ALON'S TRANSMITTING PROBLEM FOR THE HAMMING GRAPH $H(n, 3)$: TOKUSHIGE'S PROBLEM 12 AT $k = 1, 2$, AND THE BURNING NUMBER OF $H(6, 3)$

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Abstract. Tokushige, in his work on Alon's transmitting problem and a multicolour Beck–Spencer lemma, proves $\lfloor(1 - 1/q)n\rfloor + 1 \leq \mathrm{b}(H(n, q)) \leq \lfloor(1 - 1/q)n + (q + 1)/2\rfloor$ for the burning number of the Hamming graph with $q \geq 3$, conjectures that the upper bound is the truth, and closes with an open problem — Problem 12 of the journal version — whose affirmative answer would settle the conjecture for $q = 3$ and $n = 3k + 1$. It asks whether, for arbitrary $u_1, \ldots, u_{3k+1} \in \{0, 1, 2\}^{3k+1}$, some vertex w satisfies $g(u_i, w) < i$ for every i, where g maximises $|\frac{2}{3}n - d(u, \cdot)|$ over the three diagonal shifts of w. The problem is answered there for no k. We answer it affirmatively at $k = 1$ and at $k = 2$: at $k = 1$ by an exhaustive check made small by a vacuity cap, and at $k = 2$ — where the exhaustive form has 2187^5 instances — by a union bound over translates, $630 > 507 + 45 + 3$, which in fact produces at least 75 witnesses w for every input. We also determine $\mathrm{b}(H(6, 3)) = 6$ — beyond the single value in print, the smallest cell of Tokushige's conjecture at $q = 3$ settled neither by ball counting nor by Problem 12 — using a two-ball refinement of the volume bound. Along the way we record a reformulation of the problem as a band condition on letter counts, valid for every n and i, and the burning numbers $\mathrm{b}(H(n, 3))$ for $n = 4, 5, 6, 7, 8, 10, 11$. Every theorem, proposition, lemma and corollary below has been machine-checked; the closing remarks report frontier measurements made outside the formal development and are labelled as such.

1. Introduction

1.1. **Burning a Hamming graph.** The Hamming graph $H(n, q)$ has vertex set $\{0, 1, \ldots, q - 1\}^n$, two words being adjacent when they differ in exactly one coordinate; its graph distance is the Hamming distance d, and we write $\Gamma_r(v) = \{x : d(v, x) \leq r\}$ for the ball of radius r. A *burning sequence* of length b in a graph G is a tuple $(v_1, \ldots, v_b)$ of vertices with

$$\Gamma_{b-1}(v_1) \cup \Gamma_{b-2}(v_2) \cup \cdots \cup \Gamma_0(v_b) = V(G),$$

and the *burning number* $\mathrm{b}(G)$ is the least b for which one exists: a message injected at v_i in round i and spreading one step per round has reached every vertex after b rounds. For a survey of the parameter see Bonato [3].

For the hypercube $H(n, 2)$ the value is Alon's [1], stated there in the transmitting form: a sender reaches every processor of the n-cube in $\lceil n/2\rceil + 1$ rounds by a two-round scheme, and the note's result is that no scheme does it in fewer. In the vocabulary above this says $\mathrm{b}(H(n, 2)) = \lceil n/2\rceil + 1$, which is how [8, Theorem 1] records it. The combinatorial input is a lemma of Beck and Spencer [2], used in the form: given $a_1, \ldots, a_n \in \{-1, 1\}^n$ there is an $x \in \{-1, 1\}^n$ with $|a_i \cdot x| < 2i$ for every i. Alon applies it to every centre but the first and disposes of the first by replacing x with $-x$ if need be; it is that last twist whose $q = 3$ analogue is at issue in Problem 12 below.

For $q \geq 3$ Tokushige [8, Theorem 2] proves the two-sided bound

$$\left\lfloor \left(1 - \tfrac{1}{q}\right) n \right\rfloor + 1 \;\leq\; \mathrm{b}(H(n, q)) \;\leq\; \left\lfloor \left(1 - \tfrac{1}{q}\right) n + \tfrac{q+1}{2} \right\rfloor, \tag{1}$$

for every n — the statement as printed carries no hypothesis on n — the upper end being the earlier note [7], and conjectures [8, Conjecture 6] that for fixed $q \geq 3$ and large enough n the upper end is the truth. At $q = 3$, which is the case treated throughout this paper, (1) reads

$$\left\lfloor \tfrac{2n}{3} \right\rfloor + 1 \;\leq\; \mathrm{b}(H(n, 3)) \;\leq\; \left\lfloor \tfrac{2n}{3} \right\rfloor + 2, \tag{2}$$

so that for each individual n the conjecture is a single binary question: is the burning number the lower end of (2) or the upper one? The only value of $\mathrm{b}(H(n, 3))$ we have found in print is $\mathrm{b}(H(3, 3)) = 4$, and even that is not printed as a value: it is the case $q = n = 3$ of [8, Proposition 11], which reads "if $q \geq n \geq 3$ then $b(H(n, q)) = n + 1$" and whose proof passes through $\mathrm{b}(H(3, 3)) \geq 4$.

1.2. **Problem 12.** The final section of [8] isolates what its method would need in order to settle the conjecture at $q = 3$ along the residue class $n \equiv 1 \pmod 3$. Fix $k \geq 1$, put $n = 3k + 1$, and for $u, v \in \{0, 1, 2\}^n$ set

$$f(u, v) := \left| \tfrac{2}{3}(3k + 1) - d(u, v) \right|, \qquad g(u, v) := \max\{f(u, v),\ f(u, v'),\ f(u, v'')\},$$

where v' and v'' add 1 respectively 2 to every coordinate of v modulo 3. The problem, numbered 12 in the journal version, reads verbatim:

> "For $1 \leq i \leq 3k + 1$, let u_i be a given vertex of $H(3k + 1, 3)$ on $\{0, 1, 2\}^n$. Then is it true that there exists a vertex w such that $g(u_i, w) < i$ for all $1 \leq i \leq 3k + 1$?"

It is the $q = 3$ analogue of the transmitting statement Alon used at $q = 2$, strengthened by the maximum over the three diagonal shifts, and [8] answers it for no value of k; its introduction records that the author "was not able to find an appropriate extension of this tricky part for $H(n, q)$ $(q \geq 3)$". What the source does prove, in the same section, is the implication: an affirmative answer at k forces $\mathrm{b}(H(3k + 1, 3)) = 2k + 2$, the upper end of (2).

1.3. **What this paper proves.** Three statements here appear to have no published counterpart.

- *Problem 12 has an affirmative answer at* $k = 1$ (Theorem 4.1). At $n = 4$ the constraints with $i \geq 3$ hold for every w, and what remains is decided by an exhaustive check over the 81×81 pairs (u_1, u_2).
- *Problem 12 has an affirmative answer at* $k = 2$ (Theorem 4.2). At $n = 7$ the exhaustive form of the question has $2187^5 \approx 5 \cdot 10^{16}$ instances and was not run. The proof is instead a union bound over translates: four cardinalities at the zero vertex, 630 and $507, 45, 3$, with $630 > 507 + 45 + 3 = 555$, which gives at least 75 admissible w for *every* input list.
- $\mathrm{b}(H(6, 3)) = 6$ (Theorem 7.1). This is the conjectured value at the smallest $n > 3$ for which the elementary ball-volume bound does not reach it, and $6 \not\equiv 1 \pmod 3$ puts the cell outside the reach of Problem 12 as well. The lower half is a two-ball refinement of the volume bound: however the first

two balls of a length-5 burning sequence are placed they leave at least 118 of the 729 vertices uncovered, while the remaining three balls hold $73 + 13 + 1 = 87$.

One caveat belongs in the introduction rather than in a remark at the end, because it bounds what the first two of these are worth. The *consequence* Problem 12 was designed to deliver, $\mathrm{b}(H(3k+1,3)) = 2k+2$, is already available for $k \le 6$ from the ball-volume bound of Section 6 together with the upper end of (2); the first $n = 3k+1$ that counting balls misses is $n = 22$, at $k = 7$. So Theorems 4.1 and 4.2 are new as answers to Problem 12 — a strictly stronger statement, constraining all three diagonal shifts of w at once — and *not* as burning-number values. The values $\mathrm{b}(H(4,3)) = 4$ and $\mathrm{b}(H(7,3)) = 6$ that they imply (Corollary 5.5) are recorded here because they are not in print, but they are two lines of counting and we claim nothing more for them.

The supporting material is of three kinds. Section 3 records a reformulation of g as a band condition on letter counts, valid for every n and every i; it is an easy consequence of the source's definitions rather than a new theorem, but it is load-bearing, since it is what the counting of Section 4 is counting and what any attack on larger k must use. Section 5 reproduces, for every k and then for every n, the two general steps that are the source's own: the constant-word upper bound and the derivation of the value from Problem 12. Section 6 states the ball-volume bound once for all n and b and reads off $\mathrm{b}(H(n,3))$ at $n = 5, 8, 10, 11$. Section 8 collects negative controls. Section 9 reports, as measurements outside the formal development, how far each of these arguments reaches.

For each of the three statements above we ran a literature search; each returned nothing, and a search returning nothing is evidence about the search. Its extent is described in Section 11.

2. Definitions, and the integer form of g

Throughout, $q = 3$ and $V_n := \{0,1,2\}^n$, the letters being read as elements of $\mathbb{Z}/3\mathbb{Z}$, so that V_n is an abelian group and $d(u+c, v+c) = d(u,v)$ for every c. We write $\mathbf{1}$ for the all-ones word, and $v + t\mathbf{1}$ ($t \in \mathbb{Z}/3\mathbb{Z}$) for the three *diagonal shifts* of v; in the notation of [8], $v' = v + \mathbf{1}$ and $v'' = v + 2\mathbf{1}$. For $c \in \{0,1,2\}$ let

$$n_c(x) := \#\{j : x_j = c\}$$

be the c-th letter count of x, so $n_0(x) + n_1(x) + n_2(x) = n$.

The functions f and g of Problem 12 are rational. Multiplying by 3 removes the denominators: put

$$F(u,v) := |2n - 3d(u,v)|, \qquad G(u,v) := \max_{t \in \mathbb{Z}/3\mathbb{Z}} F(u, v + t\mathbf{1}),$$

both integer-valued. Then $f = F/3$ and $g = G/3$, so that for every integer m

$$g(u,v) < m \iff G(u,v) < 3m. \tag{3}$$

Every finite check below is a check on G; (3) is what keeps them out of $\mathbb{Q}$, and the results are stated in both forms.

Definition 2.1. For $k \ge 1$ write $\mathrm{Prob}(k)$ for the assertion of Problem 12 at parameter k: for every list $u_1, \ldots, u_{3k+1}$ of vertices of $H(3k+1,3)$ there is a vertex w with $g(u_i, w) < i$, equivalently $G(u_i, w) < 3i$, for all $1 \le i \le 3k+1$.

Definition 2.2. A tuple $(v_1, \dots, v_b)$ of vertices of $H(n,3)$ is a *burning sequence* if every $x \in V_n$ satisfies $d(v_i, x) \le b - i$ for some i, and $\mathrm{b}(n) := \mathrm{b}(H(n,3))$ is the least b admitting one.

Since $V_n = \Gamma_n(v)$ for any v, burning sequences of length $n+1$ exist, so $\mathrm{b}(n)$ is well defined; and a burning sequence of length b may be padded on the left by arbitrary vertices to give one of any length $c \ge b$. Consequently, to prove $\mathrm{b}(n) > m$ it suffices to refute burning sequences of the single length m. Both facts are used silently below.

The two invariances we use are immediate from the definitions and are recorded once:

$$G(u,v) = G(0, v-u), \qquad |\Gamma_r(a)| = |\Gamma_r(0)| \quad \text{for all } a. \tag{4}$$

3. The problem as a band on letter counts

The source's g is defined through three Hamming distances; every constraint in Problem 12 is, however, a constraint on the three letter counts of the single word $w - u$. The reason is one identity.

Lemma 3.1. *For all $u, w \in V_n$ and all $t \in \mathbb{Z}/3\mathbb{Z}$,*

$$d(u,\, w + t\mathbf{1}) + n_{-t}(w-u) = n.$$

Consequently $F(u, w+t\mathbf{1}) = \big|3n_{-t}(w-u) - n\big|$, and $G(u,w)$ is the largest of the three numbers $|3n_c(w-u) - n|$, $c \in \{0,1,2\}$.

Proof. A coordinate j contributes to $d(u, w+t\mathbf{1})$ exactly when $u_j \ne w_j + t$, that is exactly when $(w-u)_j \ne -t$; the coordinates split into those two classes. The second sentence follows by substituting into $F(u, w+t\mathbf{1}) = |2n - 3d(u, w+t\mathbf{1})|$ and rearranging, and the third because $-t$ runs over $\mathbb{Z}/3\mathbb{Z}$ as t does. ☐

Theorem 3.2 (the colour-band form)**.** *Let $n = 3k+1$ and $i \ge 0$. For all $u, w \in V_n$,*

$$g(u,w) < i \iff G(u,w) < 3i \iff \big(k+1-i \le n_c(w-u) \le k+i \ \ \textit{for every } c\big).$$

In particular:

(i) *the condition holds for* every w *once $i \ge 2k+1$, so a list of $3k+1$ vertices as in Problem 12 constrains only its first $2k$ members;*

(ii) *at $i = 1$ the condition says that the letter counts of $w - u$ are $k+1, k, k$ in some order, so the set of admissible w has exactly $3\,(3k+1)!/\big((k+1)!\,k!\,k!\big)$ elements.*

Proof. By Lemma 3.1 the condition $G(u,w) < 3i$ is $|3n_c(w-u) - (3k+1)| < 3i$ for all c, and for an integer n_c that is $3(k-i)+1 < 3n_c < 3(k+i)+1$, i.e. $k+1-i \le n_c \le k+i$. For (i), $i \ge 2k+1$ makes the band contain $[0, 3k+1]$, which every letter count lies in. For (ii), the band at $i=1$ is $[k, k+1]$, and three integers in $[k, k+1]$ summing to $3k+1$ are one $k+1$ and two k; the count is the number of words with such a profile. ☐

Theorem 3.2 is elementary — it is the restatement a careful reader of [8] makes — and we claim it as no more than that. We record it because the two proofs of the next section are, at bottom, counts of the sets it describes, and because the frontier measured in Section 9 is computed entirely in these terms.

4. Problem 12 at $k = 1$ and at $k = 2$

Both proofs begin from the same two reductions: the vacuity cap of Theorem 3.2(i), which leaves only $i \le 2k$ live, and the translation invariance (4), which turns a statement about pairs into a statement about the zero vertex. The second reduction is not cosmetic: deciding $\forall a, b \in V_7\,(G(a,b) \le 14)$ directly is a check over 2187^2 pairs, while $\forall b \in V_7\,(G(0,b) \le 14)$ is a check over 2187 words and the transport is one line.

Theorem 4.1. *Problem 12 has an affirmative answer at $k = 1$: for all u_1, u_2, u_3, u_4 in $\{0,1,2\}^4$ there is a $w \in \{0,1,2\}^4$ with $g(u_i, w) < i$ for $i = 1, 2, 3, 4$, equivalently with $G(u_i, w) < 3i$ for all four i.*

Proof. At $n = 4$ the largest value of $|3c - 4|$ over $c \in \{0, \ldots, 4\}$ is 8, so $G \le 8 < 9$ identically and the constraints with $i \ge 3$ hold for every w; this was verified by evaluating $G(0, \cdot)$ at all 81 vertices and transporting by (4). What remains is the assertion

$$\forall\, a, b \in V_4\ \exists\, w \in V_4 : \quad G(a,w) < 3 \ \text{ and } \ G(b,w) < 6,$$

which was decided by exhaustive evaluation: for each of the 81×81 pairs (a,b) the 81 candidates w were tested, $81^3 = 531{,}441$ triples in all. The naive reading of Problem 12 at $k = 1$ ranges over $81^5 \approx 3.5 \cdot 10^9$ instances; the vacuity cap is what removes the last two coordinates of that range. $\square$

At $k = 2$ the same route is unavailable: after the cap the live part is a statement about quadruples (u_1, u_2, u_3, u_4) with an inner search, $2187^5 \approx 5 \cdot 10^{16}$ instances, and it was not run. What replaces it is a counting argument over the four sets

$$A_i(u) := \{w \in V_7 : G(u,w) < 3i\}, \qquad B_i(u) := V_7 \setminus A_i(u),$$

each of which is by (4) a translate of $A_i(0)$ respectively $B_i(0)$ and so has a size independent of u.

Theorem 4.2. *Problem 12 has an affirmative answer at $k = 2$: for all $u_1, \ldots, u_7 \in \{0,1,2\}^7$ there is a $w \in \{0,1,2\}^7$ with $g(u_i, w) < i$ for every $i \le 7$.*

Proof. At $n = 7$ the largest value of $|3c - 7|$ over $c \in \{0, \ldots, 7\}$ is $14 < 15$, so Theorem 3.2(i) — or a direct evaluation of $G(0, \cdot)$ at the 2187 vertices, which is what was run — leaves only $i = 1, 2, 3, 4$ live. The four cardinalities involved are

$$|A_1(0)| = 630, \qquad |B_2(0)| = 507, \qquad |B_3(0)| = 45, \qquad |B_4(0)| = 3,$$

each obtained by evaluating $G(0, \cdot)$ over all 2187 words. Since $B_i(u_i)$ is a translate of $B_i(0)$,

$$\bigl|A_1(u_1) \setminus \bigl(B_2(u_2) \cup B_3(u_3) \cup B_4(u_4)\bigr)\bigr| \ \ge\ 630 - (507 + 45 + 3) \ =\ 75 \ >\ 0,$$

and any w in that difference satisfies all four live constraints simultaneously. The remaining constraints are vacuous. $\square$

The displayed inequality gives more than the theorem asks for: whatever the input list, at least 75 of the 2187 vertices are admissible. That is a step of the proof above rather than a separate statement, and it is the quantity Remark 9.1 tracks as k grows.

The four cardinalities are exactly the profile counts of Theorem 3.2: at $k = 2$ the bands are $[2,3] \subset [1,4] \subset [0,5] \subset [0,6]$, so $630 = 3 \cdot 7!/(3!\,2!\,2!)$ counts the profile $(3,2,2)$; $2187 - 1680 = 507$ with $1680 = 630 + 420 + 630$ the profiles

$(3,2,2),(3,3,1),(4,2,1)$; $45 = 42+3$ the words in which some letter occurs at least six times; and 3 the constant words. They were computed twice, once outside the proof assistant directly from the definition of g and once inside it, before and after the formal development respectively.

Two things are worth separating in Theorem 4.2. The union bound is what makes the statement provable without a search over lists, and it is not the source's method: [8] argues by the multicolour version of the floating-variable method, which it takes from Doerr and Srivastav [4], and the counting argument in the neighbouring literature (Theorem 3 of the arXiv version of [7]) yields only the asymptotic lower bound $\mathrm{b}(H(n,q)) > pn - \sqrt{2pn\log n}$, where $p = 1-1/q$. On the other hand the bound is crude, and Section 9 measures exactly how crude.

5. From Problem 12 to a burning number

This section is the source's mathematics, made precise for every k and then for every n. Nothing in it is claimed as new.

Proposition 5.1 (Tokushige). *For every n, $\mathrm{b}(H(n,3)) \le \lfloor 2n/3\rfloor + 2$; the three constant words $\bar{0},\bar{1},\bar{2}$ are the first three terms of a burning sequence of that length.*

Proof. Put $b = \lfloor 2n/3\rfloor + 2$ and let $v_1 = \bar{0}$, $v_2 = \bar{1}$, $v_3 = \bar{2}$, the remaining terms arbitrary. Since $d(\bar{c},x) = n - n_c(x)$, a vertex x lying in none of $\Gamma_{b-1}(\bar{0})$, $\Gamma_{b-2}(\bar{1})$, $\Gamma_{b-3}(\bar{2})$ has $n_0(x) \le n-b$, $n_1(x) \le n-b+1$ and $n_2(x) \le n-b+2$, so

$$n = n_0(x)+n_1(x)+n_2(x) \le 3(n-b)+3, \qquad \text{i.e.} \qquad b \le \tfrac{2n}{3}+1,$$

which contradicts $b = \lfloor 2n/3\rfloor + 2 > \frac{2n}{3}+1$. □

Proposition 5.2 (Tokushige). *For every k, $\mathrm{b}(H(3k+1,3)) \le 2k+2$.*

Proof. Proposition 5.1 at $n = 3k+1$, where $\lfloor 2n/3\rfloor + 2 = 2k+2$; the pigeonhole there reads $(k-1)+k+(k+1) < 3k+1$. □

The next statement is the implication printed in the final section of [8]. Its one ingredient is the identity below, which says that a vertex and the three diagonal shifts of another are, on average, at distance $2n/3$.

Lemma 5.3. *For all $u,v \in V_n$, $\sum_{t\in\mathbb{Z}/3\mathbb{Z}} d(u, v+t\mathbf{1}) = 2n$. Hence some t has $3d(u,v+t\mathbf{1}) \ge 2n$.*

Proof. Each coordinate j agrees with v_j+t for exactly one of the three values of t, hence contributes 2 to the sum; exchanging the order of summation gives $2n$. The second sentence is the pigeonhole. □

Theorem 5.4 (Tokushige). *If $\mathrm{Prob}(k)$ holds then $H(3k+1,3)$ has no burning sequence of length $2k+1$; consequently $\mathrm{b}(H(3k+1,3)) = 2k+2$.*

Proof. Write $n = 3k+1$ and let $(v_1,\dots,v_{2k+1})$ be a burning sequence, so that the i-th ball has radius $2k+1-i$. Apply $\mathrm{Prob}(k)$ to the list $u_i := v_{i+1}$ for $i \le 2k$, padded arbitrarily beyond that — those constraints are vacuous by Theorem 3.2(i) and are not used — obtaining w with $G(v_{i+1},w) < 3i$ for $1 \le i \le 2k$. By Lemma 5.3 choose t with $3d(v_1, w+t\mathbf{1}) \ge 2n$ and put $x = w+t\mathbf{1}$. Some ball of the sequence contains x. It is not the first, since $d(v_1,x) \ge \lceil 2n/3\rceil = 2k+1$ while that ball has radius $2k$.

If it is the i-th with $i \geq 2$, then $d(v_i, x) \leq 2k + 1 - i$, whereas $G(v_i, w) < 3(i - 1)$ and

$$2n - 3d(v_i, x) \;\leq\; F(v_i, x) \;\leq\; G(v_i, w) \;<\; 3(i - 1)$$

give $3d(v_i, x) > 2(3k + 1) - 3(i - 1) = 3(2k + 1 - i) + 2$, i.e. $d(v_i, x) > 2k + 1 - i$, a contradiction. So no burning sequence of length $2k+1$ exists; with Proposition 5.2 and the padding remark after Definition 2.2 this gives $\mathrm{b}(H(3k+1, 3)) = 2k+2$. □

Corollary 5.5. $\mathrm{b}(H(4, 3)) = 4$ *and* $\mathrm{b}(H(7, 3)) = 6$. *In particular* $H(4, 3)$ *has no burning sequence of length* 3 *and* $H(7, 3)$ *none of length* 5.

Proof. Theorem 5.4 with Theorems 4.1 and 4.2. □

We repeat that these two values are not what Theorems 4.1 and 4.2 are for. Both follow in two lines from the ball-volume bound of the next section — at $n = 4$ the balls of a length-3 sequence hold at most $33 + 9 + 1 = 43 < 81$ vertices, at $n = 7$ those of a length-5 sequence at most $939 + 379 + 99 + 15 + 1 = 1433 < 2187$ — together with Proposition 5.1. They are stated because we have found no source that tabulates any value of $\mathrm{b}(H(n, 3))$ beyond $n = 3$, not because the route through Problem 12 is needed for them.

6. The ball-volume bound, and the values it settles

Theorem 6.1. *Let* $n, b \geq 0$. *If* $\sum_{j=0}^{b-1} |\Gamma_{b-1-j}(0)| < 3^n$ *then* $H(n, 3)$ *has no burning sequence of length* b*, and hence, by the padding remark after Definition 2.2,* $\mathrm{b}(H(n, 3)) > b$*.*

Proof. The balls of a burning sequence of length b cover V_n, and by (4) the j-th has $|\Gamma_{b-1-j}(0)|$ elements wherever it is centred, so their sizes sum to at least $|V_n| = 3^n$. □

The hypothesis of Theorem 6.1 depends on n and b alone, which is what makes each individual cell a single finite evaluation; $|\Gamma_r(0)| = \sum_{j \leq r} \binom{n}{j} 2^j$. Comparing the bound with Proposition 5.1 at $b = \lfloor 2n/3 \rfloor + 1$ settles the binary question (2) at

$$n = 4, 5, 7, 8, 10, 11, 13, 16, 19$$

and at no other n with $3 \leq n \leq 22$. Two of the misses deserve naming. At $n = 3$ the sum is exactly $3^3 = 27$, so the strict inequality just fails; at $n = 6$ it is $793 \geq 729$. Thus $n = 6$ is the smallest $n > 3$ the bound misses — and not the smallest outright, a distinction that matters only because $n = 3$ is the one cell already settled in print.

Corollary 6.2. $\mathrm{b}(H(5, 3)) = 5$, $\mathrm{b}(H(8, 3)) = 7$, $\mathrm{b}(H(10, 3)) = 8$ *and* $\mathrm{b}(H(11, 3)) = 9$.

Proof. The upper halves are Proposition 5.1. For the lower halves, Theorem 6.1 applies with the four sums

$$131 + 51 + 11 + 1 = 194 < 243, \qquad 3489 + 1697 + 577 + 129 + 17 + 1 = 5910 < 6561,$$

$$44{,}515 < 59{,}049, \qquad 174{,}930 < 177{,}147,$$

each computed by enumerating the vertex set of the graph in question. The last is the tightest cell the bound reaches, with a slack of 2217. □

7. $\mathrm{b}(H(6,3)) = 6$: THE FIRST CELL BEYOND $n = 3$ THAT COUNTING BALLS MISSES

At $n = 6$ the sum of Theorem 6.1 for $b = 5$ is $473+233+73+13+1 = 793 \geq 729$, so ball counting leaves $\mathrm{b}(H(6,3)) \in \{5, 6\}$ undecided; and $6 \not\equiv 1 \pmod 3$, so Problem 12 does not reach the cell either. Beyond the single value $\mathrm{b}(H(3,3)) = 4$ of [8] it is therefore the smallest n at which the conjectured formula $\lfloor 2n/3 \rfloor + 2$ is settled by neither route. What closes it is a refinement in which the first two balls are counted jointly rather than separately.

Theorem 7.1. $\mathrm{b}(H(6,3)) = 6$*, the upper end of* (2) *and so the value the formula conjectured in* [8] *predicts.*

Proof. The upper half is Proposition 5.1, $\lfloor 12/3 \rfloor + 2 = 6$. For the lower half let $(v_1, \ldots, v_5)$ be a burning sequence of $H(6,3)$ and consider

$$U := V_6 \setminus \big(\Gamma_4(v_1) \cup \Gamma_3(v_2)\big).$$

By (4) the size of U depends only on $c := v_2 - v_1$, and an exhaustive evaluation over all 729 values of c — for each, a pass over the 729 vertices, so 531,441 evaluations in all — gives $|U| \geq 118$, the minimum being attained at $c = 111111$. The three remaining balls have $|\Gamma_2| + |\Gamma_1| + |\Gamma_0| = 73 + 13 + 1 = 87$ vertices between them, and $87 < 118$, so some vertex of U is covered by none of the five balls. Hence no burning sequence of length 5 exists. $\square$

The bound was located first by a compiled search for a length-5 burning sequence, whose level-two pruning empties the tree immediately — no v_2 whatever leaves 87 or fewer vertices uncovered — and only then turned into the counting statement above, which is what is proved. The gap $118 - 87 = 31$ is the whole slack at this cell.

8. CONTROLS

Two failure modes would not be revealed by any amount of internal consistency: a mis-transcribed definition, and an existential statement satisfied vacuously. The following are checks against both, on explicit data.

Proposition 8.1.
(i) The threshold at $i = 1$ cannot be tightened. *At $n = 4$ there is, for every $a \in V_4$, no w with $G(a, w) < 2$; at $n = 7$ there is no w with $G(0, w) < 2$, hence by* (4) *none for any a. So $g(u_1, w) < 1$ cannot be strengthened to $g(u_1, w) < 1 - \frac{1}{3}$.*
(ii) The $i = 1$ constraint is not vacuous. *Of the* 81 *vertices of $H(4,3)$,* 45 *fail $G(0, w) < 3$; of the* 2187 *vertices of $H(7,3)$,* 1557 *do.*
(iii) The bands must widen with i. *For $u_1 = 0000$, $u_2 = 0011$, $u_3 = 0022$ there is no $w \in V_4$ with $G(u_j, w) < 3$ for all three j. Hence Problem 12 is not a disguised form of a statement with a single band.*
(iv) No claimed value is too large. $H(4,3)$ *has no burning sequence of length* 3 *and* $H(7,3)$ *none of length* 5*, matching Corollary 5.5 from the other side.*

Proof. Items (i)–(iii) are exhaustive evaluations over the vertex sets named. Item (iv) is Corollary 5.5 with the padding remark after Definition 2.2. $\square$

Control (iii) was found by a bitmask sweep, which reports that 13,122 of the 81^3 ordered triples (u_1, u_2, u_3) admit no common w at the $i = 1$ band, while all 81^2 pairs do.

9. How far each argument reaches

The statements in this section were computed outside the formal development, by programs written for the purpose; they are reported as measurements, and none of them is machine-checked. They are included because the shape of a frontier is what tells the next reader which instrument to bring.

Remark 9.1 (the counting proof of Theorem 4.2). In the plain form used above — $|A_1| > \sum_{i\geq 2} |V_n \setminus A_i|$, with A_i the analogue at general k of the sets of Theorem 4.2 — the union bound holds only for $k \leq 2$: the two sides are 36 against 3 at $k = 1$ and 630 against 555 at $k = 2$, but 12,600 against 28,047 at $k = 3$. It can be sharpened, since what is needed is not $|V_n \setminus A_i|$ but $\#\{W \in A_1 : W + a \notin A_i\}$ for the worst shift a, and that count depends only on the sorted letter-count profile of a. With that sharpening the argument settles $k = 1, \ldots, 5$, with slacks

$$33, \quad 429, \quad 5793, \quad 68{,}697, \quad 374{,}163,$$

and *fails* at $k = 6$, where $|A_1| = 139{,}675{,}536$ against a worst-case bad total of 156,258,609; it keeps failing at every k from 6 to 11 that we computed. The obstruction is mathematical rather than computational, and it is visible already in the $i = 2$ term: that term alone consumes the fractions

$$0.08, \quad 0.28, \quad 0.41, \quad 0.50, \quad 0.56, \quad 0.61$$

of $|A_1|$ at $k = 1, \ldots, 6$, and what it leaves shrinks like $1/k$ over the range $k \leq 20$ we measured, while the number of further terms to be paid out of that remainder grows like $2k$. Two independent computations agree wherever both run ($k \leq 4$): direct enumeration of A_1, and a closed formula summing multinomials over 3×3 contingency tables with prescribed margins. The profile invariance the second relies on was verified by brute force over all 81 respectively 2187 shifts at $k = 1, 2$. Only $k \leq 2$ is formalised, which is why Theorems 4.1 and 4.2 stop there and this remark does not.

Remark 9.2 (two balls, off-line). The size of $|\Gamma_{r_1}(u) \cup \Gamma_{r_2}(v)|$ has a closed form in n, r_1, r_2 and $d(u, v)$: split the coordinates into the d where u, v differ and the $n - d$ where they agree and count by how many of each a candidate matches. The formula was checked against direct enumeration for all $n \leq 7$, all radii and all distances. Used in place of the enumeration in Theorem 7.1, it settles every cell the volume bound settles, plus $n = 6, 14, 22$:

n	6	14	22
maximal two-ball union	611	3,590,476	21,725,718,660
remaining balls	87	1,096,884	9,124,962,945
3^n	729	4,782,969	31,381,059,609
slack	31	95,609	530,378,004

giving $\mathrm{b}(H(14, 3)) = 11$ and $\mathrm{b}(H(22, 3)) = 16$. The second is worth naming: $22 = 3k + 1$ at $k = 7$, so it is precisely the first cell for which an affirmative answer to Problem 12 would not have been redundant. The two-ball bound fails at $n = 9, 12, 15, 17, 18, 20, 21$.

Remark 9.3 (three balls). At $n = 9$ both elementary bounds miss — the volume bound by 3328, the two-ball bound by 598. Ruling out a length-7 burning sequence needs $\Gamma_6(v_1) \cup \Gamma_5(v_2) \cup \Gamma_4(v_3)$ to leave more than $835 + 163 + 19 + 1 = 1018$ of the 19,683 vertices. Fixing $v_1 = 0$ by vertex transitivity and discarding those v_2 whose two-ball residue already exceeds $1018+|\Gamma_4|$ — a discard that is itself an inequality, so the sweep is exhaustive and not heuristic — leaves 2816 candidates, and a complete pass over the surviving pairs gives a minimum residue of $1260 > 1018$. Hence $\mathrm{b}(H(9,3)) = 8$. The same bound in closed form (the residue depends only on how the n coordinates distribute over the five agreement classes of the three centres, so a dynamic program over the $\binom{n+4}{4}$ type vectors computes the exact minimum) reproduces 1260 at $n = 9$ and 60 at $n = 6$, agreeing in both cases with independent enumerations, and settles one further cell, $\mathrm{b}(H(17,3)) = 13$, where the residue 14,702,688 beats the tail 14,597,433. With these, every $n \leq 22$ has a value except $n = 12, 15, 18, 20, 21$, and at every settled n the answer is the upper end of (2), as conjectured. In every case the minimising configuration is the three constant words — the configuration Proposition 5.1 uses.

Remark 9.4 (the resulting table). Collecting the four routes, $\mathrm{b}(H(n,3))$ for $n \leq 22$ stands as follows. Boldface marks the values proved above; the others are either [8]'s ($n = 3$) or the output of an argument whose finite step was carried out outside the formal development.

n	3	4	5	6	7	8	9	10	11	12
b	4	**4**	**5**	**6**	**6**	**7**	8	**8**	**9**	?
route	s	P	v	2	P	v	3	v	v	—

n	13	14	15	16	17	18	19	20	21	22
b	10	11	?	12	13	?	14	?	?	16
route	v	2	—	v	3	—	v	—	—	2

Routes: s = [8], P = Problem 12, v = ball volume, 2 and 3 = the two- and three-ball bounds. The five cells $n = 12, 15, 18, 20, 21$ are open, and at every settled n the value is the upper end of (2).

Remark 9.5 (the state of $n = 12$). The smallest cell left open is $n = 12$, where the three-ball residue is 27,720 against a remaining-ball total of 47,646: counting alone does not exclude a length-9 burning sequence. At the minimising configuration the residue is exactly the set of words with letter counts $(3, 4, 5)$, of size $12!/(3!\,4!\,5!) = 27{,}720$; a ball of radius 5 meets that set in at most 3010 words, and $3010 + 9969 + 2049 + 289 + 25 + 1 = 15{,}343 < 27{,}720$, so no length-9 burning sequence of $H(12,3)$ begins with the three constant words. A local search with 40 restarts over all nine centres found nothing better than 23,527 uncovered vertices of 531,441. Neither observation is a proof; both point at $\mathrm{b}(H(12,3)) = 10$.

Remark 9.6 (what was priced and not done). Three continuations were costed rather than attempted, and the costs are worth recording. (a) Problem 12 at $k = 3$ ($n = 10$) is mathematically finished by Remark 9.1, with a slack of 5793; the obstruction is that the certifying inequality is $59{,}049 \times 12{,}600$ pairs for each of five values of i, about $3.7 \cdot 10^9$ evaluations, which at the interpreted speed of the present formalisation is on the order of 10^4 CPU-hours. Formalising the profile invariance of Remark 9.1 — every shift is a coordinate permutation of one of 14 representatives — reduces this to about $9 \cdot 10^5$ evaluations, i.e. seconds, and is the reusable half. (b) Problem 12 at $k \geq 6$ is not a compute question at all: the counting argument is false there, and Remark 9.1 states exactly what a replacement has to beat. (c) The values of Remark 9.2 at $n = 14, 22$ need the union formula proved rather than evaluated ($3^{22} \approx 3.1 \cdot 10^{10}$ vertices), and those of Remark 9.3 are past 10^{12} evaluations in the present encoding.

10. Verification

Every numbered statement of Sections 2–8 has been formalised and machine-checked in Lean 4 against *mathlib* [5], with no unproved assumptions and no appeal to results quoted from the literature: the definitions, the bridge (3) between the source's rational form and the integer form, the identities of Lemmas 3.1 and 5.3, the band form of Theorem 3.2, the two answers to Problem 12, the general upper bound of Proposition 5.1, the derivation of Theorem 5.4, the volume bound of Theorem 6.1, the value $\mathrm{b}(H(6, 3)) = 6$ and every control. Everything that is stated for all n, all k or all b — Sections 3 and 5, Theorem 6.1 — is proved in the kernel with no computation. The finite checks are compiled evaluations, trusted through the standard native-evaluation axiom rather than replayed in the kernel, and they are exactly these: the vacuity caps at $n = 4$ and $n = 7$ (81 and 2187 evaluations of $G(0, \cdot)$); the 81^3 triples of Theorem 4.1; the four cardinalities of Theorem 4.2 (2187 evaluations each); the 729 relative positions of Theorem 7.1, each a pass over 729 vertices, together with the three ball sizes there; the four ball sums of Corollary 6.2, over 3^5, 3^8, 3^{10} and 3^{11} vertices; and the five evaluations behind Proposition 8.1(i)–(iii). No statement in Section 9 is machine-checked, and none is used in the proof of any numbered statement. The development is a private repository; repository reference to be added at submission.

11. Remarks on the literature search

For Problem 12 and for the value $\mathrm{b}(H(6, 3))$ we searched as follows. Semantic Scholar reports exactly two papers citing [8], namely [6] and [7], both by the same author; a positive control on an unrelated, heavily cited preprint confirmed the endpoint was answering. Of the two, [6] reproves the bound (1) for $q \geq 3$ by a different method — "our proof here is much simpler" — and does not state, discuss or resolve the problem: searches of its full text for the strings `3k+1`, `2k+2`, `prob` and `conject` hit only its bibliography. OpenAlex returns a citation count of zero for both the preprint and the journal version of [8], which contradicts the first index and which we read as an indexing gap rather than as evidence. A full-text query over the arXiv for "burning number" together with "Hamming graph" returns exactly those two papers, and a search for the identifier of [8] across a local mirror of 763,465 arXiv full texts returns the source and the same two. The OEIS has no burning-number sequence for Hamming graphs or hypercubes. Bonato's survey [3] contains no Hamming graphs and no table of small burning numbers. We found no published value of $\mathrm{b}(H(n, 3))$ other than $\mathrm{b}(H(3, 3)) = 4$, and no refinement of the ball-volume bound to two or more balls for Hamming graphs. Alon's note [1] was read in full for this paper; it is three pages, entirely about the cube, and contains no Hamming graph with $q \geq 3$ and no burning-number vocabulary.

What we did not check, and therefore do not claim as a clean negative: reviewing databases and citation indices to which we have no access; books, theses and non-arXiv venues; and anything appearing after mid-2026. All three statements we put forward are elementary in their arguments, and an elementary statement is exactly the kind that can sit unindexed in a remark of an older paper. Every negative above should be read as "not found", not as "new".

References

[1] N. Alon, *Transmitting in the n-dimensional cube*, Discrete Appl. Math. **37/38** (1992), 9–11, doi:10.1016/0166-218X(92)90121-P. The note states no numbered theorem: its result is announced in prose, and its one displayed lemma is quoted from [2]. Hence the citations to it above are to the note as a whole.

[2] J. Beck and J. Spencer, *Balancing matrices with line shifts*, Combinatorica **3** (1983), no. 3–4, 299–304, doi:10.1007/BF02579185. Not consulted directly; the statement used above is the one [1] and [8, Lemma 3] quote, and they agree.

[3] A. Bonato, *A survey of graph burning*, Contrib. Discrete Math. **16** (2021), no. 1, 185–197, doi:10.55016/ojs/cdm.v16i1.71194; arXiv:2009.10642.

[4] B. Doerr and A. Srivastav, *Multicolour discrepancies*, Combin. Probab. Comput. **12** (2003), no. 4, 365–399, doi:10.1017/S0963548303005662. Cited here only for the method [8] attributes to it; not consulted directly.

[5] The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381, doi:10.1145/3372885.3373824.

[6] H. Tanaka and N. Tokushige, *Burning numbers via eigenpolytopes — Hamming graphs, Johnson graphs, and halved cubes*, arXiv:2508.17559v1 (25 August 2025).

[7] N. Tokushige, *Burning Hamming graphs*, Graphs and Combinatorics **40** (2024), no. 6, article 111, 3 pp., doi:10.1007/s00373-024-02847-9; arXiv:2405.01347. The one theorem number cited above is that of the arXiv version, which is the version consulted.

[8] N. Tokushige, *Alon's transmitting problem and multicolor Beck–Spencer lemma*, Electron. J. Combin. **32** (2025), no. 3, Paper 3.22, 8 pp., doi:10.37236/13255; arXiv:2406.19945v3. Both were consulted, and they agree in every passage quoted or cited here. Numbers above are the journal version's, which runs a single counter through all its numbered statements; arXiv v3 instead numbers theorems, lemmas and claims on three separate counters and leaves the conjecture, the proposition and the problem unnumbered. The one collision worth flagging is benign: the two-sided bound is Theorem 2 in both.