GAUTSCHI'S CONJECTURE ON SUBRANGE JACOBI POLYNOMIALS IN THE NEGATIVE WEDGE: AN ENDPOINT OBSTRUCTION TO THE FIRST-CROSSING METHOD AND AN EFFECTIVE REDUCTION TO FINITELY MANY DEGREES

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let π_n be the monic polynomial of degree n orthogonal on $[-c, c]$, $0 < c \leq 1$, with respect to the Jacobi weight $(1-x)^\alpha(1+x)^\beta$, $-1 < \alpha < \beta$. Gautschi conjectured that $[\pi_n(-c)/\pi_n(c)]^2\,((1-c)/(1+c))^{\beta-\alpha} < 1$ for every $n \geq 1$. Botta, Castillo and Tertuliano da Silva (arXiv:2608.05963) proved it, uniformly in the degree, throughout $\beta \geq 0$, and in the remaining negative wedge, written $\alpha = -r - \lambda$, $\beta = -r + \lambda$, whenever $c^2 \leq 3/(3+r)$; their method is a first-crossing argument whose pivot is the root-sum inequality $\Sigma_N < \lambda$, and their Remark 5.2 proposes to reach the rest of the wedge through the estimate $C_{N,a} > \lambda((N + r)a^2 - N)$, where $C_{N,a}$ is defined by an exact Ward identity. We show that the proposed estimate is false at the endpoint $a = 1$, for every degree $N \geq 1$ and every point of the negative wedge: with $\Sigma_N(1)$ the classical sum of the zeros of the Jacobi polynomial, $C_{N,1} = 0$ while the estimate requires $C_{N,1} > \lambda r > 0$; equivalently, the root-sum inequality itself fails at $c = 1$ throughout the negative sector. We also run the authors' own induction from a shifted start and obtain an effective statement: if the conjecture holds at (α, β) in every degree $1 \leq n \leq n_0$ and on every subrange $[-a, a]$ with $a \leq c$, and $rc^2 \leq (n_0 + 2)(1 - c^2)$, then it holds in every degree; at $n_0 = 1$ this is exactly the published condition $c^2 \leq 3/(3 + r)$, and at every fixed $c < 1$ it reduces the degree-uniform conjecture to the explicit finite block $2 \leq n \leq \lceil rc^2/(1 - c^2)\rceil - 2$, where the source has only a non-effective finiteness statement. A numerical scan of 219,816 cells of the open region, reported as a measurement, finds no counterexample. The arithmetic of every numbered statement not labelled otherwise is machine-checked in Lean 4; the analytic inputs are the source's lemmas and enter as hypotheses.

1. Introduction

The conjecture. Let $w(x) = (1 - x)^\alpha(1 + x)^\beta$ with $-1 < \alpha < \beta$, let $0 < c \leq 1$, and let $\pi_n = \pi_n^{(\alpha,\beta)}(\,\cdot\,; c)$ be the monic polynomial of degree n with $\int_{-c}^{c} \pi_n(x)x^j w(x)\,dx = 0$ for $0 \leq j < n$. All zeros of π_n are simple and lie in $(-c, c)$. These *subrange Jacobi polynomials* were introduced by Gautschi [2]; their zeros are the nodes of the Gauss rule for the truncated Jacobi measure. In [3] Gautschi studied how the zeros move as c varies, and his variation formula led him to the following conjecture, which we quote in the form given as Conjecture 1.1 of [1]:

Conjecture (Gautschi, 2018). For every $n \geq 1$, $-1 < \alpha < \beta$, and $0 < c \leq 1$,

$$\left[\frac{\pi_n(-c)}{\pi_n(c)}\right]^2 \left(\frac{1-c}{1+c}\right)^{\beta-\alpha} < 1. \tag{1}$$

By Gautschi's variation formula, (1) is sufficient for every positive zero of π_n to move to the right as c increases. The case $c = 1$ is trivial (the second factor vanishes), so the content is $0 < c < 1$.

What is known. We know of no work on this conjecture beyond the three papers cited in this paragraph (a search of citation records and of an arXiv corpus on 2026-09-03 found none; this is a negative search result, not a claim of completeness). According to [1], Gautschi [3] recorded the sector $\alpha \leq 0 \leq \beta$ on the basis of an unpublished communication from Milovanović, and Milovanović [4] then published the criterion $\beta - \alpha \geq c|\alpha + \beta|$; we cite both results through [1], having consulted neither text. Botta, Castillo and Tertuliano da Silva [1], which we call *the source* throughout, proved the following. Write $\rho_c = (1-c)/(1+c)$.

- (Theorem 1.2 of [1]) The conjecture holds for every $n \geq 1$, $0 < c < 1$ and $0 < \alpha < \beta$. Together with the two earlier results this settles the whole range $\beta \geq 0$ (Corollary 1.3).
- (Theorem 1.4) In the *negative wedge* $-1 < \alpha < \beta < \rho_c\alpha < 0$, written

$$\alpha = -r - \lambda, \qquad \beta = -r + \lambda, \tag{2}$$

the conjecture holds for every $n \geq 1$ whenever $c^2 \leq 3/(3+r)$. Hence (Corollary 1.5) it holds for every admissible (α, β) when $0 < c \leq \sqrt{3}/2$.

- (Theorem 1.6) For every admissible triple, the conjecture holds in degree $n = 1$, and in every degree $n \geq N(\alpha, \beta, c)$ for some N depending on the triple.

What remains open, uniformly in the degree, is the source's region

$$\mathcal{O}_c = \left\{ -1 < \alpha < \beta < \rho_c \alpha < 0, \quad \alpha + \beta < -\tfrac{6(1-c^2)}{c^2} \right\}, \tag{3}$$

which in the coordinates (2) is exactly $c^2 > 3/(3+r)$ inside the negative wedge, at degrees $n \geq 2$. On effectivity the source is explicit. In its introduction: "The asymptotic argument is not effective and therefore does not itself furnish a computational stopping criterion." And in the remark following its Lemma 6.1: "The integer N in Theorem 1.6, part (2), depends on the fixed triple (α, β, c). The argument gives neither an effective value nor a bound uniform as $\beta - \alpha \to 0$ or $c \to 1$." (Statement numbers of [1] are those of the compiled e-print of version 1.)

The source's method, and its proposed continuation. Every degree-uniform result of [1] rests on one mechanism. A Pearson identity (Lemma 3.1) expresses the normalised endpoint difference d_n, whose positivity is the conjecture in degree n, through the sums Σ_n, Σ_{n+1} of the zeros; a crossing lemma (Lemma 3.4) shows that at a first zero of d_n one has $\partial d_n > 0$ provided $\Sigma_n < \lambda$, where $\lambda = (\beta - \alpha)/2$; and in the negative wedge an ensemble Ward bound (Lemma 5.1), $(N - ra^2)\Sigma_N < N\lambda a^2$ on $[-a, a]$, delivers the required *root-sum inequality* $\Sigma_N < \lambda$ as long as $a^2 \leq N/(N+r)$. The induction starts at $n = 2$ with $N = n + 1 = 3$, whence $c^2 \leq 3/(3+r)$. Remark 5.2 of the source, whose last paragraph we quote in full, says what would be needed beyond that:

> When $c^2 > 3/(3+r)$, positivity of $C_{N,a}$ alone does not imply the root-sum inequality required by the crossing lemma at every possible first crossing. In the complementary range, the same argument would require the quantitative estimate
>
> $$C_{N,a} > \lambda\big((N+r)a^2 - N\big)$$
>
> whenever the right-hand side is positive, or an independent treatment of the corresponding initial degrees. No such estimate is established here. Thus the restriction is a limitation of the present bound, not an assertion that the ensemble or first-crossing approach cannot be sharpened.

Here $C_{N,a} > 0$ is the quantity defined by the exact Ward identity $(N - ra^2)\Sigma_N = N\lambda a^2 - C_{N,a}$ (equation (5.7) of the source).

What this paper proves. Two things, both about the mechanism rather than about a new cell of $\mathcal{O}_c$.

First, the estimate proposed in Remark 5.2 is false at the endpoint (Theorem 3.5). At $c = 1$ the subrange system is the full Jacobi system, whose recurrence coefficients are classical, and the sum of the zeros telescopes to $\Sigma_N(1) = N(\beta - \alpha)/(2N + \alpha + \beta)$. In the negative sector $\alpha + \beta < 0$ this *exceeds* λ for every $N \geq 1$ (Proposition 3.2), so the root-sum inequality is false at $c = 1$; and inserting the classical value into the Ward identity gives exactly $C_{N,1} = 0$, whereas the estimate of Remark 5.2 asks for $C_{N,1} > \lambda r > 0$. Since $\Sigma_N < \lambda$ is in fact *equivalent* to Remark 5.2's estimate (Proposition 4.3), the obstruction is not a weakness of one bound but of the pivot itself: no sharpening of the ensemble bound can carry the first-crossing method to $c = 1$ at any fixed degree. A continuity argument, which we give as an ordinary remark and have not formalised, extends the failure to a punctured neighbourhood of $c = 1$; numerically it is visible already at $c = 0.999$.

Second, the source's induction, started from a shifted base, gives an effective reduction (Theorem 5.1): if the conjecture holds at (α, β) for all degrees $1 \leq n \leq n_0$ and all subranges $[-a, a]$, $0 < a \leq c$, and

$$rc^2 \leq (n_0 + 2)(1 - c^2), \tag{4}$$

then it holds in every degree. At $n_0 = 1$, where the base is the source's own Theorem 1.6(1), condition (4) is precisely $c^2 \leq 3/(3+r)$, so the reduction reproduces the published Theorem 1.4 and nothing weaker (Proposition 5.3). For general $c < 1$ the whole degree-uniform conjecture at a point of the wedge collapses to the explicit finite block $2 \leq n \leq \lceil rc^2/(1-c^2) \rceil - 2$ (Corollary 5.2), which is the "independent treatment of the corresponding initial degrees" of Remark 5.2 with a count attached. The count is honest but lossy (Section 6).

Nothing here settles a point of $\mathcal{O}_c$. A numerical scan of 18,318 points of $\mathcal{O}_c$ in degrees 1 to 12 finds the conjecture true at every one of its 219,816 cells; it is a measurement, reported as such in Section 6, and nothing else depends on it.

Related work. Two further recent preprints by one of the source's authors concern other conjectures of Gautschi and of Szegő on the *full* Jacobi polynomials — the degree dependence of their zeros [7] and the ordering of their relative extrema [8] — and do not touch the subrange endpoint inequality.

Verification. The arithmetic content of every numbered statement below that carries no "not formalised" label is machine-checked in Lean 4 with Mathlib; the analytic inputs (Lemmas 3.1, 3.4 and 5.1 of the source) are not re-proved but enter as explicit hypotheses, and the classical Jacobi recurrence coefficients enter as a definition. Section 7 lists exactly what is formal, and Appendix A reproduces the formal statements verbatim.

2. Notation and the ingredients from the source

Fix $-1 < \alpha < \beta$ and write

$$\lambda = \frac{\beta - \alpha}{2} > 0, \qquad s = \alpha + \beta, \qquad \nu = \frac{s+2}{2}, \qquad \delta = \beta - \alpha = 2\lambda.$$

For $0 < c \le 1$ let $h_n = \int_{-c}^{c} \pi_n^2 w$, let $p_n = \pi_n/\sqrt{h_n}$ be the orthonormal polynomials, and let

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) \qquad (n \ge 0)$$

be their three-term recurrence (equation (3.1) of [1]). Since $\pi_n(x) = x^n - \Sigma_n x^{n-1} + \cdots$, the sum Σ_n of the zeros of π_n satisfies $\Sigma_{n+1} = \Sigma_n + b_n$, $\Sigma_0 = 0$, i.e. $\Sigma_n = \sum_{j<n} b_j$. All of b_n, Σ_n, h_n depend on c; we write $b_n(c)$, $\Sigma_n(c)$ when the dependence matters.

Put $D_n = w(c)\pi_n(c)^2 - w(-c)\pi_n(-c)^2$ and, for $c < 1$,

$$d_n = \frac{1-c^2}{2h_n}\, D_n.$$

Since $w(-c)/w(c) = \rho_c^{\delta}$ and $\pi_n(\pm c) \ne 0$, inequality (1) in degree n is equivalent to $D_n > 0$, hence to $d_n > 0$ (Proposition 2.1 and equation (3.2) of [1]). The three analytic facts from the source that we use are:

(P) *Pearson identity* (Lemma 3.1 of [1]): for every $n \ge 0$ and $0 < c < 1$,

$$d_n = \lambda - \Sigma_n - (n+\nu) b_n = \lambda + (n+\nu-1)\Sigma_n - (n+\nu)\Sigma_{n+1}. \tag{5}$$

(X) *Crossing lemma* (Lemma 3.4): if $n \ge 1$ and at $c = c_0$ one has $d_n(c_0) = 0$, $d_{n-1}(c_0) > 0$ and $\Sigma_n(c_0) < \lambda$, then $\partial_c d_n(c_0) > 0$.

(W) *Ensemble Ward bound* (Lemma 5.1): in the negative wedge, with the coordinates (2), for every $N \ge 1$ and every $0 < a < 1$, writing $\Sigma_N(a)$ for the sum of the zeros of the degree-N monic orthogonal polynomial on $[-a, a]$,

$$(N - ra^2)\,\Sigma_N(a) < N\lambda a^2. \tag{6}$$

The proof of Lemma 5.1 in fact gives the exact identity (equation (5.7) of [1])

$$(N - ra^2)\,\Sigma_N(a) = N\lambda a^2 - C_{N,a}, \qquad C_{N,a} = (1-a^2)\Big\{\lambda\,\mathbb{E}_{N,\lambda}\sum_{i=1}^{N}\frac{x_i^2}{1-x_i^2} + r\,\mathbb{E}_{N,\lambda}\sum_{i=1}^{N}\frac{x_i^3}{1-x_i^2}\Big\} > 0, \tag{7}$$

the expectations being over the source's tilted orthogonal polynomial ensemble (its equation (5.4)) on the ordered chamber $-a < x_1 < \cdots < x_N < a$; this is the $C_{N,a}$ of Remark 5.2.

In the negative wedge the parameter restrictions are (equation (5.2) of [1])

$$0 < r < 1, \qquad 0 < \lambda < \min\{r, 1-r\}, \qquad \lambda < rc, \tag{8}$$

and $\nu = 1 - r$. We shall only use that $-1 < \alpha < \beta < 0$ forces $-2 < s < 0$, i.e. $0 < r < 1$.

The endpoint. At $c = 1$ the orthogonality is on $[-1, 1]$ and π_n is the monic Jacobi polynomial. The diagonal recurrence coefficients are then classical ([5, Table 1.1], [6, Chapter IV, §4.5]):

$$b_0(1) = \frac{\beta - \alpha}{\alpha + \beta + 2}, \qquad b_j(1) = \frac{\beta^2 - \alpha^2}{(2j + \alpha + \beta)(2j + \alpha + \beta + 2)} \quad (j \geq 1). \tag{9}$$

In the formal development, (9) is the *definition* of $b_j(1)$ (Mathlib has no orthogonal polynomials), and $\Sigma_n(1) := \sum_{j<n} b_j(1)$; the classical fact is the provenance of the definition, and every statement of Section 3 is algebra downstream of it. The $j = 0$ case is written separately because the single formula is $0/0$ at $j = 0$ when $\alpha + \beta = 0$.

3. The endpoint $c = 1$

Proposition 3.1 (classical)**.** *For $-2 < \alpha + \beta$ and every $n \geq 0$,*

$$\Sigma_n(1) = \sum_{j=0}^{n-1} b_j(1) = \frac{n(\beta - \alpha)}{2n + \alpha + \beta}.$$

Proof. Induction on n: with $t = 2n + \alpha + \beta$ and $k = \beta - \alpha$, the step is $nk/t + k(t - 2n)/(t(t + 2)) = (n+1)k/(t+2)$, valid since $t > 0$ for $n \geq 1$ and $t+2 > 0$ for $n \geq 0$ under $-2 < \alpha+\beta$, and $\beta^2-\alpha^2 = k\,(t-2n)$. The formula is the classical sum of the zeros of the Jacobi polynomial; only the telescoping is formalised. ☐

Proposition 3.2 (the sector trichotomy)**.** *Let $\alpha < \beta$ and $n \geq 1$.*

1. *If $-2 < \alpha + \beta < 0$, then $\Sigma_n(1) > \lambda$.*
2. *If $\alpha + \beta > 0$, then $\Sigma_n(1) < \lambda$.*
3. *If $\alpha + \beta = 0$, then $\Sigma_n(1) = \lambda$.*

Proof. By Proposition 3.1, $\Sigma_n(1) - \lambda = (\beta - \alpha)\big(-(\alpha + \beta)\big)/\big(2(2n + \alpha + \beta)\big)$, whose denominator is positive for $n \geq 1$ and whose numerator has the sign of $-(\alpha + \beta)$. ☐

This one line is the whole difference between the two same-sign sectors for the source's method. In the positive sector the root-sum inequality survives to the endpoint — the proof of Theorem 1.2 in [1] opens with "Since $\nu > 1$" and propagates $\Sigma_{n+1} < \lambda$ along the induction — and in the negative sector it is false there, at every degree. Part (2) also serves as a control for part (1): the sign hypothesis is not decorative.

Proposition 3.3 (the Pearson quantity vanishes at the endpoint)**.** *For $-2 < \alpha + \beta$ and every $n \geq 0$,*

$$\lambda - \Sigma_n(1) - (n + \nu)\, b_n(1) = 0.$$

Proof. Substitute Proposition 3.1 and (9); the resulting rational identity is checked by clearing the (positive) denominators $2n + \alpha + \beta$ and $2n + \alpha + \beta + 2$. ☐

Proposition 3.3 is what the Pearson identity (5) becomes at $c = 1$: the boundary term $(1-x^2)w(x)p_n(x)^2$ of the integration by parts vanishes at ± 1 because $\alpha, \beta > -1$, so the same computation on $[-1, 1]$ gives $0 = \lambda - \Sigma_n - (n + \nu)b_n$ directly (this reading is a consistency check, not part of the formal statement). The source notes that "For $c = 1$ the conjecture is immediate, since $\beta - \alpha > 0$ and $\pi_n(1) \neq 0$" — true of D_n; but the *normalised* d_n degenerates to 0 at every degree, so $c = 1$ is a genuine barrier for an argument that tracks the sign of d_n, not a trivial case of it.

Definition 3.4. For $N \geq 1$, $a \in (0, 1]$, $\lambda, r \in \mathbb{R}$ and a real number S (to be read as $\Sigma_N(a)$), put

$$C_{N,a}(\lambda, r, S) := N\lambda a^2 - (N - ra^2)\, S,$$

so that $(N - ra^2)S = N\lambda a^2 - C_{N,a}$ is the identity (7) read as a definition. For $a < 1$ and $S = \Sigma_N(a)$ this is the source's $C_{N,a}$; at $a = 1$ we take $S = \Sigma_N(1)$ of Proposition 3.1.

Theorem 3.5 (the estimate of Remark 5.2 is false at the endpoint)**.** *Let $\alpha < \beta$, $-2 < \alpha + \beta < 0$, $\lambda = (\beta - \alpha)/2$, $r = -(\alpha + \beta)/2$, and $N \geq 1$. Then*

$$C_{N,1}\big(\lambda, r, \Sigma_N(1)\big) = 0 \qquad \textit{while} \qquad \lambda\big((N + r) \cdot 1^2 - N\big) = \lambda r > 0.$$

Consequently the root-sum inequality $\Sigma_N(1) < \lambda$ is false: for every such (α, β) and every $N \geq 1$, $\neg\big(\Sigma_N(1) < \lambda\big)$.

Proof. With $t = 2N+\alpha+\beta > 0$ one has $N-r = t/2$ and $\Sigma_N(1) = N(\beta-\alpha)/t$, so $C_{N,1} = N\lambda-(t/2)\cdot N(\beta-\alpha)/t = N\lambda - N\lambda = 0$. The right-hand side is λr, a product of two positive numbers. The last sentence is Proposition 3.2(1). □

The hypotheses of Theorem 3.5 are those of the whole negative sector $-2 < \alpha+\beta < 0$, of which the negative wedge (8) is a part; in particular they hold at every point of $\mathcal{O}_c$ for every c. Read against Remark 5.2: the estimate $C_{N,a} > \lambda((N+r)a^2-N)$ is required "whenever the right-hand side is positive", and at $a = 1$ the right-hand side is $\lambda r > 0$ while the left-hand side is 0. So of the two branches offered there — a quantitative lower bound on $C_{N,a}$, or an independent treatment of the initial degrees — the first cannot be completed up to $a = 1$ at any fixed degree, whatever the sharpening of the ensemble bound. Section 5 makes the second branch effective.

Proposition 3.6 (non-vacuity)**.** $(\alpha,\beta) = (-3/5,-1/5)$ *satisfies* $-1 < \alpha < \beta < 0$*, and there* $\Sigma_2(1) = 1/4$ *while* $\lambda = 1/5$*.*

This point lies in the negative wedge for every $c > 1/2$ (one checks $\beta < \rho_c\alpha \iff \rho_c < 1/3 \iff c > 1/2$, and $\lambda < rc$ is the same condition); it exhibits the hypotheses of Theorem 3.5 as jointly satisfiable with a non-trivial conclusion.

Remark 3.7 (continuity extends the failure to a neighbourhood; not formalised). Fix (α,β) with $-2 < \alpha+\beta < 0$ and $N \geq 1$. Then there is $c_N < 1$ such that $\Sigma_N(c) > \lambda$ for all $c \in (c_N, 1)$; hence, by Proposition 4.3 below, Remark 5.2's estimate fails on the punctured neighbourhood $(c_N, 1)$ and not only at the endpoint, and $C_{N,a} \to 0$ as $a \to 1$ while the required lower bound tends to λr.

The argument is ordinary and we have not formalised it. Write $m_k(c) = \int_{-c}^{c} x^k w(x)\,dx$. Since $\alpha,\beta > -1$, $|x|^k w(x) \leq w(x)$ is integrable on $[-1,1]$, so $m_k(c) \to m_k(1)$ as $c \to 1^-$ by dominated convergence. The source (Section 3, after Corollary 3.2) writes the coefficients $q_{j,n}(c)$ of π_n as the solution of the Hankel system $\sum_{j<n} m_{j+k}(c)q_{j,n}(c) = -m_{n+k}(c)$, $0 \leq k < n$, whose matrix is positive definite, and deduces real-analyticity of $\Sigma_n = -q_{n-1,n}$ on $(0,1)$ by Cramer's rule; the same Cramer expression is continuous at $c = 1$, where the matrix is the moment matrix of the full Jacobi weight, again positive definite. So $\Sigma_N(c) \to \Sigma_N(1) > \lambda$ (Proposition 3.2), and continuity gives c_N. Numerically (Section 6) the neighbourhood is wide: $\Sigma_2/\lambda = 1.161$ at $c = 0.999$, $r = 0.7$, $\lambda = 0.12$.

4. The root-sum inequality and the Ward identity

The statements of this section are the arithmetic steps of the source's proofs of Theorems 1.2 and 1.4, stated exactly and in both directions where the source uses one; they are recorded because Theorem 5.1 is assembled from them and because Proposition 4.3 is what turns Theorem 3.5 from "the proposed estimate is unavailable at the endpoint" into "the inequality it was meant to deliver is itself false there". Throughout, $N, \lambda, r, a, S, \ldots$ are real numbers unless stated otherwise.

Lemma 4.1 (Ward bound to root-sum bound)**.** (1) *If* $\lambda > 0$*,* $N - ra^2 > 0$*,* $(N-ra^2)S < N\lambda a^2$ *and* $a^2(N+r) \leq N$*, then* $S < \lambda$*.*
(2) *If* $a^2 < 1$*, then* $a^2(N+r) \leq N$ *if and only if* $ra^2 \leq N(1-a^2)$*.*
(3) *If* $r > 0$ *and* $0 < M \leq N$*, then* $M/(M+r) \leq N/(N+r)$*.*

Proof. (1) $N\lambda a^2 \leq (N-ra^2)\lambda$ is $\lambda(a^2(N+r)-N) \leq 0$; divide by $N-ra^2 > 0$. (2) is a rearrangement. (3) $N/(N+r) - M/(M+r) = r(N-M)/((M+r)(N+r)) \geq 0$. □

Part (1) is the step "$C_{N,a} > 0$ alone implies $\Sigma_N < \lambda$ whenever $a^2 \leq N/(N+r)$" of Remark 5.2; part (2), solved for N, is what makes the block of exceptional degrees finite at every fixed $a < 1$; part (3) says each further degree can only widen the region.

Lemma 4.2 (Pearson at a crossing, and propagation)**.** (1) *Let* $r < n$ *and* $\lambda + (n-r)S_n = (n+1-r)S_{n+1}$*. Then* $S_n < \lambda$ *if and only if* $S_{n+1} < \lambda$*.*
(2) *Let* $d > 0$*,* $d = \lambda + (n+\nu-1)S_n - (n+\nu)S_{n+1}$*,* $S_n \leq \lambda$*,* $n+\nu-1 \geq 0$ *and* $n+\nu > 0$*. Then* $S_{n+1} < \lambda$*.*

Proof. (1) The hypothesis is (5) at a zero of d_n with $\nu = 1-r$; both $n-r$ and $n+1-r$ are positive, and $(n+1-r)(S_{n+1}-\lambda) = (n-r)(S_n-\lambda)$. (2) $(n+\nu)S_{n+1} < \lambda + (n+\nu-1)S_n \leq (n+\nu)\lambda$. □

The source's proof of Theorem 1.4 uses the direction "$\Sigma_{n+1} < \lambda \Rightarrow \Sigma_n < \lambda$" of (1) (its equation (5.8)); (2) is the last paragraph of its proof of Theorem 1.2.

Proposition 4.3 (the root-sum inequality is Remark 5.2's estimate)**.** *Let $N - ra^2 > 0$ and $(N - ra^2)S = N\lambda a^2 - C$. Then*

$$S < \lambda \iff C > \lambda\big((N + r)a^2 - N\big).$$

Proof. $(N - ra^2)(S - \lambda) = N\lambda a^2 - C - (N - ra^2)\lambda = \lambda((N + r)a^2 - N) - C$, and $N - ra^2 > 0$. ☐

So, with $C = C_{N,a}$ of (7), the estimate Remark 5.2 proposes is not merely sufficient for the root-sum inequality at the crossing: it is equivalent to it. (When the right-hand side is ≤ 0 the estimate is automatic from $C_{N,a} > 0$, which is Lemma 4.1(1) again.)

Proposition 4.4 (negative controls)**.**

(1) *Lemma 4.1(1) fails without its threshold hypothesis: for $N = 3$, $r = 1/2$, $a = 1$, $\lambda = 1$, $S = 11/10$ one has $N - ra^2 = 5/2 > 0$ and $(N - ra^2)S = 11/4 < 3 = N\lambda a^2$, but $a^2(N + r) = 7/2 > 3$ and $S > \lambda$.*

(2) *Lemma 4.2(1) fails without $r < n$: for $r = 2$, $n = 1$, $\lambda = 1$, $S_n = 1$, $S_{n+1} = 0$ the identity $\lambda + (n - r)S_n = (n + 1 - r)S_{n+1}$ holds ($0 = 0$), $S_{n+1} < \lambda$, and $S_n < \lambda$ fails.*

5. An effective reduction to finitely many degrees

The source's proof of Theorem 1.4 is an induction on the degree, started at $n = 2$ on the base $d_1 > 0$ (its Theorem 1.6(1)), in which the only place the size of c enters is the threshold $a_0^2 \le (n + 1)/(n + 1 + r)$ at a putative first zero a_0 of d_n, needed to extract $\Sigma_{n+1}(a_0) < \lambda$ from (6) with $N = n + 1$. Starting the same induction at $n_0 + 1$ replaces $3/(3 + r)$ by $(n_0 + 2)/(n_0 + 2 + r)$. We state this with the three analytic inputs as hypotheses, so that the statement is exactly what is machine-checked; the functions d and S are arbitrary.

Theorem 5.1 (effective reduction)**.** *Let $\lambda > 0$, $0 < r < 1$, $0 < c < 1$, let $n_0 \ge 1$ be an integer with*

$$rc^2 \le (n_0 + 2)(1 - c^2),$$

and let $d_n(a)$, $S_n(a)$ be real numbers, defined for integers $n \ge 0$ and real a, such that

- (W) *$(N - ra^2)S_N(a) < N\lambda a^2$ for all $N \ge 1$ and $0 < a \le c$;*
- (P) *$\lambda + (n - r)S_n(a) = (n + 1 - r)S_{n+1}(a)$ whenever $n \ge 1$, $0 < a \le c$ and $d_n(a) = 0$;*
- (X) *for every $n \ge 1$: if $d_n(a) > 0$ for all $0 < a \le c$, and $S_{n+1}(a) < \lambda$ at every $0 < a \le c$ with $d_{n+1}(a) = 0$, then $d_{n+1}(a) > 0$ for all $0 < a \le c$;*
- (B) *$d_n(a) > 0$ for all $1 \le n \le n_0$ and $0 < a \le c$.*

Then $d_n(a) > 0$ for every $n \ge 1$ and every $0 < a \le c$.

Proof. Induction on n. Degrees $1 \le n \le n_0$ are (B). Let $m \ge n_0$ and assume $d_m > 0$ on $(0, c]$. Let $0 < a \le c$ with $d_{m+1}(a) = 0$. Since $a^2 \le c^2 < 1$ and $m + 2 \ge n_0 + 2$,

$$ra^2 \le rc^2 \le (n_0 + 2)(1 - c^2) \le (m + 2)(1 - a^2),$$

which by Lemma 4.1(2) is the threshold $a^2(m + 2 + r) \le m + 2$; also $m + 2 - ra^2 > 0$. Hypothesis (W) with $N = m + 2$ and Lemma 4.1(1) give $S_{m+2}(a) < \lambda$; hypothesis (P) at $n = m + 1$ (note $r < 1 \le m + 1$) and Lemma 4.2(1) give $S_{m+1}(a) < \lambda$. Thus the root-sum bound holds at every zero of d_{m+1} in $(0, c]$, and (X) yields $d_{m+1} > 0$ on $(0, c]$. ☐

Corollary 5.2 (the subrange Jacobi system; not formalised beyond Theorem 5.1)**.** *Let $0 < c < 1$ and let (α, β) be a point of the negative wedge, with the coordinates (2). Suppose the conjecture holds at (α, β) for every degree $1 \le n \le n_0$ and every subrange $[-a, a]$, $0 < a \le c$, and that $rc^2 \le (n_0 + 2)(1 - c^2)$. Then it holds at (α, β, a) for every $0 < a \le c$ and every degree $n \ge 1$. In particular, at a fixed $c < 1$ the degree-uniform conjecture at (α, β) reduces to the finite block of degrees*

$$2 \le n \le \left\lceil \frac{rc^2}{1 - c^2} \right\rceil - 2,$$

which is empty exactly when $c^2 \le 3/(3 + r)$.

Proof. Apply Theorem 5.1 to $d_n(a)$ and $S_n(a) = \Sigma_n(a)$ of the subrange system on $[-a, a]$. (W) is Lemma 5.1 of [1]; (P) is its Lemma 3.1 at a zero of d_n, with $\nu = 1 - r$; (X) is the argument of its proof of Theorem 1.4: $d_{n+1} > 0$ on $(0, \lambda/r] \cap (0, c]$ by the boundary identity, a least zero $a_0 > \lambda/r$ exists by continuity and compactness if there is any zero, $\partial_a d_{n+1}(a_0) \le 0$ by minimality, and Lemma 3.4 with $d_n(a_0) > 0$ and $\Sigma_{n+1}(a_0) < \lambda$ gives $\partial_a d_{n+1}(a_0) > 0$. (B) is the hypothesis. Degree 1 always belongs to the base by Theorem 1.6(1) of [1], so the block starts at 2; the least admissible n_0 is the least integer $\ge rc^2/(1-c^2) - 2$, which is $\lceil rc^2/(1-c^2)\rceil - 2$, and it is ≤ 1 iff $rc^2 \le 3(1-c^2)$. □

The next statement is the control on Theorem 5.1: with only the source's degree-one theorem as base, the reduction returns the source's own Theorem 1.4.

Proposition 5.3 (control: $n_0 = 1$ is Theorem 1.4)**.** *For $r > 0$ and $0 < c < 1$,*

$$rc^2 \le (1+2)(1-c^2) \quad \Longleftrightarrow \quad c^2 \le \frac{3}{3+r}.$$

Proof. Both sides are $c^2(3 + r) \le 3$. □

Proposition 5.4 (what each further degree buys)**.** (1) *If $r > 0$ and $0 < m < n$, then $m/(m + r) < n/(n + r)$; in particular $3/(3 + r) < 4/(4 + r)$.*
(2) *For $r > 0$ and $0 < c < 1$ there is an integer $n_0 \ge 1$ with $rc^2 \le (n_0 + 2)(1 - c^2)$.*

Proof. (1) as in Lemma 4.1(3) with strict inequality. (2) Take $n_0 = k+1$ for any integer $k > rc^2/(1-c^2)$. □

So settling degree 2 alone, throughout the wedge and for all $a \le c$, would replace the source's Corollary 1.5 bound $c \le \sqrt{3}/2 \approx 0.866$ by $c \le 2/\sqrt{5} \approx 0.894$; settling degrees up to n_0 moves the Theorem 1.4 threshold to $c^2 \le (n_0 + 2)/(n_0 + 2 + r)$ and the Corollary 1.5 bound to $c \le \sqrt{(n_0 + 2)/(n_0 + 3)}$; and the thresholds exhaust $(0, 1)$, so at every fixed $c < 1$ only finitely many degrees are ever at stake.

Proposition 5.5 (a worked instance)**.** *At $r = 9/10$, $c = 19/20$ one has $c^2 = 361/400 > 10/13 = 3/(3 + r)$, so the point is in $\mathcal{O}_c$; $rc^2 \le (7 + 2)(1 - c^2)$ holds and $rc^2 \le (6 + 2)(1 - c^2)$ fails. Hence $n_0 = 7$ is the least admissible start: degrees $2, \dots, 7$ are the whole remaining content of the degree-uniform conjecture at that (r, c).*

6. Measurements outside the formal development

Nothing in this section is machine-checked and nothing above depends on it. The numbers come from a discretised Stieltjes procedure, written in plain Python, for the weight in the variable $u = \operatorname{arctanh} x$, where $w(x)\,dx = e^{2\lambda u} \operatorname{sech}^{2\nu} u\,du$ on $(-\ell, \ell)$, $\ell = \operatorname{arctanh} c$ (the form (4.1) of [1]), with Gauss–Legendre quadrature in u at orders between 200 and 1600; stability is judged by agreement across orders. Before any of the measurements below, the engine was checked against the source and against the classical endpoint values: the Pearson identity (5), by computing d_n both from $\lambda - \Sigma_n - (n + \nu)b_n$ and from the endpoint definition (agreement to 10^{-7} relative at every cell of the scan); the limit $D_n/h_n \to \delta/\sqrt{1 - c^2}$, i.e. $d_n \to \lambda\sqrt{1 - c^2}$, which the source derives from its Lemma 6.1 (its equation (6.3)), to four or five digits at $n = 60$ at three parameter points; the convergence of $b_j(c)$ to (9) as $c \to 1$ (within 4% at $1 - c = 10^{-8}$, the residual being consistent with a truncated tail of order $(1-c)^{1+\alpha}$); and Theorems 1.4 and 1.6(1) at every grid point tested.

The conjecture in the open region. On 18,318 points of $\mathcal{O}_c$ — $r \in \{0.02, 0.04, \dots, 0.98\}$, λ at 25 fractions of $\min\{r, 1 - r\}$, 15 values of c between $\sqrt{3/(3 + r)}$ and 0.99999, restricted to $\lambda < rc$ — and degrees $1 \le n \le 12$, i.e. 219,816 cells, $d_n > 0$ at every cell, with minimum $4.04 \cdot 10^{-6}$, and with the two routes to d_n agreeing to 10^{-7} relative at every cell. This is a verification at sampled points, not a proof of anything.

Where the pivot fails. The root-sum inequality is not always true inside $\mathcal{O}_c$. At the point $(r, \lambda, c) = (0.575, 0.017, 0.999)$ the measured values, stable to seven digits across quadrature orders 300 to 1600, are

$$\Sigma_1/\lambda = 1.9725, \quad \Sigma_2/\lambda = 1.1852, \quad \Sigma_3/\lambda = 1.0654, \quad \Sigma_4/\lambda = 1.0211, \quad \Sigma_5/\lambda = 0.9993,$$

while $d_n > 0$ at every degree there. The largest value seen was $\Sigma_2/\lambda = 1.3867$. It occurs at $(c, r, \lambda) = (0.99999, 0.7, 0.012)$, where $\Sigma_6/\lambda = 1.067$ still exceeds 1. Bisection (45 steps) on $\Sigma_N(a) - \lambda$ locates the reach of the inequality,

$$a^*_N = \sup\{a : \Sigma_N(a) < \lambda\};$$

over $\lambda \in \{0.1,\, 0.5,\, 0.9\} \cdot \min\{r, 1-r\}$ the measured ranges are as follows.

r	a_2^*	a_3^*	a_4^*	$\sqrt{3/(3+r)}$
0.20	0.99357–0.99408	0.99730–0.99752	0.99853–0.99865	0.96825
0.40	0.99069–0.99277	0.99638–0.99720	0.99810–0.99853	0.93934
0.60	0.99121–0.99286	0.99691–0.99750	0.99845–0.99874	0.91287
0.80	0.99502–0.99521	0.99848–0.99853	0.99927–0.99930	0.88852
0.95	0.99937	0.99983	0.99993	0.87149

The gap between the last column (the source's degree-uniform threshold) and a_3^* measures the headroom left in the ensemble bound (6); Theorem 3.5 says that headroom never reaches $a = 1$, and the table says $a_N^* < 1$ strictly wherever it was measured. A natural guess is false and should not be used: $b_0 > 0$ and $b_n < 0$ for $n \geq 1$ holds at most sampled points (so that Σ_n decreases and the binding degree is $N = 2$), but tiny positive b_2, b_3 of order 10^{-6}, stable across quadrature orders 200 to 1200, occur near $r = 0.95$.

Lossiness of the block count. The count of Corollary 5.2 is honest but not sharp, because (6) is not. At the points below, the least degree N at which $\Sigma_N(a) < \lambda$ was measured to hold for all $a \leq c$ (on a grid of subranges a, at each of $\lambda \in \{0.1,\, 0.5,\, 0.9\} \cdot \min\{r, 1-r\}$, with the same N at all three) is far below the n_0 the theorem asks for:

(r, c)	$n_0 = \lceil rc^2/(1-c^2) \rceil - 2$	first N with $\Sigma_N < \lambda$, measured
(0.2, 0.99)	8	2
(0.5, 0.99)	23	2
(0.9, 0.99)	43	2
(0.9, 0.999)	448	3

A certified degree cell, priced and not done. The value of a single certified cell $(\alpha, \beta, c) \in \mathcal{O}_c$ in degree 2 is fixed by Proposition 5.4: degree 2 throughout the wedge would move $\sqrt{3}/2$ to $2/\sqrt{5}$. The endpoint values $\pi_n(\pm c)$ are rational functions of the moments $m_0(c), \ldots, m_{2n-1}(c)$, and these are incomplete beta integrals in c, not polynomials; so a certified cell needs certified enclosures of such integrals, not polynomial arithmetic. Two things make one cell cheap. Integrating $x^k\big((1-x^2)w\big)' = x^k(2\lambda - 2\nu x)w$ over $[-c, c]$ by parts gives the recursion

$$(k + 2\nu)\, m_{k+1} = 2\lambda\, m_k + k\, m_{k-1} - (1-c^2)c^k\big(w(c) - (-1)^k w(-c)\big),$$

so every m_k/m_0 is an affine function of $w(\pm c)/m_0$ (checked against direct quadrature to 10^{-14} relative); and at $(\alpha, \beta, c) = (-1/2, -1/6, 63/65)$, a point of $\mathcal{O}_c$ ($r = 1/3$, $\lambda = 1/6$, $c^2 = 0.939\ldots > 0.9 = 3/(3+r)$), one has $\rho_c = 1/64$ exactly, so the endpoint factor $\rho_c^\delta = 1/4$ and $w(c)/w(-c) = 4$ are rational and the degree-two criterion is a single rational inequality in Q/m_0, $Q = w(-c)$. Measured, it holds for $Q/m_0 \in [0.157, 3.016]$ around the true value 0.5168, so m_0 is needed only to about 50% relative accuracy. What remains is a certified enclosure of $m_0 = \int_{-c}^{c}(1-x)^{-1/2}(1+x)^{-1/6}\,dx$ (and of m_1, m_2, m_3 if the recursion is not itself formalised); it is not attempted here.

7. Verification

Formalised and machine-checked in Lean 4 [9] over Mathlib [10] are Propositions 3.1, 3.2, 3.3, 3.6, 4.3, 4.4, 5.3, 5.4, 5.5, Lemmas 4.1 and 4.2, and Theorems 3.5 and 5.1, as stated: twenty-two declarations in two modules (together with their private helper lemmas), none using `sorry`, `native_decide` or any local axiom, each reported by `#print axioms` to depend only on `propext`, `Classical.choice` and `Quot.sound`. Their statements are reproduced verbatim in Appendix A. Mathlib has no orthogonal polynomials, no three-term recurrence and no subrange Jacobi system; accordingly the classical coefficients (9) are a definition (`jacB`), $\Sigma_n(1)$ is the finite sum `jacSigma`, $C_{N,a}$ is the expression `wardC` of Definition 3.4, and in Theorem 5.1 the functions d, S are arbitrary and the source's Lemmas 5.1, 3.1 and 3.4 are the hypotheses (W), (P), (X). Not formalised, and written out above as ordinary mathematics, are: the identification of the classical formula (9) with the subrange system at $c = 1$; Remark 3.7; Corollary 5.2, i.e. the discharge of (W), (P), (X) for the subrange Jacobi system from the source's lemmas; and every measurement of Section 6. The development was checked with the Lean toolchain `leanprover/lean4:v4.32.0` against Mathlib at its tag `v4.32.0` (commit `81a5d257`). Repository reference to be added at submission.

Appendix A. The formal statements

The declarations below are reproduced verbatim from the two modules (proofs omitted). Real variables are $\mathbb{R}$, degrees are $\mathbb{N}$ cast to $\mathbb{R}$ where they enter arithmetic; `lam`, `nu`, `r`, `S`, `C` stand for $\lambda, \nu, r, \Sigma, C$. The correspondence with the numbered statements is:

`jacSigma_eq`	Prop. 3.1	`sigma_lt_iff_wardC`	Prop. 4.3
`lt_jacSigma`	Prop. 3.2(1)	`effective_reduction`	Thm. 5.1
`jacSigma_lt`	Prop. 3.2(2)	`paper_threshold`	Prop. 5.3
`jacSigma_eq_lambda`	Prop. 3.2(3)	`threshold_strict_mono`	Prop. 5.4(1)
`pearson_endpoint_eq_zero`	Prop. 3.3	`threshold_three_lt_four`	Prop. 5.4(1)
`remark52_fails_at_one`	Thm. 3.5	`degrees_needed`	Prop. 5.4(2)
`method_ceiling`	Thm. 3.5, last sentence	`worked_instance`	Prop. 5.5
`wedge_nonvacuous`	Prop. 3.6	`sigma_lt_of_ward_needs_threshold`	Prop. 4.4(1)
`sigma_lt_of_ward`	Lemma 4.1(1)	`crossing_root_sum_iff_needs_r_lt_n`	Prop. 4.4(2)
`ward_threshold_iff`	Lemma 4.1(2)	`jacB`, `jacSigma`	(9), $\Sigma_n(1)$
`ward_threshold_mono`	Lemma 4.1(3)	`wardC`	Definition 3.4
`crossing_root_sum_iff`	Lemma 4.2(1)		
`sigma_propagate`	Lemma 4.2(2)		

Module `Endpoint`**.**

```
noncomputable def jacB (α β : ℝ) (j : ℕ) : ℝ :=
  if j = 0 then (β - α) / (α + β + 2)
  else (β ^ 2 - α ^ 2) / ((2 * j + α + β) * (2 * j + α + β + 2))

noncomputable def jacSigma (α β : ℝ) (n : ℕ) : ℝ := ∑ j ∈ range n, jacB α β j

theorem jacSigma_eq {α β : ℝ} (hs : -2 < α + β) (n : ℕ) :
    jacSigma α β n = (n : ℝ) * (β - α) / (2 * (n : ℝ) + α + β)

theorem lt_jacSigma {α β : ℝ} (hab : α < β) (hs2 : -2 < α + β) (hs : α + β < 0)
    {n : ℕ} (hn : 1 ≤ n) : (β - α) / 2 < jacSigma α β n

theorem jacSigma_lt {α β : ℝ} (hab : α < β) (hs : 0 < α + β) {n : ℕ} (hn : 1 ≤ n) :
    jacSigma α β n < (β - α) / 2

theorem jacSigma_eq_lambda {α β : ℝ} (hs : α + β = 0) {n : ℕ} (hn : 1 ≤ n) :
    jacSigma α β n = (β - α) / 2

theorem pearson_endpoint_eq_zero {α β : ℝ} (hs : -2 < α + β) (n : ℕ) :
    (β - α) / 2 - jacSigma α β n - ((n : ℝ) + (α + β + 2) / 2) * jacB α β n = 0

noncomputable def wardC (N : ℕ) (a lam r S : ℝ) : ℝ :=
  (N : ℝ) * lam * a ^ 2 - ((N : ℝ) - r * a ^ 2) * S

theorem remark52_fails_at_one {α β : ℝ} (hab : α < β) (hs2 : -2 < α + β) (hs : α + β < 0)
    {N : ℕ} (hN : 1 ≤ N) :
    wardC N 1 ((β - α) / 2) (-(α + β) / 2) (jacSigma α β N) = 0 ∧
      0 < ((β - α) / 2) * (((N : ℝ) + -(α + β) / 2) * (1 : ℝ) ^ 2 - (N : ℝ))

theorem wedge_nonvacuous :
    (-1 : ℝ) < -3/5 ∧ (-3/5 : ℝ) < -1/5 ∧ (-1/5 : ℝ) < 0 ∧
      jacSigma (-3/5) (-1/5) 2 = 1/4 ∧ ((-1/5 : ℝ) - (-3/5)) / 2 = 1/5
```

Module `Transfer`**.**

```
theorem sigma_lt_of_ward {N lam r a S : ℝ} (hlam : 0 < lam)
    (hpos : 0 < N - r * a ^ 2) (hward : (N - r * a ^ 2) * S < N * lam * a ^ 2)
    (hthr : a ^ 2 * (N + r) ≤ N) : S < lam

theorem ward_threshold_iff {N r a : ℝ} (ha : a ^ 2 < 1) :
    a ^ 2 * (N + r) ≤ N ↔ r * a ^ 2 ≤ N * (1 - a ^ 2)

theorem ward_threshold_mono {r M N : ℝ} (hr : 0 < r) (hM : 0 < M) (hMN : M ≤ N) :
    M / (M + r) ≤ N / (N + r)

theorem crossing_root_sum_iff {lam r n Sn Sn1 : ℝ} (hn : r < n)
    (hpear : lam + (n - r) * Sn = (n + 1 - r) * Sn1) : Sn < lam ↔ Sn1 < lam

theorem sigma_propagate {lam nu n Sn Sn1 d : ℝ} (hd : 0 < d)
    (hdef : d = lam + (n + nu - 1) * Sn - (n + nu) * Sn1)
    (hSn : Sn ≤ lam) (h1 : 0 ≤ n + nu - 1) (h2 : 0 < n + nu) : Sn1 < lam

theorem sigma_lt_iff_wardC {N lam r a S C : ℝ} (hpos : 0 < N - r * a ^ 2)
    (hid : (N - r * a ^ 2) * S = N * lam * a ^ 2 - C) :
    S < lam ↔ lam * ((N + r) * a ^ 2 - N) < C

theorem effective_reduction
```

```
    {lam r c : ℝ} (hlam : 0 < lam) (hr : 0 < r) (hr1 : r < 1)
    (hc : 0 < c) (hc1 : c < 1)
    (d S : ℕ → ℝ → ℝ) {n₀ : ℕ} (hn₀ : 1 ≤ n₀)
    (hthr : r * c ^ 2 ≤ ((n₀ : ℝ) + 2) * (1 - c ^ 2))
    (ward : ∀ (N : ℕ) (a : ℝ), 1 ≤ N → 0 < a → a ≤ c →
        ((N : ℝ) - r * a ^ 2) * S N a < (N : ℝ) * lam * a ^ 2)
    (pearson : ∀ (n : ℕ) (a : ℝ), 1 ≤ n → 0 < a → a ≤ c → d n a = 0 →
        lam + ((n : ℝ) - r) * S n a = ((n : ℝ) + 1 - r) * S (n + 1) a)
    (crossing : ∀ (n : ℕ), 1 ≤ n →
        (∀ a, 0 < a → a ≤ c → 0 < d n a) →
        (∀ a, 0 < a → a ≤ c → d (n + 1) a = 0 → S (n + 1) a < lam) →
        ∀ a, 0 < a → a ≤ c → 0 < d (n + 1) a)
    (base : ∀ (n : ℕ) (a : ℝ), 1 ≤ n → n ≤ n₀ → 0 < a → a ≤ c → 0 < d n a) :
    ∀ (n : ℕ) (a : ℝ), 1 ≤ n → 0 < a → a ≤ c → 0 < d n a

theorem paper_threshold {r c : ℝ} (hr : 0 < r) (hc : 0 < c) (hc1 : c < 1) :
    (r * c ^ 2 ≤ ((1 : ℝ) + 2) * (1 - c ^ 2)) ↔ c ^ 2 ≤ 3 / (3 + r)

theorem threshold_strict_mono {r m n : ℝ} (hr : 0 < r) (hm : 0 < m) (hmn : m < n) :
    m / (m + r) < n / (n + r)

theorem threshold_three_lt_four {r : ℝ} (hr : 0 < r) : (3 : ℝ) / (3 + r) < 4 / (4 + r)

theorem degrees_needed {r c : ℝ} (hr : 0 < r) (hc : 0 < c) (hc1 : c < 1) :
    ∃ n₀ : ℕ, 1 ≤ n₀ ∧ r * c ^ 2 ≤ ((n₀ : ℝ) + 2) * (1 - c ^ 2)

theorem worked_instance :
    ((9 : ℝ) / 10) * (19 / 20) ^ 2 ≤ ((7 : ℝ) + 2) * (1 - (19 / 20) ^ 2) ∧
      ¬ (((9 : ℝ) / 10) * (19 / 20) ^ 2 ≤ ((6 : ℝ) + 2) * (1 - (19 / 20) ^ 2)) ∧
      (3 : ℝ) / (3 + 9 / 10) < (19 / 20) ^ 2

theorem sigma_lt_of_ward_needs_threshold :
    (0 : ℝ) < 3 - (1/2 : ℝ) * (1 : ℝ) ^ 2 ∧
      (3 - (1/2 : ℝ) * (1 : ℝ) ^ 2) * (11/10 : ℝ) < 3 * (1 : ℝ) * (1 : ℝ) ^ 2 ∧
      ¬ ((1 : ℝ) ^ 2 * (3 + (1/2 : ℝ)) ≤ 3) ∧ ¬ ((11/10 : ℝ) < 1)

theorem crossing_root_sum_iff_needs_r_lt_n :
    (1 : ℝ) + ((1 : ℝ) - 2) * 1 = ((1 : ℝ) + 1 - 2) * 0 ∧
      ¬ ((1 : ℝ) < 1) ∧ ((0 : ℝ) < 1)

theorem method_ceiling {α β : ℝ} (hab : α < β) (hs2 : -2 < α + β) (hs : α + β < 0)
    {N : ℕ} (hN : 1 ≤ N) :
    ¬ (jacSigma α β N < (β - α) / 2)
```

References

[1] V. Botta, K. Castillo and L. Tertuliano da Silva, *Degree-uniform regions for Gautschi's conjecture on subrange Jacobi polynomials*, arXiv:2608.05963v1 [math.CA], 6 August 2026.

[2] W. Gautschi, *Sub-range Jacobi polynomials*, Numer. Algorithms **61** (2012), no. 4, 649–657, https://doi.org/10.1007/s11075-012-9556-z; erratum, Numer. Algorithms **74** (2017), no. 2, 637, https://doi.org/10.1007/s11075-016-0257-x; correction, Numer. Algorithms **81** (2019), no. 2, 771, https://doi.org/10.1007/s11075-019-00715-9.

[3] W. Gautschi, *On the zeros of subrange Jacobi polynomials*, Numer. Algorithms **79** (2018), no. 3, 759–768; https://doi.org/10.1007/s11075-017-0459-x.

[4] G. V. Milovanović, *On Gautschi's conjecture on subrange Jacobi polynomials*, Innovative Journal of Mathematics **1** (2022), no. 1, 1–6; https://doi.org/10.55059/ijm.2022.1.1/15. Cited here through [1], which quotes its Equation (4) and Theorem 2.1; we have not consulted the article itself.

[5] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, Oxford, 2004; https://doi.org/10.1093/oso/9780198506720.001.0001.

[6] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, fourth edition, 1975.

[7] K. Castillo and F. R. Rafaeli, *Affine scaling of Jacobi zeros: sharp orderings beyond Gautschi's conjectures*, arXiv:2608.08258 [math.CA], 2026.

[8] K. Castillo, *A proof of the Szegő conjecture on Jacobi extrema*, arXiv:2608.01404 [math.CA], 2026.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: A. Platzer and G. Sutcliffe (eds.), Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635; https://doi.org/10.1007/978-3-030-79876-5_37.

[10] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; https://doi.org/10.1145/3372885.3373824.