p = 7/10$, $q = 27$ is a witness at order 12 but not at order 13; the value $q = 20$ of* [10, Table 1]*, a witness at order 11, is not a witness at order 12; and its $q = 2$, a witness at order 2, is far from one at order 12 — the pivot sign word there is $(+,+,-,+,+,+,+,+,+,-,+,+,+)$, so $D_2, \ldots, D_8 < 0$ and $D_9, \ldots, D_{12} > 0$.*

(3) *(Non-vacuity.) At $(7/10, 27)$ the pivot sign word of H_{12} has 13 entries, so the order-12 statements there are assertions about 13 pivots and not about an empty list; and the witness predicate and the all-positive predicate are certified to disagree on both patterns at length 4 — $(+,+,+,+)$ is not a witness word and $(+,+,+,-)$ is not all-positive — so neither predicate is trivially true.*

Proof sketch. Twelve exact computations of the kind already described, plus two Boolean identities on explicit words. Item (1) is the sharp control: the negative certificates of Theorem 6.1 accumulate at 1 from both sides and reach 2/3 and 4/3, while these positive words are computed at the very endpoints 1/2 and 3/2 of the conjectured window. □

9. The published table, recomputed

For completeness, and as a check on the implementation, the ten rows of [10, Table 1] were recomputed independently.

Proposition 9.1. *For $p = 7/10$ and $q_N = 2, \frac{5}{2}, 3, 4, 5, 7, 10, 12, 15, 20$ at $N = 2, \dots, 11$ respectively, $\boldsymbol{\beta}_{p,q_N}$ is a witness at order N. Moreover the exact reduced final pivot $\ell_N = D_N/D_{N-1}$ agrees, for every one of the ten rows, with the value in the ancillary certificate distributed with* [10]*; for instance*

$$\ell_2(\boldsymbol{\beta}_{7/10,2}) = -\frac{33249216302892080000000}{157692896359568381940899612 49}.$$

Proof sketch. Twenty exact computations: ten sign words and ten exact final pivots, as in the proof of Theorem 3.1. This is [10, Theorem 5.1]; only the independent recomputation is ours. See Section 13 for what else was matched against [10] outside the formal development. □

10. The hierarchy as one scalar, and monotone firing

Everything so far reads the hierarchy one determinant at a time. This section records what holds it together, and it contains the first statements of this paper that are proved rather than computed.

Alongside (1), write for a law μ with all moments

$$T_N(\mu) = \big(\kappa_{i+j+4}(\mu)\big)_{i,j=0}^{N-1}, \qquad E_M(\mu) = \det\big(\kappa_{i+j+4}(\mu)\big)_{i,j=0}^{M} \ (M \geq 0), \qquad E_{-1} := 1, \tag{10}$$

so that T_N is exactly H_N with its first row and column deleted, T_N is $N \times N$, and $\det T_N = E_{N-1}$. We call E_M the *trailing* determinants. No source computes them.

Theorem 10.1 (Schur reduction and sign transfer)**.** *Let K be a commutative ring, $\kappa\colon \mathbb{N} \to K$ a sequence, $N \geq 0$, and suppose T_N is invertible. Put $w_N = (\kappa_3, \kappa_4, \dots, \kappa_{N+2})^{\mathsf{T}}$ and*

$$\Phi_N \;=\; \kappa_2 \;-\; w_N^{\mathsf{T}}\, T_N^{-1}\, w_N \;\in\; K. \tag{11}$$

Then

$$D_N \;=\; E_{N-1}\, \Phi_N.$$

If moreover K is an ordered field and $E_{N-1} > 0$, then $D_N < 0 \iff \Phi_N < 0$ and $D_N > 0 \iff \Phi_N > 0$; and if only $E_{N-1} \neq 0$, then $D_N = 0 \iff \Phi_N = 0$.

Proof sketch. Split H_N along its first row and column, so that it becomes a 2×2 block matrix with scalar corner κ_2, border w_N and trailing block T_N, and take the determinant of the Schur complement of T_N. The sign transfer is then immediate from $E_{N-1} > 0$. Both are formalised for an arbitrary commutative ring and an arbitrary sequence κ (Section 13); no search and no numerical input is involved. □

Remark 10.2. Three certified controls say that the hypothesis is not removable, that the identity is not the trivial $D_N = E_{N-1}\kappa_2$, and what Φ_N measures. What each certifies is a determinant; the two values of Φ_1 below are then read off Theorem 10.1. For the free Poisson law of rate 1 ($\kappa_n \equiv 1$) one has $E_1 = D_2 = 0$, so there is no Φ_2 and the identity says nothing; for the same law $E_0\kappa_2 = 1$ while $D_1 = 0$, so the correction term in (11) is genuinely present, and $\Phi_1 = 0$. For (semicircular of variance 1) $\boxplus$ (free Poisson of rate 1), whose cumulants are $\kappa_2 = 2$ and $\kappa_n = 1$ for $n \geq 3$, one gets $\Phi_1 = 1$ — the semicircular coefficient of that law's free Lévy triplet, which in the form $R_\mu(w) = \kappa_1 + \int w(1 - xw)^{-1}\, d\tau(x)$ of Section 11 is the atom $\tau(\{0\})$. That reading of Φ_N as a finite-order approximation to $\tau(\{0\})$ is suggestive, not a theorem of this paper.

Theorem 10.3 (monotone firing)**.** *Let $\kappa\colon \mathbb{N} \to \mathbb{R}$ be an arbitrary sequence and $N \geq 0$. Suppose T_{N+1} is positive semidefinite, $E_N \neq 0$ and $E_{N-1} \neq 0$. Then*

$$\Phi_{N+1} \;\leq\; \Phi_N.$$

Consequently, if in addition $\Phi_N < 0$ then $\Phi_{N+1} < 0$: on any range of orders where the trailing Hankel matrices are positive definite, the leading Hankel exclusion, once it fires, never un-fires, and the sign sequence $\operatorname{sgn} D_0, \operatorname{sgn} D_1, \dots$ changes at most once there, from $+$ to $-$.

Proof sketch. For positive semidefinite T with invertible determinant, completing the square in $(u - T^{-1}w)^{\mathsf{T}}T(u - T^{-1}w) \geq 0$ gives the variational bound

$$2\langle u, w\rangle - u^{\mathsf{T}}Tu \;\leq\; w^{\mathsf{T}}T^{-1}w \qquad \text{for every } u,$$

with equality at $u = T^{-1}w$. Now T_N is the leading block of T_{N+1} and w_N the first N entries of w_{N+1}, so padding the maximiser of the order-N problem with a zero gives a competitor for the order-$(N+1)$ problem with the same value; hence $w_N^{\mathsf{T}}T_N^{-1}w_N \leq w_{N+1}^{\mathsf{T}}T_{N+1}^{-1}w_{N+1}$, which is the assertion after subtracting from κ_2. The last sentence follows with Theorem 10.1. This is formalised over $\mathbb{R}$ for an arbitrary real sequence κ; a one-dimensional negative control ($T = (-1)$, $w = u = 1$, where the bound reads $3 \leq -1$) is certified alongside it, so the positive semidefiniteness hypothesis is not decoration. □

Theorem 10.4 (one negative square, and the trailing determinants)**.**

(1) *For $p = 1$ and for $p = 6/5$ the pivot sign word of $H_{18}(\boldsymbol{\gamma}_p)$ is $+^{13}-+^{5}$; that is, $D_0, \ldots, D_{12} > 0$ and $D_{13}, \ldots, D_{18} < 0$. At both shapes the trailing determinants satisfy $E_0, \ldots, E_{18} > 0$.*

(2) *For $p = 11/8$ the pivot sign word of $H_{22}(\boldsymbol{\gamma}_p)$ is $+^{15} - +^{7}$; that is, $D_0, \ldots, D_{14} > 0$ and $D_{15}, \ldots, D_{22} < 0$, so $\nu(11/8) = 15$. The pivot sign word of the trailing matrix $T_{23}(\boldsymbol{\gamma}_{11/8})$ is $+^{22}-$; that is, $E_0, \ldots, E_{21} > 0 > E_{22}$.*

In particular each of $H_{18}(\boldsymbol{\gamma}_1)$, $H_{18}(\boldsymbol{\gamma}_{6/5})$ and $H_{22}(\boldsymbol{\gamma}_{11/8})$ has exactly one negative square: their inertias are $(18, 1, 0)$, $(18, 1, 0)$ and $(22, 1, 0)$. And $D_{15}(\boldsymbol{\gamma}_1) < 0$, which is the negative 16×16 Hankel determinant of $(r_n)_{n\geq 2}$ recorded in [6, §5.3] *and credited to F. Lehner by a footnote in its §1.3.*

Proof sketch. Six exact rational eliminations of the kind described in the proof of Theorem 3.1, three on the leading matrices H_{18}, H_{18}, H_{22} and three on the trailing matrices T_{19}, T_{19}, T_{23}, on the moment sequence $m_n = (p)_n$. The pivot signs of an unpivoted symmetric elimination are the diagonal of a congruence $H = LDL^{\mathsf{T}}$ with L unit lower triangular, so Sylvester's law of inertia turns each word directly into the inertia claim; a word $+^{\nu} - +^{k}$ says $D_0, \ldots, D_{\nu-1} > 0$ and $D_{\nu}, \ldots, D_{\nu+k} < 0$, since after the single negative pivot every further positive pivot keeps the determinant negative. The largest of the six, order 22 at $p = 11/8$, peaks at about 4.2 GB of kernel memory.

What was a search is the shape 11/8. An exact-integer scan of all 127 rationals $p = a/b \in (\frac{1}{2}, \frac{3}{2})$ with $b \leq 20$ found no negative trailing determinant below index 22, and index 22 at exactly eight of them; 11/8 has the smallest ν among those eight and hence the cheapest check. None of the certified statements depends on the scan. □

Remark 10.5. Part (2) is the negative control that Theorem 10.3 needs. At $p = 11/8$ its hypothesis holds exactly through order 21 and fails at 22 — which still covers the whole firing transition at $\nu = 15$ — and yet the exclusion is observed to persist past order 22, which the theorem does not explain. At the FID endpoint $p = 1/2$, by contrast, both words are all-positive through order 18, so nothing fires and nothing to explain remains; that computation is also recorded in Proposition 8.1(1). Two further controls certify that the firing predicate is not vacuous: it is false at $p = 1/2$, and false at $p = 1$ when the first failing order is misdeclared as 12. The shape 11/8 lies inside $\mathcal{I}$, where non-FID was already known, so part (2) adds no new non-FID shape; it is a statement about the *shape of the sign word*, not about divisibility.

Remark 10.6. Outside the certified statements, the same two words were computed in exact integer arithmetic at each of the thirteen shapes of Table 2 and at $p = 1$, to order $N = 32$. At every one of them the leading pivot word is $+^{\nu}-^{33-\nu}$, and at every one of them the first negative trailing determinant, when there is one below order 32, occurs at an index strictly larger than ν; the earliest such index found anywhere is 22, at $p = 7/5$. So the monotone-firing hypothesis of Theorem 10.3 holds, on this evidence, across the whole transition at every shape examined. These are measurements, not certificates.

11. Completeness of the hierarchy, and the answer to the open question

This section is different in kind from the rest of the paper and is separated for that reason. Its proofs are classical analysis, written out in full but *not* machine-checked; every theorem, lemma and corollary in it carries that label in its heading. Every ingredient is a published theorem, and Remark 11.2 says which. What is new here is the answer to the question of [10], not the technique that answers it.

Throughout, μ is a Borel probability measure on $\mathbb{R}$ with moments m_n of all orders and free cumulants κ_n given by (8), and H_N, D_N are as in (1).

Theorem 11.1 (completeness of the hierarchy; not machine-checked)**.** *Let μ have moments of all orders and let its moment sequence be determinate in the Hamburger sense. Then*

$$\mu \text{ is FID} \qquad \Longleftrightarrow \qquad H_N(\mu) \succeq 0 \text{ for every } N \geq 0.$$

Proof. Two published inputs are used.

(BV) By [4], μ is FID if and only if $-\varphi_\mu$ extends to a Pick function on $\mathbb{C}^+$ — taken here in the form [6, Theorem 2.4] quotes from [4], "$-\phi_\mu$ extends to a Pick function, i.e. an analytic map from $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$"; a Pick function has a Nevanlinna representation, and the linear term vanishes here because $\varphi_\mu(z) \to \kappa_1$ at infinity, μ having a first moment. So μ is FID exactly when there is a real γ and a finite positive measure σ with

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1 + xz}{z - x}\, d\sigma(x),$$

and conversely every such pair (γ, σ) is the Voiculescu transform of a unique probability measure, which is then FID. Writing $d\tau := (1 + x^2)\, d\sigma$ and $w = 1/z$, this is the free Lévy–Khintchine form

$$R_\mu(w) = \kappa_1 + \int_{\mathbb{R}} \frac{w}{1 - xw}\, d\tau(x), \tag{12}$$

whose expansion at $w = 0$ has coefficients $\int x^{m-2}\, d\tau$ in degree $m - 1$, $m \geq 2$.

(BG) By [3, Theorem 1.3(a)], if μ admits a p-th moment then R_μ admits the Taylor expansion $R_\mu(z) = \sum_{i=0}^{p-1} \kappa_{i+1}(\mu) z^i + o(z^{p-1})$, the coefficients being the combinatorial free cumulants of (8). Write $\nu_{\boxplus}^{\gamma,\sigma}$ — the notation of [3], this superscripted ν having nothing to do with the gamma ceiling — for the FID law with pair (γ, σ). Then by [3, Proposition 2.3]: (1) if σ has a moment of order p then so does $\nu_{\boxplus}^{\gamma,\sigma}$; and (2) if in addition p is even — the only case used here — the converse holds. That σ is what [3] means by the Lévy measure is its own definition: in [3, §2], immediately after the pair (γ, σ) is introduced and shown unique, "the measure σ is said to be the Lévy measure of $\nu_*^{\gamma,\sigma}$ and $\nu_{\boxplus}^{\gamma,\sigma}$".

($\Rightarrow$) Let μ be FID with all moments. By (BV) it has a pair (γ, σ); by [3, Proposition 2.3(2)] the measure σ has moments of every even order, hence of every order, and so does the finite positive measure $\tau = (1 + x^2)\sigma$. By [3, Theorem 1.3(a)] the coefficients of (12) are then the free cumulants, so

$$\kappa_m = \int_{\mathbb{R}} x^{m-2}\, d\tau \qquad (m \geq 2), \tag{13}$$

and therefore $H_N = \big(\int x^{i+j}\, d\tau\big)_{i,j=0}^{N}$ is a moment matrix of a positive measure, hence positive semidefinite, for every N.

($\Leftarrow$) Suppose $H_N \succeq 0$ for every N. Any finitely supported vector is supported in $\{0, \dots, N\}$ for some N, so the Hankel form of the sequence $s_n := \kappa_{n+2}$ is positive semidefinite, and Hamburger's theorem [1] supplies a positive measure τ on $\mathbb{R}$ with $\int x^n\, d\tau = s_n$ for all $n \geq 0$. It is finite, $\tau(\mathbb{R}) = s_0 = \kappa_2$, and has moments of all orders, the odd absolute ones by Cauchy–Schwarz. Put $d\sigma := d\tau/(1+x^2)$, which is finite since $\sigma \leq \tau$, and $\gamma := \kappa_1 - \int x\, d\sigma$, which is finite since $|x|/(1+x^2) \leq \frac{1}{2}$. By the converse half of (BV) there is an FID probability measure μ' with pair (γ, σ). Its Lévy measure has moments of all orders — for $m \geq 2$, $\int_{|x|>1} |x|^m\, d\sigma \leq \int_{|x|>1} |x|^m\, d\tau < \infty$ — so by [3,

Proposition 2.3(1)] μ' has moments of all orders, and (13) applied to μ' gives $\kappa_m(\mu') = \int x^{m-2}\,d\tau = \kappa_m(\mu)$ for $m \geq 2$, while $\kappa_1(\mu') = \gamma + \int x\,d\sigma = \kappa_1(\mu)$. The moment–cumulant relation (8) is a bijection between sequences, so μ' and μ have the same moment sequence; μ is determinate, so $\mu' = \mu$, and μ is FID. □

Remark 11.2 (what is classical here). For *compactly supported* μ the equivalence of Theorem 11.1 is [8, Theorem 13.16] — the criterion [10] cites in its Introduction and uses, in the necessary direction and with the compactness hypothesis, as its own Lemma 2.1. That theorem (p. 220 of the book) is a four-way equivalence for "a probability measure μ on $\mathbb{R}$ with compact support", and its first two clauses are "μ is infinitely divisible" and "the sequence $(\kappa_n)_{n\geq 1}$ of free cumulants of μ is conditionally positive definite"; its third is the compactly supported free Lévy–Khintchine form $\mathcal{R}(z) = \kappa_1 + \int_{\mathbb{R}} z(1-xz)^{-1}\,d\rho(x)$ with ρ finite and compactly supported, which is (12) in that case. "Conditionally positive definite" is defined there (Notation 13.10) as $\sum_{n,m=1}^{r} \alpha_n \bar{\alpha}_m \kappa_{n+m} \geq 0$ for all r and all $\alpha \in \mathbb{C}^r$, which is exactly $H_N \succeq 0$ for every N. The hypothesis there is on the measure and nothing else: no algebra of operators and no state enters the statement. At the level of states on $\mathbb{R}\langle x\rangle$, with "freely infinitely divisible" meaning that the functional ϕ^t with $R_{\phi^t} = tR_\phi$ is positive for every $t > 0$, the corresponding statement is in [2]. What the proof above supplies is the measure-level statement for unbounded support, and determinacy is exactly what carries the moment-level conclusion back to measures. We claim no novelty for Theorem 11.1, and it is not only its ingredients that are published: the statement itself is asserted, without proof, in footnote 2 to Remark 3.2 of [5] (p. 373), where it reads

> "If a measure μ has a compact support, the free infinite divisibility is equivalent to the conditional nonnegative definiteness of free cumulants. This equivalence can be extended to a measure with finite moments of all orders when the moment problem is determinate."

One of the two authors of [5] is the author of [6]. What the proof above adds is the proof, and a statement in the form the rest of this section needs.

One dependency is taken in the form in which [3, §2] states it rather than from a numbered statement of [4]: the surjectivity half of (BV), that every pair (γ, σ) with σ finite positive *is* the Voiculescu transform of a probability measure. [3, §2] attributes the classification to [4] and states it in both directions, in these words (quoted from the arXiv version):

> "μ is $\boxplus$ (resp. $*$)-infinitely divisible if and only if there exists a real number γ and a positive finite measure on the real line (abbreviated from now on into p.f.m.) σ such that $R_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{z+t}{1-tz}\,d\sigma(t)$ [...]. Moreover, in this case, such a pair (γ, σ) is unique, and we denote μ by $\nu_{\boxplus}^{\gamma,\sigma}$ (resp. $\nu_*^{\gamma,\sigma}$)."

Surjectivity is then the Bercovici–Pata map $\nu_*^{\gamma,\sigma} \mapsto \nu_{\boxplus}^{\gamma,\sigma}$ of [3, §2] being defined on *all* $*$-infinitely divisible laws, which is how that paper uses it. We did not consult [4] itself: every use of it in this paper is quoted from [6] or from [3], and no statement of it is cited here by number. Readers who want the primary form should look at Theorem 5.10 of [4], which is what the literature cites for this correspondence; we have not checked what that theorem's own statement contains.

Lemma 11.3 (endpoint positivity; not machine-checked)**.** *For every* $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$ *and every* $N \geq 0$, $D_N(\boldsymbol{\gamma}_p) > 0$.

Proof. $\boldsymbol{\gamma}_p$ is FID there by [6, Corollary 1.3(1)], so $H_N \succeq 0$ and $D_N \geq 0$ by the forward half of Theorem 11.1. Suppose $D_N = 0$ with N least. Then H_{N-1} is positive definite and H_N is positive semidefinite and singular; take $c \neq 0$ in its kernel. Its last coordinate is nonzero, since otherwise the truncation of c would lie in the kernel of H_{N-1}. For every $M > N$, padding c with zeros gives a vector $\tilde{c}$ with $\tilde{c}^{\mathsf{T}} H_M \tilde{c} = c^{\mathsf{T}} H_N c = 0$, and a positive semidefinite form vanishing at a vector has that vector in its kernel, so $H_M \tilde{c} = 0$; reading off coordinates, $\sum_{k=0}^{N} c_k \kappa_{i+k+2} = 0$ for $0 \leq i \leq M - N$. Letting $M \to \infty$, the sequence (κ_{n+2}) satisfies a linear recurrence of order $N+1$ with nonzero leading coefficient, hence $|\kappa_n| \leq C(n+1)^N R^n$ for some C, R. Then $C(z) = 1 + \sum_{n\geq 1} \kappa_n z^n$ is analytic near

0, and the functional equation $M(z) = C\big(zM(z)\big)$ has, by the analytic implicit function theorem applied to $F(z,w) = C(zw) - w$ at $(0,1)$ where $\partial_w F = -1 \neq 0$, an analytic solution near 0, which must be the formal series M. So $m_n = (p)_n$ would grow at most geometrically, which is false. □

Corollary 11.4 (the gamma law; not machine-checked)**.** *Write $\nu'(p)$ for the least N with $H_N(\boldsymbol{\gamma}_p) \not\succeq 0$, and ∞ if there is none. Then*

$$\nu'(p) < \infty \qquad \Longleftrightarrow \qquad \boldsymbol{\gamma}_p \text{ is not FID.}$$

In particular $\nu'(p) < \infty$ for every $p \in \mathcal{I}$ and $\nu'(p) = \infty$ for $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$. Moreover $\nu'(p) = \nu(p)$ whenever $D_j(\boldsymbol{\gamma}_p) \neq 0$ for all $j < \nu'(p)$, and that holds for all but countably many $p > 0$.

Proof. The moments $m_n(\boldsymbol{\gamma}_p) = (p)_n$ satisfy $\sum_n m_{2n}^{-1/2n} = \infty$, so the moment sequence of $\boldsymbol{\gamma}_p$ is determinate by Carleman's criterion [1], and Theorem 11.1 applies; the two "in particular" clauses are then [6, Corollary 1.3(1)]. For the last sentence, suppose $D_j > 0$ for all $j < N_0$ where $N_0 = \nu'(p)$. Then H_{N_0-1} is positive definite by Sylvester's criterion, so by eigenvalue interlacing H_{N_0} has at most one non-positive eigenvalue; that eigenvalue cannot be 0, since then H_{N_0} would be positive semidefinite, so $D_{N_0} < 0$ and $\nu(p) = N_0$. Finally each $p \mapsto D_j(\boldsymbol{\gamma}_p)$ is a polynomial with integer coefficients — immediate by induction from (8) and $m_n = (p)_n \in \mathbb{Z}[p]$ — and it is not identically zero because $D_j(\boldsymbol{\gamma}_{1/2}) > 0$ by Lemma 11.3; so $\bigcup_j \{p : D_j(\boldsymbol{\gamma}_p) = 0\}$ is countable. □

Theorem 11.5 (endpoint escape; not machine-checked)**.** *For every N there is $\varepsilon_N > 0$ such that $D_j(\boldsymbol{\gamma}_p) > 0$ for all $j \le N$ and all p with $|p - \frac{1}{2}| < \varepsilon_N$; consequently $\nu(p) > N$ and $\nu'(p) > N$ there. The same holds at $\frac{3}{2}$.*

Proof. Each $p \mapsto D_j(\boldsymbol{\gamma}_p)$ is a polynomial, hence continuous, and $D_j(\boldsymbol{\gamma}_{1/2}) > 0$ for $j \le N$ by Lemma 11.3; take ε_N to be the smallest of the finitely many continuity radii. Sylvester's criterion then makes H_j positive definite for $j \le N$, so $\nu'(p) > N$. □

Theorem 11.6 (ν is unbounded; not machine-checked)**.** $\sup\{\nu(p) : p > 0,\ \nu(p) < \infty\} = \infty$. *Consequently, by Lemma 2.5, for infinitely many orders N there is a beta law $\boldsymbol{\beta}_{p,q}$ that is a witness at order N: the leading Hankel hierarchy strictly enlarges the excluded region at infinitely many orders.*

Proof. Fix N. By Theorem 11.5 choose $\varepsilon_N > 0$ with $D_j(\boldsymbol{\gamma}_p) > 0$ for $j \le N$ on $(\frac{1}{2}, \frac{1}{2} + \varepsilon_N)$, shrinking so that $\frac{1}{2} + \varepsilon_N \le \frac{2}{3}$. That interval is uncountable, while $\bigcup_j \{p : D_j(\boldsymbol{\gamma}_p) = 0\}$ is countable by Corollary 11.4, so pick p in it with $D_j(\boldsymbol{\gamma}_p) \neq 0$ for every j. Since $(\frac{1}{2}, \frac{2}{3}) \subset \mathcal{I}$, [6, Corollary 1.3(1)] makes $\boldsymbol{\gamma}_p$ non-FID, so $\nu'(p) < \infty$ by Corollary 11.4, and then $\nu(p) = \nu'(p) < \infty$; while $\nu(p) > N$ by Theorem 11.5. As N was arbitrary, ν takes arbitrarily large finite values. For the consequence, let p be such a shape and $N = \nu(p)$; then Lemma 2.5 gives a q_0 beyond which $\boldsymbol{\beta}_{p,q}$ is a witness at order N. □

Remark 11.7. Theorem 11.6 answers the first clause of the sentence quoted from the Discussion of [10] in Section 1, at *infinitely many* orders. It does not answer it at *every* order: the two facts available — that ν takes every value from 13 to 25 (Theorem 5.1) and that it takes arbitrarily large values (Theorem 11.6) — do not combine into every order, and the gap between them is a real one. The same Discussion of [10] ends by suggesting that a full classification "requires either all-order asymptotics or a direct analytic study of the Voiculescu transform". Theorem 11.1 is the all-order statement, and it turns out to be available from the criterion [10] already cites, once determinacy is supplied.

Remark 11.8 (Hasebe's conjecture, reframed). Corollary 11.4 makes [6, Conjecture 5.6(3)] — that $\boldsymbol{\gamma}_p$ is not FID for every $p \in (\frac{1}{2}, \frac{3}{2})$ — *equivalent* to the statement that the leading Hankel hierarchy of $\boldsymbol{\gamma}_p$ fails at some finite order for every such p, i.e. that $\nu'(p) < \infty$ on $(\frac{1}{2}, \frac{3}{2})$, and hence (off a countable exceptional set) that $\nu(p) < \infty$ there. So at each of the 28 shapes of Theorem 6.1 the

failure of the hierarchy at a finite order is not merely one sufficient certificate of non-FID among others: it is entailed by non-FID, whatever the route to it. Conversely, a single $p \in (\frac{1}{2}, \frac{3}{2})$ with $H_N(\boldsymbol{\gamma}_p) \succeq 0$ for every N would disprove the conjecture.

It is worth being precise about which direction Section 12 uses. Its proof of the conjecture, Corollary 12.8, needs only the elementary implication "FID $\Rightarrow H_N \succeq 0$" of Lemma 2.1, so it rests on nothing in this section. What the equivalence adds is the assurance that the instrument could not have failed for a bad reason: if the hierarchy of $\boldsymbol{\gamma}_p$ had been positive semidefinite at every order for some $p \in (\frac{1}{2}, \frac{3}{2})$, that would have refuted the conjecture rather than merely put it out of reach of this method. So if the conjecture is true, then at every single shape some order fires; that one vector fires on a whole interval of shapes at once is an extra, and it is what Section 12 supplies.

Remark 11.9 (no previous bound on ν). It is worth saying why ν was not already known to be finite anywhere except at $p = 1$. The non-FID half of [6, Corollary 1.3(1)] is a consequence of that paper's Theorem 5.1 — "[t]he assertion on non free infinite divisibility of $\boldsymbol{\gamma}_p$ is not a consequence of Theorem 1.2, but of Theorem 5.1", it says — and the proof of Theorem 5.1 is purely analytic: it continues the Cauchy transform onto a helix-like Riemann surface, reads its asymptotics at 0, and analyses the compositional inverse of F_μ to produce a point $w_\alpha \in (0, \infty)$ with $\varphi_\mu(w_\alpha) \in \mathbb{C}^+$, contradicting the Bercovici–Voiculescu characterisation (his Theorem 2.4, quoted there from [4]). The hypothesis $\alpha \in \mathcal{I}$ is used only in that last step, and only to fix a parity of the sheet index: the proof splits into the cases $\alpha \in (1 - \frac{1}{2n}, 1 - \frac{1}{2n+1})$ and $\alpha \in (1 + \frac{1}{2n+1}, 1 + \frac{1}{2n})$, which is exactly $\alpha \in \mathcal{I}$, and of the first [6] notes that it "is equivalent to $(2n-1)\pi < \theta(\alpha) < 2n\pi$", where $\theta(\alpha) = \big(|\alpha - 1|^{-1} - 1\big)\pi$; that inequality is what puts the point produced by the argument on a sheet lying over the upper half plane, and the author's own footnote "Here the assumption $\alpha \in \mathcal{I}$ is used" is attached to precisely that step. No Hankel determinant appears anywhere in that proof, and it yields no bound on the order at which a Hankel test would fire; [6] computes Hankel determinants only in §5.3 (a footnote in §1.3 points forward to it), and there only at $p = 1$ and for the beta and beta-prime families. So the finiteness of ν' throughout $\mathcal{I}$, which Corollary 11.4 supplies, restates nothing in [6].

Remark 11.10 (the second clause of the question is already settled). The beta law $\boldsymbol{\beta}_{p,q}$ is supported on $[0, 1]$, hence compactly supported, and for compactly supported measures [8, Theorem 13.16] is an equivalence: such a measure is FID if and only if its free cumulants are conditionally positive definite, i.e. if and only if $H_N \succeq 0$ for every N. [10] cites that equivalence in its Introduction — "we use the equivalent conditional-positive-definiteness criterion for free cumulants of compactly supported measures", with a pointer to Chapter 13 of [8] — and uses it, in the necessary direction only, as its own Lemma 2.1. Read in the other direction its contrapositive says exactly

> every non-FID beta law fails $H_N \succeq 0$ at some finite N — that is, is eventually separated by a finite Hankel matrix.

This is the second clause of the sentence quoted in Section 1, which [10] lists among what its table does not establish. It is not open; it follows from the criterion the paper itself cites. We record this in the interest of accuracy and with no criticism intended of a result whose first clause is genuinely deep. One caveat belongs with it: the determinant form is not immediate. Failure of $H_N \succeq 0$ gives a negative *principal* minor, and that the *leading* one is negative needs the non-degeneracy argument in the proof of Corollary 11.4. The tests of [10] are stated as $H_N \succeq 0$, so the sentence as written is settled.

Remark 11.11 (a falsification test; measurement). Theorem 11.1 together with [6, Corollary 1.3(1)] does not merely assert something about $(\frac{1}{2}, \frac{2}{3})$: it forces $\nu'(p) < \infty$ at *every* $p \in \mathcal{I}$. A single $p \in \mathcal{I}$ with all $D_N(\boldsymbol{\gamma}_p) > 0$ up to a large order would be strong evidence against it. We tested the mediant of each of the first sixteen intervals of $\mathcal{I}$ — the simplest rational strictly inside — in exact integer arithmetic to order 45. Below 1 the values are $\nu(3/5) = 24$, $\nu(7/9) = 16$, $\nu(11/13) = 15$ and then 14 at each of 15/17, 19/21, 23/25, 27/29, 31/33; above 1 they are $\nu(7/5) = 16$ and then 13 at each of 11/9, 15/13, 19/17, 23/21, 27/25, 31/29, 35/33. No refutation: ν is finite at all sixteen and

settles at 14 approaching 1 from below and at 13 from above. That is also the first quantitative picture of ν on the accumulation of $\mathcal{I}$ at 1 — the ceiling is flat there, and its growth is entirely an endpoint phenomenon. All sixteen shapes lie inside $\mathcal{I}$, where non-FID was already known, so they are consistency data and not new non-FID instances; and they are measurements, computed outside the formal development.

12. One vector for an interval of shapes

Every certificate so far fixes a rational parameter and computes a determinant, and a shape-by-shape computation can never exhaust an interval. This section changes the certificate. The failure of $H_N \succeq 0$ has a witness that is one vector rather than one elimination, and a vector is the kind of object that can be kept fixed while the law moves.

The quadratic form of a fixed vector is a polynomial in the shape.

Lemma 12.1. *For every $n \geq 1$ the free cumulant $\kappa_n(\boldsymbol{\gamma}_p)$ is a polynomial in p with integer coefficients. Consequently, for every $N \geq 0$ and every $u \in \mathbb{Z}^{N+1}$, the quadratic form Q of* (5) *lies in $\mathbb{Z}[p]$.*

Proof. The moments $m_n(\boldsymbol{\gamma}_p) = (p)_n = \prod_{i<n}(p+i)$ lie in $\mathbb{Z}[p]$, and the recursion (8) expresses κ_n as an integer polynomial in $m_1, \dots, m_n$ and $\kappa_1, \dots, \kappa_{n-1}$; induct on n. The second claim is then immediate from (5), all u_i being integers. (This is the same observation used at the end of the proof of Corollary 11.4, there applied to D_j rather than to Q.) □

Remark 12.2 (degrees; measurement). Outside the certified statements, computed exactly for $n \leq 42$: $\kappa_n(\boldsymbol{\gamma}_p)$ has zero constant term, all its other coefficients are *positive* integers, the linear one is $(n-1)!$, and $\deg_p \kappa_n(\boldsymbol{\gamma}_p) = \lfloor n/2 \rfloor$ for $2 \leq n \leq 42$ (with $\kappa_1 = p$, the only index at which that formula fails); the first values are $\kappa_1 = \kappa_2 = p$, $\kappa_3 = 2p$, $\kappa_4 = 6p + p^2$, $\kappa_5 = 24p + 10p^2$, $\kappa_6 = 120p + 82p^2 + 4p^3$. Two consequences of that measurement, neither of them needed below but both worth recording: for $n \leq 42$, $\kappa_n(\boldsymbol{\gamma}_p) > 0$ at every real $p > 0$, which is what one would expect and which no computation in this paper had established off the finitely many tested shapes; and $\deg Q \leq N + 1$ in (5) for $N \leq 20$, so the certificate polynomial is small — degree 14 at $N = 13$, degree 21 at $N = 20$ — even though the matrix is not. The certificates below do not use the bound: each supplies its Q and its degree explicitly, and the identity that is checked would fail if either were wrong.

Bernstein certificates. The sign of Q on a closed interval is certified in the standard way of real algebraic geometry, by exhibiting the polynomial in the Bernstein basis of that interval with coefficients of one sign. We state the form that is used, for the reader's convenience and because it is what is formalised; there is nothing new in it.

Proposition 12.3 (the engine)**.** *Let $N \geq 0$, let $u \in \mathbb{Z}^{N+1}$, and let $Q \in \mathbb{Z}[X]$ be the polynomial* (5) *attached to u. Let $a < b$ and $c > 0$ be integers, let $d \geq 0$, and suppose integers $B_0, \dots, B_d$ satisfy*

$$c^d (b-a)^d\, Q(X) \;=\; \sum_{i=0}^{d} B_i\, (cX - a)^i\, (b - cX)^{d-i} \qquad \text{in } \mathbb{Z}[X], \qquad B_i < 0 \text{ for all } i. \tag{14}$$

Then $Q(p) < 0$, and hence $H_N(\boldsymbol{\gamma}_p) \not\succeq 0$ and $\boldsymbol{\gamma}_p$ is not FID, for every real p *with $\frac{a}{c} \leq p \leq \frac{b}{c}$.*

Proof. Fix such a p and put $X_0 = cp - a \geq 0$, $Y_0 = b - cp \geq 0$; then $X_0 + Y_0 = b - a > 0$. Every term $B_i X_0^i Y_0^{d-i}$ of the right-hand side of (14) at $X = p$ is ≤ 0, and at least one is < 0: if $X_0 > 0$ the term $i = d$ is $B_d X_0^d < 0$, and if $X_0 = 0$ then $Y_0 = b - a > 0$ and the term $i = 0$ is $B_0 Y_0^d < 0$. So the right-hand side is negative, and $c^d(b-a)^d > 0$ forces $Q(p) < 0$. Since $Q(p) = u^{\mathsf{T}} H_N(\boldsymbol{\gamma}_p) u$ with u real, the matrix $H_N(\boldsymbol{\gamma}_p)$ is not positive semidefinite, and $\boldsymbol{\gamma}_p$ — which has moments of all orders for every $p > 0$ — is not FID by Lemma 2.1 in the finite-moment form of Remark 2.2. □

Two things make this usable. The hypothesis (14) is an identity between integer polynomials and a list of integer signs, so it is decidable outright: no root isolation, no interval arithmetic, no analysis. And it is complete in a weak but sufficient sense — the Bernstein coefficients of a polynomial negative on $[\frac{a}{c}, \frac{b}{c}]$ need not all be negative, but degree elevation and subdivision preserve the certificate and eventually produce one, so a failure is a failure of the first attempt and not of the method. Neither certificate below needed subdivision.

Remark 12.4 (where the vector comes from). Suppose $\nu(p_0) = N$, so that $D_{N-1}(\boldsymbol{\gamma}_{p_0}) > 0 > D_N(\boldsymbol{\gamma}_{p_0})$, and let u be the last column of the adjugate of $H_N(\boldsymbol{\gamma}_{p_0})$. Then $H_N u = D_N e_N$, so

$$u^{\mathsf{T}} H_N(\boldsymbol{\gamma}_{p_0})\, u \;=\; D_N(\boldsymbol{\gamma}_{p_0})\, u_N \;=\; D_N(\boldsymbol{\gamma}_{p_0})\, D_{N-1}(\boldsymbol{\gamma}_{p_0}) \;<\; 0,$$

the last cofactor u_N being D_{N-1}. So a shape at which the hierarchy first fails at order N hands over a vector that certifies the failure, and continuity in p does the rest. The vector is cleared to integers by the diagonal congruence that rescales $\boldsymbol{\gamma}_{P/D}$ by D, under which κ_n is multiplied by D^n and H_N by a congruence, so that the moments $\prod_{i<n}(P + iD)$ and every minor are integers. None of this enters the certified statements: (14) takes u as data, and a wrong u would simply fail the identity.

The two certificates.

Theorem 12.5 (an interval around the exponential law)**.** *For every real p with $\frac{49}{50} \le p \le \frac{6}{5}$, the free-cumulant Hankel matrix $H_{13}(\boldsymbol{\gamma}_p) = \big(\kappa_{i+j+2}(\boldsymbol{\gamma}_p)\big)_{i,j=0}^{13}$ is not positive semidefinite. Hence $\boldsymbol{\gamma}_p$ is not freely infinitely divisible for any such p.*

Proof sketch. Proposition 12.3 with $N = 13$, $(a, b, c) = (49, 60, 50)$ and $d = 14$. The vector u is the one of Remark 12.4 at $p_0 = 1$, where $\nu(1) = 13$ by Proposition 7.1: fourteen integers of 104 to 117 digits. The Bernstein data $B_0, \dots, B_{14}$ are fifteen integers of 252 to 257 digits, all negative, and the identity certified is

$$50^{14} \cdot 11^{14}\, Q(X) \;=\; \sum_{i=0}^{14} B_i\, (50X - 49)^i\, (60 - 50X)^{14-i}.$$

What is checked in one evaluation is: the cumulant polynomials $\kappa_1, \dots, \kappa_{28}$ of $\boldsymbol{\gamma}_p$ over $\mathbb{Z}[p]$, computed from $m_n = (p)_n$ by (8); the contraction (5) of the resulting H_{13} against u; the polynomial identity above; the fifteen signs; and two controls, $Q(\frac{1}{2}) > 0$ and $Q(\frac{3}{2}) > 0$. All of it is decided by the proof kernel on integer arithmetic (Section 13). □

Corollary 12.6. *Infinitely many of the closed gaps of $\mathcal{I}$ consist entirely of shapes at which $\boldsymbol{\gamma}_p$ is not FID: all of $\big[\frac{2k}{2k+1}, \frac{2k+1}{2k+2}\big]$ with $k \ge 25$ and all of $\big[\frac{2n+3}{2n+2}, \frac{2n+2}{2n+1}\big]$ with $n \ge 2$.*

Proof. Consecutive intervals of the first union in (3) leave the gap $[\frac{2k}{2k+1}, \frac{2k+1}{2k+2}]$, and $\frac{2k}{2k+1} \ge \frac{49}{50}$ exactly when $k \ge 25$; consecutive intervals of the second union leave the gap $[\frac{2n+3}{2n+2}, \frac{2n+2}{2n+1}]$, and $\frac{2n+2}{2n+1} \le \frac{6}{5}$ exactly when $n \ge 2$. Both families of gaps therefore lie inside $[\frac{49}{50}, \frac{6}{5}]$, where Theorem 12.5 applies. □

Theorem 12.7 (the interval $[\frac{2}{3}, \frac{4}{3}]$)**.** *For every real p with $\frac{2}{3} \le p \le \frac{4}{3}$, the matrix $H_{20}(\boldsymbol{\gamma}_p)$ is not positive semidefinite. Hence $\boldsymbol{\gamma}_p$ is not freely infinitely divisible for any such p.*

Proof sketch. Proposition 12.3 with $N = 20$, $(a, b, c) = (2, 4, 3)$ and $d = 21$. The vector is the one of Remark 12.4 at $p_0 = \frac{2}{3}$, where $\nu(\frac{2}{3}) = 20$ by Theorem 4.1(3): twenty-one integers of 508 to 531 digits. The Bernstein data are twenty-two negative integers of 1064 to 1073 digits, and the identity certified is

$$3^{21} \cdot 2^{21}\, Q(X) \;=\; \sum_{i=0}^{21} B_i\, (3X - 2)^i\, (4 - 3X)^{21-i},$$

checked together with the same two controls $Q(\frac{1}{2}) > 0$, $Q(\frac{3}{2}) > 0$ after the cumulant polynomials $\kappa_1, \dots, \kappa_{42}$ have been computed over $\mathbb{Z}[p]$. This is the one computation in the paper that the proof

kernel does not carry out itself: the $\mathbb{Z}[p]$ recursion to depth 42 is estimated at 50 GB inside the kernel, and the certificate is checked by compiled evaluation instead, at the cost of one further axiom (Section 13). Theorem 12.5 is the same statement at the largest order the kernel can afford. □

Corollary 12.8 ([6, Conjecture 5.6(3)], and the classification)**.** *$\boldsymbol{\gamma}_p$ is not freely infinitely divisible for any $p \in (\frac{1}{2}, \frac{3}{2})$. Consequently $\boldsymbol{\gamma}_p$ is freely infinitely divisible if and only if $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$.*

Proof. Taking $n = 1$ in each union of (3) gives $(\frac{1}{2}, \frac{2}{3}) \subset \mathcal{I}$ and $(\frac{4}{3}, \frac{3}{2}) \subset \mathcal{I}$, so [6, Corollary 1.3(1)] makes $\boldsymbol{\gamma}_p$ non-FID for $p \in (\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2})$, while Theorem 12.7 covers $[\frac{2}{3}, \frac{4}{3}]$. These three sets exhaust $(\frac{1}{2}, \frac{3}{2})$. The second sentence adds the positive half of [6, Corollary 1.3(1)], that $\boldsymbol{\gamma}_p$ is FID on $(0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$. □

Corollary 12.9 (where the hierarchy fires)**.** *With ν' as in Corollary 11.4, and p ranging over $(0, \infty)$, one has $\nu'(p) < \infty$ exactly for $p \in (\frac{1}{2}, \frac{3}{2})$. On $[\frac{2}{3}, \frac{4}{3}]$ the bound is uniform and certified, $\nu'(p) \le 20$; on $[\frac{49}{50}, \frac{6}{5}]$ it is $\nu'(p) \le 13$; on the two remaining intervals $(\frac{1}{2}, \frac{2}{3})$ and $(\frac{4}{3}, \frac{3}{2})$ finiteness comes from Corollary 11.4, hence from the material of Section 11, which is not machine-checked.*

Proof. The two certified bounds are Theorems 12.7 and 12.5. For $p \in (\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2}) \subset \mathcal{I}$, non-FID is [6, Corollary 1.3(1)] and finiteness of ν' is then Corollary 11.4. For $p \notin (\frac{1}{2}, \frac{3}{2})$, $\boldsymbol{\gamma}_p$ is FID by the same corollary of [6], so $H_N \succeq 0$ for every N by Theorem 11.1 and $\nu'(p) = \infty$. □

Remark 12.10 (why order 20, and why this interval). The order is forced by the left endpoint. Because $\nu(\frac{2}{3}) = 20$ (Theorem 4.1(3)), $H_N(\boldsymbol{\gamma}_{2/3}) \succeq 0$ for every $N < 20$, so no vector at an order below 20 can certify anything at $p = \frac{2}{3}$; and $\frac{2}{3}$ has to be covered, because the first interval of $\mathcal{I}$ is $(\frac{1}{2}, \frac{2}{3})$, open at that end. Order 20 is therefore the smallest that can close the conjecture this way, and the certificate attains it. Outside the certified statements, a search over all rational reference shapes $a/b \in (\frac{1}{2}, \frac{3}{2})$ with $b \le 20$ and all orders $13 \le N \le 22$ produced exactly one certificate covering $[\frac{2}{3}, \frac{4}{3}]$: the one above.

The kernel-checked interval $[\frac{49}{50}, \frac{6}{5}]$ is likewise close to the best its vector can do. The degree-14 polynomial Q of Theorem 12.5 has roots near 0.97271 and 1.20981 — decimals reported for orientation only, and not part of any certified statement — so $[\frac{49}{50}, \frac{6}{5}]$ is nearly the whole interval on which that Q is negative. Enlarging it means raising the order, and raising the order means running the $\mathbb{Z}[p]$ recursion deeper, which is precisely what the kernel cannot afford (Section 13).

Remark 12.11 (what the vector route gives up). A witness vector at order N shows that $H_N(\boldsymbol{\gamma}_p) \not\succeq 0$; it says nothing about smaller orders, so it yields no analogue of Definition 2.3 and no value of ν. The two instruments are complementary in exactly this way:

	leading-minor word	witness vector
what it certifies	$D_0, \dots, D_{N-1} > 0 > D_N$	$H_N(\boldsymbol{\gamma}_p) \not\succeq 0$
what that means	$\nu(p) = N$	$\nu'(p) \le N$
shapes per check	one rational	a closed interval of reals
work	elimination, $O(N^3)$ on exact rationals	one contraction, $O(N^2)$

For Hasebe's conjecture minimality is irrelevant, which is why the cheaper instrument settles it; for the questions of Sections 3–5, which are about *which* order fires first, it is useless.

Remark 12.12 (controls). Four checks stand behind these two theorems, all of them decided rather than argued. (i) The certificate machinery is exercised on a case where the answer is known and on a case where it must fail: the polynomial $X - 1$ with $B = (-2, -1)$ satisfies (14) on $[0, \frac{1}{2}]$ and is *rejected* on $[0, 2]$, where $X - 1$ is not negative; and a coefficient list with a non-negative entry is rejected outright. (ii) The two shape-parameter controls, $Q(\frac{1}{2}) > 0$ and $Q(\frac{3}{2}) > 0$, are part of the same certified evaluation as (14) itself: at the two endpoints of the range where [6, Corollary 1.3(1)] proves $\boldsymbol{\gamma}_p$ *is* FID, each witness vector gives a positive value, as it must — a negative one there would have contradicted a published theorem. (iii) The $\mathbb{Z}[p]$ cumulants, evaluated at $p = \frac{2}{3}$ and at

$p = 1$, are certified equal to the exact-rational cumulants that every other theorem in this paper uses, to depth 20; the polynomial layer is thus tied to the arithmetic layer and not merely to itself. (iv) Outside the formal development, each Q was recomputed by a second implementation from free cumulants obtained directly as fractions, at both endpoints and eleven interior points — thirteen exact agreements, all negative — and found positive at $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{2}, 2$, outside the interval.

Remark 12.13 (what this does not settle). Corollary 12.8 proves the statement of [6, Conjecture 5.6(3)]; it does not answer the neighbouring question of [6] about the *method*. His Theorem 5.1 is a criterion for non-FID read off the local form of a density: if the restriction of μ to $(x_0-\delta, x_0+\delta)$ has density $c(x-x_0)^{\alpha-1}(1+f(x))$ for $x > x_0$ and 0 for $x < x_0$, with f analytic and small in a sector in the sense stated there, and if $\alpha \in \mathcal{I}$, then μ is not FID. The exponent α is what the shape p becomes for $\boldsymbol{\gamma}_p$, and the hypothesis $\alpha \in \mathcal{I}$ is the same set (3). Immediately before Problem 5.7 the paper says "One may expect that the proof of Theorem 5.1 also applies to any $\alpha \in (\frac{1}{2}, \frac{3}{2})$, but just a slight modification seems not sufficient for that purpose", and Problem 5.7 then reads, in full, "Does Theorem 5.1 extend to arbitrary $\alpha \in (\frac{1}{2}, \frac{3}{2})$?" That is a question about the analytic argument, and nothing here bears on it. Nothing here says anything about the Cauchy transform of $\boldsymbol{\gamma}_p$, and a proof that ran through the Voiculescu transform would presumably also give the rate at which ν diverges (Remark 14.1), which this one does not. Nor does the interval method say anything about the beta laws, for which [6, Conjecture 5.6(1)] remains open; Section 14 says what would be needed.

13. Verification

Every claim of Sections 3–9, Section 10 and Section 12 — about the pivot word of a matrix, the exact value of a pivot at a given rational parameter, the value of one of the small determinants of Remark 10.2, the two structural identities of Section 10, or a Bernstein certificate together with the statement about an interval of real shapes that it supports — is verified in Lean 4 (toolchain `v4.32.0`): 159 theorems across 22 modules, beside a definitions module and an aggregator. Of these, 107 are closed by kernel evaluation (`decide +kernel`) on exact rational or integer arithmetic and three small Boolean identities (Proposition 8.1(3) and Remark 12.12(i)) by plain `decide`; exactly one, the order-20 certificate behind Theorem 12.7, is closed by compiled evaluation (`native_decide`); and the remaining 48 are proved against the Mathlib library — the structural results of Section 10, Theorem 10.1 over an arbitrary commutative ring and Theorem 10.3 over $\mathbb{R}$ for an arbitrary sequence of cumulants, together with the evaluation homomorphism and the interval argument of Section 12, which turn a certified identity between integer coefficient lists into a statement about the real matrix $H_N(\boldsymbol{\gamma}_p)$ and its positive semidefiniteness in the library's sense. What is *not* machine-checked is exactly what is announced as cited, elementary or classical: Lemma 2.1 and the limit (6), both quoted from the literature; the transfer Lemma 2.5 deduced from the latter; the continuity argument in Theorem 3.2(1); the textbook linear algebra of (9) that turns a certified pivot sign word into the signs of the leading minors $D_0, \dots, D_N$, and Sylvester's law of inertia, which turns the same word into the inertia claim of Theorem 10.4 — what the kernel decides at a given parameter is the pivot sign word itself, and everything said about determinants there is read off it through (9); in Section 12, Lemma 12.1 for general n (the certificates need only the finitely many cumulant polynomials they compute, and those they compute rather than assume), the adjugate identity of Remark 12.4, which explains the vectors but certifies nothing, the interval arithmetic of Corollary 12.6, and the half of Corollary 12.8 quoted from [6]; and the whole of Section 11, for which the library carries neither moment-problem theory nor free probability, so that formalising it is a programme rather than a check. The arithmetic layer's definitions module imports nothing at all — not Mathlib, not Batteries, not the standard library — and the modules that carry its theorems add only a small module defining an annotation attribute used for bookkeeping. It represents a rational number as a bare pair of arbitrary-precision integers kept coprime with positive denominator, and a polynomial in the shape as a bare list of integer coefficients, so that moments, cumulants, eliminations and the $\mathbb{Z}[p]$ recursion of Section 12 reduce inside the kernel without the normalisation of a library type. That layer holds

111 of the 159 theorems, and `#print axioms` on 110 of them reports that the theorem *does not depend on any axioms*: not `sorryAx`, and also not `propext`, `Classical.choice` or `Quot.sound`. The exception is the order-20 certificate, whose compiled-evaluation proof introduces one axiom of its own, asserting that the compiled evaluation of that Boolean returned true; Theorem 12.7 and Corollary 12.8 depend on it, and nothing else in this paper does. The 48 Mathlib-side theorems carry Mathlib's three axioms (`propext`, `Classical.choice`, `Quot.sound`) and, for the one just named, that fourth. There is no floating-point arithmetic in any statement or proof, and no decimal decides a sign: decimals appear in this paper only where a search is reported (Remarks 3.3, 4.2, 5.3, 12.10 and 14.1) or a resource cost is quoted, and every parameter that carries a theorem is a rational number or a rational endpoint, every sign attached to it re-derived exactly.

The whole development checks in about 53 minutes, that being the sum of the separately measured per-module elaboration times rather than one end-to-end run. Among the eliminations the heaviest is the $\nu = 25$ column of Theorem 5.1, which measured 5.4 GB of kernel memory on an idle machine and 6.5 GB inside the batch run; Theorem 10.4 at $p = 11/8$ measured 4.2 GB. The polynomial layer of Section 12 is heavier still: the module carrying the order-13 certificate and the controls of Remark 12.12(i) and (iii) measured 144 CPU-seconds and a peak of 7.31 GB, and the module that turns it into the statements about $H_N(\boldsymbol{\gamma}_p)$ over $\mathbb{R}$ measured 9.2 CPU-seconds and 2.89 GB. The reason Theorem 12.7 is not checked by the kernel is in those numbers. The $\mathbb{Z}[p]$ cumulant recursion to depth M costs about $M^5/120$ integer multiplications, since the truncated power $[z^j]M(z)^k$ has degree j in p; two kernel probes measured 2.0 GB at depth 20 and 7.7 GB at depth 28. Those two points are in the ratio 3.9, where the multiplication count would predict $(28/20)^5 = 5.4$: over that range memory grows like $M^{4.0}$, not like M^5, so the count is an upper bound on what memory does and not a fit to it. Extrapolating from the depth-28 probe to the depth 42 that order 20 requires gives 39 GB at the measured exponent and 58 GB at M^5; the "50 GB" quoted elsewhere in this paper is the order of magnitude those two bracket, not a measurement, and either of them is far past the budget available here. Order 13, at depth 28, gives the largest kernel-checked interval this method buys today; Section 14 prices two routes to a kernel-checked $[\frac{2}{3}, \frac{4}{3}]$.

Two independent implementations were used. All parameter searches — the 617-point grid and the bisections in q at $p = 7/10$ and $p = 2/3$, the scan of ν in p, and the scan of the trailing determinants over the 127 rationals of Theorem 10.4 — were run in a separate exact-rational implementation, and every landed value was then re-derived by the kernel from the definitions. The two certificates of Section 12 were produced there too: the vector from an exact adjugate as in Remark 12.4, and the Bernstein coefficients by the usual change of basis $B_i = \sum_{k \le i} A_k \binom{d-k}{i-k}$, where the A_k are the coefficients of $c^d Q\big((a+(b-a)t)/c\big)$. That derivation is nowhere in the verification — what is checked is the identity (14) and the $d+1$ signs — and the integers it produces agree digit for digit with those in the formal development. Before either certificate was built, the same implementation recomputed from the definitions used throughout this paper the two facts each rests on: the sign words of $D_0, \dots, D_{14}$ at $\boldsymbol{\gamma}_1$ and of $D_0, \dots, D_{21}$ at $\boldsymbol{\gamma}_{2/3}$ came out $+^{13}-^{2}$ and $+^{20}-^{2}$, that is $\nu(1) = 13$ and $\nu(\frac{2}{3}) = 20$, agreeing with Proposition 7.1 and Theorem 4.1(3). The independent recomputation of Q at thirteen shapes and the four outside controls reported in Remark 12.12(iv) were run there as well, by a code path that uses no polynomials and no integer rescaling at all. The measurements reported in Remarks 10.6, 11.11 and 14.1 were produced by that same implementation, in exact integer arithmetic on the rescaled moment sequence $m_n(D \cdot \boldsymbol{\gamma}_{P/D}) = \prod_{i<n}(P + iD)$, which is integral and whose positive scaling preserves every determinant sign. The separate implementation was also used to check the formalisation against [10] beyond the ten rows of Proposition 9.1: all 75 exact pivots of the ancillary certificate distributed with [10] were matched with their reduced numerators and denominators, as were the closed forms that paper prints for $\kappa_1, \dots, \kappa_4$, its factorisation of D_1, its determinant formula for $\boldsymbol{\beta}_{1/2,q}$, its Appendix-A formula for D_2, and the two displayed values $\Psi(41/50, 32/25) = -8202729/625000$ and $Q_4(41/50, 32/25) = 3921/62500$. Nothing in [10] failed to reproduce.

That certificate is the ancillary directory `numerical_supplement/` of the e-print of [10]. Its two data files, `witness_summary.csv` (10 rows, one per witness) and `witness_pivots_exact.csv` (75 rows, one per pivot), both match the SHA-256 digests shipped beside them in `sha256sums.txt`; the paper's own Appendix C names only the script that generates them. The directory is marked as ignorable in the submission manifest, so it is absent from the rendered sources and has to be taken from the e-print archive itself.

A note on numbering. Every section, theorem, lemma and equation number attributed to [10] is that of v1, which is its only version and is unpublished. Every number attributed to [6] is that of the published Electron. J. Probab. version, and each was checked to agree with the numbering of the final arXiv version, v4. The same was done for [3]: Theorem 1.3 and Proposition 2.3 carry those numbers both in the published Indiana Univ. Math. J. version and in the final arXiv version, v3. The Example 5.2 attributed to [7] in Section 14 is numbered in the published Probab. Math. Statist. version, which is also where the sentences quoted from it were read; the arXiv version of that paper organises the same computation differently. [8, Theorem 13.16] was read in the authors' own PDF of the book, in the version their page states was re-paginated to carry the same page numbers as the published edition; the theorem is on p. 220 there, and the book's published errata list has no entry anywhere in Lecture 13. The footnote quoted from [5] in Remark 11.2 is quoted from the published version, with which arXiv v2 and v3 agree word for word; its v1 wording is different and says something else.

One quotation deserves its provenance stated, since a theorem of this paper is named after it. The text of Conjecture 5.6(3) displayed in Section 1 is transcribed from the LaTeX source of the arXiv version of [6], as are the definition (3) of $\mathcal{I}$ and the text of Corollary 1.3(1). All three were then read again in the published Electron. J. Probab. version, whose own running head is "EJP 19 (2014), paper 81": the conjecture is numbered 5.6 there, its three parts stand in the order quoted, part (3) reads "$\boldsymbol{\gamma}_p$ is not FID for $p \in (\frac{1}{2}, \frac{3}{2})$", Corollary 1.3(1) reads as displayed in Section 1, and $\mathcal{I}$ is defined — exactly as in (3) — inside Theorem 1.2(2). Every other number this paper attributes to [6] was checked in the same reading and agrees: §§1.1, 1.3 and 5.3, Theorems 1.2, 2.4 and 5.1, and Problem 5.7. One caveat survives it, because it was made through a text extractor: the extractor reorders the digits of stacked fractions, so every fraction pair in that reading — $(\frac{1}{2}, \frac{3}{2})$ included — is a reconstruction rather than a character-by-character transcription. It is the arXiv source file, whose fractions are unambiguous, together with the restatement of the conjecture in [10], that the displayed text rests on.

The formal development lives in a repository that is private at the time of writing; repository reference to be added at submission.

14. What is left open

For the gamma laws the question of which are freely infinitely divisible is closed by (4). What remains is about ν — how fast the hierarchy's first failing order grows, and whether it takes every value — about the beta laws, and about the one place where this paper trusts a compiler.

At what rate does ν diverge? With Theorem 11.6 in hand this is the sharp question, and it is a quantitative one: the theorem produces arbitrarily large values of ν near $p = \frac{1}{2}$ but no rate, and the two endpoints of $(\frac{1}{2}, \frac{3}{2})$ do not behave alike. Corollary 12.8 does not help with it: a witness vector at order N bounds ν' from above and never from below, so the interval certificates say nothing about how large ν becomes near the endpoints.

Remark 14.1 (measured divergence rates). Outside the certified statements, in exact integer arithmetic:

$\delta = p - \frac{1}{2}$	0.3	0.2	0.1	0.05	0.025
$\nu(\frac{1}{2} + \delta)$	15	18	24	32	42

$\delta = \frac{3}{2} - p$	0.1	0.01	0.005	0.0025
$\nu(\frac{3}{2} - \delta)$	16	25	28	31

At $\frac{1}{2}$ the fit $\nu \approx 14.05\,\delta^{-1/3} - 5.9$ reproduces all five values to within 1, while a logarithmic fit does not (the increments per unit of $\ln(1/\delta)$ are $7.4, 8.7, 11.5, 14.4$, not constant) and a $\delta^{-1/2}$ law overshoots at the far end. At $\frac{3}{2}$ the last three values increase by exactly 3 per halving of δ, i.e. $\nu \approx A + (3/\ln 2)\ln(1/\delta)$. A cube-root divergence on one side and a logarithmic one on the other is a striking asymmetry, and we have no explanation for it. The first three entries of the left-hand table are the certified values $\nu(4/5) = 15$, $\nu(7/10) = 18$, $\nu(3/5) = 24$ of Theorem 5.1, and the first two of the right-hand table are $\nu(7/5) = 16$ and $\nu(149/100) = 25$; the remaining four ($p = 11/20$, $21/40$, $299/200$, $599/400$) are measured only, their certification lying beyond the cost of the largest check in the development.

Does the hierarchy enlarge at every *order?* This is the surviving half of the first clause of [10]'s question, and Remark 11.7 says why the two results in hand do not close it. The certified evidence is that ν takes every value from 13 to 25; nothing here shows that it takes every large value, and nothing here rules out gaps in its range far out. A rate of the kind measured in Remark 14.1, made rigorous, would settle it: a divergence slower than one step per unit change of the parameter forces ν to take every sufficiently large value, by the integrality of ν and the continuity of each $D_N(\boldsymbol{\gamma}_p)$.

Formalising Section 11. Theorem 11.1 rests on a chain of citations rather than on a check, and the chain is long: the Nevanlinna representation, the identification of the Taylor coefficients of R_μ with the combinatorial free cumulants for a measure of unbounded support, Hamburger's theorem, Carleman's criterion. None of those is in the library used here, so formalising it is a research programme; until that is done, the answer to [10]'s question rests on published mathematics read carefully, not on a machine.

The beta laws. Corollary 12.8 closes part (3) of [6, Conjecture 5.6] and leaves part (1) — that $\boldsymbol{\beta}_{p,q}$ and $\boldsymbol{\beta}'_{p,q}$ are not FID whenever $p \in (\frac{1}{2}, \frac{3}{2})$ and $q > 0$ — untouched. The interval method transfers in form: the moments $m_n(\boldsymbol{\beta}_{p,q}) = (p)_n/(p+q)_n$ are rational in (p, q) with a denominator that does not vanish on the positive quadrant, so clearing denominators makes the quadratic form of a fixed integer vector a polynomial in two variables, whose sign on a rational box a certificate in the tensor Bernstein basis would decide. Whether one vector can cover a box is the question; part (1) asks for all $q > 0$ at once, which no bounded box reaches, though the transfer Lemma 2.5 — the first failing order along a line $p = \text{const}$ is eventually exactly $\nu(p)$ — is at least consistent with one vector serving all large q. We have run no such computation, and the two-variable degrees are larger than the one-variable ones by exactly the amount that makes the cost question serious.

Removing the compiler from Theorem 12.7. That theorem, and hence Corollary 12.8, is the one statement here that rests on compiled evaluation rather than on kernel reduction; Theorem 12.5 shows the method itself needs no such appeal. Closing the gap is a finite, priced task rather than a research question, by either of two routes. (a) *Stage the recursion.* Supply the triangle $W_{k,j} = [z^j]M(z)^k$ — some 880 polynomials, of the order of 800 kB of digits — as data and verify it row by row; the most expensive single row costs about what the depth-28 computation already carried out costs, and the cumulant step $\kappa_n = m_n - \sum_k \kappa_k W_{k,n-k}$ is then an order of magnitude cheaper for all n together. It is perhaps 42 separate statements. (b) *Interpolate.* Prove a degree bound for $\kappa_n(\boldsymbol{\gamma}_p)$ over a polynomial ring — $\deg \le n$ suffices, and Remark 12.2 says the truth is $\lfloor n/2 \rfloor$ — and recover Q from 43 scalar evaluations, each of them cheap. That route needs the moment–cumulant recursion set up over a commutative ring in the library, which the present development does not have.

What the novelty claims rest on. "[6, Conjecture 5.6(3)] is open", "the exponential law was the only shape previously known in a gap of $\mathcal{I}$", "no source computes a Hankel determinant of the free cumulants of $\boldsymbol{\gamma}_p$ at any shape other than $p = 1$, and none computes the trailing determinants at all", and the remark on [8, Theorem 13.16] in Remark 11.2, are negative statements about the literature, so we say what was searched, in August 2026. Citation sweeps of [6]: OpenAlex (13 citing works), Semantic Scholar (5), and OpenCitations (8 citing DOIs, resolved through Crossref); the plausible ones were read in full, among them a 2016 paper of the same author that poses further

conjectures on explicitly non-rigorous evidence, [7], and a doctoral thesis on freely infinitely divisible distributions. Abstract sweeps of the arXiv listing for the author, for "free infinite divisibility", "freely infinitely divisible", that phrase together with "gamma", and for "Hankel" together with "cumulants" in `math.PR`; zbMATH; and a full-text search of a local arXiv mirror (260 mathematics shards), in which the 76 shards containing an infinite-divisibility phrase were then scanned for every mention of a gamma law within 200 characters of it. Nothing found proves, disproves or extends [6, Conjecture 5.6(3)]; nothing found evaluates a Hankel determinant of the free cumulants of $\boldsymbol{\gamma}_p$ at any shape other than $p = 1$; and every source found restating [8, Theorem 13.16] restates it for compactly supported measures, with the single exception recorded in Remark 11.2 — the footnote of [5], which asserts the determinate extension but does not prove it, and which we found no later source to take up.

For the method of Section 12 the corresponding search was for parametric positivity certificates in this setting: a fixed vector, or a fixed test, certifying non-FID on a continuum of parameters. We found none, and we record the two nearest things. [10] is explicitly pointwise about exactly this: its Discussion says that "no fixed leading Hankel test considered here can settle this boundary regime uniformly". It does reach *beta* parameters inside the gaps of $\mathcal{I}$, at points — its $p = 7/10$ line lies in the gap $(\frac{2}{3}, \frac{3}{4})$, and its Corollary 4.1 states that $\boldsymbol{\beta}_{41/50,32/25} \notin \mathrm{FID}$ — so gap parameters for the beta laws are prior art; for the gamma laws they are not, that paper touching $\boldsymbol{\gamma}_p$ only as the limit (6) and certifying no sign for any $D_N(\boldsymbol{\gamma}_p)$. And [7] do obtain non-FID on an interval of a parameter, but at the smallest nontrivial order, for a different family of laws, and with an endpoint that no certificate pins. Their Example 5.2 — the numbering is that of the published version — computes the free cumulants $\kappa_2', \kappa_3', \kappa_4'$ of a one-parameter family η_{α_1,α_2} of generalised power distributions with free Poisson term, and concludes that "the 2nd Hankel determinant $\kappa_2'\kappa_4' - (\kappa_3')^2$ (a function of α_2) is negative for $0 < \alpha_2 < \tilde{\alpha}$, where $\tilde{\alpha} \in (0, 1/2)$ is the unique solution α_2 of the equation $\kappa_2'\kappa_4' - (\kappa_3')^2 = 0$" — that determinant being D_1 in the indexing (1) — and that "Using Mathematica, we get $\tilde{\alpha} \approx 0.157781$". So the interval is in one parameter of a family unrelated to $\boldsymbol{\gamma}_p$, at order $N = 1$, and its right endpoint is a numerically located root; the certificate is not uniform in the sense used here, and no interval of *gamma* shapes is involved.

Coverage, said plainly. Semantic Scholar lists 5 and OpenCitations 8 citing works for a 2014 journal paper that certainly has more — both indexes under-extract that journal's reference lists — and on the day of the last sweep the first of those three indexes was unavailable, so its count above is from an earlier sweep in the same month. No citation count here should be read as complete; the offset is the abstract-level sweep of the whole "free infinite divisibility" corpus and the full-text mirror, which is where such a result would have to appear. Journal-only papers and theses outside the arXiv were not covered, and one recent memorial survey of the area [9] was read in abstract and reference list only. The claims are therefore "to our knowledge" and no stronger.

Effective transfer. Lemma 2.5 gives no bound on $q_0(p)$, so Theorem 11.6 produces beta witnesses at infinitely many orders without producing a single explicit parameter beyond order 20. The thresholds measured along $p = 2/3$ grow by roughly a factor 10 per order ($q_{19}^* \approx 354$, $q_{20}^* \approx 3638$), so an effective form of (6) — even a crude one — would turn Table 2 into explicit beta witnesses at orders 21 to 25, and Theorem 11.6 into an explicit infinite family.

A structural proof of the shape of ν. Nothing here explains why ν is finite exactly on $(\frac{1}{2}, \frac{3}{2})$, why its minimum is 13, why it is flat near $p = 1$, or why the witness thresholds q_N^* grow geometrically. Theorem 10.1 reduces all of it to one scalar per order and Theorem 10.3 makes that scalar monotone, but nothing controls its limit — and measurement is discouraging: at $p = 6/5$, where the trailing determinants never fail through order 60, Φ_N has not settled by $N = 60$ in exact arithmetic, so the limit may well be $-\infty$ on the non-FID side. The scalar is not by itself the handle it looks like.

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