# HOW TIGHT IS THE FOUR-HYPERPLANE COUNTEREXAMPLE? EXACT BOUNDS FOR ITS POLYNOMIAL SYSTEM

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ABSTRACT. Soberón has settled the last open case of Grünbaum's 1960 equipartition problem by producing a mass in $\mathbb{R}^4$ that no four hyperplanes equipartition. The computational core of that proof is a system of five polynomials $f_0, \ldots, f_4$ in six variables, of degree at most 4 in each variable, attached to a quaternionic chart of $SO(4)$: what the theorem needs is exactly that the five have no common zero on the box $Q = [-1,1]^6$, and the source proves this with a subdivision certificate of 38 857 dyadic boxes checked in fixed-point interval arithmetic. We study that polynomial system as an object in its own right. First, the nonvanishing is re-verified in exact integer arithmetic inside a formal proof: the passage from the certificate to the conclusion — the power-to-Bernstein change of basis, de Casteljau subdivision, and the convexity bound on a box — is machine-checked, and the only thing computed rather than proved is the sign of $38\,857 \cdot 5^6 = 607\,140\,625$ integers. Second, we determine the size of the system: $\sup_Q |f_s| = 3064$ exactly for $s = 0, 1, 2, 3$, attained at explicit corners of $Q$, and $\sup_Q |f_4| \le 1440$. The two constants are the exact maxima of the Bernstein coefficient arrays of the $f_s$ on $Q$; for $f_4$, and only for $f_4$, that bound is not attained, and its sup norm is left open. Third, with $S = (3064, 3064, 3064, 3064, 1440)$ we show that the nonvanishing is tight:

$$\inf_{z \in Q} \max_{0 \le s \le 4} |f_s(z)|/S_s \le 1/3064 \approx 3.26 \cdot 10^{-4},$$

the value at the origin of the chart, where the five values are $(-1, 0, 0, 0, 0)$. No point of $\{-1, 0, 1\}^6$ does better, and unformalised searches get no further than 1/3187, so the constant is within about 4% of the best value we can find. Soberón's counterexample therefore clears zero by about three parts in ten thousand of its own scale — which is why a subdivision proof of it cannot be small. Every theorem and proposition here is machine-checked in Lean 4; the searches that calibrate the constant are not, and are labelled where they occur.

## 1. INTRODUCTION

**Equipartition by hyperplanes.** A *mass* in $\mathbb{R}^d$ is a finite Borel measure that vanishes on every hyperplane. A family of $k$ hyperplanes $H_1, \ldots, H_k$ cuts $\mathbb{R}^d$ into $2^k$ open orthants, and it *equipartitions* a mass $\mu$ when each orthant carries the fraction $2^{-k}$ of $\mu(\mathbb{R}^d)$. Following Ramos [6], write $\Delta(j,k)$ for the smallest dimension $d$ such that for every collection of $j$ masses in $\mathbb{R}^d$ there are $k$ affine hyperplanes cutting each of the $j$ masses into $2^k$ equal pieces; determining $\Delta(j,k)$ is the Grünbaum–Hadwiger–Ramos problem, and the definition is stated in this form in the survey [7]. Grünbaum [3, §4, Remark (v)] asked in 1960 whether $\Delta(1,d) = d$, that is, whether $d$ hyperplanes always suffice for one mass in $\mathbb{R}^d$: he observes there that the answer is yes in the plane, even for orthogonal lines, that orthogonality fails in general in $E^n$, and asks whether dropping it repairs the statement. Hadwiger [4] answered $d = 3$ positively in 1966, and Avis [5] answered every $d \ge 5$ negatively, by concentrating the mass near the moment curve. The case $d = 4$ resisted both routes for sixty-six years.

In August 2026 Soberón [1] settled it: $\Delta(1,4) = 5$, so Grünbaum's question has a negative answer in dimension four, and with it Ramos' conjecture [6] that $\Delta(j,k) = \lceil (2^k - 1)j/k \rceil$, whose value at $(j,k) = (1,4)$ is $\lceil 15/4 \rceil = 4$. (What [1] proves is $\Delta(1,4) > 4$; the matching upper bound $\Delta(1,4) \le \Delta(2,3) = 5$ was already known, and [1] quotes it to pin the value.) The counterexample is a perturbed Gaussian,

$$d\mu_\varepsilon(x) = \big(1 + \varepsilon\, \eta(\|x\|^2)\, (P_A(x) + P_B(x))\big)\, d\gamma(x), \tag{1}$$

with $P_A$ cubic and $P_B$ quartic and $\eta$ a cutoff, and the whole proof turns on one finite statement about those two polynomials.

**The finite core.** Every equipartition of the Gaussian $\gamma$ by four hyperplanes is given by an orthonormal frame through the origin, so a Walsh-coefficient analysis of (1) reduces "$\mu_\varepsilon$ still has an equipartition" to "five explicit contractions of the symmetric tensors of $P_A$ and $P_B$ have a common zero on $O(4)$". A symmetry reduction then replaces $O(4)$ by a chart: parametrise $SO(4)$ by pairs of unit quaternions, normalise so that the first coordinate of each is largest in modulus, and one lands on the cube $[-1,1]^6$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$. What remains — the source's Theorem 2, after the reduction of its §3 — is the assertion that five explicit polynomials $f_0, \dots, f_4$ in six variables have *no common zero on* $Q = [-1,1]^6$. Section 2 writes them out.

That assertion is proved in [1] by exhibiting a binary subdivision tree of 38 857 dyadic subboxes of $Q$, each carrying an integer covector $w \in \mathbb{Z}^5$ that has strictly positive inner product with all $5^6 = 15\,625$ tensor-product Bernstein control vectors of that box; since every value of a polynomial on a box is a convex combination of its control values, the map $(f_0, \dots, f_4)$ cannot vanish there. The tree and its covectors are distributed with [1] as the ancillary package [2], whose checker works with an outward interval enclosure on the fixed scale $2^{52}$ in 128-bit integer arithmetic (§7).

**What is settled here.** The five polynomials are the object of this paper. Three things are proved about them; all three are machine-checked in Lean 4, in the sense made precise in §7.

The first is the nonvanishing itself, in exact arithmetic and inside a formal proof.

**Theorem A** (= Theorem 4.1)**.** *The five polynomials $f_0, \dots, f_4$ have no common zero on $Q = [-1,1]^6$.*

The statement is the source's, and we claim no priority for it; what is new is the arithmetic and the trust boundary. Every coefficient handled here is an exact integer — subdivision is the integer de Casteljau step, scaled by 16 per bisection, so no rounding occurs anywhere — and the passage from the accepted certificate to the conclusion, which is where the mathematics of the method sits, is a proof rather than a program: the power-to-Bernstein change of basis, the two subdivision identities, the six-axis lift and the convexity argument are all machine-checked, and the only step that is computed is the sign of $38\,857 \cdot 5^6 = 607\,140\,625$ integers (§4).

The second and third are quantitative, and are not in [1], which proves nonvanishing and stops there. No bound on the size of the $f_s$ is stated anywhere in [1]. The words *margin* and *tight* do not occur in its text at all; the one supremum it takes, $\sup_Q \|z_\varepsilon(Q)\| \le C\,|\varepsilon|$, belongs to the implicit-function step and not to the polynomial system; and the constants 3064, 1440, 142 953 984 and 67 184 640 below occur nowhere in it, the last two occurring nowhere in the ancillary package [2] either. The checker of [2] does compute the two root maxima internally, as the componentwise normalisation described in §4, but it neither prints them nor uses them as a bound.

**Theorem B** (= Theorems 5.1 and 5.2)**.** *On $Q$ one has $|f_s| \le 3064$ for $s = 0,1,2,3$ and $|f_4| \le 1440$; these are exactly the maxima of the Bernstein coefficient arrays of the $f_s$ on $Q$. For $s = 0,1,2,3$ the bound is attained at an explicit corner of $Q$, so $\sup_Q |f_s| = 3064$ exactly. For $f_4$ it is not attained (Remark 5.4), and $\sup_Q |f_4|$ is not determined here.*

**Theorem C** (= Theorem 6.3)**.** *Put $S = (3064, 3064, 3064, 3064, 1440)$. Then*

$$m \;:=\; \inf_{z \in Q}\, \max_{0 \le s \le 4} \frac{|f_s(z)|}{S_s} \;\le\; \frac{1}{3064}.$$

The three fit together. The sup bounds are what make the third statement mean anything: without a scale, "the five polynomials come within 1 of a common zero" is a statement about the normalisation of $A$ and $B$ and not about the geometry. With the scale, the number is intrinsic (Remark 6.2), and it says that Soberón's counterexample clears zero by about three parts in ten thousand. The extremal point is the origin of the chart, where the frame is the standard basis and the five values are $(-1,0,0,0,0)$; no point of $\{-1,0,1\}^6$ does better (Remark 6.5), and unformalised searches (§6) get no further than 1/3187, within about 4%. That the near-miss sits at the origin is legible in the construction: the origin of the chart sees exactly the multilinear coefficients of the two tensors, and [1] presents $P_A$ with a single multilinear monomial and $P_B$ with none (Remark 2.3).

The margin is also the number that governs the cost of *any* subdivision proof of the nonvanishing. The Bernstein enclosure of a polynomial on a box of side $h$ overestimates its range by $O(h^2)$ under repeated bisection [8], so separating a ratio of 1 : 3064 needs boxes of side of order $2^{-6}$ before constants are put in; the certificate of [1] in fact goes to depth 10 or 11 per coordinate. The point is not the exponent but that 1/3064 forbids a small certificate, without anyone looking at the tree.

**What is not claimed.** This paper does not machine-check $\Delta(1,4)=5$. It machine-checks the finite statement that [1] reduces that theorem to, and states the reduction as quoted. Three gaps are left open, and §8 prices them: the analytic half of [1] (Walsh coefficients, an implicit function theorem on $O(4)\times\mathbb{R}^{10}$, a uniform Taylor estimate, a compactness argument); its symmetry reduction from $O(4)$ to the chart, which is finite and is the natural next target; and, inside the formal development, the identification of the 2096 integer coefficients used here with the tensor contractions they are meant to be — an identification carried out by an independent computation outside the proof assistant and checked against the source's own data files, but not itself formalised (§2).

## 2. The polynomial system

Everything attributed to [1] in this section — the two polynomials, the quaternionic chart, and the list of the five contractions — is from §3 of [1]. The two displayed polynomials are transcribed from it; the expansion of the five contractions in the monomial basis was recomputed here from those two displays alone, and is compared below against the data file of [2].

**The two polynomials.** Write $x=(x_0,x_1,x_2,x_3)$ for a point of $\mathbb{R}^4$. The polynomials appearing in (1) are

$$\begin{aligned} P_A(x) = {} & -13x_0^3+36x_0^2x_1-9x_0^2x_2-15x_0^2x_3+15x_0x_1^2-45x_0x_2^2+42x_0x_3^2 \\ & -8x_1^3-30x_1^2x_2+18x_1x_2^2-\tfrac{6}{5}x_1x_2x_3+18x_1x_3^2-3x_2^3-30x_2x_3^2-13x_3^3, \end{aligned} \tag{2}$$

$$\begin{aligned} P_B(x) = {} & -24x_0^2x_1x_3-24x_0^2x_3^2+36x_0x_1x_3^2-12x_0x_2^2x_3 \\ & -12x_1^2x_2x_3-6x_1^2x_3^2-12x_1x_2^2x_3+12x_1x_2x_3^2-24x_2x_3^3. \end{aligned} \tag{3}$$

Let $A$ be the unique symmetric trilinear form on $\mathbb{R}^4$ with $A(x,x,x)=P_A(x)$ and $B$ the unique symmetric 4-linear form with $B(x,x,x,x)=P_B(x)$; concretely, the entry of $A$ at an index triple is the coefficient of the corresponding monomial in (2) divided by the number of distinct orderings of that triple, and likewise for $B$. Every entry of $B$ obtained this way is an integer, and so is every entry of $A$ except the six orderings of the index triple $\{1,2,3\}$, where the single term $-\frac{6}{5}x_1x_2x_3$ gives $A_{123}=-\frac{1}{5}$. We therefore work throughout with $5A$ and $5B$, which are integral. Scaling a multilinear form by a positive constant scales its contractions by that constant and moves no zero.

**The chart.** Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, with basis $e_0=1$, $e_1=i$, $e_2=j$, $e_3=k$. For $(x,y)=(x_1,x_2,x_3,y_1,y_2,y_3)\in Q=[-1,1]^6$ put

$$p=1+x_1i+x_2j+x_3k, \qquad r=1+y_1i+y_2j+y_3k, \qquad q_m=p\,e_m\,\bar r \quad (m=0,1,2,3).$$

The four $q_m$ are pairwise orthogonal of common length $\|p\|\|r\|$, so after normalisation they are the columns of the element of $SO(4)$ that the double cover $\mathrm{Spin}(4)=S^3\times S^3\to SO(4)$, $(p,r)\mapsto(x\mapsto px\bar r)$, attaches to $(p/\|p\|,\ r/\|r\|)$. The reduction in [1] concludes that *every possible common zero of the five contractions on $SO(4)$ is represented by a point of $Q$ in this chart*: the tensors are symmetric, so multiplying $p$ or $r$ by a quaternionic unit — which sign-permutes the coordinates and the columns — changes no vanishing, and one may therefore assume the first coordinate of each of $p,r$ is largest in modulus and rescale it to 1. Each $q_m$ is bilinear: of degree 1 in $x$ and 1 in $y$, and of degree at most 1 in each single variable.

**Definition 2.1.** The five *chart polynomials* are the functions $Q \to \mathbb{R}$

$$f_0 = 5A(q_1, q_2, q_3), \quad f_1 = 5A(q_0, q_2, q_3), \quad f_2 = 5A(q_0, q_1, q_3), \quad f_3 = 5A(q_0, q_1, q_2),$$
$$f_4 = 5B(q_0, q_1, q_2, q_3),$$

so that for $s \le 3$ the polynomial $f_s$ is the contraction of $5A$ against the three frame vectors other than $q_s$.

Because each $q_m$ is bilinear, $f_0, \dots, f_3$ have degree 3 in $x$ and 3 in $y$ and degree at most 3 in each single variable, while $f_4$ has degree 4 in $x$ and 4 in $y$ and degree at most 4 in each single variable. All five therefore lie in the space $V$ of real polynomials in six variables of degree at most 4 in each variable, $\dim V = 5^6 = 15\,625$; this is the ambient space used throughout, with $f_0, \dots, f_3$ sitting in it by degree elevation. Expanded in the monomial basis of $V$ the five have $316, 316, 316, 316$ and 832 nonzero coefficients, all integers — 2096 integers in all. These five counts are what the expansion computed here gives, and they are also the five entries of the `term_counts` field of the file `final_chart_polynomials.json` of [2]; the expansion agrees with that file coefficient for coefficient, with the one bookkeeping difference that [2] records its fifth component as $B(q_0, q_1, q_2, q_3)$ where Definition 2.1 uses $5B$, so the fifth agrees after the factor 5.

*Remark* 2.2 (the formal boundary). Everything proved below is proved about *those* 2096 *integers*. That they are the coefficients of the contractions of Definition 2.1 is established outside the formal development, by an independent implementation of the symmetric-tensor and quaternion algebra applied to (2) and (3), whose output agrees with the corresponding data file of [2] in the sense just described. Closing that gap inside the proof assistant needs a small verified polynomial layer and is a matter of a few hundred lines with no computation; it was not attempted. See §8.

*Remark* 2.3 (the origin). At $x = y = 0$ one has $p = r = 1$ and hence $q_m = e_m$: the frame is the standard basis. So the five values at the origin are $5A_{123}, 5A_{023}, 5A_{013}, 5A_{012}$ and $5B_{0123}$, i.e. the four multilinear coefficients of $5A$ and the one of $5B$. Inspecting (2) and (3): the only multilinear monomial present anywhere is $-\frac{6}{5}x_1x_2x_3$, giving $5A_{123} = -1$, and the four others vanish. This is Proposition 4.3 below. It is not an accident of presentation. The introduction of [1] formulates the object of its search as a pair of homogeneous polynomials "such that there exists no orthonormal change of basis that makes the four multilinear terms of $P_A$ vanish and the single multilinear term of $P_B$ vanish", and immediately after displaying (2) and (3) it notes that "the presentation above has a single multilinear term for $P_A$ and none for $P_B$". The origin of the chart is exactly the point at which that observation is read off.

## 3. Bernstein arrays on a box

We recall the three facts about the Bernstein form that the proofs use, in the shape in which they are used. Let $b_k(t) = \binom{4}{k} t^k (1-t)^{4-k}$ for $0 \le k \le 4$ be the Bernstein basis of degree 4 on $[0, 1]$.

Let $R = \prod_{j=1}^{6} [\ell_j, h_j] \subseteq \mathbb{R}^6$ be a nondegenerate box and $g \in V$. Writing $u_j = (z_j - \ell_j)/(h_j - \ell_j)$ for the affine coordinates of $z \in R$, there are unique reals $c_\alpha(g, R)$, indexed by $\alpha \in \{0, \dots, 4\}^6$, with

$$g(z) = \sum_\alpha c_\alpha(g, R) \prod_{j=1}^{6} b_{\alpha_j}(u_j) \qquad (z \in R). \tag{4}$$

We call $(c_\alpha(g, R))_\alpha$ the *Bernstein array* of $g$ on $R$ and the values $c_\alpha$ its *control values*. Since each $b_k$ is nonnegative on $[0, 1]$ and $\sum_k b_k \equiv 1$, the products in (4) are a partition of unity by nonnegative functions on $R$, and (4) exhibits $g(z)$ as a convex combination of the control values. Two consequences:

**Lemma 3.1** (separation). *Let $g_0, \dots, g_4 \in V$ and $w \in \mathbb{R}^5$. If $\sum_s w_s c_\alpha(g_s, R) > 0$ for every $\alpha \in \{0, \dots, 4\}^6$, then $\sum_s w_s g_s(z) > 0$ for every $z \in R$; in particular $g_0, \dots, g_4$ have no common zero in $R$.*

**Lemma 3.2** (range bound). *Let $g \in V$ and $K \geq 0$. If $|c_\alpha(g, R)| \leq K$ for every $\alpha$, then $|g(z)| \leq K$ for every $z \in R$.*

The third fact is that the Bernstein array of a half of $R$ is computed from that of $R$ by an explicit linear map. Write $L, U$ for the $5 \times 5$ matrices

$$L = \frac{1}{16}\begin{pmatrix} 16 & & & & \\ 8 & 8 & & & \\ 4 & 8 & 4 & & \\ 2 & 6 & 6 & 2 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \qquad U = \frac{1}{16}\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ & 2 & 6 & 6 & 2 \\ & & 4 & 8 & 4 \\ & & & 8 & 8 \\ & & & & 16 \end{pmatrix}.$$

Then the Bernstein array of $g$ on the lower (resp. upper) half of $R$ along axis $j$ is obtained from that of $g$ on $R$ by applying $L$ (resp. $U$) along axis $j$: this is the de Casteljau subdivision step at the midpoint. Likewise, the passage from the monomial coefficients of $g$ on $Q = [-1, 1]^6$ to the Bernstein array of $g$ on $Q$ is the sixfold application, one axis at a time, of a fixed $5 \times 5$ matrix $P$; the substitution $t \mapsto 2t - 1$ is built into it.

**Integer scaling.** All three matrices become integral after clearing denominators: $16L$, $16U$ and $6P$ have integer entries, and the three scalings are positive, so they change no sign and no zero. The development therefore works throughout with the scaled arrays

$$C_\alpha(g, R) \;=\; 6^6 \cdot 16^d \cdot c_\alpha(g, R), \tag{5}$$

where $d$ is the depth at which the box $R$ occurs in the subdivision: the entries are integers when $g$ has integer monomial coefficients, at the root the array is $6^6 = 46\,656$ times the true Bernstein array of $g$ on $Q$, and each bisection multiplies by a further 16. At the maximum depth 60 reached by the certificate this is a factor of about $2^{240}$, which is why the arithmetic has to be either exact or carefully enclosed — and why 128-bit registers do not suffice for the exact route.

## 4. The nonvanishing, in exact integer arithmetic

**The certificate and the checker.** The certificate is a preorder traversal of a binary tree, distributed with [1] as the file `certificate/grunbaum_certificate.txt` of [2] and re-encoded here as a string: a byte `0`–`5` is an internal node that bisects the box along that coordinate, its two children following immediately and the lower half first; a byte `L` begins a leaf and is followed by five comma-separated integers, the covector of that leaf. Neither the child order nor the normalisation of the next paragraph is stated in the body of [1]; both are specified in the file `certificate/CERTIFICATE_FORMAT.md` of [2], which says "left child before right child" and "the left child is the lower half and the right child is the upper half". The same file's header line records `witness_scale` $= 2^{52}$, and the covector entries are indeed integers of modulus at most $2^{52}$, with equality attained.

The checker verified here carries, at each node, the five integer arrays $C_\alpha(f_s, R)$ of (5) for the current box $R$. It descends by applying $16L$ or $16U$ along the indicated axis, and at a leaf it reads the covector $w$ and tests

$$\sum_{s=0}^{4} w'_s\, C_\alpha(f_s, R) \;>\; 0 \quad \text{for all } \alpha \in \{0, \dots, 4\}^6, \qquad w' = (180w_0, 180w_1, 180w_2, 180w_3, 383w_4). \tag{6}$$

The rescaling of the last component has the following origin. The checker of [2] "divides every component by the maximum absolute value of that component's exact root Bernstein coefficients", a positive componentwise normalisation. For the five polynomials of Definition 2.1 those maxima are

$$M_s := \max_\alpha |C_\alpha(f_s, Q)| = 142\,953\,984 \quad (s \le 3), \qquad M_4 = 67\,184\,640, \tag{7}$$

on the integer scale (5), so $M_s/6^6$ is 3064 for $s \le 3$ and 1440 for $s = 4$; their ratio is exactly 383/180. Multiplying the normalised covector $(w_s/M_s)_s$ by the positive constant $180M_0 = 383M_4 = 25\,731\,717\,120$ clears the denominators and gives $w'$. One bookkeeping point: [2] builds its fifth component from $B$ rather than $5B$, so the fifth maximum it computes is $M_4/5$; the two factors of 5 cancel and the cleared covector $w'$ is the same. The values (7) are themselves machine-checked here, and sharply (Theorem 5.1), so the normalisation is a verified property of the data and not an assumption about it. We stress that the pair $(180, 383)$ is not part of the theorem: by Lemma 3.1 the leaf test is sound for *any* fixed $w'$, and the constants matter only in that the certificate does not pass without them.

**Theorem 4.1.** *The five chart polynomials of Definition 2.1 have no common zero on $Q = [-1,1]^6$: for all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$ with $|x_1|, |x_2|, |x_3|, |y_1|, |y_2|, |y_3| \le 1$, the five values $f_0, \ldots, f_4$ at that point are not all zero.*

*Proof sketch.* The proof has a computed half and a proved half, and the division between them is the point of the section.

*Computed.* A single deterministic pass over the certificate string. It performs $2 \cdot 38\,856$ subdivision steps, each applying one of the two integer matrices $16L, 16U$ along one axis to each of five arrays of 15 625 integers, and 38 857 leaf tests, each evaluating (6) at all 15 625 control indices: 607 140 625 strict inequalities between integers in total, on integers of a few hundred bits. There is no search: the tree, the axis at each node and the covector at each leaf are all read from the certificate of [2]. All arithmetic is exact — $\mathbb{Z}$ throughout, no fixed scale, no rounding — so the conclusion does not depend on the outward enclosure or on the scale $2^{52}$ used by the checker of [2]. The pass is the one thing here whose result is accepted from a computation rather than derived (§7); it takes about 1 h 47 m. What the kernel receives from it is a single Boolean; the counts 38 856, 38 857 and 607 140 625 just quoted are properties of the certificate string, read off it outside the kernel and agreeing with the tree statistics printed in [1] and [2].

*Proved.* Everything else, by induction on the walk. The induction hypothesis at a node mentions no geometry: it knows only the five *functions* $g_0, \ldots, g_4$ on the unit cube $[0,1]^6$ that the node's five integer arrays denote in the Bernstein form (4). At a leaf, (6) together with Lemma 3.1 gives $\sum_s w'_s g_s(u) > 0$, which contradicts $g_0(u) = \cdots = g_4(u) = 0$. At an internal node bisecting axis $k$, the de Casteljau identity says the children denote $16\, g_s(u$ with $u_k \mapsto u_k/2)$ and $16\, g_s(u$ with $u_k \mapsto (1+u_k)/2)$; and for a point $u$ of the cube either $u_k \le 1/2$, and $u$ is the image of a point of the cube under the first substitution, or $u_k \ge 1/2$ and it is the image of one under the second. So the recursion covers the cube *by construction*, and no hypothesis that the tree tiles $Q$ is needed anywhere. At the root, the identity relating the monomial coefficients of $f_s$ to $6^6$ times its Bernstein array on $Q$ transports the conclusion back to $Q$ through $z = 2u - 1$. Each of the three change-of-basis identities is proved once for one axis — a linear-algebra identity between two degree-4 polynomials in one variable — and lifted to all six axes by a single lemma that pulls a scalar out of one level of the nesting. □

**Controls.** Exhaustive computations are only as good as the predicate they evaluate, so the checker is guarded by negative controls: statements that are *false*, checked by the same pipeline that proves the theorem.

**Proposition 4.2.** (1) *The zero covector separates nothing: the leaf test* (6) *with $w = 0$ fails already on the root box.*
(2) *The covector attached to the certificate's first leaf fails the leaf test on the root box.*

(3) *The same covector passes the leaf test on its own box — the box reached by taking the lower half at each of the certificate's first nineteen bisections, namely* $[-1, -\frac{3}{4}]$ *in each of* $x_1, x_2, x_3, y_1, y_3$ *and* $[-1, -\frac{7}{8}]$ *in* $y_2$*.*
(4) *Negating that covector makes the same leaf reject.*

Item (2) is the one that shows the subdivision is doing work rather than decorating a fact that one covector could establish on the whole cube; a count made outside the formal statement finds 7674 of the 15 625 root control indices on the wrong side. Items (3) and (4) show respectively that (2) is a statement about the box and not about the covector, and that the smallest possible corruption of the certificate is detected. The next proposition is a control of a different kind — it refutes a strengthening of Theorem 4.1 — and is also the input to §6.

**Proposition 4.3.** *At the origin of the chart,* $f_0 = -1$ *and* $f_1 = f_2 = f_3 = f_4 = 0$*. In particular no four of the five polynomials have the property of Theorem 4.1: none of the five may be dropped.*

*Proof sketch.* Formally, five lookups of a single coefficient: at the origin the Bernstein form is not needed and the value of $f_s$ is the constant term of its monomial expansion, so the claim is the evaluation of five entries of the coefficient arrays, done by exact integer computation and cast to $\mathbb{R}$ through the six-axis evaluation lemma. Conceptually it is Remark 2.3: the origin sees exactly the multilinear coefficients of $5A$ and $5B$, and only $5A_{123} = -1$ is nonzero. □

## 5. The size of the system

The sup bounds come from the mirror image of Lemma 3.1. Lemma 3.2 applied to the *root* array alone — no subdivision, no certificate — bounds $f_s$ on all of $Q$ by the largest modulus of its Bernstein coefficients there, and those maxima turn out to be exactly $6^6$ times two small integers.

**Theorem 5.1.** *For every* $z \in Q = [-1, 1]^6$*,*

$$|f_0(z)|\,,\ |f_1(z)|\,,\ |f_2(z)|\,,\ |f_3(z)| \ \le\ 3064, \qquad |f_4(z)| \ \le\ 1440.$$

*Moreover the maxima* (7) *of the integer root arrays* $C_\alpha(f_s, Q) = 6^6 c_\alpha(f_s, Q)$ *are exact:*

$$\max_\alpha |C_\alpha(f_s, Q)| = 142\,953\,984 = 6^6 \cdot 3064 \quad (s \le 3), \qquad \max_\alpha |C_\alpha(f_4, Q)| = 67\,184\,640 = 6^6 \cdot 1440,$$

*in the sense that the bound holds and the bound one smaller fails, for each of the five components.*

*Proof sketch. Computed.* Two exhaustive scans of the same five root arrays that Theorem 4.1 starts from: for each of the five components, all 15 625 entries are compared against $\pm K$ with $K$ as displayed, and then against $\pm(K - 1)$. The first scan succeeds for all $5 \cdot 15\,625 = 78\,125$ entries and the second fails, for each component; both are exact integer comparisons. *Proved.* Lemma 3.2, in the same six-axis form as Lemma 3.1: the one-axis statement is that a value of the Bernstein form lies between the smallest and the largest of its five control values, proved from nonnegativity of the $b_k$ and $\sum_k b_k = 1$, and it is lifted through the six axes. Applied to the root array it gives $6^6\,|f_s(z)| \le K$ on $Q$, and $142\,953\,984/6^6 = 3064$, $67\,184\,640/6^6 = 1440$. □

For four of the five components the Bernstein bound is not merely a bound. The corners of $Q$ are the points at which one Bernstein basis function per axis equals 1 and the others vanish, so the value of $f_s$ at a corner is one of the 64 control values $c_\alpha(f_s, Q)$ with $\alpha \in \{0, 4\}^6$; if the largest modulus over all 15 625 control values happens to be attained at such an $\alpha$, the bound of Theorem 5.1 is the exact sup norm. It is.

**Theorem 5.2.** *With coordinates* $(x_1, x_2, x_3, y_1, y_2, y_3)$*,*

$$f_0(-1,-1,1,-1,1,1) = -3064, \qquad f_1(-1,-1,1,1,1,-1) = -3064,$$
$$f_2(-1,-1,1,1,-1,1) = +3064, \qquad f_3(-1,-1,1,-1,-1,-1) = +3064.$$

*Hence* $\sup_Q |f_s| = 3064$ *exactly, for* $s = 0, 1, 2, 3$*.*

*Proof sketch. Computed.* Four evaluations of a polynomial with 2096 integer coefficients at a point of $\{-1,1\}^6$, carried out in $\mathbb{Z}$: the development carries an exact integer evaluator for the monomial form, so no real or floating-point arithmetic enters. *Proved.* A cast lemma identifying that integer evaluator with the real evaluation of $f_s$ at the image of the integer point, proved through the same six-axis lift as everything else; then Theorem 5.1 for the reverse inequality. □

**Proposition 5.3.** 3063 *is not a bound for* $|f_0|$ *on* $Q$*, and the hypothesis* $z \in Q$ *is not decoration:* $f_2(2,2,2,2,2,2) = -69\,420$*, more than twenty times the bound that holds inside the box.*

*Proof sketch.* The first is Theorem 5.2 for $f_0$. The second is one more exact integer evaluation, at the integer point $(2,2,2,2,2,2)$, through the same evaluator and cast lemma. □

*Remark* 5.4 (the fifth bound). For $f_4$ the Bernstein bound 1440 is *not* attained. Two computations outside the formal development locate the truth: an exhaustive exact evaluation over the 64 corners of $Q$ gives $\max|f_4| = 800$ there, and a multi-start pattern search in double precision over all of $Q$ reaches only about 1272, so the bound is roughly 13% loose. Neither number is machine-checked — both are stated here as measurements, not as theorems — and nothing below depends on them: a loose $S_4$ only weakens the margin statement of §6. The exact sup norm of $f_4$ on $Q$ is left open (§8).

## 6. How tight the nonvanishing is

**Definition 6.1.** With $S = (S_0, \ldots, S_4) = (3064, 3064, 3064, 3064, 1440)$ the constants of Theorem 5.1, the *relative margin* of the system is

$$m \;:=\; \inf_{z\in Q}\; \max_{0\le s\le 4} \frac{|f_s(z)|}{S_s} \;\in\; [0,1].$$

*Remark* 6.2. The quantity $m$ does not depend on the normalisation of $A$ and $B$, nor on the factor 5 of Definition 2.1. If $f_s$ is replaced by $\lambda_s f_s$ with $\lambda_s > 0$, its Bernstein array on $Q$ and hence the constant $S_s$ are multiplied by $\lambda_s$ as well, and the ratio $|f_s|/S_s$ is unchanged. This is what makes $m$, rather than $\inf\max_s |f_s|$, the right measure of how narrowly the counterexample works.

**Theorem 6.3.** *There is a point of* $Q$ *— the origin of the chart — at which* $|f_s| \le S_s/3064$ *for every* $s$*. Consequently*

$$m \;\le\; \frac{1}{3064} \;\approx\; 3.26\cdot 10^{-4}.$$

*Proof sketch.* Nothing is computed here that was not computed already. By Proposition 4.3 the value vector at the origin is $(-1,0,0,0,0)$; by Theorem 5.1 the scale is $S$; and $1 \le 3064/3064$ while $0 \le S_s/3064$ for the other four components. The formal statement is the exhibition of that point together with the five inequalities. □

*Remark* 6.4 (the qualitative converse). $m > 0$. Indeed $z \mapsto \max_s |f_s(z)|/S_s$ is continuous and $Q$ is compact, so the infimum is attained, and by Theorem 4.1 it is attained at a point where not all five values vanish. This argument is not part of the formal development — it is the one place in this paper where compactness is used — and, more importantly, it is ineffective: it yields no explicit $\varepsilon$ with $m \ge \varepsilon$. An explicit lower bound is what the analytic half of [1] needs, and it is the main thing this paper leaves open (§8).

*Remark* 6.5 (the origin is optimal among integer points). No point of $\{-1,0,1\}^6$ gives a better bound than Theorem 6.3. At such a point all five values are integers, since the $f_s$ have integer coefficients; by Theorem 4.1 they are not all zero, so some $|f_s| \ge 1$, and then $\max_s |f_s|/S_s \ge 1/\max_s S_s = 1/3064$. The origin attains this. So improving the constant requires a genuinely rational point — as the searches below confirm. (The argument combines Theorems 4.1 and 5.1 with the integrality of the coefficients; the combination itself is not part of the formal development. A sweep over all 729 points of $\{-1,0,1\}^6$, made outside it, finds the origin to be the unique minimiser.)

**How close is $1/3064$ to the truth?** Two searches were run, neither of them part of the formal development, and both are reported here only to calibrate the constant.

An exact-rational coordinate descent on the dyadic grid of denominator 1024 reaches the point $z^* = (13, 4, -6, -22, 1, 2)/1024$, at which the ratio is exactly

$$\max_{0\le s\le 4} \frac{|f_s(z^*)|}{S_s} = \frac{70\,289\,946\,485\,184\,855}{220\,784\,468\,132\,211\,195\,904} = \frac{1}{3141.05\ldots}.$$

A multi-start pattern search in double precision over all of $Q$ (1500 starts, three random seeds) reaches $3.13787 \cdot 10^{-4} = 1/3186.9\ldots$, at $z \approx (0.0305, -0.0227, 0.0186, -0.0200, -0.0242, -0.0498)$, a point about 0.05 from the origin in the sup norm; the three seeds land within 1.3% of one another. At each of their optima the four ratios $|f_s|/S_s$, $s \le 3$, come out equal to within a percent or better — at one of the three they agree to ten digits — so the minimax is balanced across the four cubic components, while where $|f_4|/S_4$ falls below them varies a great deal from one optimum to the next. That last sentence is an observation about the output of a floating-point search, not a theorem, and nothing in this paper rests on it.

So the kernel-checked constant $1/3064$ is within about 4% of the apparent infimum, and the near-miss really is a neighbourhood of the origin.

*Remark* 6.6 (why the certificate is large). The subdivision tree of [1] has 38 857 leaves and 38 856 internal nodes, maximum total depth 60, and maximum depth $10, 10, 10, 10, 10, 11$ in the six coordinates respectively; these are counted here from the same string the checker consumes, and the leaf count and depth agree with the figures printed by the checker of [2]. Theorem 6.3 explains the order of magnitude without looking at the tree at all. The Bernstein enclosure of a polynomial on a box of side $h$ overestimates its range by $O(h^2)$: this is the classical quadratic convergence of the Bernstein range bound under repeated bisection, proved by Garloff [8] — his Theorem 4 (p. 48) states the enclosure error at bisection level $m$ as $\varepsilon(k-1)k^{-2}m^{-2}$, and the estimate its proof actually establishes is $\varepsilon(k-1)k^{-2} \cdot 2^{m+1}(2^m+1)(2^m-1)^{-4} = O(4^{-m}) = O(h^2)$ for $h = 2^{-m}$; the remark following that proof records that the same argument gives quadratic convergence in $r \ge 3$ variables. So separating a ratio of $1 : 3064$ requires boxes of side of order $2^{-6}$ before constants are supplied; the observed $2^{-10}$ to $2^{-11}$ per coordinate is the same statement with the constants put back. A subdivision proof of Theorem 4.1 cannot be small.

## 7. Verification

Every theorem and proposition above is machine-checked in Lean 4 [10] against Mathlib [11]; the two lemmas of §3 are machine-checked in the coordinates in which they are used, namely on the unit cube, the passage to a general box $R$ being the affine change of variables that defines the Bernstein array in the first place. The numbered remarks are commentary; each says where its content sits relative to the formal development, and several of them sit outside it. The checker of §4 — the nested integer arrays, the three $5 \times 5$ transforms, the certificate parser and the walk — is written in a fragment that imports nothing, in particular no part of Mathlib, so that it compiles to a shared library and runs as native code; its meaning, and every inequality above, is then supplied by ordinary proofs in a second layer that does import Mathlib. The exhaustive computations are discharged by Lean's `native_decide`, which evaluates the compiled decision procedure and records the outcome as a named axiom. It is worth being exact about where that boundary lies, because it is narrow. The soundness of the subdivision argument, the power-to-Bernstein change of basis, the two de Casteljau identities, the six-axis lift, the separation lemma 3.1, the range bound 3.2 and the exact integer evaluation at a point are all ordinary proofs, whose axiom sets contain only Mathlib's `propext`, `Classical.choice` and `Quot.sound`. Native evaluation is confined to *finite data*: for Theorem 4.1, the single Boolean saying that the walk consumes the whole certificate and every leaf test (6) holds; for Theorem 5.1, the ten Booleans saying that the five root arrays lie in $[-K, K]$ and not in $[-(K-1), K-1]$; for Theorem 5.2 and Proposition 5.3, five integer evaluations at integer points; for Proposition 4.3, five coefficient lookups; and for Proposition 4.2, four Boolean leaf tests. Theorem 6.3 adds no computation of its own. No statement uses `sorry`, and no module of the development declares an axiom. The costs are asymmetric: the certificate walk takes 1 h 47 m of wall time (1.75 CPU-hours, 7.2 GB peak memory) and is a one-off, while every other module — the controls and all of §5 and §6 — re-checks in under fifteen seconds. The method is not new: Muñoz and Narkawicz formalised multivariate Bernstein range enclosures in PVS and built verified proof strategies for polynomial global optimisation on them [9], and what is done here is a re-implementation of that idea in another system, aimed at one particular object. Repository reference to be added at submission.

Independently of Lean, and before it, the data and the certificate were reproduced outside the proof assistant: the five chart polynomials were derived from (2) and (3) by an implementation written here and agree with the corresponding file of [2] as described in §2; the C checker of [2] was compiled and reproduces its published output, including its minimum leaf margin 121 251 517; and an exact 384-bit integer replay of the whole certificate, with no interval arithmetic and no fixed scale, certifies all 38 857 leaves with minimum margin 35 780 923 117 598 146 560 on the scale (5) and with the covector of (6). None of these three is part of the formal development; they are recorded because they are what the formal development was checked against.

One point about the second of them deserves care, because it is what makes the exact replay a strengthening rather than a repetition. The file `CERTIFICATE_FORMAT.md` of [2] specifies that its checker stores an outward interval enclosure of the normalised root control data on the fixed scale $2^{52}$, in the form $L[c,\alpha] \le 2^{52} \cdot C[c,\alpha] \le U[c,\alpha]$, rounds subdivision endpoints outward, and accepts a leaf only when the integer $\sum_c w[c] \cdot (L[c,\alpha]$ if $w[c] \ge 0$ else $U[c,\alpha])$ — a rigorous lower bound for the true value, evaluated in signed `__int128` — is strictly positive at every control index. Acceptance by that test therefore implies strict positivity of the exact value, so it is a sufficient condition for what is proved here and not an extra assumption; the exact replay and the Lean development test that exact positivity directly, with no enclosure and no fixed scale, and so inherit no rounding hypothesis from it. This description is read from `CERTIFICATE_FORMAT.md` and from compiling and running the program, not from a line-by-line audit of its 2.4 MB source.

## 8. What remains

**Question 8.1.** Give an explicit $\varepsilon > 0$ with $\max_s |f_s(z)|/S_s \ge \varepsilon$ for all $z \in Q$.

This is the statement the analytic half of [1] actually consumes — a uniform margin, not bare nonvanishing — and Theorem 6.3 bounds it only from above; by Remark 6.4 such an $\varepsilon$ exists, and by §6 it is at most $3.184 \cdot 10^{-4}$, the exactly computed value at the dyadic point $z^*$. The route is clear and was priced but not taken: a second pass over the same certificate that accumulates the minimum leaf margin instead of a Boolean, with the target value already known from the exact replay quoted in §7. What makes it a project rather than a run is that the soundness induction has to be redone: the bound a leaf contributes is its margin divided by $6^6\, 16^d \|w\|_1$, and the depth $d$ is not constant over the tree, so the induction must carry the running scale as well as the covector.

**Question 8.2.** Formalise the symmetry reduction from $O(4)$ to the chart $Q$.

This is a finite argument — the double cover $\mathrm{Spin}(4) \to SO(4)$, and the normalisation that multiplies each of $p, r$ by a quaternionic unit so that its first coordinate is largest in modulus — about signed permutations of quaternion coordinates. Together with Theorem 4.1 it would upgrade the statement from one about a chart to one about $O(4)$, which is the form the rest of [1] uses. Two smaller gaps are of the same kind: the identification of Remark 2.2, which needs a verified multivariate polynomial layer and no computation at all, and the fifth sup norm of Remark 5.4. The analytic half of [1] is of a different order and is not a target here.

Finally, the machinery is not specific to this system. Everything in §3 is proved for an arbitrary tensor of multidegree at most $(4, \ldots, 4)$ in six variables; only the arity is baked in. Any polynomial positivity question over a box with a Bernstein subdivision certificate can be replayed the same way, with the same narrow trust boundary — one Boolean evaluation on the certificate the author of the certificate supplies.

## References


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[2] The ancillary package of the e-print [1]. It carries the two tensors $5A$ and $B$ and the five chart polynomials as data files (`data/final_tensors.json`, `data/final_chart_polynomials.json`), the subdivision certificate in a documented text format (`certificate/grunbaum_certificate.txt`, `certificate/CERTIFICATE_FORMAT.md`, `certificate/LEAF_BOXES_FORMAT.md`, `grunbaum_leaf_boxes.csv`), and two C checkers for it (`verify_grunbaum_tensors.c`, `checkers/check_grunbaum_text.c`) with their expected output (`expected_output.txt`).

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