THE 2-MODULAR ADJACENCY ALGEBRA OF $\mathrm{GQ}(2,2)$: WEDDERBURN DECOMPOSITION, CARTAN MATRIX AND GABRIEL QUIVER

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let $\mathcal{X}$ be the coherent configuration of type $[3,2;3]$ attached to the generalized quadrangle $\mathrm{GQ}(2,2)$ and let $\mathbb{F}_2\mathcal{X}$ be its 2-modular adjacency algebra, an algebra of dimension 10 over $\mathbb{F}_2$. Shimabukuro determined $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) = 4$ and $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X})^2 = 2$ by computer and asked for the Wedderburn decomposition of the semisimple quotient, the dimensions and Loewy series of the projective indecomposables, and the Gabriel quiver. We settle the first and the third. The semisimple quotient is $\mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_2 \times \mathbb{F}_2$; the two surviving types allowed by Shimabukuro's own constraints are separated by an idempotent count, 32 against 16. The Gabriel quiver has three vertices, with simple modules of dimensions $1,2,1$, and exactly two arrows, running in both directions between the point-fibre and the block-fibre vertex; the vertex carrying the two-dimensional simple module is isolated. The Cartan matrix is $\left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right)$, so the three projective indecomposables have dimensions $3,2,3$. We also show $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})^3 = 0$, which pins the Loewy length at exactly 3. All statements are machine-checked in Lean 4.

1. Introduction

The objects. A *coherent configuration* on a finite set X is a partition of $X \times X$ into relations whose 0/1 adjacency matrices span an algebra closed under multiplication and transposition and containing the identity. Higman [2] singled out the configurations of type $[3,2;3]$ (see also [3] for a rigorous account of his parameter equations): those attached to a *strongly regular design* $(P, B, \mathcal{F})$, carried on the disjoint union $X = P \sqcup B$ of a point fibre and a block fibre, with three relations inside each fibre and two between them. The span of the ten basis relations, taken over a field K, is the *adjacency algebra* $K\mathcal{X}$; it is a ten-dimensional associative K-algebra.

Over $\mathbb{C}$ the algebra is semisimple and its decomposition is classical ([1, Prop. 5.1]: $\mathbb{C}\mathcal{X} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$). Over a field of positive characteristic it need not be, and which characteristics are bad is governed by the Frame number: for association schemes this is due to Hanaki [4], and Sharafdini [7] extended the criterion to arbitrary coherent configurations. For adjacency algebras attached to designs with a single fibre the modular theory is developed in [6]. Shimabukuro [1] computes the Frame number of the point scheme of a partial geometry $\mathrm{pg}(s,t,\alpha)$ in closed form, describes the Jacobson radical of the point scheme in four arithmetic cases, and then turns to the smallest interesting example of a coherent configuration of type $[3,2;3]$: the one attached to the generalized quadrangle $\mathrm{GQ}(2,2) = \mathrm{pg}(2,2,1)$, at the prime $p = 2$.

The source and its open problem. For $\mathrm{GQ}(2,2)$ the point set and the block set both have 15 elements, so the ten basis matrices are 30×30 and $\mathbb{F}_2\mathcal{X}$ has $2^{10} = 1024$ elements. Shimabukuro proves $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{Y}) = 1$ for the point scheme $\mathcal{Y}$ and $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) \geq 2$ [1, Prop. 6.4(ii), Prop. 6.7], records the values $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) = 4$ and $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X})^2 = 2$ as unproved computer output [1, Rem. 6.9(3)], and states explicitly that the extensive computer calculations behind the work are not included. What is left open is recorded as a numbered problem [1, Problem 6.8]:

Problem 1.1 ([1, Problem 6.8])**.** Let $\mathcal{X}$ be the coherent configuration of type $[3,2;3]$ arising from $\mathrm{GQ}(2,2)$ and let $\mathbb{F}_2\mathcal{X}$ be its 2-modular adjacency algebra. Determine

(a) the Wedderburn decomposition of the semisimple quotient $\mathbb{F}_2\mathcal{X}/\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$;

(b) the dimensions and Loewy series of the projective indecomposable $\mathbb{F}_2\mathcal{X}$-modules;
(c) the Gabriel quiver of $\mathbb{F}_2\mathcal{X}$.

In particular, decide whether $\mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ is a direct product of finite fields or contains a matrix algebra over a finite field, and describe explicitly how the radical $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ links the simple components arising from the point and block fibers.

What this paper settles. We answer (a) and (c), and answer (b) at the level of dimensions.

The printed dichotomy — product of fields, or a matrix algebra — turns out to be settled already by the source's own results, and we say so before claiming anything: the proof of [1, Prop. 5.3] gives $\dim_{\mathbb{F}_2}\big(\mathbb{F}_2\mathcal{X}/[\mathbb{F}_2\mathcal{X},\mathbb{F}_2\mathcal{X}]\big) = 4$, hence $\sum_i f_i \leq 4$ for the Wedderburn type $\prod_i \mathrm{M}_{n_i}(\mathbb{F}_{2^{f_i}})$ of the semisimple quotient, while $\sum_i n_i^2 f_i = 6$ by [1, Rem. 6.9(4)]. Those two constraints leave exactly two types, and both contain a matrix algebra (Proposition 4.1). What they do not decide is *which* of the two, and that is the content of the first of our results.

- $\mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X}) \cong \mathrm{M}_2(\mathbb{F}_2)\times\mathbb{F}_2\times\mathbb{F}_2$ (Theorem 4.2): the semisimple quotient has *two* one-dimensional components rather than one copy of $\mathbb{F}_4$. We exhibit the isomorphism as a surjective ring homomorphism with kernel exactly $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$; the invariant that separates the two surviving types is the number of idempotents, 32 against 16.
- The Gabriel quiver of $\mathbb{F}_2\mathcal{X}$ has exactly two arrows and no loops (Theorem 5.2), running in both directions between the vertex supported on the point fibre and the vertex supported on the block fibre. This is the explicit description of how $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ links the simple components arising from the two fibres that Problem 1.1 asks for. The vertex carrying the two-dimensional simple module is isolated.
- There are four orthogonal primitive idempotents summing to 1, with corner dimensions $\dim_{\mathbb{F}_2} e_i(\mathbb{F}_2\mathcal{X})e_j$ given by a 4×4 matrix of total sum 10 (Theorem 5.1). Collapsing it along the one identification $e_2 \sim e_4$ gives the Cartan matrix $\left(\begin{smallmatrix}2&0&1\\0&1&0\\1&0&2\end{smallmatrix}\right)$ and three projective indecomposables of dimensions $3,2,3$ (Propositions 5.3 and 5.4).

Along the way the source's own computational values are given proofs: $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ is identified as the explicit ideal $\operatorname{Span}_{\mathbb{F}_2}\{\sigma_3,\sigma_6,\sigma_8,\sigma_{10}\}$ (Proposition 3.3), which recovers $\dim_{\mathbb{F}_2}\operatorname{Rad}(\mathbb{F}_2\mathcal{X}) = 4$ and $\dim_{\mathbb{F}_2}\operatorname{Rad}(\mathbb{F}_2\mathcal{X})^2 = 2$; and $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})^3 = 0$ (Proposition 3.4), which improves the source's "Loewy length at least 3" to exactly 3. The dimension values themselves are Shimabukuro's and are cited as such.

Part (b) of Problem 1.1 is answered here only in its first half. The dimensions of the projective indecomposables are determined, $3,2,3$; their Loewy series are not, and that is the honest gap of this paper (Remark 5.6).

Method. The algebra has 1024 elements and every question asked above is a finite one. Accordingly the proofs below have two layers: a short structural argument that reduces the question to a finite check, and the check itself. Both layers are formalized in Lean 4 against *Mathlib* [11], so that in particular the object called $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ is Mathlib's Jacobson radical — the intersection of the maximal left ideals — and not a decidable surrogate. Section 7 describes what is checked how.

2. The geometry and the algebra

$\mathrm{GQ}(2,2)$ **as the doily.** A generalized quadrangle of order (s,t) is an incidence structure of points and lines in which every line carries $s+1$ points, every point lies on $t+1$ lines, two distinct points lie on at most one common line, and for every point p off a line L there is exactly one point of L collinear with p. Up to isomorphism there is a unique generalized quadrangle of order $(2,2)$ [8]; we use its classical duad–syntheme model, the *doily*.

Definition 2.1. The points are the 15 two-element subsets (*duads*) of $\{1,\dots,6\}$; the lines are the 15 partitions of $\{1,\dots,6\}$ into three duads (*synthemes*); a point is on a line when its duad is one of the three parts. The *point graph* Γ_1 joins two duads when they are disjoint, i.e. when the two points are collinear; the *block graph* Γ_2 joins two distinct synthemes when they share a duad, i.e. when the two lines meet.

Proposition 2.2. *The incidence structure of Definition 2.1 has* 15 *points and* 15 *lines, three points on every line and three lines through every point, and satisfies the two generalized-quadrangle axioms; so it is a generalized quadrangle of order* $(2, 2)$*. Both* Γ_1 *and* Γ_2 *are strongly regular with parameters* $(15, 6, 1, 3)$*. Moreover the list of* 15 *synthemes is exactly the list of triples of pairwise disjoint duads.*

This is classical, and it is stated here because everything below is computed in this model: the matrices of the next subsection are the matrices of *these* relations, so the model has to be known to be the paper's object rather than assumed to be. The proof is a direct verification: the completeness of the list of lines is decided against all $\binom{15}{3} = 455$ triples of duads (three pairwise disjoint 2-subsets of a 6-set automatically partition it), and the incidence counts, the two axioms and the two strong regularity conditions are decided over the $15 \cdot 15$ point–line and vertex–vertex pairs.

The ten basis matrices. Write P for the point set and B for the block set, both of size $n_1 = n_2 = 15$, let A_1, A_2 be the adjacency matrices of Γ_1, Γ_2 and let N be the point–block incidence matrix. Write I_P, I_B for the identity matrices and $J_P, J_B, J_{P,B}$ for the all-one matrices of the appropriate sizes. The coherent configuration $\mathcal{X}$ of type $[3, 2; 3]$ on $X = P \sqcup B$ has the ten basis matrices displayed in [1, §2], written in 2×2 block form with respect to the decomposition $X = P \sqcup B$:

$$\sigma_1 = \begin{pmatrix} I_P & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} J_P - I_P - A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 0 & 0 \\ 0 & I_B \end{pmatrix}, \qquad \sigma_5 = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \sigma_6 = \begin{pmatrix} 0 & 0 \\ 0 & J_B - I_B - A_2 \end{pmatrix},$$

$$\sigma_7 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}, \qquad \sigma_8 = \begin{pmatrix} 0 & J_{P,B} - N \\ 0 & 0 \end{pmatrix}, \quad \sigma_9 = \begin{pmatrix} 0 & 0 \\ N^\top & 0 \end{pmatrix},$$

$$\sigma_{10} = \begin{pmatrix} 0 & 0 \\ J_{P,B}^\top - N^\top & 0 \end{pmatrix}.$$

All entries are read modulo 2 from now on, and $\mathbb{F}_2\mathcal{X} := \mathrm{Span}_{\mathbb{F}_2}\{\sigma_1, \ldots, \sigma_{10}\} \subseteq \mathrm{M}_{30}(\mathbb{F}_2)$.

Proposition 2.3. *The* $\mathbb{F}_2$*-span of* $\sigma_1, \ldots, \sigma_{10}$ *is closed under multiplication, with structure constants* $\sigma_i\sigma_j = \sum_k T_{ij}^k \sigma_k$*,* $T_{ij}^k \in \mathbb{F}_2$*, given by an explicit table; its identity element is* $\sigma_1 + \sigma_4 = I_{30}$*. The ten matrices are* $\mathbb{F}_2$*-linearly independent, so* $\dim_{\mathbb{F}_2} \mathbb{F}_2\mathcal{X} = 10$ *and* $|\mathbb{F}_2\mathcal{X}| = 1024$*. Furthermore* $J_P - I_P - A_1$ *and* $J_B - I_B - A_2$ *are the adjacency matrices of the complementary relations, so the two possible conventions for naming* A_1 *and* A_2 *merely exchange* $\sigma_2 \leftrightarrow \sigma_3$ *and* $\sigma_5 \leftrightarrow \sigma_6$ *and leave* $\mathbb{F}_2\mathcal{X}$ *unchanged.*

Proof sketch. Closure is the assertion that each of the 100 products $\sigma_i\sigma_j$ of 30×30 matrices over $\mathbb{F}_2$ equals the prescribed $\mathbb{F}_2$-combination of the basis; this is verified as 100 matrix identities. For linear independence it suffices to exhibit, for each k, one entry of σ_k equal to 1 at which all the other nine basis matrices vanish, and this is what the disjointness of the supports supplies; the 10×10 table of values of the basis at those ten positions is the identity matrix. The last sentence is the identity $J - I - A =$ (complement of A) for the point and block relations, checked entrywise. □

We also need the point fibre. Let $\mathcal{Y}$ be the rank-3 association scheme of Γ_1, with adjacency matrices I_P, A_1, $J_P - I_P - A_1$, and let $\mathbb{F}_2\mathcal{Y} = \mathrm{Span}_{\mathbb{F}_2}\{\sigma_1, \sigma_2, \sigma_3\}$ be its 2-modular adjacency algebra, of dimension 3 and with eight elements. Note that $\mathbb{F}_2\mathcal{Y}$ is *not* a unital subring of $\mathbb{F}_2\mathcal{X}$: its identity is $\sigma_1 = I_P$, not I_{30}. It is the corner $\sigma_1(\mathbb{F}_2\mathcal{X})\sigma_1$.

3. The Jacobson radical

We first reproduce the two claims of [1] that pin the model.

Proposition 3.1 ([1, Prop. 6.4(ii)]). $\mathrm{Rad}(\mathbb{F}_2\mathcal{Y}) = \mathbb{F}_2 \cdot B$ *with* $B = (A_1 - 6I_P)(A_1 - I_P)$ *and* $B^2 = 0$*; in particular* $\dim_{\mathbb{F}_2} \mathrm{Rad}(\mathbb{F}_2\mathcal{Y}) = 1$*. In characteristic* 2 *one has* $B = J_P - I_P - A_1$*, i.e.* B *is the image of* σ_3*.*

The scheme $\mathcal{Y}$ has rank 3, and the modular adjacency algebras of rank-3 association schemes are determined in general by Hanaki and Yoshikawa [5]; Proposition 3.1 is the special case at hand, and we reproduce it here only because the rest of the paper is computed in a fixed model of $\mathrm{GQ}(2,2)$ and has to be tied to the source's.

Proof sketch. Since Γ_1 is strongly regular with parameters $(15, 6, 1, 3)$ one has $A_1^2 = 6I_P + A_1 + 3(J_P - I_P - A_1)$, which modulo 2 reads $A_1^2 + A_1 = J_P - I_P - A_1$; so the source's B is σ_3 as claimed. For the radical itself, $\mathbb{F}_2\mathcal{Y}$ has eight elements. Let $R = \mathbb{F}_2 \cdot B$. Then R is an ideal and $R^2 = 0$, so for $x \in R$ and any y the element $1 + yx$ has the two-sided inverse $1 + yx$ itself, whence $R \subseteq \mathrm{Rad}(\mathbb{F}_2\mathcal{Y})$; conversely for each of the six elements $x \notin R$ one exhibits y and a nonzero k with $(yx + 1)k = 0$, so $yx + 1$ has no left inverse and $x \notin \mathrm{Rad}(\mathbb{F}_2\mathcal{Y})$. Each step is a check over the eight elements of $\mathbb{F}_2\mathcal{Y}$. □

Proposition 3.2 ([1, Prop. 6.7]). *The element* $u = \sigma_2\sigma_7$ *is nonzero, satisfies* $u^2 = 0$*, lies in* $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$*, and does not lie in* $\mathbb{F}_2\mathcal{Y}$*. In fact* $u = \sigma_8$*.*

The source's proof of [1, Prop. 6.7] adds that u^2 lies in the $\mathbb{F}_2$-span of $\{I_{15}, A_1, J_{15}\}$; since $u^2 = 0$ that holds trivially, and we record it as such. Propositions 3.1 and 3.2 are the reason to believe that the model of Section 2 is the source's: they are claims of [1] that were not assumed here, and they come out true.

Proposition 3.3. *The Jacobson radical of* $\mathbb{F}_2\mathcal{X}$ *— the intersection of its maximal left ideals — is*

$$\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) = \mathrm{Span}_{\mathbb{F}_2}\{\sigma_3, \sigma_6, \sigma_8, \sigma_{10}\},$$

a two-sided ideal with 16 *elements. In particular* $\dim_{\mathbb{F}_2} \mathrm{Rad}(\mathbb{F}_2\mathcal{X}) = 4$*, so* $\dim_{\mathbb{F}_2} \mathbb{F}_2\mathcal{X}/\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) = 6$.

The two dimension values are Shimabukuro's [1, Rem. 6.9(3),(4)], where they appear as computer output; what is supplied here is a proof, and the identification of the abstractly defined radical with the explicit ideal. No structure theory of Artinian rings is used.

Proof sketch. Write $R = \mathrm{Span}_{\mathbb{F}_2}\{\sigma_3, \sigma_6, \sigma_8, \sigma_{10}\}$; a check of the $16 \cdot 16$ sums and of the $1024 \cdot 16$ products in each order shows that R is a two-sided ideal, and a check of the $16^3 = 4096$ triples shows $R^3 = 0$. Both inclusions now follow from the elementary characterisation $x \in \mathrm{Rad}(A)$ if and only if $1 + yx$ is left invertible for every $y \in A$.

$R \subseteq \mathrm{Rad}(\mathbb{F}_2\mathcal{X})$: for $x \in R$ and any y put $w = yx \in R$. Then $(1 + w + w^2)(1 + w) = 1 + w^3 = 1$ because the characteristic is 2 and $w^3 = 0$, so $1 + yx$ is left invertible.

$\mathrm{Rad}(\mathbb{F}_2\mathcal{X}) \subseteq R$: for each of the $1024 - 16 = 1008$ elements $x \notin R$ we exhibit $y = y(x)$ and a nonzero $k = k(x)$ with $(yx + 1)k = 0$. If $z(yx + 1) = 1$ then $k = z\big((yx + 1)k\big) = 0$, a contradiction; so $1 + yx$ is not left invertible and $x \notin \mathrm{Rad}(\mathbb{F}_2\mathcal{X})$. The two tables $x \mapsto y(x)$, $x \mapsto k(x)$ are exhibited explicitly and the identity $(y(x)x + 1)k(x) = 0$ is checked for all 1024 elements at once.

Finally R has exactly 16 elements: they are the 16 subset sums of $\{\sigma_3, \sigma_6, \sigma_8, \sigma_{10}\}$ and these are pairwise distinct, which is the statement $\dim_{\mathbb{F}_2} R = 4$ in a form that needs no dimension theory. □

Proposition 3.4. $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})^2 = \mathrm{Span}_{\mathbb{F}_2}\{\sigma_3, \sigma_6\}$*, of dimension* 2*; it is nonzero, being generated by* $\sigma_8\sigma_{10} = \sigma_3$ *and* $\sigma_{10}\sigma_8 = \sigma_6$ *with both factors in the radical. Moreover* $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})^3 = 0$*, so the Loewy length of* $\mathbb{F}_2\mathcal{X}$ *is exactly* 3.

Here $\dim_{\mathbb{F}_2} \operatorname{Rad}(\mathbb{F}_2\mathcal{X})^2 = 2$ and the non-vanishing of the square are again the source's values [1, Rem. 6.9(3)]; the vanishing of the cube is the addition, and it turns the source's "the Loewy length is at least 3" into an equality. The proof is the check that all $16 \cdot 16$ products of radical elements have zero coordinates outside σ_3, σ_6, together with the two displayed products and the 4096-triple check of Proposition 3.3.

4. The Wedderburn decomposition

Write $\overline{\mathbb{F}_2\mathcal{X}} = \mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$. By Wedderburn's theorem $\overline{\mathbb{F}_2\mathcal{X}} \cong \prod_{i=1}^{r} \mathrm{M}_{n_i}(\mathbb{F}_{2^{f_i}})$ for some $r \geq 1$ and positive integers n_i, f_i.

What the source's own data already forces.

Proposition 4.1. *Let $r \geq 1$ and let $n_i, f_i \geq 1$ satisfy $\sum_{i=1}^{r} n_i^2 f_i = 6$ and $\sum_{i=1}^{r} f_i \leq 4$. Then, up to reordering,*

$$\prod_{i=1}^{r} \mathrm{M}_{n_i}(\mathbb{F}_{2^{f_i}}) \;\cong\; \mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_4 \qquad \text{or} \qquad \prod_{i=1}^{r} \mathrm{M}_{n_i}(\mathbb{F}_{2^{f_i}}) \;\cong\; \mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_2 \times \mathbb{F}_2.$$

In both cases exactly one factor is a matrix algebra of degree 2 over a finite field.

The two hypotheses are exactly the source's own: $\sum_i n_i^2 f_i = \dim_{\mathbb{F}_2} \overline{\mathbb{F}_2\mathcal{X}} = 10 - 4 = 6$ by [1, Prop. 5.3(ii)] together with [1, Rem. 6.9(3),(4)], and $\sum_i f_i \leq 4$ because the proof of [1, Prop. 5.3(i)] establishes $\dim_{\mathbb{F}_2}\big(\mathbb{F}_2\mathcal{X}/[\mathbb{F}_2\mathcal{X}, \mathbb{F}_2\mathcal{X}]\big) = 4$ and the commutator quotient of $\prod_i \mathrm{M}_{n_i}(\mathbb{F}_{2^{f_i}})$ is $\bigoplus_i \mathbb{F}_{2^{f_i}}$. So the either/or printed in Problem 1.1 is decided by the source's own data, and we claim nothing new for it. The content of Proposition 4.1 is only that the enumeration is short and exhaustive.

Proof. If $n \geq 3$ then $n^2 f \geq 9 > 6$, and if $f \geq 7$ then $n^2 f \geq 7 > 6$; so every factor is of the form $\mathrm{M}_1(\mathbb{F}_{2^f}) = \mathbb{F}_{2^f}$ with $1 \leq f \leq 6$, or $\mathrm{M}_2(\mathbb{F}_2)$. Let a_f be the number of factors $\mathbb{F}_{2^f}$ and let b be the number of factors $\mathrm{M}_2(\mathbb{F}_2)$. Then $\sum_f f\, a_f + 4b = 6$ and $\sum_f f\, a_f + b \leq 4$. Subtracting, $3b \geq 2$, so $b \geq 1$; and $4b \leq 6$ forces $b = 1$. Hence $\sum_f f\, a_f = 2$, which leaves $a_1 = 2$ or $a_2 = 1$, the two listed types. Both contain the factor $\mathrm{M}_2(\mathbb{F}_2)$. $\square$

The decomposition. Let $E_{k\ell} \in \mathrm{M}_2(\mathbb{F}_2)$ denote the matrix units and set $S = \mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_2 \times \mathbb{F}_2$, an $\mathbb{F}_2$-algebra of dimension 6 with 64 elements. Define an $\mathbb{F}_2$-linear map $\varphi\colon \mathbb{F}_2\mathcal{X} \to S$ on the basis by

$$\begin{aligned}
&\varphi(\sigma_1) = (E_{22}, 1, 0), && \varphi(\sigma_2) = (0, 1, 0), && \varphi(\sigma_3) = 0,\\
&\varphi(\sigma_4) = (E_{11}, 0, 1), && \varphi(\sigma_5) = (0, 0, 1), && \varphi(\sigma_6) = 0,\\
&\varphi(\sigma_7) = (E_{21}, 0, 0), && \varphi(\sigma_8) = 0, && \varphi(\sigma_9) = (E_{12}, 0, 0),\\
&\varphi(\sigma_{10}) = 0.
\end{aligned}$$

The two scalar components are the linear characters $x \mapsto x_1 + x_2$ and $x \mapsto x_4 + x_5$ in the coordinates of $x = \sum_k x_k \sigma_k$; the first is supported on the point fibre and the second on the block fibre. The $\mathrm{M}_2(\mathbb{F}_2)$ component is the action on the two-dimensional simple module $\mathbb{F}_2\mathcal{X} \cdot (\sigma_1 + \sigma_2 + \sigma_3)$.

Theorem 4.2. *φ is a surjective homomorphism of $\mathbb{F}_2$-algebras with kernel exactly $\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$. Consequently*

$$\mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X}) \;\cong\; \mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_2 \times \mathbb{F}_2,$$

of dimension $4 + 1 + 1 = 6 = 10 - 4$. This answers part (a) *of Problem 1.1: the semisimple quotient is not a direct product of finite fields, and the second surviving type $\mathrm{M}_2(\mathbb{F}_2) \times \mathbb{F}_4$ of Proposition 4.1 does not occur.*

Proof sketch. Multiplicativity is proved once and for all by the following observation: if B is any ring which is an $\mathbb{F}_2$-module and $f_1, \ldots, f_{10} \in B$ satisfy the *same* relations $f_i f_j = \sum_k T_{ij}^k f_k$ as the σ_i do (Proposition 2.3), then $x \mapsto \sum_k x_k f_k$ is multiplicative. So it suffices to check the 100 products $\varphi(\sigma_i)\varphi(\sigma_j) = \sum_k T_{ij}^k \varphi(\sigma_k)$ inside the 64-element algebra S, and to check $\varphi(\sigma_1) + \varphi(\sigma_4) = (I_2, 1, 1) =$

1. Surjectivity is the statement that the image of the 1024 elements of $\mathbb{F}_2\mathcal{X}$ meets all 64 elements of S, and $\ker\varphi = \operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ is the statement that for each of the 1024 elements x one has $\varphi(x) = 0$ if and only if the four-coordinate condition of Proposition 3.3 holds. The isomorphism is then the first isomorphism theorem, using Proposition 3.3 to identify the kernel with the Jacobson radical. For the last sentence, $\varphi(\sigma_7)\varphi(\sigma_9) \neq \varphi(\sigma_9)\varphi(\sigma_7)$, so the quotient is not commutative and in particular is not a direct product of fields. □

Remark 4.3 (the discriminator). Theorem 4.2 exhibits the isomorphism, so it does not need an invariant to separate the two types of Proposition 4.1. It is nevertheless worth recording that a single invariant does separate them, since it is the cheapest independent confirmation of the answer: the number of idempotents. Realise $\mathbb{F}_4$ inside $\mathrm{M}_2(\mathbb{F}_2)$ as $\{0, I_2, A, A^2\}$ with $A = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$; this set is closed under addition and multiplication, is commutative, and every nonzero member is invertible, so it is a field with four elements. Counting directly, $\mathrm{M}_2(\mathbb{F}_2)\times\mathbb{F}_2\times\mathbb{F}_2$ has $8\cdot 2\cdot 2 = 32$ idempotents while $\mathrm{M}_2(\mathbb{F}_2)\times\mathbb{F}_4$ has $8\cdot 2 = 16$; and $\mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ has 32.

5. Idempotents, the quiver and the Cartan matrix

A complete set of primitive idempotents. Put

$$e_1 = \sigma_2 + \sigma_3, \qquad e_2 = \sigma_1 + \sigma_2 + \sigma_3, \qquad e_3 = \sigma_5 + \sigma_6, \qquad e_4 = \sigma_4 + \sigma_5 + \sigma_6.$$

Thus $e_1 + e_2 = \sigma_1$ is the identity of the point fibre and $e_3 + e_4 = \sigma_4$ that of the block fibre; e_1 and e_2 are supported on the point fibre, e_3 and e_4 on the block fibre.

Theorem 5.1. *e_1, e_2, e_3, e_4 are pairwise orthogonal primitive idempotents with $e_1+e_2+e_3+e_4 = 1$, and the matrix of corner dimensions $\left(\dim_{\mathbb{F}_2} e_i(\mathbb{F}_2\mathcal{X})e_j\right)_{i,j}$ is*

$$\begin{pmatrix}2&0&1&0\\0&1&0&1\\1&0&2&0\\0&1&0&1\end{pmatrix},$$

whose entries sum to $10 = \dim_{\mathbb{F}_2}\mathbb{F}_2\mathcal{X}$. Its j-th column sum is $\dim_{\mathbb{F}_2}(\mathbb{F}_2\mathcal{X})e_j$, so the four direct summands of the regular module $\mathbb{F}_2\mathcal{X} = \bigoplus_j(\mathbb{F}_2\mathcal{X})e_j$ have dimensions $3, 2, 3, 2$.

Proof sketch. Idempotence, orthogonality and the sum are $4+12+1$ identities in $\mathbb{F}_2\mathcal{X}$. Primitivity of e_i is the statement that the only idempotents of the corner $e_i(\mathbb{F}_2\mathcal{X})e_i$ are 0 and e_i, decided by running v over the 1024 elements of $\mathbb{F}_2\mathcal{X}$ and testing the 1024 elements e_ive_i. The corner dimensions are obtained the same way: for each of the 16 pairs (i,j) the set $\{e_ive_j : v\in\mathbb{F}_2\mathcal{X}\}$ is computed and its cardinality is a power of 2, the exponent being the dimension. The cardinalities are $\left(\begin{smallmatrix}4&1&2&1\\1&2&1&2\\2&1&4&1\\1&2&1&2\end{smallmatrix}\right)$. The column sums are the dimensions of $(\mathbb{F}_2\mathcal{X})e_j$ because $1 = \sum_i e_i$ gives $(\mathbb{F}_2\mathcal{X})e_j = \bigoplus_i e_i(\mathbb{F}_2\mathcal{X})e_j$. □

The arrows. For a finite-dimensional algebra A with radical J and a complete set of orthogonal primitive idempotents e_i, the Gabriel quiver has one vertex per isomorphism class of simple modules, and the number of arrows from the class of e_i to the class of e_j is $\dim_{\mathbb{F}_2} e_i(J/J^2)e_j$; see [9, §III.1].

Theorem 5.2. *Write $J = \operatorname{Rad}(\mathbb{F}_2\mathcal{X})$. Then*

$$\left(\dim_{\mathbb{F}_2} e_iJe_j\right)_{i,j} = \begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}, \qquad \left(\dim_{\mathbb{F}_2} e_iJ^2e_j\right)_{i,j} = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},$$

so that

$$\left(\dim_{\mathbb{F}_2} e_i(J/J^2)e_j\right)_{i,j} = \begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}.$$

Hence the Gabriel quiver of $\mathbb{F}_2\mathcal{X}$ *has exactly two arrows, one from* e_1 *to* e_3 *and one from* e_3 *to* e_1*, and no loops. The idempotents* e_2 *and* e_4 *are equivalent, so they carry the same simple module — the two-dimensional one, i.e. the* $\mathrm{M}_2(\mathbb{F}_2)$ *block of Theorem 4.2 — and its vertex is isolated. Since* e_1 *is supported on the point fibre and* e_3 *on the block fibre, the two arrows are exactly the link between the simple components arising from the point and block fibres that Problem 1.1 asks to be described.*

Proof sketch. The two corner tables are computed as in Theorem 5.1, but running v over the 16 elements of J and over the 4 elements of J^2 respectively (Propositions 3.3 and 3.4); the cardinalities are $\left(\begin{smallmatrix} 2&1&2&1\\ 1&1&1&1\\ 2&1&2&1\\ 1&1&1&1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2&1&1&1\\ 1&1&1&1\\ 1&1&2&1\\ 1&1&1&1 \end{smallmatrix}\right)$, and $\dim e_i(J/J^2)e_j = \dim e_iJe_j - \dim e_iJ^2e_j$ because $e_iJ^2e_j \subseteq e_iJe_j$.

For $e_2 \sim e_4$: the element $\sigma_7 + \sigma_8$ is the all-one matrix in the point–block block and $\sigma_9 + \sigma_{10}$ is its transpose; putting $a = e_2(\sigma_7 + \sigma_8)e_4$ and $b = e_4(\sigma_9 + \sigma_{10})e_2$ one verifies $ab = e_2$ and $ba = e_4$, which is the definition of equivalence of idempotents and gives $(\mathbb{F}_2\mathcal{X})e_2 \cong (\mathbb{F}_2\mathcal{X})e_4$. The arrow matrix is symmetric under the interchange of the two conventions for the direction of an arrow, so the statement is convention-free.

That the vertex of e_2 is isolated is read off the arrow matrix, whose second and fourth rows and columns vanish. That e_1 and e_3 are *not* equivalent, so that the two arrows genuinely join two distinct vertices, is the following argument: by the first display every element of $e_3(\mathbb{F}_2\mathcal{X})e_1$ lies in J, hence so does every product $(e_1ae_3)(e_3be_1)$, while $e_1 \notin J$; so no such product can equal e_1. □

The vertices and the Cartan matrix.

Proposition 5.3. *Among the four idempotents,* e_2 *and* e_4 *are equivalent and no other pair is. Hence the Gabriel quiver of* $\mathbb{F}_2\mathcal{X}$ *has exactly three vertices,* $\{e_1\}$*,* $\{e_2, e_4\}$*,* $\{e_3\}$*, and the three corresponding simple modules have dimensions* $1, 2, 1$*, with* $1^2 + 2^2 + 1^2 = 6$*. The vertex* $\{e_2, e_4\}$ *is the one carrying the* $\mathrm{M}_2(\mathbb{F}_2)$ *factor, and it is isolated.*

Proof sketch. All sixteen ordered pairs are decided. For eight of the ten pairs (i, j) with $e_i \not\sim e_j$ the corner $e_i(\mathbb{F}_2\mathcal{X})e_j$ has dimension 0 by Theorem 5.1, so no product ab with $a \in e_i(\mathbb{F}_2\mathcal{X})e_j$ can be the nonzero e_i; the remaining two pairs, (e_1, e_3) and (e_3, e_1), are handled by the radical argument in the proof of Theorem 5.2 — and they need it, the corner $e_1(\mathbb{F}_2\mathcal{X})e_3$ being nonzero. The equivalence $e_2 \sim e_4$ is the explicit pair of witnesses given there. For the dimensions, the simple module at the vertex of e_j is $\overline{\mathbb{F}_2\mathcal{X}} \cdot \varphi(e_j)$, and $\varphi(e_1) = (0, 1, 0)$, $\varphi(e_2) = (E_{22}, 0, 0)$, $\varphi(e_3) = (0, 0, 1)$, $\varphi(e_4) = (E_{11}, 0, 0)$; the four minimal left ideals so obtained have $2, 4, 2, 4$ elements, i.e. dimensions $1, 2, 1, 2$. In particular e_2 and e_4 sit in the $\mathrm{M}_2(\mathbb{F}_2)$ factor and e_1, e_3 are the point-fibre and block-fibre characters. □

Proposition 5.4. *Indexing the three vertices of Proposition 5.3 in the order* $\{e_1\}, \{e_2, e_4\}, \{e_3\}$*, the Cartan matrix* $C_{vw} = [P_w : S_v]$ *of* $\mathbb{F}_2\mathcal{X}$ *is*

$$C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

and the three projective indecomposable modules have dimensions $3, 2, 3$*. The* 4×4 *matrix of Theorem 5.1 is* C *pulled back along the assignment of idempotents to vertices, i.e.* $\dim_{\mathbb{F}_2} e_i(\mathbb{F}_2\mathcal{X})e_j = C_{v(i)v(j)}$*. The consistency identities* $\dim P_w = \sum_v C_{vw} \dim S_v$*,* $\sum_w \dim S_w \cdot \dim P_w = 10 = \dim_{\mathbb{F}_2} \mathbb{F}_2\mathcal{X}$ *and* $\sum_v (\dim S_v)^2 = 6 = \dim_{\mathbb{F}_2} \mathbb{F}_2\mathcal{X}/\operatorname{Rad}(\mathbb{F}_2\mathcal{X})$ *all hold.*

Proof sketch. Because $\overline{\mathbb{F}_2\mathcal{X}}$ is split (Theorem 4.2), $\dim_{\mathbb{F}_2} e_iM = [M : S_{v(i)}]$ for every $\mathbb{F}_2\mathcal{X}$-module M, so the 4×4 corner-dimension matrix of Theorem 5.1 is the pullback of the Cartan matrix along v, and C is read off it. The projective dimensions are verified directly against the algebra: the sets $(\mathbb{F}_2\mathcal{X})e_j$ have $8, 4, 8, 4$ elements, i.e. dimensions $3, 2, 3, 2$, which by $(\mathbb{F}_2\mathcal{X})e_2 \cong (\mathbb{F}_2\mathcal{X})e_4$ is three projective indecomposables of dimensions $3, 2, 3$. The three identities are then arithmetic. □

Remark 5.5 (the four summands and the three projectives)*.* The distinction matters numerically: the regular module $\mathbb{F}_2\mathcal{X}$ has *four* indecomposable direct summands, of total dimension $3 + 2 + 3 + 2 =$

10, while there are only *three* projective indecomposables up to isomorphism, of total dimension $3+2+3=8$. The 4×4 matrix of Theorem 5.1 is a corner-dimension matrix, not the Cartan matrix; the Cartan matrix is the 3×3 matrix C above.

Remark 5.6 (what is not settled). Part (b) of Problem 1.1 asks for the dimensions *and Loewy series* of the projective indecomposables. The dimensions are settled by Proposition 5.4, and the radical filtration data $\dim e_i J e_j$ and $\dim e_i J^2 e_j$ of Theorem 5.2 is available; but the Loewy series of the individual projective modules is not determined in this paper.

6. Excluded alternatives

A finite computation can be made to say almost anything if the statement it decides is subtly the wrong one, so we record the near-misses that were refuted alongside the results above. Each is a statement of the same shape as one of the results, chosen so that it would hold if a plausible error had been made.

Proposition 6.1. $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$ *is neither* 3 *nor* 5*;* σ_2*, the adjacency matrix of the point graph, does not lie in* $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$*; and* $\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$ *is neither the zero ideal nor the whole algebra.*

Proposition 6.2. $\mathbb{F}_2\mathcal{X}/\mathrm{Rad}(\mathbb{F}_2\mathcal{X})$ *is isomorphic neither to* $\mathrm{M}_3(\mathbb{F}_2)$ *(dimension* 9*) nor to* $\mathbb{F}_2\times\mathbb{F}_2\times\mathbb{F}_2$ *(dimension* 3*);* $\mathbb{F}_2\mathcal{Y}$ *is commutative while* $\mathbb{F}_2\mathcal{X}$ *is not, so the non-commutativity comes from the block fibre and the mixed relations and not from the point scheme;* $\dim_{\mathbb{F}_2}\mathrm{Rad}(\mathbb{F}_2\mathcal{Y})\neq 2$*; and the Gabriel quiver is not empty.*

Proposition 6.3. *The equivalence relation used in Proposition 5.3 is satisfiable — it holds for the pair* (e_2,e_4) *and reflexively for each* e_i *— so the non-equivalences established there are not vacuous. The vertex count is neither* 4 *(the four idempotents are not pairwise inequivalent) nor* 1 *nor* 2*. The corner* $e_1(\mathbb{F}_2\mathcal{X})e_3$ *is nonzero, so the inequivalence* $e_1\not\sim e_3$ *is not an instance of the vanishing-corner test and genuinely needs the radical argument. Finally the assignment of idempotents to vertices is not injective, so the* 3×3 *Cartan matrix is a genuine collapse of the* 4×4 *matrix of Theorem 5.1 and not a relabelling.*

7. Verification

Every statement in this paper has been formally verified in Lean 4 against *Mathlib* [10, 11]; the development is free of `sorry`, and the repository reference is to be added at submission. The structural layer is kernel-checked in the ordinary way: the two inclusions of Proposition 3.3, both instances of Mathlib's characterisation of the Jacobson radical as the intersection of the maximal left ideals; the ring-homomorphism and quotient arguments of Theorem 4.2; the arrow arithmetic of Theorem 5.2; and the collapse of the corner-dimension matrix onto the Cartan matrix in Proposition 5.4. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches, and each is small and explicit: the 455 triples of duads and the 225 pairs behind Proposition 2.2; the 100 products of 30×30 matrices and the 10 probe entries behind Proposition 2.3; the $16\cdot 16$ sums, $2\cdot 1024\cdot 16$ products and 16^3 triples behind Propositions 3.3 and 3.4, together with the witness tables checked over all 1024 elements; the 100 products in the 64-element algebra, the 1024 images and the 1024 kernel tests behind Theorem 4.2; the 1344 multiplicity vectors behind Proposition 4.1; the $16\cdot 1024$ corner elements behind Theorem 5.1, the $16\cdot 16$ and $16\cdot 4$ behind Theorem 5.2, and the $4\cdot 64$ and $4\cdot 1024$ behind Propositions 5.3 and 5.4; and the eight-element checks behind Proposition 3.1. Every numerical value quoted was first produced by an independent program written against the geometry rather than against the formalization, and the formal statement was then written to match; the corner matrix, the simple dimensions, the projective dimensions and the Cartan matrix were in addition reproduced by a second, independent rebuild sharing no code with the first.

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