THE RANKS OF PRIMITIVES IN THE GREEDY REGULAR CONVOLUTIONS

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Abstract. For each positive integer d, Snellman's *greedy convolution of block-length d* is the bounded homogeneous regular convolution obtained by partitioning $\mathbb{N}^+$ greedily into arithmetic progressions $c, 2c, \dots, rc$ with $r \le d$. The head c of such a progression is a *primitive* element and r is its *rank*. Snellman tabulates the least primitive of each rank for $2 \le d \le 16$, records that 36 of those 135 cells were not filled by his search, and asks whether a primitive of rank $d-1$ exists at all. We prove that none does, for any block-length: a primitive whose rank falls short of d has rank at most $d-2$, so the rank spectrum is contained in $\{1, \dots, d-2\} \cup \{d\}$. The proof is four lines and uses only the greedy rule; it settles the fifteen cells of the table that lie in the rank $d-1$ column, and it proves one clause of Snellman's conjecture for block-length 4. The same step yields three arithmetic corollaries the source does not state: a primitive of rank $r < d$ is divisible by some t with $1 < t \le d$; consequently every integer whose least prime factor exceeds d is a primitive of full rank d; and a primitive of rank $d-2$ is divisible by d. Finally, we fill the remaining twenty-one unfilled cells with explicit least witnesses, the largest being 5 000 000 at block-length 10 and rank 8, so that all 135 cells of the table are settled and the rank spectrum for $d \le 16$ is exactly $\{1, \dots, d-2\} \cup \{d\}$. Every numbered result is machine-checked in Lean 4.

1. Introduction

Regular convolutions. An *A-convolution* on the ring of arithmetical functions is given by a map A assigning to each $n \in \mathbb{N}^+$ a set $A(n)$ of divisors of n, through $(f *_A g)(n) = \sum_{e \in A(n)} f(e) g(n/e)$. Narkiewicz [2] classified the *regular* ones: they are exactly the convolutions obtained by attaching to each prime p a decomposition π_p of $\mathbb{N}$ into arithmetic progressions $\{0, t, 2t, \dots, rt\}$ containing 0, no two of which share a positive integer, and declaring $\prod p_i^{a_i} \in A(\prod p_i^{b_i})$ when for every i one has $a_i \le b_i$ with a_i, b_i in the same block of π_{p_i}. The convolution is *homogeneous* when all the π_p coincide, and *bounded* when the block lengths are bounded. The Dirichlet convolution is homogeneous and unbounded; the unitary convolution is homogeneous and bounded with all blocks of length one. For the general theory see [5].

For a bounded homogeneous regular convolution, a single partition π of $\mathbb{N}^+$ into finite progressions $\{c, 2c, \dots, rc\}$ carries all the data [1, Definition 5]: the head c of a block is a *primitive* element, the number r of its elements is the *rank* of c, and an integer $n = kc$ lying in the block headed by c has *height* $\mathbf{h}_d(n) = k$. Thus $\mathbb{N}^+ = \bigsqcup_{k=1}^{d} kM$ with M the set of primitive elements and d the bound on the block lengths.

The greedy convolutions. Fix $d \ge 1$. Snellman [1, Definition 7] defines the *greedy convolution of block-length d* by building the partition greedily. In his formulation one grows a rooted tree $T_d(n)$ on the vertices $\{0, 1, \dots, n\}$: $T_d(0)$ is the single vertex 0, and $T_d(n+1)$ is obtained from $T_d(n)$ by attaching $n+1$ as a leaf to a branch $0 \to k \to 2k \to \cdots \to mk$, extending it to $0 \to k \to \cdots \to mk \to (m+1)k$, whenever $n+1 = (m+1)k$ with $m+1 \le d$; among several such branches the one with the *smallest* k is chosen; and if there is none, $n+1$ is attached to the root, starting a new branch. Deleting the root turns the branches into the blocks of a partition of $\mathbb{N}^+$, and every block is a progression $c, 2c, \dots, rc$ with $r \le d$, so the construction is a bounded homogeneous regular convolution of block-length d. Block-length 1 gives the unitary convolution and block-length 2 the "ternary" convolution, a name coined in Gawell's thesis [3] (see [1, Lemma 9 and Remark 4]), and

these are the only two greedy convolutions in which all blocks have the same length [1, Corollary 13], by the theorem that for $d > 2$ some block is short [1, Theorem 12].

The three statistics are then read off the tree [1, Lemma 10]: n is primitive exactly when it is adjacent to the root; the rank of a primitive is the length of its branch, counted in edges, so it is a number in $\{1, \dots, d\}$; and the height $\mathbf{h}_d(n)$ is the distance from the root to n. We write $\rho_d(c)$ for the rank of a primitive c, and set $\rho_d(c) = 0$ when c is not primitive.

The question. In [1, Table 8] the source tabulates, for $2 \le d \le 16$ and $1 \le r \le d$, the least primitive of rank r for block-length d — 135 cells — announcing it with the sentence "A zero indicates the we have not found a primitive of the given type". Thirty-six of the cells are printed 0. Fifteen of them, one in each row, are the cell $r = d - 1$; whether that cell is empty is the first of the four questions the source asks about ranks [1, Questions 5–8]:

Question 1.1 ([1, Question 5])**.** Consider the greedy convolution of block-length $d > 2$. Are there primitives of rank $d - 1$?

The others ask for the ranks of primitives in general, for the smallest primitive of a given rank, and for the proportion and distribution of the ranks.

Very little was known. For $d = 2$ the greedy convolution is the ternary one [1, Lemma 9], whose blocks are the pairs $\{m, 2m\}$ [1, Theorem 3], so every block has length 2 and no primitive has rank $1 = d-1$; for $d = 3$ the block sizes are 3 and 1 [1, Theorem 14(i)], so no primitive has rank $2 = d-1$. For $d = 4$ the emptiness of the rank-3 cell is one of the three clauses of a conjecture:

Conjecture 1.2 ([1, Conjecture 1])**.** Let M denote the primitives of rank 4 for the greedy convolution of length 4.

- The primitives of rank one and two are all divisible by 8 or 9.
- "Most" primitives of rank one or two belong to $8M$ or $9M$.
- There are no primitives of rank 3.

The narrative text of [1] hedges the same clause: for block-length 4 there are "primitives of rank one, rank 2, and full rank 4, but seemingly no primitives of rank 3". The nearest general theorem is [1, Theorem 12], credited there to Gawell, that for $d > 2$ some block is short — that some length below d occurs, not which lengths are excluded.

Results. Question 1.1 has the answer *no*, at every block-length, and the reason is a property of the greedy rule rather than of the arithmetic of any particular d.

The rank spectrum (Section 3). If a primitive c has rank $r < d$ then $r \le d-2$ (Theorem 3.2). Hence no primitive has rank $d-1$, for any d, and the rank of a primitive always lies in $\{1, \dots, d-2\} \cup \{d\}$. The proof occupies four lines and uses only two features of the greedy rule: a height never exceeds the block-length, and the rule takes the largest admissible placement index.

The arithmetic that comes with it (Section 3). The same step exhibits a multiplier: a primitive of rank $r < d$ admits an m with $r + 2 \le m \le d$ and $m \mid (r + 1)c$, hence a divisor t of c with $1 < t \le d$ (Theorem 3.4). Two consequences: *every* integer with no divisor in $[2, d]$ — in particular every integer whose least prime factor exceeds d — is a primitive of full rank d (Corollary 3.5), which is an explicit infinite supply of full-rank primitives at every block-length; and a primitive of rank $d-2$ is divisible by d (Corollary 3.6), which is visible in every rank-$(d-2)$ cell of the table.

Block-length four (Section 4). The third clause of Conjecture 1.2 is the case $d = 4$ of Theorem 3.2 and is therefore a theorem (Corollary 4.1). The first clause remains open; Theorem 3.4 gives its unconditional weakening ($2 \mid c$ or $3 \mid c$ at rank one, $4 \mid c$ at rank two), and we verify the clause itself for every primitive below $4 \cdot 10^6$.

The completed table (Section 5). The twenty-one printed zeros outside the rank $d - 1$ column are search-limit artefacts, and we fill each with its least witness (Theorem 5.1), the largest being 5 000 000 at block-length 10 and rank 8. Together with Theorem 3.2 this settles all 135 cells and

shows that for $d \le 16$ the rank spectrum is exactly $\{1, \dots, d-2\} \cup \{d\}$; that answers [1, Questions 6 and 7] for those block-lengths.

Section 6 records what was searched and two traps in reading the source, and Section 7 describes the formal verification.

2. Preliminaries

Throughout, $d \ge 1$ is the block-length and c, n, r, m, j denote positive integers.

2.1. **The greedy rule as a recursion on heights.** The whole construction is carried by the height function $\mathbf{h}_d \colon \mathbb{N}^+ \to \{1, \dots, d\}$. Because a branch grows in the order $k, 2k, 3k, \dots$, the branch headed by k has length exactly $j-1$ at the moment $n = jk$ is placed if and only if $\mathbf{h}_d(ik) = i$ for every $1 \le i \le j-1$ (the case $i = 1$ says that k is primitive). Since $k = n/j$, choosing the smallest admissible head k is choosing the largest admissible index j. The source's rules therefore read:

Definition 2.1. Call j *admissible* for n if $2 \le j \le \min(d, n)$, $j \mid n$, and $\mathbf{h}_d\big(i \cdot (n/j)\big) = i$ for every $1 \le i \le j-1$. Then $\mathbf{h}_d(n)$ is the largest admissible j if one exists, and $\mathbf{h}_d(n) = 1$ otherwise; in the latter case n is *primitive*.

This is a recursion on n: the conditions on j refer only to the values $\mathbf{h}_d(i \cdot (n/j))$ with $i \cdot (n/j) \le n - n/j < n$. Three facts about it are all we use, and each is immediate from the search that defines $\mathbf{h}_d$.

Lemma 2.2. *For $d \ge 1$ and $n \ge 1$, $\mathbf{h}_d(n) \le \min(d, n)$; and $\mathbf{h}_d(n) \ge 1$.*

Lemma 2.3 (maximality of the greedy rule)**.** *If j is admissible for n, then $j \le \mathbf{h}_d(n)$.*

Lemma 2.4. *If $\mathbf{h}_d(n) \ge 2$ then $\mathbf{h}_d(n) \mid n$.*

Definition 2.5. For $c \ge 1$ put

$$\rho_d(c) \;=\; \max\big\{\, r \le d \;:\; \mathbf{h}_d(ic) = i \text{ for all } 1 \le i \le r \,\big\},$$

the length of the branch headed by c, with $\rho_d(c) = 0$ when no $r \ge 1$ qualifies, i.e. when c is not primitive.

So $1 \le \rho_d(c) \le d$ exactly when c is primitive, and ρ_d agrees with the rank of [1, Lemma 10].

2.2. **The matrix description.** Snellman also describes the same partition by a sweep over the columns of the multiplication table [1, Lemma 8]: form the $d \times N$ matrix $A^{(d,N)} = (ij)$; starting at column $c = 2$, run r from 1 to d and, as soon as $a_{r,c}$ has occurred in an earlier column, zero out $a_{\ell,c}$ for all $\ell \ge r$; the surviving columns are the blocks with head $\le N$. Equivalently: visit $c = 1, 2, 3, \dots$ in order and let an unclaimed c claim $c, 2c, 3c, \dots$ until it meets a claimed multiple or reaches length d. This description is division-free and much faster to run; the computations of Section 5 use it for two of the fifteen block-lengths, and Proposition 5.5 records the check that the two descriptions agree on the relevant range.

3. The rank spectrum

Lemma 3.1. *Let $d \ge 1$ and let $c \ge 1$ be primitive with $r = \rho_d(c) < d$. Then*

$$\mathbf{h}_d\big((r+1)c\big) \;\ge\; r + 2.$$

Proof. Write $n = (r+1)c$. By Definition 2.5 we have $\mathbf{h}_d(ic) = i$ for $1 \le i \le r$, while $\mathbf{h}_d(n) \ne r+1$, since otherwise $r+1 \le d$ would qualify in Definition 2.5 and the rank of c would be at least $r+1$. On the other hand $r+1$ is admissible for n: indeed $2 \le r+1 \le d$, $r+1 \le n$, $(r+1) \mid n$, $n/(r+1) = c$, and $\mathbf{h}_d(ic) = i$ for $1 \le i \le r$. Lemma 2.3 gives $\mathbf{h}_d(n) \ge r+1$, and with $\mathbf{h}_d(n) \ne r+1$ this forces $\mathbf{h}_d(n) \ge r+2$. □

In words: the branch headed by c stops at rc, so $(r+1)c$ went somewhere else; but it *could* have extended that branch, so the greedy rule sent it to a strictly smaller head $b < c$, and $(r+1)c = mb$ with $b < c$ forces $m > r+1$.

Theorem 3.2. *Let $d \geq 1$ and let $c \geq 1$ be primitive.*

(i) *If $\rho_d(c) < d$ then $\rho_d(c) + 2 \leq d$.*
(ii) *If $d \geq 2$ then $\rho_d(c) \neq d-1$:* no primitive has rank $d-1$, at any block-length.
(iii) *If $d \geq 2$ then $\rho_d(c) \in \{1, \ldots, d-2\} \cup \{d\}$.*

Proof. Put $r = \rho_d(c)$ and suppose $r < d$. By Lemma 3.1, $\mathbf{h}_d((r+1)c) \geq r+2$, and by Lemma 2.2, $\mathbf{h}_d((r+1)c) \leq d$. Hence $r+2 \leq d$, which is (i). For (ii), $\rho_d(c) = d-1$ would be a rank $< d$ with $\rho_d(c) + 2 = d+1 > d$, contradicting (i). For (iii), a primitive has $1 \leq \rho_d(c) \leq d$, and if the rank is not d then it is $< d$, so (i) bounds it by $d-2$. □

That is the whole argument. It is worth saying explicitly what it does *not* use: no computation, no arithmetic beyond divisibility, and no hypothesis on d beyond $d \geq 2$ in (ii)–(iii). In particular the fifteen cells of the source's table in the rank $d-1$ column are proved empty rather than searched, and the statement extends to block-lengths the table does not reach.

Remark 3.3. Theorem 3.2 is a property of the greedy rule, not of bounded homogeneous regular convolutions in general. For $d = 4$, take the partition of $\mathbb{N}^+$ whose first block is $\{1, 2, 3\}$ — closed there, so that 4 must start a block of its own — and which is completed greedily thereafter with blocks of length at most 4: it is a partition of $\mathbb{N}^+$ into progressions $c, 2c, \ldots, rc$ with $r \leq 4$, it has blocks of length 4 (for instance $\{4, 8, 12, 16\}$), and its primitive 1 has rank $3 = d-1$. By Narkiewicz's classification [1, Theorem 2] this partition defines a bounded homogeneous regular convolution, so no statement of the form "rank $d-1$ does not occur" can hold for that class. This remark is an illustration and lies outside the formal development: that the completed partition has maximal block length 4 was checked by an auxiliary program, on the heads up to $3 \cdot 10^5$.

The multiplier produced inside Lemma 3.1 is itself a placement index, so it divides the integer it placed. That gives the arithmetic content of the same step.

Theorem 3.4. *Let $d \geq 1$ and let $c \geq 1$ be primitive with $r = \rho_d(c) < d$. Then*

(i) *there is an m with $r+2 \leq m \leq d$ and $m \mid (r+1)c$;*
(ii) *consequently c has a divisor t with $1 < t \leq d$.*

Proof. Take $m = \mathbf{h}_d((r+1)c)$. Lemma 3.1 gives $m \geq r+2$, Lemma 2.2 gives $m \leq d$, and $m \geq 2$ with Lemma 2.4 gives $m \mid (r+1)c$; this is (i). For (ii) put $g = \gcd(m, r+1)$ and $t = m/g$. Cancelling g in $m \mid (r+1)c$ leaves $t \mid \big((r+1)/g\big)c$ with $\gcd\big(t, (r+1)/g\big) = 1$, so $t \mid c$. Finally $g \leq r+1 < r+2 \leq m$ gives $t = m/g > 1$, and $t \leq m \leq d$. □

Corollary 3.5. *Let $d \geq 1$ and $c \geq 1$ have no divisor t with $1 < t \leq d$ — for instance, let c be any integer whose least prime factor exceeds d. Then c is primitive and $\rho_d(c) = d$.*

Proof. If $\mathbf{h}_d(c) \geq 2$ then by Lemmas 2.2 and 2.4 the height $\mathbf{h}_d(c)$ is a divisor of c with $1 < \mathbf{h}_d(c) \leq d$, which is excluded; so $\mathbf{h}_d(c) = 1$ and c is primitive. If $\rho_d(c) < d$, Theorem 3.4(ii) produces a divisor of c in $(1, d]$, again excluded. Hence $\rho_d(c) = d$. □

This is an explicit infinite family of full-rank primitives at every block-length; the source has such a description for $d = 2$ and $d = 3$ only [1, Section 6], through the identifications of the primitives at $d = 2$ with the OEIS entry A003159 [1, Theorem 3] and of the full-rank primitives at $d = 3$ with A339690 [1, Theorem 14].

Corollary 3.6. *Let $d \geq 3$ and let $c \geq 1$ be primitive with $\rho_d(c) = d-2$. Then $d \mid c$.*

Proof. Here $r = d-2 < d$, so Theorem 3.4(i) gives m with $d = r+2 \leq m \leq d$, that is $m = d$, and $d \mid (r+1)c = (d-1)c$. Since $\gcd(d, d-1) = 1$, $d \mid c$. □

Corollary 3.6 is confirmed by every rank-$(d-2)$ entry of the completed table of Section 5:

$$6 = 3\cdot 2,\quad 8 = 4\cdot 2,\quad 40 = 5\cdot 8,\quad 72 = 6\cdot 12,\quad 504 = 7\cdot 72,\quad 96 = 8\cdot 12,\quad 34\,560 = 9\cdot 3840,$$
$$5\,000\,000 = 10\cdot 500\,000,\quad 99 = 11\cdot 9,\quad 5880 = 12\cdot 490,\quad 16\,900 = 13\cdot 1300,$$
$$21\,952 = 14\cdot 1568,\quad 332\,640 = 15\cdot 22\,176,\quad 724\,416 = 16\cdot 45\,276.$$

It also explains, in part, why the ranks just below $d-1$ are so thin. In a census made by an auxiliary program, outside the formal development, at block-length 16 and $c \le 2.5\cdot 10^7$ there are 7 primitives of rank 13 and 13 of rank 14, against 5 371 894 of rank 16.

4. Block-length four

Corollary 4.1. *For the greedy convolution of block-length* 4 *there are no primitives of rank* 3. *This is the third clause of Conjecture 1.2.*

Proof. The case $d = 4$ of Theorem 3.2(ii). □

The first clause of Conjecture 1.2 — that a primitive of rank one or two is divisible by 8 or by 9 — remains open, and Theorem 3.4 says exactly how much of it is unconditional. At $d = 4$ a primitive of rank 1 has an $m \in \{3, 4\}$ with $m \mid 2c$, whence $3 \mid c$ or $2 \mid c$; a primitive of rank 2 has $m = 4$ with $4 \mid 3c$, whence $4 \mid c$. The step from $4 \mid c$ to $8 \mid c$ or $9 \mid c$ is what is missing.

Proposition 4.2. *For block-length* 4*, among the primitives* $c \le 4\cdot 10^6$ *there are exactly* 189 241 *of rank one and* 214 380 *of rank two, and every one of them is divisible by* 8 *or by* 9.

Proof (exhaustive computation). The greedy partition of $1, \dots, 1.6\cdot 10^7$ is computed by the column sweep of Section 2, which makes the rank of every $c \le 4\cdot 10^6$ final, and the two counts and the divisibility test are read off it. This extends the published evidence, which is the 2598-term b-file of OEIS A382748 [6] and reaches primitives up to 5000, by a factor of about 800. It is a search, not a proof of the clause. □

5. The first primitive of each rank

For $1 \le r \le d$ and a bound B write $F_d(r, B)$ for the least primitive $c \le B$ with $\rho_d(c) = r$, and $F_d(r, B) = 0$ if there is none. A non-zero value of $F_d(r, B)$ is the least primitive of rank r outright, every smaller candidate having been examined, and we then write $F_d(r)$ for it. Snellman's table reports values of $F_d(r, B)$ for an unstated B, with 0 where the search failed. Table 1 is the completed table: an entry in bold is a cell printed 0 in the source, and an entry — is the cell $r = d-1$, proved empty by Theorem 3.2(ii).

Theorem 5.1. *For each block-length* $d \in \{8, 9, 10, 12, 13, 14, 15, 16\}$ *and each rank* r *listed below, the least primitive of rank* r *is*

$F_8(5) = 3360$	$F_9(7) = 34\,560$	$F_{10}(8) = 5\,000\,000$	$F_{12}(5) = 1120$
$F_{12}(9) = 5775$	$F_{12}(10) = 5880$	$F_{13}(5) = 1120$	$F_{13}(9) = 5775$
$F_{13}(11) = 16\,900$	$F_{14}(5) = 1815$	$F_{14}(9) = 161\,051$	$F_{14}(11) = 101\,088$
$F_{14}(12) = 21\,952$	$F_{15}(9) = 369\,754$	$F_{15}(11) = 67\,600$	$F_{15}(12) = 3120$
$F_{15}(13) = 332\,640$	$F_{16}(7) = 2187$	$F_{16}(11) = 22\,113$	$F_{16}(13) = 1\,286\,250$
$F_{16}(14) = 724\,416$			

These are the twenty-one cells printed 0 *in* [1, Table 8] *outside the rank* $d-1$ *column. Together with Theorem 3.2 it follows that every one of the* 135 *cells of that table is settled and that for* $2 \le d \le 16$ *the rank spectrum is exactly* $\{1, \dots, d-2\} \cup \{d\}$.

Proof (exhaustive computation). Each value is a witness, and each is verified least by an exhaustive sweep of an initial segment. Concretely, the greedy partition of $1, \dots, dB_d$ is computed, which makes the rank of every $c \le B_d$ final, and $F_d(r, B_d)$ is read off for every $r \le d$; the bounds used are

$$B_d = 10^5 \ \big(d \in \{2, \dots, 9, 11, 12, 13\}\big),\quad B_{10} = 5\cdot 10^6,\quad B_{14} = 2\cdot 10^5,\quad B_{15} = 4\cdot 10^5,\quad B_{16} = 1.3\cdot 10^6.$$

TABLE 1. The first primitive of each rank for the greedy convolutions of block-length 2 to 16. Bold: a cell printed 0 in [1, Table 8] and filled here. —: proved empty for every block-length (Theorem 3.2).

$d\backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	—	1														
3	6	—	1													
4	9	8	—	1												
5	9	8	40	—	1											
6	12	80	10	72	—	1										
7	12	80	10	35	504	—	1									
8	15	12	105	14	**3360**	96	—	1								
9	15	16	105	12	896	729	**34 560**	—	1							
10	18	16	15	162	14	63	567	**5 000 000**	—	1						
11	18	16	15	162	14	63	567	616	99	—	1					
12	21	20	125	18	**1120**	16	144	99	**5775**	**5880**	—	1				
13	21	20	125	18	**1120**	16	144	99	**5775**	143	**16 900**	—	1			
14	24	20	125	18	**1815**	352	22	99	**161 051**	143	**101 088**	**21 952**	—	1		
15	24	28	20	168	21	180	18	99	**369 754**	143	**67 600**	**3120**	**332 640**	—	1	
16	27	24	25	32	21	112	**2187**	20	880	143	**22 113**	195	**1 286 250**	**724 416**	—	1

Every value in Table 1 is below its bound, and the value 0 appears in the output only in the cell $r = d-1$. Sixteen of the twenty-one witnesses — all but the five at block-lengths 10 and 16 — were in addition computed from the height recursion of Definition 2.1 directly, at the bounds $B_d = 5 \cdot 10^4$ for $d \in \{2, \dots, 9, 11, 12, 13\}$, $B_{14} = 2 \cdot 10^5$ and $B_{15} = 4 \cdot 10^5$; the remaining five rest on the column sweep, that is, on [1, Lemma 8], together with Proposition 5.5. The emptiness of the cells $r = d-1$ is *not* part of this computation: it is Theorem 3.2(ii). □

The rank-8 cell at block-length 10 is the extreme one: the least primitive of rank 8 is $5\,000\,000 = 2^6 \cdot 5^7$, and in the whole range $c \le 4 \cdot 10^7$ only four primitives of rank 8 occur at all (again a count made outside the formal development). Corollary 3.6 accounts for the size of several of the new entries, since it forces $d \mid c$ in the rank-$(d-2)$ cell.

Reproduction of the published values.

Proposition 5.2. *All* 99 *non-zero cells of* [1, Table 8] *are reproduced by the computation described in Theorem 5.1, at every block-length* $2 \le d \le 16$.

This is the source's own data, recomputed from the definition rather than copied: twice outside the formal development by two independent programs, once from the tree recursion and once from the column sweep, and then inside it by both routes.

Proposition 5.3. *The height recursion of Definition 2.1 reproduces the five matrices* $A^{(d,20)}$ *for* $d = 1, \dots, 5$ *printed in* [1]*; of* [1, Table 7]*, at* $d = 4$*, the printed rank-one row (*15 *entries), both printed rank-two rows (*30 *entries) and the first two printed rank-four rows (*30 *entries); the first* 100 *terms of OEIS A382749 and the terms up to* 200 *of OEIS A382748* [6]*, two of the sequences registered from* [1]*; and* [1, Remark 5]*, that* 32 *is primitive at* $d = 4$ *although* $32/4 = 8$ *is primitive. The full b-files of A382748 (*2598 *terms) and A382749 (*5000 *terms) were matched outside the formal development.*

Proposition 5.4. *For every block-length* $3 \le d \le 16$*, the least primitive of rank one below* 100 *is* $3(d+1)/2$ *for odd* d *and* $3(d+2)/2$ *for even* d*, as* [1, Lemma 21] *proves for all* $d > 2$.

Proposition 5.5. *The tree recursion of Definition 2.1 and the column sweep of* [1, Lemma 8] *assign the same rank to every* $c \le 3000$*, at every block-length* $d \le 16$.

Propositions 5.2–5.5 are checks, not results: the first two confirm published data, the third confirms a published lemma, and the fourth confirms, on the range actually used, an equivalence the source proves.

Negative controls. A computation of this shape fails silently if a hypothesis is vacuous, if a truncation is too short, or if the proved theorem and the evaluated model are different objects. The following were checked against those failure modes.

Proposition 5.6. (i) *$\rho_4(8) = 2 = d - 2$ and 8 is primitive: the bound of Theorem 3.2(i) is attained, so "rank $d - 2$ is also empty" is false.*

(ii) *$\rho_4(9) = 1$ and 9 is primitive: not every primitive has full rank, so Corollary 3.5 cannot drop its hypothesis.*

(iii) *5 has no divisor t with $1 < t \le 4$, so the hypothesis of Corollary 3.5 is satisfiable; the corollary predicts, with no computation, that 5 is primitive of rank 4, and evaluating Definition 2.1 gives the same. The same example shows that the hypothesis $\rho_d(c) < d$ of Theorem 3.4 cannot be dropped.*

(iv) *$\mathbf{h}_4(12) = 2$ and $\rho_4(12) = 0$: primitivity is a real hypothesis.*

(v) *At block-length 8, no $c < 3360$ has rank 5; so any value smaller than 3360 in that cell of Table 1 is refuted.*

(vi) *At block-length 8, the ranks of all $c \le 3360$ read off the height table of length $8 \cdot 3360$ are unchanged when that table is doubled in length — the truncation assumption every entry of Table 1 makes.*

(vii) *For $d = 17, \dots, 24$, block-lengths the source does not tabulate, no primitive below 50 000 has rank $d - 1$, as Theorem 3.2(ii) predicts; a single hit would refute it.*

6. Provenance

Two features of the source deserve a warning, since either one would silently derail a reader.

First, the preprint has been revised four times and its internal numbering moves with the revisions, so every reference above is to version 4, submitted 14 August 2026. The rank table carries the same 135 entries in all four versions — v1 (3 April 2025), v2 (27 April 2025), v3 (29 April 2025) and v4 — but it is Table 3 in v1 and v2, Table 6 in v3 and Table 8 in v4; likewise Question 1.1 is Question 26, 31, 35 and 5 in the four versions. A reader working from a mirror that holds an earlier version therefore finds the data under none of the labels used here. What v3 added is the third clause of Conjecture 1.2: v1 and v2 state only the first two.

Second, a file distributed with the source of version 4 describes OEIS A382747 as [1, Table 8] "linearized as a triangle read by rows". It is not: A382747 is "Greedy partition of the positive integers into arithmetic progressions of length at most 4", a table by rows of length 4, which is also what the body of the preprint says about it. No OEIS entry contains Table 8, and no entry tabulates ranks for block-length 5 or more; the eleven OEIS sequences citing the preprint all concern block-length at most 4 or single-prime sieves. Taking the ancillary description at face value would make the whole of Section 5 look like already-published data.

The searches behind the claims of novelty were these. A full-text sweep of an arXiv mirror (85 GB of extracted text) for *greedy convolution*, *regular convolution*, *arithmetical convolution*, *Narkiewicz*, *block-length*, *unitary convolution*, *ternary convolution* and *selective sifting* returned 2942 hits in 1014 papers, and for *Gavell*, *Gawell*, *Snellman*, *primitives of rank*, *greedy partition*, *homogeneous convolution*, *Sita Ramaiah* and *Yocom* in addition; only [1] itself discusses greedy convolutions or ranks of primitives, and the three genuine neighbours (arXiv:2202.13512 [4], arXiv:2405.01278, arXiv:2503.11671) concern general A-functions, cyclotomic polynomials and Ramanujan sums and state nothing about ranks. Citation indexes list two citations of [1] in all, neither about ranks, and no journal version: arXiv carries no journal reference, OpenAlex records the item as a preprint with arXiv as its only location, and it has not appeared in the *Journal of Integer Sequences*. The classical literature cannot contain Theorem 3.2 for the reason given in Remark 3.3: block lengths are unconstrained for a non-greedy bounded homogeneous regular convolution.

The one place where a general statement about the rank spectrum could have hidden is Gawell's thesis [3], which [1] cites both for the name "ternary" and for its Theorem 12. It is not hard to obtain:

its full text is in Stockholm University's public course archive. It is where the ternary convolution is named — "We call this convolution *ternary convolution*" (p. 20) — and its two statements about which block lengths occur are the following, quoted in its own language, in which a *chain* is a block together with 0, so that a chain of length ℓ is a block of rank $\ell - 1$.

> **Proposition 3.6.** *If all chains in a Hasse diagram of posets $\{p^i | i \geq 0\}$ have the same length, then the length have to be 2, 3 or the diagram will be composed of only one infinite chain.* (p. 16)
>
> **Proposition 3.7.** *All diagrams, except for the one with just one chain of infinite length (i.e. the Dirichlet convolution), must contain chains of length 2 or 3.* (p. 17)

In the present notation: a partition all of whose blocks have one and the same rank is the unitary, the ternary or the Dirichlet one — [1, Corollary 13] is its bounded case — and every partition but Dirichlet's has a block of rank 1 or 2. Neither excludes a rank, and neither is parametrised by a block-length: the partitions the thesis builds greedily are built at one *fixed* chain length ("Let all chains be of length $r > 2$", p. 16, and the ternary case $r = 3$, p. 20), never with the length merely bounded above by d. Across the arXiv mirror the thesis is cited exactly once, by [4], which says nothing about ranks.

The literature searches, the OEIS entries, the citation indexes and the thesis were all read or re-run on 2026-08-29.

7. Verification

Every numbered lemma, theorem, corollary and proposition of this paper has been formally verified in Lean 4 against *Mathlib* [7, 8], and the development is free of `sorry`. Three things are not part of it and are marked as such where they appear: Remark 3.3, the census after Corollary 3.6, and the count of rank-8 primitives in Section 5; all three are auxiliary-program output, quoted as context. The mathematical layer is kernel-checked with no appeal to compiled evaluation: the height recursion of Definition 2.1, Lemmas 2.2–2.4 and 3.1, Theorems 3.2 and 3.4, Corollaries 3.5, 3.6 and 4.1, and the lemma identifying the memoised rank table with Definition 2.5 all depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: the fifteen rows of Table 1 behind Theorem 5.1 and Proposition 5.2, the count of Proposition 4.2, the reproductions of Propositions 5.3 and 5.4, the agreement of Proposition 5.5, and the controls of Proposition 5.6; those statements rest on the correctness of the compiler and of the evaluation. The values behind them were produced outside the formal development first, by five independently written implementations (three in Python and two in C) covering both of the source's descriptions of the partition, and the formal statements were then written against them. The complete run costs about eight minutes of compilation. Repository reference to be added at submission.

References

[1] J. Snellman, *Greedy Regular Convolutions*, arXiv:2504.02795v4 (submitted 14 August 2026; v1 3 April 2025). All numbering cited here is that of version 4.

[2] W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math. **10** (1963), no. 1, 81–94.

[3] E. Gawell, *Rings of arithmetic functions with regular convolutions*, Examensarbete i matematik 20 poäng, advisor J. Snellman; Examensarbeten i matematik 2005:8, Matematiska institutionen, Stockholms universitet, 2005. (The bibliography of [1] spells the author's name "Gavell".)

[4] J. V. Burnett and A. Taylor, *On a General Class of A-functions*, arXiv:2202.13512 (2022).

[5] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Pure and Applied Mathematics **126**, Marcel Dekker, New York, 1989.

[6] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A003159, A339690, A382747, A382748, A382749, read 2026-08-29.

[7] L. de Moura and S. Ullrich, *The Lean 4 Theorem Prover and Programming Language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635.

[8] The mathlib Community, *The Lean mathematical library*, in: Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.