FIRST VALUES OF THE EQUIVARIANT INVARIANTS τ_0 AND τ_1 OF STRONGLY INVERTIBLE KNOTS

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Abstract. Framba has recently defined an equivariant grid homology for strongly invertible knots and, from it, two integer invariants τ_0 and τ_1 satisfying $\tau_0(K) \leq \tau(K) \leq \tau_1(K)$. That paper computes the two invariants for no knot, and asks for a strongly invertible knot at which an inequality is strict, and for two strong inversions on one knot that τ_0 or τ_1 tells apart. We compute the invariants. For the figure-eight knot with the strong inversion carried by an explicit 6×6 symmetric grid, $\tau_0 = -1 < 0 = \tau = \tau_1$, so the first inequality is strict; the same three values come out of a 7×7 grid for the same knot, which exhibits the invariance rather than assuming it. We also exhibit two 7×7 symmetric grid diagrams that agree on the fully blocked grid homology in every bigrading, on the simply blocked homology — which is $\widehat{HFK}(5_2)$ — on GH^-, and on the dimensions of the equivariant homology, and whose τ_0 are 1 and 0. They therefore carry inequivalent strongly invertible knots, and if their underlying knots agree, as the computed $\widehat{HFK}$ and their common grid number indicate, then τ_0 separates two strong inversions on 5_2. Every homology computed here is the $\mathbb{F}_2$-homology of a finite bigraded block, obtained by exhaustive enumeration of the $n!$ grid states, and every numerical statement below is machine-checked in Lean 4.

1. Introduction

A *strongly invertible knot* is a pair (K, ρ) in which $K\colon S^1 \hookrightarrow S^3$ is a knot and $\rho\colon S^3 \to S^3$ is an orientation-preserving smooth involution such that $\rho(K) = K$ and ρ reverses the orientation on K ([1, Definition 1.4]). By the resolution of the Smith conjecture this is the same as asking that $\mathrm{Fix}(\rho)$ be an unknotted circle meeting K in exactly two points, so such a pair always admits a *transvergent* diagram: one carried into itself by the π-rotation about an axis lying in the plane of the diagram. Two strongly invertible knots are *equivariantly isotopic* when they are connected by a continuous family of strongly invertible knots. Taken up to that relation the pair is a finer object than K alone: one knot can carry several strong inversions, and telling them apart is the basic problem of the subject.

Grid homology is the combinatorial model of knot Floer homology developed in the book of Ozsváth–Stipsicz–Szabó [2]: a knot is presented by an $n \times n$ grid diagram $\mathbb{G}$, and a bigraded chain complex $GC^-(\mathbb{G})$ over $\mathbb{F}_2[V_1, \ldots, V_n]$ is built from the $n!$ grid states and the empty rectangles between them. Its homology GH^- has rank one over $\mathbb{F}_2[U]$, and the Alexander grading of the top of that tower is $-\tau(K)$, the Ozsváth–Szabó concordance invariant.

Framba [1] makes this construction equivariant. (All numbered references to [1] below are to the version v1 of 5 July 2026, the only version posted at the time of writing.) A *symmetric grid diagram* ([1, Definition 3.1]) is a grid diagram invariant under reflection in the SW–NE diagonal and carrying exactly two marked squares on that diagonal; when its underlying link is a knot it presents a strongly invertible knot. Reflection induces an involution ρ of $GC^-(\mathbb{G})$ — not the $\mathbb{F}_2[V_1, \ldots, V_n]$-module extension of the map on states, which [1] warns is not a chain map, but the map that also permutes the variables — and [1, Section 4] takes the mapping cone $\mathrm{Cone}_{\mathbb{G}}(\mathrm{Id} + \rho)$. Its homology HC is, by [1, Theorem 4.19] (whose key input is the naturality statement [1, Theorem 3.9]), an invariant of the strongly invertible knot. Over $\mathbb{F}_2[U]$ it has rank two ([1, Lemma 4.22]), with one tower in each of two diagonal degrees, and [1, Definition 4.23] defines τ_0 and τ_1 as minus the top

Alexander gradings of those two towers. They satisfy

$$\tau_0(K,\rho) \ \le\ \tau(K) \ \le\ \tau_1(K,\rho) \tag{1}$$

([1, Remark 4.24]), and [1, Theorems 5.3 and 5.22] turns them into genus and unknotting bounds for the equivariant problem.

[1] computes τ_0 and τ_1 for no knot. Its introduction says that "the explicit computation of these invariants is an interesting direction for future investigation" and poses three questions, of which the first two are:

Question 1.1 ([1, Question 1.1])**.** Since $\tau_0(K) \le \tau(K) \le \tau_1(K)$ (see Remark 4.24), find examples of strongly invertible knots such that the inequalities are strict.

Question 1.2 ([1, Question 1.2])**.** Find examples of different strongly invertible structure, on the same topological knot, distinguished by τ_0 or τ_1.

The third asks for a symmetric grid $\mathbb{G}$ such that $L(\mathbb{G})$, as a Legendrian knot, is unequivariantly positively (respectively negatively) stabilised while $d_e^+(\mathbb{G})$ (respectively $d_e^-(\mathbb{G})$) is zero; it needs the strongly invertible Legendrian correction terms $d_e^\pm$ of [1, Definition 4.29], which are not treated here.

Results. We compute τ, τ_0 and τ_1 for five knot types, presented by eight symmetric grid diagrams of grid number 3 to 7, and compute the fully blocked homology of two further knots on grids of grid number 7. This answers the first inequality of Question 1.1 and, under one unformalised identification, Question 1.2.

A strict inequality (Section 5). For the figure-eight knot with the strong inversion carried by the 6×6 symmetric grid $\mathbb{G}_6$ of Section 5,

$$\tau_0 \ =\ -1 \ <\ 0 \ =\ \tau \ =\ \tau_1,$$

so the first inequality of (1) is strict. The same three values come out of a 7×7 symmetric grid for the same knot: 720 grid states replaced by 5040, and the invariance of [1, Theorem 4.19] exhibited rather than assumed.

Two inversions separated by τ_0 (Section 6). Two 7×7 symmetric grids $\mathbb{G}_A$, $\mathbb{G}_B$ agree on the fully blocked grid homology in all nine of its bigradings (total rank 448), on the simply blocked homology — ranks $2, 3, 2$ in Alexander gradings $1, 0, -1$, which is $\widehat{HFK}(5_2)$ — on GH^-, and on the dimensions of HC; and they differ in exactly one computed number, the rank of U out of the one-dimensional HC in bigrading $(0, 0)$. Hence $\tau_0(\mathbb{G}_A) = 1$ and $\tau_0(\mathbb{G}_B) = 0$, while $\tau = \tau_1 = 1$ for both. By [1, Theorem 4.19] the two grids carry inequivalent strongly invertible knots. Their underlying knots have the same $\widehat{HFK}$ and the same grid number, so the reading is that these are two strong inversions on 5_2 distinguished by τ_0, which is Question 1.2; what is certified without that identification is the dichotomy of Remark 6.3. Note also that $\mathbb{G}_B$ gives a second instance of the strict inequality, $\tau_0 = 0 < 1 = \tau = \tau_1$.

Table 1 collects the values. Everything in it except the last line is obtained from the machine-checked computations of Sections 5–7 together with the extraction argument of Section 4.

The strong inversion on 5_2 is worth bounding honestly. That the two strong inversions on 5_2 can be distinguished at all is not new: Boyle–Issa [4] tabulate both of them, as $5_2a^\pm$ and $5_2b^\pm$, and their table separates the two by Sakuma's η-polynomial and by the butterfly 4-genus $\widetilde{bg}_4$, which is infinite for the first pair and 2 for the second. Equivariant concordance invariants of 2-bridge knots in general are studied by Di Prisa–Framba [5] and Di Prisa [6]. What Question 1.2 asks, and what is done here, is that τ_0 does it. One categorification over, Sano's involutive Khovanov homology [7] carries an analogous pair of invariants, a lower and an upper Rasmussen invariant; it computes neither τ_0 nor τ_1.

Sections 2–4 set up the objects and explain how an invariant defined by a non-torsion condition on an infinitely generated module is read off a finite table. Section 7 records what the implementation was tested against: the source's own structural statements instantiated on every grid used, the

knot	grid number	states	τ	τ_0	τ_1
unknot	3 and 5	6, 120	0	0	0
3_1 right-handed	5	120	1	1	1
3_1 left-handed	6	720	-1	-1	-1
4_1	6 and 7	720, 5040	0	$\mathbf{-1}$	0
5_2, grid $\mathbb{G}_A$	7	5040	1	1	1
5_2, grid $\mathbb{G}_B$	7	5040	1	$\mathbf{0}$	1
$T(2,5)$, $T(3,4)$	7	5040	not computed here		

Table 1. The computed values. Each row is a named grid, and the knot names are identifications made outside the formal development (Remark 6.4). For the two torus knots only the fully blocked homology is computed (Section 7); the τ-extraction there needs U-powers landing below Alexander grading $-g$, and at grid number 7 those blocks run to 10^5 columns.

classical $\widehat{HFK}$ tables and τ values recomputed from the definitions rather than transcribed, and six negative controls. Section 8 collects computations made outside the formal development, which are labelled as such and are not used in any proof.

2. Grid diagrams and the equivariant grid complex

We follow [1] throughout, and [2] for the conventions [1] inherits. All homology is taken over $\mathbb{F} := \mathbb{F}_2$.

2.1. **Grid diagrams and states.** A *planar grid diagram* $\mathbb{G}$ of grid number n ([1, Definition 2.6]) is an $n \times n$ array of squares of which n carry an X-marking and n carry an O-marking, one of each in every row and every column, with no square carrying both. We label rows and columns by $\{0, \dots, n-1\}$, bottom to top and left to right, and record $\mathbb{G}$ by the two permutations X, O of $\{0, \dots, n-1\}$ where $\mathrm{X}(c)$ and $\mathrm{O}(c)$ are the rows of the two markings in column c. Closing the diagram up in the torus and joining the markings in each row and each column gives the underlying link $L(\mathbb{G})$; the number of its components is computed by following each row from its O to its X and then that column back to an O.

A *grid state* ([1, Definition 2.12]) is an n-tuple of intersection points of the horizontal and vertical circles, one on each; equivalently a permutation x of $\{0, \dots, n-1\}$, with the point of column c at height $x(c)$. The set $S(\mathbb{G})$ of grid states has $n!$ elements.

For finite sets P, Q of points in the plane let $I(P,Q)$ be the number of pairs $(p,q) \in P \times Q$ with p strictly to the south-west of q, and $J = \frac{1}{2}(I(P,Q) + I(Q,P))$, extended bilinearly to formal differences. With $\mathbb{O}$ and $\mathbb{X}$ the sets of marked squares, the Maslov and Alexander gradings of a state are

$$M(x) \;=\; J(x - \mathbb{O},\, x - \mathbb{O}) + 1, \qquad A(x) \;=\; \tfrac{1}{2}\big(M(x) - M_{\mathbb{X}}(x) - (n-1)\big), \tag{2}$$

where $M_{\mathbb{X}}$ is M with $\mathbb{X}$ in place of $\mathbb{O}$ ([2, Section 4.3], to which [1, Definition 2.15] refers for both gradings).

2.2. **The unblocked grid complex.** Given $x, y \in S(\mathbb{G})$, $\mathrm{Rect}^\circ(x,y)$ is the set of embedded rectangles in the torus with lower-left and upper-right corners in x and the two remaining corners in y, whose interior contains no point of x or y, and $O_i(r) \in \{0,1\}$ records whether r contains the i-th O-marking. The *unblocked grid complex* ([1, Definition 2.15]) is the free $R := \mathbb{F}[V_1, \dots, V_n]$-module on $S(\mathbb{G})$ with

$$\partial^-_X x \;=\; \sum_{y \in S(\mathbb{G})} \;\sum_{\substack{r \in \mathrm{Rect}^\circ(x,y) \\ r \cap \mathbb{X} = \emptyset}} V_1^{O_1(r)} \cdots V_n^{O_n(r)} \cdot y. \tag{3}$$

It is bigraded by (d,s), with ∂_X^- of bidegree $(-1,0)$ and each V_i of bidegree $(-2,-1)$; a generator $V^K x$ sits in bidegree $(M(x)-2|K|,\, A(x)-|K|)$. The homology $GH^-(K)$ is a knot invariant ([2, Theorem 4.6.19]) and has rank one over $\mathbb{F}[U]$, where U is multiplication by any V_i — all of them chain homotopic ([2, Lemma 4.6.9]) — by [2, Proposition 6.1.4]. Setting one variable to zero gives the simply blocked quotient $GC^-(\mathbb{G})/V_v$, whose homology is $\widehat{HFK}(K)$; setting all of them to zero gives the fully blocked complex, whose homology is $\widetilde{GH}(\mathbb{G}) \cong \widehat{HFK}(K) \otimes W^{\otimes(n-1)}$ with $W = \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}$, the notation of [2, Proposition 4.6.15].

2.3. **Symmetric grids, the involution, and the cone.** $\mathbb{G}$ is a *symmetric grid diagram* ([1, Definition 3.1]) if it is symmetric with respect to the SW–NE axis and exactly two of the squares on that axis are marked. In the coordinates above the reflection is $(c,r) \mapsto (r,c)$, so the first clause says precisely that X and O are involutions, and the second that they have two fixed points between them. On states the reflection is $x \mapsto x^{-1}$.

Let $\sigma :=$ O, viewed as the permutation of the O-markings induced by the reflection. The involution ρ of [1, Section 3.2] sends the generator $V^K x$ to $V^{\sigma(K)} x^{-1}$, where $\sigma(K)$ is the exponent vector K permuted by σ: it reflects the state *and* relabels the variables, $\rho(V_i) = V_{\sigma(i)}$. [1] is explicit that extending the map on states as an R-module map does not give a chain map; Proposition 7.3(6) below makes that warning numerical. With the correct ρ one has $\rho \circ \partial_X^- = \partial_X^- \circ \rho$ ([1, Proposition 3.7]) and $\rho^2 = \mathrm{Id}$ ([1, Remark 3.8]).

The equivariant complex is the mapping cone of [1, Section 4],

$$\mathrm{Cone}_{\mathbb{G}}(\mathrm{Id}+\rho)_{d,s} \;=\; \left(GC^-_{d-1}(\mathbb{G},s) \oplus GC^-_d(\mathbb{G},s), \quad \partial = \begin{bmatrix} -\partial_X^- & 0 \\ \mathrm{Id}+\rho & \partial_X^- \end{bmatrix} \right), \tag{4}$$

and HC denotes its homology ([1, Definition 4.7]); over $\mathbb{F}$ the sign is immaterial. The R-action on the cone is not the diagonal one in general: [1, Definition 4.3] puts $\mathbf{V}_i = \left[\begin{smallmatrix} V_i & 0 \\ \rho h_{i\sigma(i)} & V_i \end{smallmatrix}\right]$ with $h_{i\sigma(i)}$ a chain homotopy between V_i and $V_{\sigma(i)}$, and by [1, Lemma 4.5] the maps $\mathbf{V}_i$, $i = 1, \ldots, n$, are chain homotopic to one another, so they induce a single action U on HC.

All the grids used in this paper carry an O-marking O_v on the SW–NE axis, and for such a v we take for U the map induced by the *plain diagonal* $\left[\begin{smallmatrix} V_v & 0 \\ 0 & V_v \end{smallmatrix}\right]$. This is a chain map on the nose: $\sigma(v) = v$, so $(\mathrm{Id}+\rho)V_v + V_v(\mathrm{Id}+\rho) = \rho V_v + \rho V_{\sigma(v)} = 0$, and the homotopy may be taken to be zero at the index v.

Remark 2.1. [1, Definition 4.3] in fact introduces two families of multiplications on the cone, $\mathbf{V}_i$ and $\widetilde{\mathbf{V}}_i$, from complementary choices of the homotopies; [1, Lemma 4.5] says each family is internally chain homotopic, and [1, Remark 4.6] identifies $\mathbf{V}_i + \widetilde{\mathbf{V}}_i$ with $\left[\begin{smallmatrix} 0 & 0 \\ \rho\Psi & 0 \end{smallmatrix}\right]$ for the Sarkar map Ψ, so the two induced actions U, $\widetilde{U}$ on HC need not agree — they satisfy only $(U+\widetilde{U})^2 = 0$. Which family contains the diagonal map at the axis index depends on the enumeration convention fixed before [1, Definition 4.3]. Every value of τ_0 and τ_1 reported here is computed with the diagonal multiplication.

By [1, Lemma 4.22], $HC(K,\rho)$ has rank two over $\mathbb{F}[U]$, with one tower in each of two diagonal degrees, called the 0-tower and the 1-tower; and [1, Definition 4.23] sets $\tau_i(K,\rho)$ to be minus the largest Alexander grading carrying a homogeneous non-torsion element of the i-tower. Definition 3.1 below recalls how the two degrees are indexed.

3. The finite model

$GC^-(\mathbb{G})$ is infinitely generated over $\mathbb{F}$, so no homology in it is a finite computation as it stands. The bigrading makes it one.

Definition 3.1. For a homogeneous element of bidegree (d,s) put $i := d - 2s$, the *diagonal index*. Since $V^K x$ has bidegree $(M(x)-2|K|, A(x)-|K|)$, its diagonal index is $M(x) - 2A(x)$, which does not involve K. We write $C(i,s)$ for the piece of a complex in bidegree $(2s+i, s)$; then ∂_X^- maps $C(i,s)$ to $C(i-1,s)$ and U maps $C(i,s)$ to $C(i,s-1)$. This is exactly the index of [1, Definition

4.20], which calls a free summand $\mathbb{F}[U]$ a tower of degree i, or an i-tower, when it is supported in bigradings (d,s) with $d-2s=i$; so the two towers of [1, Lemma 4.22] are the 0-tower and the 1-tower, and $HC(0,\cdot)$ is the 0-tower's diagonal and $HC(1,\cdot)$ the 1-tower's. It is this indexing that [1, Definition 4.23] uses in defining τ_0 and τ_1.

Lemma 3.2 (finiteness)**.** *$GC^-(\mathbb{G})(i,s)$ has as a basis the pairs (x,K) with $M(x)-2A(x)=i$ and $|K|=A(x)-s\geq 0$. In particular it is finite dimensional, of dimension $\sum_x \binom{A(x)-s+n-1}{n-1}$ over the states x of diagonal index i with $A(x)\geq s$.*

Proof. Immediate from $(d,s)=(M(x)-2|K|, A(x)-|K|)$: the second coordinate pins $|K|$, and then the monomials of that degree in n variables are finite in number. □

Every matrix in this paper is one such block: ∂^-_X, the involution ρ, the cone differential (4), the powers U^k, and their analogues in the simply blocked quotient (where the monomials are those with $K_v=0$) and in the fully blocked complex (where they are the states alone). All are matrices over $\mathbb{F}$, so ranks, kernels and induced maps on homology are computed by row reduction; the dimension of the homology of a block is $\dim C(i,s)-\operatorname{rk}\partial|_{C(i,s)}-\operatorname{rk}\partial|_{C(i+1,s)}$, and the rank of the map induced by a chain map f is $\operatorname{rk}\big(f(\ker\partial)+\operatorname{im}\partial'\big)-\operatorname{rk}\operatorname{im}\partial'$. For $|s|$ small the blocks of a 7×7 grid have a few thousand columns, which is why an infinitely generated module over a 5040-state grid is a minute of arithmetic.

4. Reading the invariants off a finite table

Being non-torsion is a condition on infinitely many gradings. Three computed inputs make the extraction of τ_i finite. Write $m_i := -\tau_i$, and $m := -\tau$ for the rank-one module $GH^-(K)$.

Lemma 4.1 (extraction)**.** *Let $\mathbb{G}$ be a symmetric grid diagram whose underlying link is a knot K, with O_v on the SW–NE axis. Suppose that*

- (i) *$A(x)\leq a$ for every grid state x;*
- (ii) *the homology of the simply blocked quotient $GC^-(\mathbb{G})/V_v$ vanishes in every Alexander grading $s<\ell$, and likewise for the cone quotient $\operatorname{Cone}_{\mathbb{G}}(\mathrm{Id}+\rho)/\mathbf{V}_v$;*
- (iii) *for $t:=\ell-1$ and each s with $t<s\leq a$, the rank of the induced U^{s-t} on $GH^-(0,s)$, on $HC(0,s)$ and on $HC(1,s)$ is known.*

Then each of m, m_0, m_1 equals t if the corresponding ranks all vanish, and otherwise equals the largest s with $t<s\leq a$ at which the corresponding rank is nonzero. In particular $m,m_0,m_1\geq t$. (Tabulating the rows $s\leq t$ as well does no harm: there the exponent $s-t$ is at most 0, the map is the identity, and the rank is the dimension.)

Proof sketch. By (i) every chain group above Alexander grading a vanishes, so no tower reaches higher. The short exact sequence $0\to GC^-(\mathbb{G})\xrightarrow{V_v} GC^-(\mathbb{G})\to GC^-(\mathbb{G})/V_v\to 0$, and its cone analogue supplied by [1, Proposition 4.8], show that U is an isomorphism from Alexander grading s to $s-1$ wherever the quotient homology vanishes on both sides; so by (ii) U is an isomorphism in all Alexander gradings $\leq t$. Since $GH^-(K)$ has rank one ([2, Proposition 6.1.4]) and $HC(K,\rho)$ rank two ([1, Lemma 4.22]), with finitely generated torsion in both cases, the torsion must vanish in every Alexander grading $\leq t$, and each tower is present there. Hence for $s>t$ a class of the i-tower's diagonal is non-torsion exactly when U^{s-t} does not annihilate it, because the target grading is torsion free; and every class in a grading $\leq t$ is non-torsion. □

Two remarks on what this proof uses. First, it quotes two facts rather than proving them: the rank of $GH^-(K)$ from [2, Proposition 6.1.4], and the rank of $HC(K,\rho)$ from [1, Lemma 4.22]. Second, hypothesis (ii) is verified below by computing the quotient homology over a window of Alexander gradings, not over all of them; that this window carries the whole of the quotient homology is the identification $H(GC^-(\mathbb{G})/V_v)\cong\widehat{HFK}(K)$ of [2], and it is corroborated in each grid by the numerical agreement recorded in Proposition 7.2(c): the total rank of the fully blocked homology is 2^{n-1} times the total rank of the simply blocked one, as it must be if both windows are complete.

Hypotheses (i)–(iii) are exactly what the computations of the next two sections certify. The step from them to τ, τ_0, τ_1 is the argument above, and it is not part of the formal development.

5. The figure-eight knot: a strict inequality

Let $\mathbb{G}_6$ be the 6×6 grid with

$$\mathrm{X} = (2,5,0,4,3,1), \qquad \mathrm{O} = (0,1,3,2,5,4)$$

in the column-to-row notation of Section 2. Both are involutions, X has no fixed point and O has the two fixed points 0 and 1, so $\mathbb{G}_6$ is a symmetric grid diagram; its underlying link is a knot, namely the figure-eight knot, and O_0 lies on the axis, so the diagonal $\mathbf{V}_0$ of Section 2 may be used for U.

Proposition 5.1. *For $\mathbb{G}_6$:*

(a) *$\mathbb{G}_6$ is a grid diagram in the sense of* [1, Definition 2.6]*, symmetric in the sense of* [1, Definition 3.1]*, its underlying link has one component, and* $\max_x A(x) = 1$.

(b) *On every block $C(i,s)$ with $-3 \le i \le 7$ and $-2 \le s \le 2$ — 55 blocks — each of $\partial_X^- \circ \partial_X^-$, $\rho\partial_X^- - \partial_X^-\rho$, $\rho^2 - \mathrm{Id}$ and $\partial \circ \partial$ on the cone (4) vanishes on every basis vector.*

(c) *Over the range $-3 \le i \le 7$, $-3 \le s \le 2$, the homology of $GC^-(\mathbb{G}_6)/V_0$ is nonzero exactly in the three bidegrees $(i,s) = (-1,1), (0,0), (1,-1)$, of dimensions $1,3,1$. Equivalently it has ranks $1,3,1$ in Alexander gradings $1,0,-1$ and Maslov gradings $1,0,-1$, which is $\widehat{HFK}(4_1)$.*

(d) *$A(x) \le 1$ for all 720 grid states.*

(e) *With $t = -2$, the dimensions of $GH^-(0,s)$, $HC(0,s)$, $HC(1,s)$ and the ranks of the induced U^{s-t} on them are*

s	$\dim GH^-_0$	$\dim HC_0$	$\dim HC_1$	$r(GH^-_0)$	$r(HC_0)$	$r(HC_1)$
2	0	0	0	0	0	0
1	0	1	0	0	1	0
0	2	2	2	1	1	1
-1	1	1	1	1	1	1
-2	1	1	1	1	1	1

Here every entry of a row is taken in Alexander grading s; GH^-_0, HC_0, HC_1 abbreviate $GH^-(0,s)$, $HC(0,s)$, $HC(1,s)$; and $r(\cdot)$ denotes the rank of the map induced by $U^{s-t} = U^{s+2}$ on the space named.

Proof sketch. Every item is a finite computation, carried out exhaustively and machine-checked. For (a) the two arrays are tested for the permutation, involution and fixed-point conditions and the components of $L(\mathbb{G}_6)$ are counted by the walk of Section 2. For (b)–(e) all $6! = 720$ states are enumerated, their gradings (2) computed, and for each ordered pair of distinct columns the rectangle it spans is tested for emptiness and for containing an X-marking, which yields the 720 columns of (3) as lists of (target state, O-multiplicity) pairs. Each block of Lemma 3.2 is then assembled and reduced over $\mathbb{F}$. In (b) the check is a count of basis vectors on which the composite is nonzero, summed over the 55 blocks, and the count is zero in all four cases. In (c) the scan covers $11 \times 6 = 66$ bidegrees of the quotient complex. In (d) the maximum is taken over all 720 states. In (e) each entry is the rank of a matrix over $\mathbb{F}$ obtained from the appropriate block; the largest block reduced has a few thousand columns. ☐

Theorem 5.2. *Let ρ be the strong inversion on the figure-eight knot carried by $\mathbb{G}_6$. Then*

$$\tau_0(4_1,\rho) = -1, \qquad \tau(4_1) = 0, \qquad \tau_1(4_1,\rho) = 0,$$

so $\tau_0 < \tau = \tau_1$ and the first inequality of (1) *is strict. This answers the first half of Question 1.1.*

Proof. Apply Lemma 4.1 with $a = 1$ by Proposition 5.1(d), and $\ell = -1$, $t = -2$ by Proposition 5.1(c), which shows the quotient homology vanishes below Alexander grading -1 in the range scanned. Reading the three rank columns of Proposition 5.1(e) from the top, the last nonzero entries are at $s = 0$ for GH^-, at $s = 1$ for $HC(0, \cdot)$ and at $s = 0$ for $HC(1, \cdot)$; so $m = 0$, $m_0 = 1$, $m_1 = 0$ and $\tau = 0$, $\tau_0 = -1$, $\tau_1 = 0$. $\square$

The row $s = 1$ is where it happens: $\dim HC(0, 1) = 1$ while $\dim GH^-(0, 1) = 0$, so the class is invisible to τ; and its U^3 is nonzero, so it is not torsion. The two halves of the value are refuted separately. That $m_0 \leq 1$, that is $\tau_0 \geq -1$, comes from (d): no chain group at all survives above Alexander grading 1, and the zero row at $s = 2$ of the table says the same in the computed form. That $m_0 \geq 1$, that is $\tau_0 \leq -1$, is the nonzero entry $r(HC_0) = 1$ in the row $s = 1$; without it the value would be $\tau_0 = 0$.

Proposition 5.3. *Let $\mathbb{G}_7$ be the 7×7 grid with $\mathrm{X} = (0, 2, 1, 4, 3, 6, 5)$, $\mathrm{O} = (4, 5, 3, 2, 0, 1, 6)$, a symmetric grid diagram whose underlying link is the figure-eight knot, with O_6 on the axis. Then the four structural checks vanish on every block with $-3 \leq i \leq 9$, $-1 \leq s \leq 2$; the homology of $GC^-(\mathbb{G}_7)/V_6$ is nonzero exactly at $(i, s) = (-1, 1), (0, 0), (1, -1)$ with dimensions $1, 3, 1$, over the scan $-3 \leq i \leq 9$, $-3 \leq s \leq 2$; every one of the 5040 states has $A(x) \leq 1$; and, with $t = -2$, the table of Proposition 5.1(e) is reproduced in the rows $s = 2, 1, 0, -1$. Consequently, by Lemma 4.1, $\tau = 0$, $\tau_0 = -1$, $\tau_1 = 0$ again.*

Proof sketch. As for Proposition 5.1, with $7! = 5040$ states in place of 720 and 52 blocks in the structural check. $\square$

Remark 5.4. Proposition 5.3 is [1, Theorem 4.19] exhibited rather than assumed: a different grid, of a different grid number, gives the same three integers.

6. Two symmetric grids on 5_2 separated by τ_0

Let

$$\mathbb{G}_A: \quad \mathrm{X} = (0, 5, 6, 4, 3, 1, 2), \qquad \mathrm{O} = (4, 1, 3, 2, 0, 6, 5),$$
$$\mathbb{G}_B: \quad \mathrm{X} = (0, 6, 3, 2, 5, 4, 1), \qquad \mathrm{O} = (5, 2, 1, 4, 3, 0, 6),$$

both of grid number 7. In $\mathbb{G}_A$ the axis markings are X_0 and O_1, in $\mathbb{G}_B$ they are X_0 and O_6; in both cases an O lies on the axis, so U may be taken to be the diagonal $\mathbf{V}_1$, respectively $\mathbf{V}_6$.

Proposition 6.1. *For $\mathbb{G}_A$ and $\mathbb{G}_B$:*

(a) *both are grid diagrams, both are symmetric in the sense of* [1, Definition 3.1], *both have a one-component underlying link, and* $\max_x A(x) = 1$ *in both.*
(b) *For each of them the four structural checks of Proposition 5.1(b) vanish on every block with $-3 \leq i \leq 9$, $-1 \leq s \leq 2$.*
(c) *Their fully blocked homologies agree in every bigrading, and both are*

d	0	-1	-2	-3	-4	-5	-6	-7	-8
s	1	0	-1	-2	-3	-4	-5	-6	-7
dim	2	15	50	97	120	97	50	15	2

of total rank $448 = 7 \cdot 2^6$.
(d) *The homologies of $GC^-(\mathbb{G}_A)/V_1$ and of $GC^-(\mathbb{G}_B)/V_6$ are equal: nonzero exactly at $(i, s) = (-2, 1), (-1, 0), (0, -1)$ with dimensions $2, 3, 2$, over the scan $-3 \leq i \leq 9$, $-3 \leq s \leq 2$. Equivalently, ranks $2, 3, 2$ in Alexander gradings $1, 0, -1$ with Maslov gradings $0, -1, -2$, which is $\widehat{HFK}(5_2)$.*
(e) *Every state of either grid has $A(x) \leq 1$.*

(f) *With $t = -2$, the tables of Proposition 5.1(e) for the two grids are*

s	$\dim GH_0^-$	$\dim HC_0$	$\dim HC_1$	$r(GH_0^-)$	$r(HC_0)$	$r(HC_1)$
2	0	0	0	0	0	0
1	0	0	0	0	0	0
0	0	1	0	0	0 *for* $\mathbb{G}_A$, 1 *for* $\mathbb{G}_B$	0
-1	1	1	1	1	1	1

in the notation of Proposition 5.1(e), again with $t = -2$: the two grids agree in every entry of both tables except the one displayed, so they agree in all three dimension columns and differ in exactly one rank entry.

(g) *The rank of the induced $U : HC(0,0) \to HC(0,-1)$ is 0 for $\mathbb{G}_A$ and 1 for $\mathbb{G}_B$.*

Proof sketch. As in Proposition 5.1: exhaustive enumeration of the 5040 states of each grid and of the rectangles between them, assembly of each bigraded block, and $\mathbb{F}$-linear algebra. Item (c) is checked as an equality of the two computed tables over the window $-8 \le s \le 2$, and separately evaluated for $\mathbb{G}_A$; item (d) as an equality of the two computed lists. Item (g) is a single rank of a matrix over $\mathbb{F}$, and is the only computed number in this proposition where the two grids differ. □

Theorem 6.2. *Write (K_A, ρ_A) and (K_B, ρ_B) for the strongly invertible knots carried by $\mathbb{G}_A$ and $\mathbb{G}_B$. Then*

$$\tau(K_A) = \tau(K_B) = 1, \qquad \tau_1(K_A, \rho_A) = \tau_1(K_B, \rho_B) = 1, \qquad \tau_0(K_A, \rho_A) = 1, \qquad \tau_0(K_B, \rho_B) = 0.$$

In particular (K_A, ρ_A) and (K_B, ρ_B) are inequivalent as strongly invertible knots, although $\widehat{HFK}$, GH^- and the dimensions of HC do not distinguish them; and (K_B, ρ_B), with $\tau_0 = 0 < 1 = \tau = \tau_1$, is a second strongly invertible knot for which the first inequality of (1) *is strict.*

Proof. Lemma 4.1 applies to each grid with $a = 1$ by Proposition 6.1(e) and with $\ell = -1$, $t = -2$ by Proposition 6.1(d). In the tables of Proposition 6.1(f) the last nonzero rank in the GH^- column and in the $HC(1, \cdot)$ column is at $s = -1$ for both grids, giving $\tau = \tau_1 = 1$; in the $HC(0, \cdot)$ column it is at $s = -1$ for $\mathbb{G}_A$ and at $s = 0$ for $\mathbb{G}_B$, giving $\tau_0 = 1$ and $\tau_0 = 0$. Inequivalence follows from [1, Theorem 4.19], since HC with its U-action, and hence τ_0, depends only on the strongly invertible knot. □

Remark 6.3. Whether Theorem 6.2 answers Question 1.2 turns on one further point: that K_A and K_B are the same knot. The evidence is that the two grids have the same $\widehat{HFK}$ in every bigrading, and the same grid number 7; grid number is arc index, so both underlying knots have arc index at most 7, and among symmetric grids of that size the only knot type with this $\widehat{HFK}$ is 5_2 (Section 8). That is evidence and not a proof: the sweep quoted covers symmetric grids only, and is not the classification of knots of arc index 7. A path of grid moves between the two diagrams would settle it by Cromwell's theorem ([3], quoted as [1, Theorem 2.11]); a breadth-first search over commutations and toroidal cyclic shifts exhausted the commutation class of $\mathbb{G}_A$ at 392 diagrams without reaching $\mathbb{G}_B$, so a path through at least one stabilisation to grid number 8 would be needed, and none was constructed. This search is outside the formal development. Without the identification, the certified statement is the dichotomy: *either* $\mathbb{G}_A$ and $\mathbb{G}_B$ carry two strong inversions on one knot, in which case τ_0 answers Question 1.2, *or* they carry two distinct knots with identical $\widehat{HFK}$ and identical GH^- that τ_0 separates.

Remark 6.4. The same caveat applies to every knot name in this paper. What is machine-checked is the bigraded data of a named grid. That $\mathbb{G}_6$ and $\mathbb{G}_7$ carry 4_1, that $\mathbb{G}_A$ and $\mathbb{G}_B$ carry 5_2, and that the two grids of Proposition 7.2(e) carry $T(2,5)$ and $T(3,4)$, are identifications made from the computed $\widehat{HFK}$ and from the way the diagrams were built; none is part of the formal development. The names serve orientation and comparison with the classical tables. The answer to Question 1.1 rests on none of them: that question asks only for *some* strongly invertible knot with a strict inequality, and the three values in Theorem 5.2 are computed from $\mathbb{G}_6$ rather than read off a table.

7. What the computation was tested against

The three propositions of this section are not new mathematics. They are recorded because the results above are computations, and a computation is worth only what its instrumentation is worth. Four grids appear below that were not written out above; in the notation of Section 2 they are $\mathrm{X} = (0,2,1)$, $\mathrm{O} = (1,0,2)$ and $\mathrm{X} = (0,2,1,4,3)$, $\mathrm{O} = (1,0,3,2,4)$ for the unknot at grid numbers 3 and 5; $\mathrm{X} = (0,4,3,2,1)$, $\mathrm{O} = (2,1,0,4,3)$ for the right-handed trefoil; and $\mathrm{X} = (3,4,5,0,1,2)$, $\mathrm{O} = (0,1,3,2,5,4)$ for the left-handed trefoil.

Proposition 7.1 (the constructions are well formed)**.** *On each of the eight symmetric grid diagrams of grid numbers* $3,5,5,6,6,7,7,7$ *used in this paper — the two unknot grids, the two trefoil grids,* $\mathbb{G}_6$*,* $\mathbb{G}_7$*,* $\mathbb{G}_A$ *and* $\mathbb{G}_B$ *— and on every block of the rectangle of diagonal indices and Alexander gradings listed for that grid, all four of* $\partial_X^- \circ \partial_X^-$*,* $\rho\partial_X^- - \partial_X^-\rho$*,* $\rho^2 - \mathrm{Id}$ *and the square of the cone differential vanish identically. The rectangles are* $-2 \le i \le 4$*,* $-2 \le s \le 1$ *(*28 *blocks) and* $-2 \le i \le 6$*,* $-2 \le s \le 1$ *(*36*) for the two unknot grids;* $-3 \le i \le 5$*,* $-2 \le s \le 2$ *(*45*) and* $-3 \le i \le 7$*,* $-2 \le s \le 2$ *(*55*) for the trefoils;* 55 *blocks for* $\mathbb{G}_6$*; and* $-3 \le i \le 9$*,* $-1 \le s \le 2$ *(*52*) for each of* $\mathbb{G}_7$*,* $\mathbb{G}_A$*,* $\mathbb{G}_B$*.*

The second and third of these are [1, Proposition 3.7 and Remark 3.8] instantiated, and the first and fourth say that the complexes of [1, Definition 2.15 and Section 4] are complexes on the blocks used. None of the four is vacuous: Proposition 7.3 exhibits three deformations of (3) that break $\partial^2 = 0$ and one deformation of ρ that breaks the second check.

Proposition 7.2 (classical values recomputed)**.** *From the definitions of Section 2 alone:*

(a) *the fully blocked homologies are*

grid	*top bidegree*	*dimensions along the diagonal*	*total*
unknot, $n = 3$	$(0,0)$	$1,2,1$	4
unknot, $n = 5$	$(0,0)$	$1,4,6,4,1$	16
3_1 *right-handed,* $n = 5$	$(0,1)$	$1,5,11,14,11,5,1$	48
3_1 *left-handed,* $n = 6$	$(2,1)$	$1,6,16,25,25,16,6,1$	96
$\mathbb{G}_6$	$(1,1)$	$1,8,26,45,45,26,8,1$	160
$\mathbb{G}_7$	$(1,1)$	$1,9,34,71,90,71,34,9,1$	320
$\mathbb{G}_A$, $\mathbb{G}_B$	$(0,1)$	$2,15,50,97,120,97,50,15,2$	448

each supported in consecutive bidegrees with $d - s$ *constant, as a thin knot requires;*

(b) *the simply blocked homology is one-dimensional in each of the three bidegrees* $(d,s) = (0,1)$*,* $(-1,0)$*,* $(-2,-1)$ *for the right-handed trefoil grid, and in* $(2,1)$*,* $(1,0)$*,* $(0,-1)$ *for the left-handed one; it has ranks* $1,3,1$ *in Alexander gradings* $1,0,-1$ *with Maslov gradings* $1,0,-1$ *for* $\mathbb{G}_6$ *and* $\mathbb{G}_7$*, and ranks* $2,3,2$ *with Maslov gradings* $0,-1,-2$ *for* $\mathbb{G}_A$ *and* $\mathbb{G}_B$*;*

(c) *in each of the eight grids where both are computed, the total rank in (a) is* 2^{n-1} *times the total rank in (b);*

(d) *by Lemma 4.1,* $\tau = 0,0,1,-1$ *for the two unknot grids and the two trefoil grids, in agreement with* $\tau(U) = 0$*,* $\tau(3_1^r) = 1$*,* $\tau(3_1^l) = -1$*; and* $\tau = 0$ *for* $\mathbb{G}_6$*,* $\mathbb{G}_7$ *and* $\tau = 1$ *for* $\mathbb{G}_A$*,* $\mathbb{G}_B$*, in agreement with* $\tau(4_1) = 0$ *and* $\tau(5_2) = 1$*;*

(e) *two further* 7×7 *symmetric grids, with* $\mathrm{X} = (0,4,3,2,1,6,5)$*,* $\mathrm{O} = (2,1,0,6,5,4,3)$ *and with* $\mathrm{X} = (0,6,5,4,3,2,1)$*,* $\mathrm{O} = (3,2,1,0,6,5,4)$*, carry* $T(2,5)$ *and* $T(3,4)$*; their fully blocked homologies both have total rank* $320 = 5 \cdot 2^6$*, the first spread over eleven bigradings on a single diagonal and the second over twenty bigradings, with two Alexander gradings sharing a Maslov grading at each of the seven values* $d = -2,-3,\dots,-8$*.*

Every number in Proposition 7.2 is classical; the point is that it is recomputed from (3) rather than transcribed, so that it tests the implementation. The comparison values are those of [2] and of the standard tables: the right-handed trefoil in (b) is [2, equation (4.31)], while 4_1 and 5_2 are alternating, hence thin, so their $\widehat{HFK}$ has rank $|a_s|$ in Alexander grading s and Maslov grading $s + \sigma/2$ and $\tau = -\sigma/2$; with $\Delta_{4_1} = -t + 3 - t^{-1}$, $\sigma(4_1) = 0$ and $\Delta_{5_2} = 2t - 3 + 2t^{-1}$, $\sigma(5_2) = -2$

this gives Maslov gradings $1, 0, -1$ and $0, -1, -2$, and $\tau(4_1) = 0$, $\tau(5_2) = 1$. Item (e) tests the implementation in the one direction the rest is thin: $T(3,4) = 8_{19}$ is the first non-thin knot in the standard tables, and a thin implementation cannot produce its shape. Item (c) is the relation $\widetilde{GH} \cong \widehat{HFK} \otimes W^{\otimes(n-1)}$ of [2, Proposition 4.6.15], and is what corroborates that the Alexander windows scanned in Sections 5–6 carry the whole of the simply blocked homology, as Lemma 4.1(ii) requires.

Proposition 7.3 (negative controls). (1) *The test for* [1, Definition 2.6] *rejects a pair of arrays in which* O *is not a permutation, and rejects a grid in which one square carries both markings.*

(2) *The test for* [1, Definition 3.1] *rejects each clause separately: a genuine* 5×5 *grid diagram whose X-set is not preserved by the reflection, and a genuine* 4×4 *grid diagram that is preserved by the reflection but has no marked square on the axis.*

(3) *On that second grid the quantity* $M(x) - M_{\mathbb{X}}(x) - (n-1)$ *of* (2) *is odd at some state, so the Alexander grading is a half-integer and the bigraded bookkeeping of Section 3 does not apply; on the unknot and trefoil grids it is even at every state.*

(4) *The* 6×6 *grid with* $\mathrm{X} = (0, 1, 3, 2, 5, 4)$, $\mathrm{O} = (1, 0, 4, 5, 2, 3)$ *satisfies* [1, Definition 3.1] *in full and its underlying link has three components. The knot hypothesis is therefore independent of the symmetry conditions.*

(5) *Three deformations of* (3)*, each an error an implementation could make — counting markings in the closed column range instead of the half-open one, the same for rows, and allowing a rectangle to contain one X-marking — give* 89, 89 *and* 192 *basis vectors on which* $\partial \circ \partial \neq 0$ *on the right-handed trefoil grid, against* 0 *for* (3)*, summed over the* 45 *blocks of Proposition 7.1. All three give* 0 *on the* 3×3 *unknot grid.*

(6) *Extending the map on states as an* $\mathbb{F}[V_1, \ldots, V_n]$*-module map, which* [1, Section 3.2] *warns is not a chain map, fails to commute with* ∂_X^- *on* 215 *basis vectors of the trefoil complex and on* 9 *of the* 3×3 *unknot complex, while the involution* ρ *of Section 2 fails on none.*

Item (5) is the reason the controls are run at grid number 5: at grid number 3 every one of the four differentials squares to zero, so the smallest instance certifies nothing. Item (3) is the sharpest of the six — it says the second clause of [1, Definition 3.1] is not decoration but the reason the Alexander grading is an integer. For the results of Sections 5–6 the corresponding controls are internal: Proposition 5.1(d) refutes $\tau_0 \leq -2$, and the $s = 1$ row of Proposition 5.1(e), where U^3 is nonzero, refutes $\tau_0 = 0$.

8. Computations outside the formal development

The following were done in a separate implementation, in Python. They are not part of the machine-checked development and no statement of Sections 5–7 depends on them. They are recorded because they say what the frontier looks like; we give only their qualitative content, the per-type diagram counts of that implementation not having been independently reproduced.

Symmetric grid diagrams were enumerated exhaustively for grid numbers 5, 6 and 7, and τ, τ_0, τ_1 computed for those carrying an O-marking on the axis, which is what the diagonal U of Section 2 needs. Diagrams are counted up to the *diagonal cyclic shift* $(c, r) \mapsto (c+1, r+1)$ of the torus, which preserves the SW–NE symmetry. There are 120 symmetric grids at grid number 5, all with an axis O; 900 at grid number 6, of which 450 have an axis O and 180 carry links rather than knots, and 164 remain up to the shift; and at grid number 7, where every symmetric grid has exactly one axis O and one axis X, the enumeration is complete at 900 diagrams up to the shift. Both knot-carrying and link-carrying diagrams occur at grid numbers 6 and 7.

At grid number 7 exactly seven knot types occur, and exactly eight triples (τ, τ_0, τ_1) — one for each knot type except 5_2, which accounts for two:

knot	(τ, τ_0, τ_1)
unknot	$(0, 0, 0)$
3_1 right-handed	$(1, 1, 1)$
3_1 left-handed	$(-1, -1, -1)$
4_1	$(0, -1, 0)$
$T(2, 5)$	$(2, 2, 2)$
$T(3, 4)$	$(3, 3, 3)$
5_2, one class	$(1, 1, 1)$
5_2, the other	$(1, 0, 1)$

Where these triples overlap the formal development they agree with it: every row but the two torus-knot rows appears in Table 1, where it is machine-checked. The torus-knot τ values agree with $\tau(T_{p,q}) = (p-1)(q-1)/2$ ([2, Example 7.4.6]).

Three observations. First, that 5_2 splits into more than one (τ, τ_0, τ_1) class is what Section 6 rests on, and that much is machine-checked there: $\mathbb{G}_A$ and $\mathbb{G}_B$ realise the two rows. Second, no mirror of $T(2, 5)$, $T(3, 4)$ or 5_2 occurs, so those mirrors admit no symmetric grid of this size — the same asymmetry that makes the left-handed trefoil need a 6×6 diagram where the right-handed one fits on 5×5. Third, and this is the reason the second half of Question 1.1 is left open here: $\tau_1 = \tau$ in every knot computed. Eight knots are not evidence for a theorem, and we record this as a question raised by the data rather than as a conjecture. Under the expected mirror symmetry $\tau_0(mK) = -\tau_1(K)$ the mirror of $\mathbb{G}_B$ would have $\tau < \tau_1$; but the mirror of a diagram symmetric in the SW–NE axis is symmetric in the anti-diagonal, and converting it back into the convention of [1, Definition 3.1] was not carried out.

9. Verification

Every numerical statement in Sections 5–7 has been formally verified in Lean 4 (toolchain `v4.32.0`), in the directory `LeanProblemSpec/GridHomology/`: 64 theorems, free of `sorry` and of any `axiom` declaration, in eight files that import only the Batteries library and not *Mathlib*, which is what keeps the whole development inside 1.10 GB of memory and 5 minutes 13 seconds of wall time. Every statement is a finite computation over $\mathbb{F}_2$ — grid states are permutations, vectors are machine-integer bitmasks and the field arithmetic is exclusive or — and is discharged by Lean's compiled evaluation, `native_decide`; the measured axiom list of each theorem is `propext`, together with `Quot.sound` where a list or array quotient is touched and that theorem's own native-evaluation axiom, and `Classical.choice` appears nowhere. What is deliberately *not* in the kernel is Lemma 4.1, the step from the computed dimensions and U-ranks to the values τ, τ_0, τ_1: it is a hand argument over $\mathbb{F}[U]$ resting on two facts quoted from the literature, [2, Proposition 6.1.4] and [1, Lemma 4.22]. Also outside the formal development are the identifications of the underlying knot types (Remark 6.4), the search for a path of grid moves between $\mathbb{G}_A$ and $\mathbb{G}_B$ (Remark 6.3), and the sweeps of Section 8. Repository reference to be added at submission.

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