CERTIFIED PERIOD-DOUBLING THRESHOLDS OF THE GRADIENT-DESCENT CUBIC MAP $(1 + a)x - ax^3$, FROM a_4 TO a_{512}

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Abstract. Fixed-step gradient descent with step size a on the double-well loss $\frac{1}{4}(x^2 - 1)^2$ is the odd cubic map $g_a(x) = (1 + a)x - ax^3$. Hofmann (arXiv:2607.04993) uses it as the scalar core of a hierarchy of solvable learning models: the edge of stability $a = 1$ is its first flip bifurcation, and the period-doubling thresholds a_m — the parameters at which the attracting m-cycle has multiplier -1 — organise everything past the edge. He proves $a_2 = \sqrt{5} - 1$ and prints a_4, a_8, a_{16}, a_{32} to eight significant digits, "solved numerically", with no error bound; a_4 and a_8 are also algebraic numbers of degrees 32 and 3200 in Zhang's tables for the map $rx - x^3$. We prove, with every step checked by a proof assistant, that for each $m \in \{4, 8, 16, 32, 64, 128, 256, 512\}$ there is a parameter a_m carrying a cycle of *exact* period m with multiplier exactly -1, that the pair (cycle point, parameter) is the *unique* one in an explicit closed ball of radius 2^{-180} ($m \le 64$), 2^{-360}, 2^{-700} or 2^{-1400} ($m = 128, 256, 512$), and that a_m lies in a stated interval of width 10^{-48}. The values $a_{64}, \dots, a_{512}$ appear nowhere in the literature we could find, and a_{16}, a_{32} only to eight digits. From the brackets we obtain the strict ordering $\sqrt{5} - 1 < a_4 < \cdots < a_{512}$ and rigorous enclosures of the first seven Feigenbaum gap ratios $(a_m - a_{m/2})/(a_{2m} - a_m)$, three of them to three decimals and four to six, all below $\delta = 4.6692016\dots$ and the seventh within $4 \cdot 10^{-6}$ of it; nothing is claimed about the limit. The certificate is a Krawczyk-type test whose Lipschitz input is obtained by exact divided differences along two orbits, with no calculus, so that a period-n certificate is a recursion of length n over rational intervals; a second period-4 flip parameter at $1.5488\dots$ and a root of Zhang's polynomial at $1.9676\dots$ are certified as controls that the uniqueness and the algebraic characterisation are local statements.

1. Introduction

Gradient descent with a fixed step size $a > 0$ on the one-dimensional double-well loss $\ell(x) = \frac{1}{4}(x^2 - 1)^2$ is the iteration $x \mapsto x - a\,\ell'(x)$, that is, the cubic map

$$g_a(x) = (1 + a)\,x - a\,x^3. \tag{1}$$

Hofmann's paper [1] (§2, "The Cubic Map") takes this map as the scalar prototype of finite-step learning dynamics: the fixed points ± 1 are the minima, they are attracting exactly for $0 < a < 1$, and $a = 1$ — the "edge of stability" — is the first flip bifurcation, at which the fixed point $x = 1$ gives birth to a two-cycle. Everything the paper says about the post-edge regime of this model (the two-cycle, sharpness "hovering", the accumulation of period doublings, the chaotic band and the exactly solvable endpoint $a = 2$) is a statement about the bifurcation structure of (1). That structure is classical: in the normalisation $f_\alpha(z) = \alpha z^3 + (1 - \alpha)z$ of Rogers and Whitley [3], g_a is linearly conjugate, up to a global sign, to f_{a+2} ([1], Proposition 7).

The objects of this paper are the *period-doubling thresholds*. Following [1], Appendix A.4, for $m = 2, 4, 8, \dots$ the threshold a_m is a solution of

$$g_a^m(x_0) = x_0, \qquad (g_a^m)'(x_0) = -1, \tag{2}$$

"following the attracting branch": the parameter at which the period-m cycle born at the previous threshold loses stability and the period-$2m$ cycle appears. For $m = 2$ the source solves (2) in closed form, $a_2 = \sqrt{5} - 1$ (its Proposition 3). For the rest it says, verbatim (Appendix A.4):

> For the later thresholds, the equations are solved numerically. The first values are $2 \to 4:$ $a_2 = \sqrt{5} - 1 = 1.2360679\dots$, $4 \to 8:$ $a_4 = 1.2880317\dots$, $8 \to 16:$ $a_8 = 1.2992279\dots$, $16 \to 32:$ $a_{16} = 1.3016289\dots$, $32 \to 64:$ $a_{32} = 1.3021432\dots$.

> The consecutive gaps shrink geometrically. More precisely, $\frac{a_4-a_2}{a_8-a_4}$, $\frac{a_8-a_4}{a_{16}-a_8}$, $\frac{a_{16}-a_8}{a_{32}-a_{16}}$, … approach the Feigenbaum constant. The cascade accumulates at $a_\infty = 1.3022834\ldots$.

Those eight-digit numbers, with no error bound and no value for any ratio, are the whole of what the source asserts about the thresholds beyond $m = 2$.

What is known. The map $rx - x^3$, which is (1) with $r = 1 + a$, is one of the three maps in Zhang's *Cycles of the logistic map* [2] (its §IV, eq. (27)). Zhang computes minimal polynomials of onset and bifurcation parameters of n-cycles (through a characteristic equation in cyclic polynomials of the orbit points) and prints, in his Table IX, the bifurcation parameter of the 4-cycle reached by two period doublings as 2.2880317545 with a minimal polynomial of degree 32, and that of the 8-cycle reached by three as 2.2992279397 with degree 3200; the polynomials themselves are on his web site. So a_4 and a_8 are known algebraic numbers, printed to ten decimals, and his computation stops at $m = 8$. Rogers and Whitley [3] give the $m = 2$ threshold ($\alpha = 1 + \sqrt{5}$ in their normalisation) and state that they made no attempt to verify the Feigenbaum behaviour[1]. The rigorous-numerics literature on period doubling is about renormalisation: Lanford's computer-assisted proof of the Feigenbaum conjectures [6] and its successors bound the universal constants and the fixed point of the doubling operator, not the thresholds of a particular family. We found no rigorous — interval-arithmetic or proof-assistant-checked — enclosure of a flip parameter of a specific one-dimensional map, no value of a_{16} or a_{32} beyond the eight digits above, and no value of a_{64} or any later threshold at any precision (Section 8 records the searches).

What is proved here. Write $M_n(a,x) = \prod_{j<n} g_a'(g_a^j(x))$ for the multiplier, which is $(g_a^n)'(x)$, and $\mathrm{Flip}_n(a,x)$ for the system (2) with $m = n$.

(i) For each $m \in \{4, 8, 16, 32, 64, 128, 256, 512\}$ there are $a_m, x_m \in \mathbb{R}$ with $\mathrm{Flip}_m(a_m, x_m)$, with $g^k_{a_m}(x_m) \neq x_m$ for every proper divisor k of m (the cycle has *exact* period m), and with a_m in the interval of width 10^{-48} printed in Table 1. Moreover the pair (x_m, a_m) is the *only* solution of Flip_m in an explicit closed sup-norm ball about an explicit rational centre, of radius 2^{-180} for $m \leq 64$ and $2^{-360}, 2^{-700}, 2^{-1400}$ for $m = 128, 256, 512$ (Theorems 4.1–4.8).

(ii) $\sqrt{5} - 1 < a_4 < a_8 < \cdots < a_{512}$, and the first seven gap ratios $(a_m - a_{m/2})/(a_{2m} - a_m)$ lie in the intervals of Table 2: three decimals for the three ratios the source names, six decimals for the next four (Theorems 5.1–5.4).

(iii) Controls. A second period-4 flip parameter at $1.5488\ldots$, inside a periodic window above the accumulation point, is certified in the same form (Theorem 6.1), so the uniqueness in (i) is a statement about the ball and not about the family; Zhang's degree-32 polynomial has exactly one real root in each of the two certified period-4 brackets and a further root at $1.9676\ldots$ (Theorem 6.2); and four deliberately broken certificates fail the test (Theorem 6.3).

The numbers a_4 and a_8 are not new — they agree with Zhang's to every digit he prints, and the 59-digit root in his data file lies inside our a_8 bracket — but the certificates, the uniqueness and the exact-period statements are. For a_{16} and a_{32} the new content is the rigorous determination to 48 decimals of numbers known to eight; for $a_{64}, \ldots, a_{512}$ and for all seven ratio brackets we know of no prior value. We claim nothing about universality: the brackets are consistent with the ratios increasing towards Feigenbaum's $\delta = 4.669201609\ldots$ [4, 5, 7], and that is all.

Method. Each threshold is certified by a Krawczyk-type test [8] for the two-variable system $F(x,a) = (g_a^n(x) - x,\ M_n(a,x) + 1)$: a rational centre, a radius, a rational 2×2 preconditioner Y, and three inequalities between rationals computed from them, which imply that $p \mapsto p - YF(p)$ is a contraction of the ball into itself. The Lipschitz constant is obtained by slope arithmetic, a standard device of interval analysis, rather than from an interval derivative of the n-fold composition: we track *exact divided differences* of the orbit and of the multiplier along two orbits at once (Section 3); each is a one-step polynomial identity carried by induction, so no calculus enters, and a period-n certificate is a recursion

[1] The statements attributed here to Rogers–Whitley 1983 (p. 14: the two-cycle eigenvalue $7 + 4\alpha - 2\alpha^2$ and the flip at $\alpha = 1 + \sqrt{5}$; p. 15: "no attempt has been made to verify this") are quoted from the research record of this project and not from the paper, whose full text was not available to us. The review Zbl 0531.58033 confirms the family $f_\alpha(z) = \alpha z^3 + (1-\alpha)z$ on $[-1, 1]$, $0 < \alpha \leq 4$, and [1], Section 2.7, places $a_2 = \sqrt{5} - 1$ at $\alpha = 1 + \sqrt{5}$.

of length n over rational intervals with outward dyadic rounding. The ingredients are standard; what is new is their formalisation and the certificates themselves. All proofs are checked in Lean 4 (Section 8); the only computations not replayed by the proof kernel are the evaluations of the rational recursions, which are delegated to the compiled runtime and recorded as such.

2. The map, its two-cycle, and the flip condition

Throughout, g_a is (1), $g_a'(x) = 1 + a - 3ax^2$, g_a^n is the n-fold iterate, and

$$M_n(a,x) = \prod_{j=0}^{n-1} g_a'\big(g_a^j(x)\big), \qquad M_0 = 1.$$

Proposition 2.2 below says $M_n(a,x) = (g_a^n)'(x)$, so that "multiplier -1" in (2) means what it says.

Definition 2.1. For $n \geq 1$, $\mathrm{Flip}_n(a,x)$ is the statement $g_a^n(x) = x$ and $M_n(a,x) = -1$. A pair with $\mathrm{Flip}_n(a,x)$ is a *flip pair of period* n, and a a *flip parameter*. The cycle of x has *exact period* n if $g_a^k(x) \neq x$ for every proper divisor k of n; for $n = 2^j$ this is the condition at $k = 1, 2, \ldots, 2^{j-1}$, and it implies that the minimal period of x is n.

Proposition 2.2 (the multiplier is the derivative)**.** *For all $a, x \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $y \mapsto g_a^n(y)$ is differentiable at x with derivative $M_n(a,x)$.*

Proof. Induction on n with the chain rule; $g_a'(z) = 1 + a - 3az^2$ is the derivative of g_a. ☐

2.1. **The two-cycle in closed form.** For $a > 1$ put $s = \sqrt{1 + 3/a}$, $t = \sqrt{1 - 1/a}$ and $x_\pm = \frac{1}{2}(s \pm t)$. This is the two-cycle of [1], Proposition 2 ("Period-two orbit"); the source derives it from $x_- + x_+ = s$, $x_- x_+ = 1/a$.

Proposition 2.3. *For $a > 1$: $g_a(x_-) = x_+$, $g_a(x_+) = x_-$, and $g_a^2(x_-) = x_-$.*

Proof. One algebraic lemma does the work and avoids any case analysis on square roots: if $a \neq 0$, $s^2 = 1 + 3/a$ and $y^2 = sy - 1/a$, then $g_a(y) = s - y$ (since $y^3 = s^2 y - s/a - y/a$). Both $x_\pm$ are roots of $T^2 - sT + 1/a$. ☐

Proposition 2.4 (two-cycle multiplier)**.** *For $a > 1$, $M_2(a, x_-) = 9 - 2(1 + a)^2$.*

Proof. $M_2(a, x_-) = g_a'(x_-) g_a'(x_+)$, and $a\, x_- x_+ = 1$, $a(x_-^2 + x_+^2) = a + 1$ reduce it to a polynomial identity. This is equation (1) of [1]; in the Rogers–Whitley normalisation $\alpha = a + 2$ it reads $7 + 4\alpha - 2\alpha^2$. ☐

Proposition 2.5 (the first threshold)**.** *For $a > 1$, $M_2(a, x_-) = -1$ if and only if $a = \sqrt{5} - 1$; and $\mathrm{Flip}_2(\sqrt{5} - 1,\, x_-)$ holds at $a = \sqrt{5} - 1$.*

Proof. From Proposition 2.4, $M_2 = -1 \iff (1 + a)^2 = 5$, and $1 + a > 0$. The second clause is Proposition 2.3 at $a = \sqrt{5} - 1 > 1$; it says the hypothesis of the "if and only if" is not vacuous. This is [1], Proposition 3. ☐

Proposition 2.6 (controls for the two-cycle)**.** *(a) At $a = 1$, $x_- = x_+ = 1$ and $g_1(1) = 1$: the two cycle points collide at the fixed point, so the hypothesis $a > 1$ above is load-bearing. (b) For $a > 1$, $x_- \neq x_+$: the two-cycle is not a fixed point. (c) $M_2(1, 2) \neq g_1'(2)^2$: the multiplier depends on the orbit, not only on its initial point.*

Proof. (a) $s = 2$, $t = 0$. (b) $t > 0$. (c) $M_2(1, 2) = 460$ and $g_1'(2)^2 = 100$. ☐

3. Exact slopes and the certificate

The system to be solved for a threshold is

$$F(x,a) = \big(g_a^n(x) - x,\; M_n(a,x) + 1\big) = 0, \qquad F : \mathbb{R}^2 \to \mathbb{R}^2. \tag{3}$$

A Krawczyk test needs, on a box B around an approximate solution, a bound K such that $G = \mathrm{id} - Y \circ F$ satisfies $\|G(p) - G(q)\|_\infty \leq K \|p - q\|_\infty$ on B. We produce it algebraically.

3.1. **Divided differences along two orbits.** Fix two pairs (a,x) and (a',x') and write $u_j = g_a^j(x)$, $v_j = g_{a'}^j(x')$. Define

$$A_j = (1+a) - a\,(u_j^2 + u_j v_j + v_j^2), \qquad B_j = v_j - v_j^3,$$
$$U_0 = 1,\ V_0 = 0, \qquad U_{j+1} = A_j U_j, \quad V_{j+1} = A_j V_j + B_j,$$

and, with $d_j = g_a'(u_j)$ and $Q_j = M_j(a',x')$,

$$C_j = -3a\,(u_j+v_j)\,U_j, \qquad D_j = (1-3v_j^2) - 3a\,(u_j+v_j)\,V_j,$$
$$W_0 = Z_0 = 0, \qquad W_{j+1} = W_j d_j + Q_j C_j, \quad Z_{j+1} = Z_j d_j + Q_j D_j.$$

Proposition 3.1 (slope identities)**.** *For all $a,x,a',x' \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$g_a^n(x) - g_{a'}^n(x') = (x-x')\,U_n + (a-a')\,V_n, \tag{4}$$
$$M_n(a,x) - M_n(a',x') = (x-x')\,W_n + (a-a')\,Z_n. \tag{5}$$

Proof. Induction on n. The step of (4) is the identity $g_a(u) - g_{a'}(v) = (u-v)\big((1+a) - a(u^2+uv+v^2)\big) + (a-a')(v-v^3)$; the step of (5) is a two-term linear combination of the induction hypotheses. No mean value theorem, no derivative and no side condition: the identities hold for any two pairs. □

3.2. **The interval recursion.** A rational interval is a pair $I=[\ell,h]$ with $\ell \le h$ in $\mathbb{Q}$; $x \in I$ means $\ell \le x \le h$ in $\mathbb{R}$. Sums, differences, products and squares of intervals are computed by the usual endpoint formulas, and $\mathrm{round}_p(I)$ widens I outwards to endpoints with denominator 2^p. Each operation is proved to preserve membership. A *frame* is an 8-tuple of intervals $(\mathbf{u},\mathbf{v},\mathbf{p},\mathbf{q},\mathbf{U},\mathbf{V},\mathbf{W},\mathbf{Z})$; given intervals $\mathbf{A},\mathbf{A'}$ for the two parameters and $\mathbf{X},\mathbf{X'}$ for the two points, the initial frame is $(\mathbf{X},\mathbf{X'},[1,1],[1,1],[1,1],[0,0],[0,0],[0,0])$ and one step evaluates the recursions for u,v,M,Q,U,V,W,Z of the previous subsection in interval arithmetic, rounding every component outwards to 2^{-p}. Write $\mathrm{fr}_p^n(\mathbf{A},\mathbf{A'},\mathbf{X},\mathbf{X'})$ for the frame after n steps.

Proposition 3.2 (soundness of the recursion)**.** *If $a\in\mathbf{A}$, $a'\in\mathbf{A'}$, $x\in\mathbf{X}$, $x'\in\mathbf{X'}$, then for every n the eight reals $g_a^n(x)$, $g_{a'}^n(x')$, $M_n(a,x)$, $M_n(a',x')$, U_n,V_n,W_n,Z_n lie in the corresponding components of $\mathrm{fr}_p^n(\mathbf{A},\mathbf{A'},\mathbf{X},\mathbf{X'})$.*

3.3. **The certificate.** A *certificate* is a 9-tuple of rationals $C=(n,p,a_c,x_c,r,y_{11},y_{12},y_{21},y_{22})$ with $n,p\in\mathbb{N}$: the period, the rounding precision in bits, a centre (x_c,a_c), a radius r, and a preconditioner $Y = \left(\begin{smallmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{smallmatrix}\right)$. Its ball is $B_C = \{(x,a)\in\mathbb{R}^2 :\ |x-x_c|\le r,\ |a-a_c|\le r\}$, with interval sides $\mathbf{X}_C=[x_c-r,x_c+r]$ and $\mathbf{A}_C=[a_c-r,a_c+r]$. Two numbers are *computed* from C by the recursion:

- the *residual bound* β_C: run fr_p^n on the degenerate box $\mathbf{A}=\mathbf{A'}=[a_c,a_c]$, $\mathbf{X}=\mathbf{X'}=[x_c,x_c]$, form the residual intervals $\mathbf{R}_1=\mathbf{u}-x_c$, $\mathbf{R}_2=\mathbf{p}+1$, and let β_C be the larger of the magnitude bounds of $y_{11}\mathbf{R}_1+y_{12}\mathbf{R}_2$ and $y_{21}\mathbf{R}_1+y_{22}\mathbf{R}_2$ (the magnitude bound of $[\ell,h]$ is $\max(|\ell|,|h|)$);
- the *contraction bound* K_C: run fr_p^n on $\mathbf{A}=\mathbf{A'}=\mathbf{A}_C$, $\mathbf{X}=\mathbf{X'}=\mathbf{X}_C$, form

  $$\mathbf{c}_{00} = 1-\big(y_{11}(\mathbf{U}-1)+y_{12}\mathbf{W}\big), \qquad \mathbf{c}_{01} = -\big(y_{11}\mathbf{V}+y_{12}\mathbf{Z}\big),$$
  $$\mathbf{c}_{10} = -\big(y_{21}(\mathbf{U}-1)+y_{22}\mathbf{W}\big), \qquad \mathbf{c}_{11} = 1-\big(y_{21}\mathbf{V}+y_{22}\mathbf{Z}\big),$$

  and let K_C be the larger of the two row sums of magnitude bounds.

Theorem 3.3 (soundness of the certificate)**.** *Let C be a certificate with $r>0$, $K_C<1$, $\beta_C+K_Cr\le r$ and $y_{11}y_{22}-y_{12}y_{21}\ne 0$. Then there is exactly one $(x,a)\in B_C$ with $\mathrm{Flip}_n(a,x)$.*

Proof. Let $G(p)=p-YF(p)$ with F as in (3). For $P=(x,a)$, $P'=(x',a')$ in B_C, Proposition 3.1 gives $G(P)_1-G(P')_1=(x-x')\big(1-(y_{11}(U_n-1)+y_{12}W_n)\big)-(a-a')(y_{11}V_n+y_{12}Z_n)$ and similarly for the second coordinate, and Proposition 3.2 puts the four coefficients into $\mathbf{c}_{00},\dots,\mathbf{c}_{11}$; hence $\|G(P)-G(P')\|_\infty \le K_C\|P-P'\|_\infty$. The degenerate run bounds $\|G(c)-c\|_\infty = \|YF(c)\|_\infty \le \beta_C$ at the centre $c=(x_c,a_c)$. With $\beta_C+K_Cr\le r$ the map G sends the closed ball into itself and is a contraction there, so it has exactly one fixed point in B_C; since Y is injective, fixed points of G are exactly the zeros of F, i.e. the flip pairs of period n. □

TABLE 1. The certified thresholds. For each m the theorem states $d_m < a_m < d_m + 10^{-48}$ with d_m the printed number; r is the radius of the ball in which the flip pair is unique, and p the rounding precision of the recursion. The last column is what [1], Appendix A.4, prints. $a_2 = \sqrt{5} - 1$ is exact (Proposition 2.5). The bottom row is the window threshold of Theorem 6.1.

m	d_m	r	p	source
2	$\sqrt{5} - 1 = 1.236067977499789696409173668\ldots$	—	—	$1.2360679\ldots$
4	1.288031754482842761954342435445572514441033845073	2^{-180}	420	$1.2880317\ldots$
8	1.299227939650204094163177321224660033964990043692	2^{-180}	420	$1.2992279\ldots$
16	1.301628914037081195197756509066654739334743731359	2^{-180}	420	$1.3016289\ldots$
32	1.302143271581458122996650293284747114084794699858	2^{-180}	420	$1.3021432\ldots$
64	1.302253437742128565172079349083254326345677410797	2^{-180}	420	—
128	1.302277032259641670362172426105624664109218639849	2^{-360}	760	—
256	1.302282085496493499044942323804032998555673048225	2^{-700}	1450	—
512	1.302283167745712275624657981558236703751436265303	2^{-1400}	2900	—
4 (window)	1.548831219278772124652324551492506085547307398534	2^{-180}	420	—

Theorem 3.4 (exact period)**.** *Let C be a certificate with $r \geq 0$ and $k \in \mathbb{N}$. If the* $\mathbf{u}$*-component of* $\mathrm{fr}_p^k(\mathbf{A}_C, \mathbf{A}_C, \mathbf{X}_C, \mathbf{X}_C)$ *is disjoint from $\mathbf{X}_C$, then $g_a^k(x) \neq x$ for every $(x, a) \in B_C$.*

Proof. Proposition 3.2 with $(a', x') = (a, x)$: $g_a^k(x)$ lies in an interval that misses $\mathbf{X}_C \ni x$. □

The hypotheses of Theorems 3.3 and 3.4 are decidable statements about rationals. A certified threshold is therefore: a certificate C, the evaluation of those statements, and the two theorems.

4. The thresholds

For each m in Table 1 the development contains a certificate C_m with period m, precision p and radius r as tabulated, a centre (x_c, a_c) with denominator 2^{200} ($m \leq 64$), 2^{380}, 2^{720}, 2^{1420} ($m = 128, 256, 512$), and a preconditioner with entries of denominator 2^{64} in every case. The centres were computed outside the proof assistant by Newton's method on (3) with an exact forward-mode Jacobian (in Python decimal arithmetic at 90 digits for $m \leq 32$; in PARI/GP at 423 and again at 905 digits for $m \geq 64$, a run that also reproduced the former), for $m \geq 64$ the cycle point x_c being chosen among the m points of the cycle to maximise $\min_{k|m,\, k<m} |g_a^k(x) - x|$; the preconditioner is the inverse Jacobian rounded to 64 bits. None of this enters the proof: the proof consists of the evaluation of the decidable hypotheses of Theorems 3.3 and 3.4 for the tabulated data.

In the following statements B_m denotes the ball B_{C_m} and d_m the number printed in Table 1. "Unique in B_m" means: exactly one pair $(x, a) \in B_m$ satisfies $\mathrm{Flip}_m(a, x)$. Since $|a - a_c| \leq r < 10^{-54}$ for that pair and the bracket is proved from $a_c \pm r$, the centre a_c itself lies in the bracket.

Theorem 4.1 (a_4)**.** *There are a, x with $\mathrm{Flip}_4(a, x)$, $g_a(x) \neq x$, $g_a^2(x) \neq x$ and $d_4 < a < d_4 + 10^{-48}$; the flip pair of period 4 is unique in B_4 (radius 2^{-180}).*

Theorem 4.2 (a_8)**.** *There are a, x with $\mathrm{Flip}_8(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, 4$, and $d_8 < a < d_8 + 10^{-48}$; the flip pair of period 8 is unique in B_8 (radius 2^{-180}).*

Theorems 4.1 and 4.2 re-certify published numbers: Zhang's Table IX prints $1 + a_4 = 2.2880317545$ and $1 + a_8 = 2.2992279397$, and the 59-digit value

$$2.2992279396502040941631773212246600339649900436923049094263$$

in his file `cr8b.txt` lies inside the bracket of Theorem 4.2: it agrees with $1 + d_8$ in all 48 certified decimals and continues with a 3. What is new in the two theorems is the certificate. The next two go beyond every published digit.

Theorem 4.3 (a_{16})**.** *There are a, x with $\mathrm{Flip}_{16}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, 4, 8$, and $d_{16} < a < d_{16} + 10^{-48}$; the flip pair of period 16 is unique in B_{16} (radius 2^{-180}).*

Table 2. The seven gap ratios. The bracket is the theorem; the value in the third column is computed from the certified brackets outside the formal development (they determine it to about 47 decimals) and is printed only for comparison with $\delta = 4.66920160910299067185 3\ldots$ [7].

i	ratio	certified bracket	value
1	$(a_4 - a_2)/(a_8 - a_4)$	$(4.641,\ 4.642)$	4.641203785602
2	$(a_8 - a_4)/(a_{16} - a_8)$	$(4.663,\ 4.664)$	4.663183925891
3	$(a_{16} - a_8)/(a_{32} - a_{16})$	$(4.667,\ 4.668)$	4.667909342684
4	$(a_{32} - a_{16})/(a_{64} - a_{32})$	$(4.668925,\ 4.668926)$	4.668925024224
5	$(a_{64} - a_{32})/(a_{128} - a_{64})$	$(4.669142,\ 4.669143)$	4.669142338225
6	$(a_{128} - a_{64})/(a_{256} - a_{128})$	$(4.669188,\ 4.669189)$	4.669188918894
7	$(a_{256} - a_{128})/(a_{512} - a_{256})$	$(4.669198,\ 4.669199)$	4.669198890752
	δ (Feigenbaum)		4.669201609103

Theorem 4.4 (a_{32})**.** *There are a, x with $\mathrm{Flip}_{32}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, 4, 8, 16$, and $d_{32} < a < d_{32} + 10^{-48}$; the flip pair of period 32 is unique in B_{32} (radius 2^{-180}).*

Theorem 4.5 (a_{64})**.** *There are a, x with $\mathrm{Flip}_{64}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, \ldots, 32$ (k a power of two), and $d_{64} < a < d_{64} + 10^{-48}$; the flip pair of period 64 is unique in B_{64} (radius 2^{-180}).*

Theorem 4.6 (a_{128})**.** *There are a, x with $\mathrm{Flip}_{128}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, \ldots, 64$ (k a power of two), and $d_{128} < a < d_{128} + 10^{-48}$; the flip pair of period 128 is unique in B_{128} (radius $2^{-360} < 5 \cdot 10^{-109}$).*

Theorem 4.7 (a_{256})**.** *There are a, x with $\mathrm{Flip}_{256}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, \ldots, 128$ (k a power of two), and $d_{256} < a < d_{256} + 10^{-48}$; the flip pair of period 256 is unique in B_{256} (radius $2^{-700} < 2 \cdot 10^{-211}$).*

Theorem 4.8 (a_{512})**.** *There are a, x with $\mathrm{Flip}_{512}(a, x)$, $g_a^k(x) \neq x$ for $k = 1, 2, \ldots, 256$ (k a power of two), and $d_{512} < a < d_{512} + 10^{-48}$; the flip pair of period 512 is unique in B_{512} (radius $2^{-1400} < 4 \cdot 10^{-422}$).*

Proof of Theorems 4.1–4.8. Uniqueness is Theorem 3.3 for C_m. Of its four hypotheses, $r > 0$ and $\det Y \neq 0$ are checked by normalisation of the literal rationals; $K_{C_m} < 1$ and $\beta_{C_m} + K_{C_m} r \leq r$ are closed decidable propositions whose truth value is obtained by evaluating the recursion of Section 3 — m frame steps at precision p over the ball and m more at the centre — in the compiled runtime. Existence and the bracket follow from uniqueness: the unique pair lies in B_m, so $a_c - r \leq a \leq a_c + r$, and $d_m < a_c - r$, $a_c + r < d_m + 10^{-48}$ are inequalities between explicit rationals (normalised by the kernel for $m \leq 64$, where the denominators are 2^{200}; evaluated in the runtime for $m \geq 128$, where they exceed 2^{256}). Exact period is Theorem 3.4 at each proper divisor k, the disjointness of the k-step orbit interval over B_m from $\mathbf{X}_{C_m}$ being again a decided proposition. □

The values of K_C and β_C that the evaluations return are not part of any theorem, but they explain the parameters: for $m \leq 32$ (and the window certificate) $K_C < 8 \cdot 10^{-17}$ and $\beta_C < 3 \cdot 10^{-61}$ against $r = 2^{-180} \approx 6.6 \cdot 10^{-55}$; for $m = 64$, $K_C \approx 1.9 \cdot 10^{-6}$, $\beta_C \approx 3.1 \cdot 10^{-62}$; for $m = 128$, $K_C \approx 1.0 \cdot 10^{-12}$, $\beta_C \approx 1.9 \cdot 10^{-115}$; for $m = 256$, $K_C \approx 6.1 \cdot 10^{-15}$, $\beta_C \approx 7.4 \cdot 10^{-218}$; for $m = 512$, $K_C \approx 3.8 \cdot 10^{-14}$, $\beta_C/r \approx 4.7 \cdot 10^{-7}$. Why the radius must shrink with the period is discussed in Section 7.

5. Ordering and the gap ratios

Because the brackets of Table 1 are disjoint and increasing, and $\sqrt{5} - 1$ is bracketed in the kernel from $(\sqrt{5})^2 = 5$, the ordering and the gap ratios are arithmetic consequences. The statements are existential, as in the formal development: "there are flip pairs of the stated periods whose parameters satisfy …"; by Theorems 4.1–4.8 the parameters may be taken to be the certified a_m, and in fact the inequalities hold for any parameters in the brackets.

Theorem 5.1 (the first three ratios)**.** *There are flip pairs $(a_4, x_4), (a_8, x_8), (a_{16}, x_{16}), (a_{32}, x_{32})$ of periods $4, 8, 16, 32$ with $\sqrt{5} - 1 < a_4 < a_8 < a_{16} < a_{32}$ and*

$$4.641 < \frac{a_4 - (\sqrt{5} - 1)}{a_8 - a_4} < 4.642, \qquad 4.663 < \frac{a_8 - a_4}{a_{16} - a_8} < 4.664, \qquad 4.667 < \frac{a_{16} - a_8}{a_{32} - a_{16}} < 4.668.$$

Theorem 5.2 (the fourth and fifth ratios)**.** *There are flip pairs of periods* $16, 32, 64, 128$ *with* $a_{16} < a_{32} < a_{64} < a_{128}$ *and*

$$4.668925 < \frac{a_{32} - a_{16}}{a_{64} - a_{32}} < 4.668926, \qquad 4.669142 < \frac{a_{64} - a_{32}}{a_{128} - a_{64}} < 4.669143.$$

Theorem 5.3 (the sixth ratio)**.** *There are flip pairs of periods* $64, 128, 256$ *with* $a_{64} < a_{128} < a_{256}$ *and* $4.669188 < (a_{128} - a_{64})/(a_{256} - a_{128}) < 4.669189$.

Theorem 5.4 (the seventh ratio)**.** *There are flip pairs of periods* $128, 256, 512$ *with* $a_{128} < a_{256} < a_{512}$ *and* $4.669198 < (a_{256} - a_{128})/(a_{512} - a_{256}) < 4.669199$.

Proof of Theorems 5.1–5.4. Take the pairs of Theorems 4.1–4.8. Each bracket has width 10^{-48} while consecutive gaps are at least 10^{-6}, so the differences are positive and each ratio bound reduces, after clearing the positive denominator, to a polynomial inequality in the bracketed quantities, which the `nlinarith` tactic closes from the bracket hypotheses. For the first ratio the bracket

$$2.236067977499789696409173668731 < \sqrt{5} < 2.236067977499789696409173668732$$

enters, proved from $(\sqrt{5})^2 = 5$. The margins are wide: the bracket widths contribute about 10^{-48} to each ratio against margins of at least $2 \cdot 10^{-8}$ from the stated endpoints, which is why six decimals are available at no extra cost. □

Remark 5.5. The seven ratios increase and stay below δ, and the last is less than $4 \cdot 10^{-6}$ below it. This is consistent with, and is in no way a proof of, the universality the source invokes; Feigenbaum's constant enters this paper only as a number to compare with [4, 5, 6]. Likewise the certified $a_{512} = 1.30228316\ldots$ is consistent with the printed $a_\infty = 1.3022834\ldots$, and nothing about the accumulation point is proved here: it is a renormalisation fixed point, not a solution of a finite system such as (3).

6. Controls, and the cross-check against Zhang's polynomial

6.1. **Uniqueness is about the ball.** The map g_a has further period-4 flip parameters outside the principal cascade. One of them lies above the accumulation point, in a periodic window: it is the bifurcation parameter of the 4-cycle in the row "4" of Zhang's Table IX (unprimed there, i.e. a cycle not produced by period doubling), printed as 2.5488312193 with a degree-32 minimal polynomial.

Theorem 6.1 (a second period-4 flip parameter)**.** *Let*

$$d_w = 1.548831219278772124652324551492506085547307398534.$$

There are a, x *with* $\mathrm{Flip}_4(a, x)$, $g_a(x) \neq x$, $g_a^2(x) \neq x$ *and* $d_w < a < d_w + 10^{-48}$*, and the flip pair of period* 4 *is unique in the ball* B_w *of radius* 2^{-180} *about the corresponding centre. Consequently there are two distinct parameters* $a \neq a'$ *each carrying a period-*4 *flip pair, and the period-*4 *flip parameter of Theorem 4.1 is not* $\sqrt{5} - 1$.

Proof. The same certificate shape as in Section 4. The two consequences compare the brackets of Theorems 4.1 and 6.1 and the bracket for $\sqrt{5}$. □

So the uniqueness clauses of Section 4 cannot be read as "the unique period-m flip parameter of the family": enlarging the ball of Theorem 4.1 to $[1.28, 1.55]$ would make the statement false.

6.2. **The published period-4 polynomial.** Zhang's file `cr4b.txt` prints the period-4 bifurcation polynomial of $rx - x^3$ as a product of a degree-8 and a degree-32 factor in r, the latter even in r. Substituting $r = 1 + a$ into the degree-32 factor gives the integer polynomial

$$P(a) = 8192\, a^{32} + 262144\, a^{31} + 3676160\, a^{30} + \cdots - 6492540731392,$$

whose 33 coefficients are listed in Appendix A; we re-derived it independently by resultants (an odd-symmetry reduction of the period-4 conditions to symmetric functions of a two-cycle of g_a^2, then two resultants in PARI/GP) and the coefficients agree exactly, but nothing in the formal development depends on that derivation. What is checked is the following.

Theorem 6.2 (cross-check)**.** *P has exactly one real root in the closed interval $[d_4,\, d_4 + 10^{-48}]$ of Theorem 4.1, exactly one in $[d_w,\, d_w + 10^{-48}]$ of Theorem 6.1, and at least one in*

$$[\,1.9675562012733241127,\ 1.9675562012733241128\,].$$

Proof. On each interval $[\ell, h]$: $P(\ell)$ and $P(h)$ have opposite signs (Horner evaluation in $\mathbb{Q}$), so the intermediate value theorem gives a root; and the exact divided difference $(P(x)-P(y))/(x-y)$, computed by a Horner recursion for the pair, is enclosed over $[\ell, h]^2$ in an interval not containing 0, so P is injective there. No derivative and no Sturm sequence is used. The third interval, at $a \approx 1.9676$, is far above the accumulation point: the published polynomial vanishes at parameters that are not cascade thresholds, so "root of P" is strictly weaker than "a_4". ☐

6.3. **Breaking the certificate.** Each of the following modifies exactly one component of a passing certificate and is refuted by the same decision procedure that the real certificates pass.

Theorem 6.3 (negative controls)**.** *(a) Rounding the preconditioner of C_{64} to entries with denominator 2^8 gives $y_{12} = y_{22} = 0$: the matrix is singular, and moreover $K \geq 1$. (b) C_{64} with the radius enlarged from 2^{-180} to 2^{-160} has $K \geq 1$. (c) C_{128} with the radius enlarged from 2^{-360} to 2^{-320} has $K \geq 1$. (d) C_{64} with the centre moved by $2^{-100} \approx 7.9 \cdot 10^{-31}$ in the parameter coordinate fails $\beta + Kr \leq r$.*

Proof. Evaluation of the recursion for the modified data. The returned values are $K \approx 3.18$ in (a), $K \approx 2.03$ in (b), $K \approx 1.11$ in (c), and $\beta \approx 7.9 \cdot 10^{-31}$ against $r \approx 6.6 \cdot 10^{-55}$ in (d). ☐

Control (d) says that the 48 printed decimals are carried by the certificate and not by the way the centre is written down: a perturbation in the 31st decimal is detected. Controls (b) and (c) say the radius is not a free parameter, and (a) that the preconditioner must resolve $y_{22} = -2/\det J$, a quantity that shrinks by a factor δ at each cascade level.

7. Measurements outside the formal development

The following are observations from running the certificates; none is a theorem of the development and none is claimed as one.

Why the radius shrinks. The interval enclosure of one step encloses $(1+a)u - a\,u \cdot u^2$ term by term, so a box of width ρ around an orbit point u comes out with width about $((1+a) + 3au^2)\rho \approx 5.7\rho$, whereas the true derivative factor is $|g_a'(u)| = |1 + a - 3au^2|$, whose product around a flip cycle is exactly 1. Interval arithmetic does not see the cancellation, and the measured loss is 2.51 bits of radius per orbit step, at every period tested: the certificate at $m = 64$ fails at 2^{-161} ($K = 1.008$) and passes at 2^{-162}; at $m = 128$ it fails at 2^{-320} ($K = 1.105$) and passes at 2^{-321}. A period-n certificate therefore needs $r < 2^{-2.51n}$ and a rounding precision of about $2 \cdot 2.51n$ bits. The parameters of C_{512} ($r = 2^{-1400}$, $p = 2900$) were chosen from this rule before that certificate was run, and it passed with $K \approx 3.8 \cdot 10^{-14}$. A centred (mean-value) enclosure of each step would carry the true derivative factor and remove the exponential; it is the natural next step and was not needed for $m \leq 512$.

The preconditioner is not the limit. The determinant of the inverse Jacobian falls by a factor δ per level ($1.2 \cdot 10^{-2}$ at $m = 4$ down to $2.5 \cdot 10^{-7}$ at $m = 512$), but the width the preconditioner needs grows only by $\log_2 \delta \approx 2.22$ bits per level: 12 bits suffice at $m = 64$ and 128, 18 at $m = 256$, and 8 bits fail everywhere because the rounded matrix becomes singular (Theorem 6.3(a)). The 64-bit preconditioner used throughout has some 46 bits of margin at $m = 256$.

Cost. The eight modules of the development total about 1820 lines. Checking the five certificates with $m \leq 32$ takes 26 s wall and 3.1 GB; the certificates for $m = 64$ and 128 together 16 s, $m = 256$ 12 s, and $m = 512$ 32 s, all near 3 GB, on a single workstation. Net of the fixed import cost the evaluation grows by a factor of about 5.5 per level (the step count and the bignum size both double). The Newton centres and the sanity checks in PARI/GP and Python took a few CPU-minutes.

8. Verification

Every proposition and theorem of Sections 2–6 is stated and proved in Lean 4 (toolchain v4.32.0) over Mathlib, in eight modules of the development `GdCubicCascade`; Appendix A reproduces the statements verbatim. Mathlib supplies real numbers, derivatives, the intermediate value theorem and the contraction mapping principle; the rational interval layer and the Krawczyk theorem in the form used (componentwise Lipschitz hypothesis, contraction of a closed ball in $\mathbb{R}^n$ with the sup norm) are part of the same development. We ran `#print axioms` on all 46 declarations of Appendix A. The fifteen declarations of the reasoning layer — Propositions 2.2–2.6, 3.1, 3.2 and Theorems 3.3, 3.4, i.e. `mlt_isDeriv`, `g_xlo`, `g_xhi`, `orb_two_xlo`, `two_cycle_multiplier`, `flip_two_iff`, `flip_two_sqrt5`, the three `control_*` declarations of `Core`, `orb_sub`, `mlt_sub`, `holds_frames`, `KCert.sound` and `KCert.not_period` — depend on `propext`, `Classical.choice` and `Quot.sound` only, and are checked entirely by the kernel. The remaining 31 declarations (the certified thresholds, the uniqueness statements, the ratio theorems, the window and polynomial cross-checks, and the negative controls) additionally depend on the auxiliary axioms that Lean's `native_decide` introduces, one per call: each asserts that the compiled evaluation of a closed decidable proposition about rationals returned `true`. These calls are exactly the evaluations described in the proofs above (the inequalities $K_C < 1$, $\beta_C + K_C r \le r$, the orbit-interval disjointness tests, the sign and slope tests of Theorem 6.2, the bracket comparisons for $m \ge 128$, and the refutations of Theorem 6.3); the reasoning that turns them into the stated theorems is kernel-checked. No declaration depends on `sorryAx`. Repository reference to be added at submission.

The prior-art searches behind the novelty claims were run on 2026-09-03: the source and its appendix (a text search of the e-print for a_{64}, a_{128} and the digit strings 1.30225, 1.30227, 1.30228 finds only the three occurrences of a_∞); Zhang's data directory (`cr4b.txt` and `cr8b.txt` exist, `cr16b.txt` and `cr32b.txt` return HTTP 404, so his algebraic route stops at $m = 8$); a full-text search of a local mirror of the arXiv for the eight-digit prefixes of a_{64}, a_{128}, a_{256} in both normalisations (twenty hits, all unrelated numerals); the OEIS for the digit strings of every threshold (no matches; the OEIS holds no dynamical constant of a cubic map); and the arXiv listing for period doubling with interval arithmetic (renormalisation-side papers only).

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Appendix A. The Lean statements

The statements below are reproduced verbatim from the development (line breaks in a few long lines adjusted to the page width; proofs omitted after `:=`). The definitions they refer to: `g a x` is $g_a(x)$, `gd a x` is $g_a'(x)$, `orb a x n` is $g_a^n(x)$, `mlt a x n` is $M_n(a, x)$, `Flip n a x` is $\mathrm{Flip}_n(a, x)$; `sm`, `tm`, `xlo`, `xhi` are s, t, x_-, x_+ of Section 2; `sU`, `sV`, `sW`, `sZ` are U, V, W, Z and `frames p A A' X X' n` is fr_p^n; `x ∈ᵢ I` is interval membership; `KCert` is the certificate with fields `n prec ac xc rr y11 y12 y21 y22`, `C.ctr` the centre (x_c, a_c) as an element of `Fin 2 → ℝ` (coordinate `0` is x, coordinate `1` is a), `C.K`, `C.beta` the computed bounds, `C.orbItv k` the $\mathbf{u}$-component of the k-step frame over the ball, `C.ballX` the interval $\mathbf{X}_C$, and `Disj` interval disjointness; `Metric.closedBall C.ctr C.rr` is B_C in the sup norm of `Fin 2 → ℝ`. `Pflip` is P evaluated by Horner's rule from `flipCoeffs`, the coefficient list in decreasing degree:

```
def flipCoeffs : List ℚ :=
  [8192, 262144, 3676160, 29020160, 134042624, 297529344, -310726656, -4209573888,
   -11278110688, -3058728192, 58073901432, 143218185808, 32698957864, -470341834912,
   -826148854353, 210502169358, 2329654033646, 1963111682880, -3049364160174, -6433869731396,
   335662049659, 10824031953812, 7252610135061, -9188590271574, -13456271207080, 2672794065552,
   13881587071056, 3441296503200, -10036356681472, -6828249517568, 3346359861248, 5022476779520,
   -6492540731392]
```

The certificates `c4`, `c8`, `c16`, `c32`, `cw`, `c64`, `c128`, `c256`, `c512` are literal `KCert` values with the periods, precisions and radii of Table 1; their centres are rationals with the denominators stated in Section 4 (numerators of up to 428 decimal digits, not reproduced here), and their preconditioner entries have denominator 2^{64}. The control certificates are `c64w8` (`c64` with `y11 := -28/2^8, y12 := 0, y21 := 9/2^8, y22 := 0`), `c64wide` (`c64` with `rr := 1/2^160`), `c64off` (`c64` with `ac := c64.ac + 1/2^100`) and `c128wide` (`c128` with `rr := 1/2^320`). Each block is headed by the statement of this paper it corresponds to.

Module `GdCubicCascade/Core.lean`**.** Proposition 2.2 (`mlt_isDeriv`):

```
theorem mlt_isDeriv (a x : ℝ) : ∀ n : ℕ, HasDerivAt (fun y : ℝ => orb a y n) (mlt a x n) x
```

Proposition 2.3 (`g_xlo`):

```
theorem g_xlo (ha : 1 < a) : g a (xlo a) = xhi a :=
```

Proposition 2.3 (`g_xhi`):

```
theorem g_xhi (ha : 1 < a) : g a (xhi a) = xlo a :=
```

Proposition 2.3 (`orb_two_xlo`):

```
theorem orb_two_xlo (ha : 1 < a) : orb a (xlo a) 2 = xlo a :=
```

Proposition 2.4 (`two_cycle_multiplier`):

```
theorem two_cycle_multiplier (ha : 1 < a) :
    mlt a (xlo a) 2 = 9 - 2 * (1 + a) ^ 2 :=
```

Proposition 2.5 (`flip_two_iff`):

```
theorem flip_two_iff (ha : 1 < a) : mlt a (xlo a) 2 = -1 ↔ a = Real.sqrt 5 - 1 :=
```

Proposition 2.5 (`flip_two_sqrt5`):

```
theorem flip_two_sqrt5 : Flip 2 (Real.sqrt 5 - 1) (xlo (Real.sqrt 5 - 1)) :=
```

Proposition 2.6 (`control_edge_degenerate`):

```
theorem control_edge_degenerate : xlo 1 = 1 ∧ xhi 1 = 1 ∧ g 1 1 = 1 :=
```

Proposition 2.6 (`control_two_cycle_nondegenerate`):

```
theorem control_two_cycle_nondegenerate {a : ℝ} (ha : 1 < a) : xlo a ≠ xhi a :=
```

Proposition 2.6 (`control_mlt_not_power`):

```
theorem control_mlt_not_power : mlt 1 2 2 ≠ gd 1 2 ^ 2 :=
```

Module `GdCubicCascade/Slope.lean`**.** Proposition 3.1 (`orb_sub`):

```
theorem orb_sub (a x a' x' : ℝ) : ∀ n : ℕ,
    orb a x n - orb a' x' n = (x - x') * sU a x a' x' n + (a - a') * sV a x a' x' n
```

Proposition 3.1 (`mlt_sub`):

```
theorem mlt_sub (a x a' x' : ℝ) : ∀ n : ℕ,
    mlt a x n - mlt a' x' n = (x - x') * sW a x a' x' n + (a - a') * sZ a x a' x' n
```

Proposition 3.2 (`holds_frames`):

```
theorem holds_frames {a x a' x' : ℝ} {A A' X X' : Itv} (p : ℕ)
    (ha : a ∈ᵢ A) (ha' : a' ∈ᵢ A') (hx : x ∈ᵢ X) (hx' : x' ∈ᵢ X') :
    ∀ n : ℕ, Frame.Holds a x a' x' n (frames p A A' X X' n)
```

Module `GdCubicCascade/Krawczyk.lean`**.** Theorem 3.3 (`sound`):

```
theorem sound (hr : 0 < C.rr) (hK : C.K < 1) (hstep : C.beta + C.K * C.rr ≤ C.rr)
    (hdet : C.y11 * C.y22 - C.y12 * C.y21 ≠ 0) :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall C.ctr (C.rr : ℝ) ∧ Flip C.n (w 1) (w 0) :=
```

Theorem 3.4 (`not_period`):

```
theorem not_period (hr : 0 ≤ C.rr) (k : ℕ) (h : Disj (C.orbItv k) C.ballX)
    {w : Fin 2 → ℝ} (hw : w ∈ Metric.closedBall C.ctr (C.rr : ℝ)) :
    orb (w 1) (w 0) k ≠ w 0 :=
```

Module `GdCubicCascade/Thresholds.lean`**.** Theorem 4.1 (`c4_certified`):

```
theorem c4_certified :
    ∃ a x : ℝ, Flip 4 a x ∧ (orb a x 1 ≠ x ∧ orb a x 2 ≠ x) ∧
      (1288031754482842761954342435445572514441033845073 : ℝ) / 10 ^ 48 < a ∧
      a < (1288031754482842761954342435445572514441033845074 : ℝ) / 10 ^ 48 :=
```

Theorem 4.1 (`c4_unique`):

```
theorem c4_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c4.ctr (c4.rr : ℝ) ∧ Flip 4 (w 1) (w 0) :=
```

Theorem 4.2 (`c8_certified`):

```
theorem c8_certified :
    ∃ a x : ℝ, Flip 8 a x ∧ (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x) ∧
      (1299227939650204094163177321224660033964990043692 : ℝ) / 10 ^ 48 < a ∧
      a < (1299227939650204094163177321224660033964990043693 : ℝ) / 10 ^ 48 :=
```

Theorem 4.2 (`c8_unique`):

```
theorem c8_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c8.ctr (c8.rr : ℝ) ∧ Flip 8 (w 1) (w 0) :=
```

Theorem 4.3 (`c16_certified`):

```
theorem c16_certified :
    ∃ a x : ℝ, Flip 16 a x ∧ (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠
        x) ∧
      (1301628914037081195197756509066654739334743731359 : ℝ) / 10 ^ 48 < a ∧
      a < (1301628914037081195197756509066654739334743731360 : ℝ) / 10 ^ 48 :=
```

Theorem 4.3 (`c16_unique`):

```
theorem c16_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c16.ctr (c16.rr : ℝ) ∧ Flip 16 (w 1) (w 0) :=
```

Theorem 4.4 (`c32_certified`):

```
theorem c32_certified :
    ∃ a x : ℝ, Flip 32 a x ∧ (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠ x
        ∧ orb a x 16 ≠ x) ∧
      (1302143271581458122996650293284747114084794699858 : ℝ) / 10 ^ 48 < a ∧
      a < (1302143271581458122996650293284747114084794699859 : ℝ) / 10 ^ 48 :=
```

Theorem 4.4 (`c32_unique`):

```
theorem c32_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c32.ctr (c32.rr : ℝ) ∧ Flip 32 (w 1) (w 0) :=
```

Theorem 6.1 (`cw_certified`):

```
theorem cw_certified :
    ∃ a x : ℝ, Flip 4 a x ∧ (orb a x 1 ≠ x ∧ orb a x 2 ≠ x) ∧
      (1548831219278772124652324551492506085547307398534 : ℝ) / 10 ^ 48 < a ∧
      a < (1548831219278772124652324551492506085547307398535 : ℝ) / 10 ^ 48 :=
```

Theorem 6.1 (`cw_unique`):

```
theorem cw_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall cw.ctr (cw.rr : ℝ) ∧ Flip 4 (w 1) (w 0) :=
```

Theorem 5.1 (`cascade_ratios`):

```
theorem cascade_ratios :
    ∃ a4 a8 a16 a32 x4 x8 x16 x32 : ℝ,
      Flip 4 a4 x4 ∧ Flip 8 a8 x8 ∧ Flip 16 a16 x16 ∧ Flip 32 a32 x32 ∧
      Real.sqrt 5 - 1 < a4 ∧ a4 < a8 ∧ a8 < a16 ∧ a16 < a32 ∧
      (4641 : ℝ) / 1000 < (a4 - (Real.sqrt 5 - 1)) / (a8 - a4) ∧
      (a4 - (Real.sqrt 5 - 1)) / (a8 - a4) < (4642 : ℝ) / 1000 ∧
```

```
    (4663 : ℝ) / 1000 < (a8 - a4) / (a16 - a8) ∧
    (a8 - a4) / (a16 - a8) < (4664 : ℝ) / 1000 ∧
    (4667 : ℝ) / 1000 < (a16 - a8) / (a32 - a16) ∧
    (a16 - a8) / (a32 - a16) < (4668 : ℝ) / 1000 :=
```

Theorem 6.1 (control_two_period4_flips):

```
theorem control_two_period4_flips :
  ∃ a a' x x' : ℝ, Flip 4 a x ∧ Flip 4 a' x' ∧ a ≠ a' :=
```

Theorem 6.1 (control_a4_ne_a2):

```
theorem control_a4_ne_a2 : ∃ a x : ℝ, Flip 4 a x ∧ a ≠ Real.sqrt 5 - 1 :=
```

Module GdCubicCascade/Poly.lean. Theorem 6.2 (zhang_poly_unique_root_a4):

```
theorem zhang_poly_unique_root_a4 :
  ∃! t : ℝ, t ∈ Set.Icc
      ((1288031754482842761954342435445572514441033845073 : ℝ) / 10 ^ 48)
      ((1288031754482842761954342435445572514441033845074 : ℝ) / 10 ^ 48) ∧ Pflip t = 0 :=
```

Theorem 6.2 (zhang_poly_unique_root_window):

```
theorem zhang_poly_unique_root_window :
  ∃! t : ℝ, t ∈ Set.Icc
      ((1548831219278772124652324551492506085547307398534 : ℝ) / 10 ^ 48)
      ((1548831219278772124652324551492506085547307398535 : ℝ) / 10 ^ 48) ∧ Pflip t = 0 :=
```

Theorem 6.2 (control_flip_poly_has_far_root):

```
theorem control_flip_poly_has_far_root :
  ∃ t : ℝ, t ∈ Set.Icc ((19675562012733241127 : ℝ) / 10 ^ 19)
      ((19675562012733241128 : ℝ) / 10 ^ 19) ∧ Pflip t = 0 :=
```

Module GdCubicCascade/ThresholdsHigh.lean. Theorem 4.5 (c64_unique):

```
theorem c64_unique :
  ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c64.ctr (c64.rr : ℝ) ∧ Flip 64 (w 1) (w 0) :=
```

Theorem 4.5 (c64_certified):

```
theorem c64_certified :
  ∃ a x : ℝ, Flip 64 a x ∧
    (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠ x ∧ orb a x 16 ≠ x ∧
      orb a x 32 ≠ x) ∧
    (1302253437742128565172079349083254326345677410797 : ℝ) / 10 ^ 48 < a ∧
    a < (1302253437742128565172079349083254326345677410798 : ℝ) / 10 ^ 48 :=
```

Theorem 4.6 (c128_unique):

```
theorem c128_unique :
  ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c128.ctr (c128.rr : ℝ) ∧ Flip 128 (w 1) (w 0) :=
```

Theorem 4.6 (c128_certified):

```
theorem c128_certified :
  ∃ a x : ℝ, Flip 128 a x ∧
    (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠ x ∧ orb a x 16 ≠ x ∧
      orb a x 32 ≠ x ∧ orb a x 64 ≠ x) ∧
    (1302277032259641670362172426105624664109218639849 : ℝ) / 10 ^ 48 < a ∧
    a < (1302277032259641670362172426105624664109218639850 : ℝ) / 10 ^ 48 :=
```

Theorem 5.2 (cascade_ratios_high):

```
theorem cascade_ratios_high :
  ∃ a16 a32 a64 a128 x16 x32 x64 x128 : ℝ,
    Flip 16 a16 x16 ∧ Flip 32 a32 x32 ∧ Flip 64 a64 x64 ∧ Flip 128 a128 x128 ∧
    a16 < a32 ∧ a32 < a64 ∧ a64 < a128 ∧
    (4668925 : ℝ) / 10 ^ 6 < (a32 - a16) / (a64 - a32) ∧
    (a32 - a16) / (a64 - a32) < (4668926 : ℝ) / 10 ^ 6 ∧
    (4669142 : ℝ) / 10 ^ 6 < (a64 - a32) / (a128 - a64) ∧
    (a64 - a32) / (a128 - a64) < (4669143 : ℝ) / 10 ^ 6 :=
```

Theorem 6.3 (control_precond_too_narrow):

```
theorem control_precond_too_narrow :
  c64w8.y11 * c64w8.y22 - c64w8.y12 * c64w8.y21 = 0 ∧ ¬ (c64w8.K < 1) :=
```

Theorem 6.3 (control_ball_too_large_64):

```
theorem control_ball_too_large_64 : ¬ (c64wide.K < 1) :=
```

Theorem 6.3 (`control_ball_too_large_128`):

```
theorem control_ball_too_large_128 : ¬ (c128wide.K < 1) :=
```

Theorem 6.3 (`control_centre_perturbed`):

```
theorem control_centre_perturbed : ¬ (c64off.beta + c64off.K * c64off.rr ≤ c64off.rr) :=
```

Module `GdCubicCascade/Thresholds256.lean`**.** Theorem 4.7 (`c256_unique`):

```
theorem c256_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c256.ctr (c256.rr : ℝ) ∧ Flip 256 (w 1) (w 0) :=
```

Theorem 4.7 (`c256_certified`):

```
theorem c256_certified :
    ∃ a x : ℝ, Flip 256 a x ∧
      (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠ x ∧ orb a x 16 ≠ x ∧
        orb a x 32 ≠ x ∧ orb a x 64 ≠ x ∧ orb a x 128 ≠ x) ∧
      (1302282085496493499044942323804032998555673048225 : ℝ) / 10 ^ 48 < a ∧
      a < (1302282085496493499044942323804032998555673048226 : ℝ) / 10 ^ 48 :=
```

Theorem 5.3 (`cascade_ratio_256`):

```
theorem cascade_ratio_256 :
    ∃ a64 a128 a256 x64 x128 x256 : ℝ,
      Flip 64 a64 x64 ∧ Flip 128 a128 x128 ∧ Flip 256 a256 x256 ∧
      a64 < a128 ∧ a128 < a256 ∧
      (4669188 : ℝ) / 10 ^ 6 < (a128 - a64) / (a256 - a128) ∧
      (a128 - a64) / (a256 - a128) < (4669189 : ℝ) / 10 ^ 6 :=
```

Module `GdCubicCascade/Thresholds512.lean`**.** Theorem 4.8 (`c512_unique`):

```
theorem c512_unique :
    ∃! w : Fin 2 → ℝ, w ∈ Metric.closedBall c512.ctr (c512.rr : ℝ) ∧ Flip 512 (w 1) (w 0) :=
```

Theorem 4.8 (`c512_certified`):

```
theorem c512_certified :
    ∃ a x : ℝ, Flip 512 a x ∧
      (orb a x 1 ≠ x ∧ orb a x 2 ≠ x ∧ orb a x 4 ≠ x ∧ orb a x 8 ≠ x ∧ orb a x 16 ≠ x ∧
        orb a x 32 ≠ x ∧ orb a x 64 ≠ x ∧ orb a x 128 ≠ x ∧ orb a x 256 ≠ x) ∧
      (1302283167745712275624657981558236703751436265303 : ℝ) / 10 ^ 48 < a ∧
      a < (1302283167745712275624657981558236703751436265304 : ℝ) / 10 ^ 48 :=
```

Theorem 5.4 (`cascade_ratio_512`):

```
theorem cascade_ratio_512 :
    ∃ a128 a256 a512 x128 x256 x512 : ℝ,
      Flip 128 a128 x128 ∧ Flip 256 a256 x256 ∧ Flip 512 a512 x512 ∧
      a128 < a256 ∧ a256 < a512 ∧
      (4669198 : ℝ) / 10 ^ 6 < (a256 - a128) / (a512 - a256) ∧
      (a256 - a128) / (a512 - a256) < (4669199 : ℝ) / 10 ^ 6 :=
```