THREE-FAULT RE-ROOTING IN DENSE GAUSSIAN NETWORKS: AN OBSTRUCTED FAULT SET FOR EVERY DIAMETER, AND THE CENSUS OF OBSTRUCTED TRIPLES

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Abstract. The dense Gaussian network G_k is the circulant graph $C_N(k, k+1)$ on $N = 2k^2 + 2k + 1$ nodes, of degree four and diameter k. Albader, Al-Mulla and Hassan (arXiv:2606.16954) make one-to-all broadcasting tolerate a set F of faulty nodes by *re-rooting* it at a node whose hop distance from every faulty node is the diameter k; writing B_k for the set of nodes at distance k from 0, such a node exists exactly when $\bigcap_{f\in F}(f + B_k) \neq \emptyset$. They prove that every one- or two-fault set admits a re-rooted source, for every k, exhibit one three-fault set in G_3 that does not, and leave the three-fault case there. We show that for every $k \geq 3$ the network G_k contains a three-fault set with no re-rooted source. For $k \geq 16$ this is a counting argument valid in any circulant: every re-rootable pair (a, b) — one for which $\{0, a, b\}$ is re-rootable — is the image of a triple of boundary nodes under $(v, p, q) \mapsto (v - p, v - q)$, so $|B_k|^3 + 3N < N^2$ forces an obstructed pair, and the input $|B_k| \leq 4k+2$ comes from a two-sided lattice description of hop distance in G_k. For $3 \leq k \leq 15$ the explicit set $\{0,\ k + 3,\ 3k + 3\}$ is obstructed; enumeration extends this rule to $k \leq 60$ and shows that for $k \leq 30$ it is the lexicographically first obstructed triple, but a proof for all k is not given. We also determine the exact number $T_3(k)$ of obstructed three-fault sets for $k = 3, \ldots, 10$ (50, 738, 4636, 18700, 57630, 148190, 334488, 684216) and tabulate the enumerated values to $k = 30$, where the obstructed triples become the majority at $k = 14$. Every theorem is verified in Lean 4 against *Mathlib*: the counting theorem, the lattice characterisation and the case $k \geq 16$ are checked by the kernel alone, the finite parts by compiled evaluation.

1. Introduction

The objects. Fix an integer $k \geq 1$ and put $N = 2k^2 + 2k + 1 = k^2 + (k + 1)^2$. The *dense Gaussian network* G_k is the circulant graph $C_N(k, k + 1)$: its nodes are the residues $\mathbb{Z}_N$, and node n is adjacent to $n \pm k$ and $n \pm (k + 1)$ modulo N. It is the Cayley graph of the quotient $\mathbb{Z}[i]/(k + (k + 1)i)$ with respect to the four units $\pm 1, \pm i$, the two pictures being matched by the label

$$\varphi(x + yi) \equiv kx + (k + 1)y \pmod{N};$$

the nodes are represented by the Gaussian integers $x + yi$ with $|x| + |y| \leq k$, exactly N of them, and G_k is four-regular, vertex-transitive and of diameter k [6, 8, 10]. Among circulants of degree four and diameter k it has the largest possible number of nodes, which is why it is called dense (the source's Section 3.1, eq. (1); for the double-loop and multi-loop literature see the surveys [13, 14]). Write $d(u, v)$ for the *hop distance*, the least number of edges on a path from u to v, and

$$B_k = \{v \in \mathbb{Z}_N : d(0, v) = k\}$$

for the set of nodes at distance k — the diameter — from node 0; by vertex-transitivity the set of nodes at distance k from a node f is the translate $f + B_k$. In the lattice picture B_k is the image of the ℓ^1-sphere $|x| + |y| = k$, and $|B_k| = 4k$ (the source's Section 3.1: "there are $4d$ nodes at graph distance d from any given node").

The source, and where it stops. Albader, Al-Mulla and Hassan [1] make one-to-all broadcasting in G_k tolerate node failures by *re-rooting*. In the standard broadcast from a source S the nodes at distance k from S receive the message in the last of the k parallel steps and never forward it, so a fault there is harmless. Given a set F of faulty nodes the method selects a new source; in the source's words (its Section 5, p. 9), "we identify a new source node NS, referred to as the re-rooted source, such that its

graph distance from each faulty node is exactly k", routes the message from S to NS and broadcasts from NS. Such a node exists exactly when

$$\bigcap_{f \in F} (f + B_k) \neq \emptyset, \tag{1}$$

and this is the condition the source works with: "A valid re-rooted source for the three faulty nodes would need to belong to all three graph-distance-3 boundaries" (its Proposition 1). We call F *re-rootable* when (1) holds and *obstructed* otherwise.

The source proves that a re-rooted source always exists for at most two faults. Its Lemma 1 ("Boundary Difference Coverage") shows $\varphi(B) - \varphi(B) = \mathbb{Z}_N$ for the boundary $B = B_k$, by exhibiting two sides of the lattice square whose labels form a block of $2k + 2$ consecutive residues and a third side whose labels are spaced $2k+1$ apart; its Theorem 1 deduces that for any two faulty nodes F_1, F_2 "there exists a node NS such that $d(F_1, NS) = k$ and $d(F_2, NS) = k$". It then shows that this is where the guarantee ends. Its Proposition 1 reads "The two-fault re-rooting guarantee does not generally extend to three faulty nodes", and the proof computes, at $k = 3$ ($N = 25$),

$$B_3(0) = \{2, 5, 9, 10, 11, 12, 13, 14, 15, 16, 20, 23\}, \qquad F = \{0, 6, 12\}, \qquad B_3(F_1) \cap B_3(F_2) \cap B_3(F_3) = \emptyset.$$

The paragraph after the proof is explicit about the scope:

> "This counterexample explains why the present work focuses on one- and two-node fault scenarios. The proposed method is not claimed to solve arbitrary three-fault configurations in degree-4 dense Gaussian networks. Instead, the result establishes a sharp and well-defined reliability guarantee: for any one or two faulty nodes, a valid re-rooted source always exists, whereas for three faulty nodes such a source may not exist."

One obstructed triple, at one value of k; nothing about any other k, and no count. The author's later preprints keep the question open. arXiv:2606.21134 [4] (its Theorem 4) gives a second $k = 3$ example, "the three faults $(-3, 0)$, $(-1, 0)$, and $(1, 0)$", and concludes that "three-fault success depends on the triple-intersection geometry"; arXiv:2606.21132 [3] "establishes the complete constant-time certificate selection for $|F| \leq 2$ and leaves the $q \geq 3$ interacting-cap classification as a separate problem"; arXiv:2606.18715 [2] makes the two-fault source selection constant-time and does not touch three faults; arXiv:2606.17528 [5] asks a different three-fault question (the depth of an edge-minimum repair) and states a conjecture about it. Statement, section and page numbers of [1] are throughout those of its arXiv e-print as compiled by us, not of the journal version, which we did not consult; see the bibliography.

So two questions are left open by the source: for which k does an obstructed three-fault set exist, and how many are there?

Results. Throughout, a *three-fault set* is a three-element subset of $\mathbb{Z}_N$; by vertex-transitivity it may be normalised to $\{0, a, b\}$ with $a, b \neq 0$, $a \neq b$ (Section 2).

- **Existence for every** k (Theorem 5.2). For every $k \geq 3$ the network G_k contains three nodes with no common node at distance k from all of them. For $k \geq 16$ the proof is a counting argument (Proposition 3.1) that holds in any circulant: the re-rootable normalised pairs (a, b) are the image of B_k^3 under $(v, p, q) \mapsto (v-p, v-q)$, so they number at most $|B_k|^3$, while the pairs with $a, b \neq 0$, $a \neq b$ number more than $N^2 - 3N$. The input $|B_k| \leq 4k + 2$ is Proposition 4.2, a consequence of the lattice description of hop distance (Theorem 4.1), and $(4k+2)^3 + 3N < N^2$ holds exactly when $k \geq 16$. For $3 \leq k \leq 15$ the triple $\{0, k + 3, 3k + 3\}$ is exhibited.
- **An explicit obstructed triple** (Theorem 6.1). For $3 \leq k \leq 15$ the set $\{0,\ k + 3,\ 3k + 3\}$ — in lattice coordinates the three nodes 0, $-2 + 3i$, $3i$ — is obstructed, and it is obstructed even with respect to the lattice ring $R_k \supseteq B_k$ of Proposition 4.2. At $k = 3$ it is the source's $\{0, 6, 12\}$. Enumeration (Remark 6.3) finds the same triple obstructed for every $k \leq 60$, and lexicographically first among the obstructed triples containing 0 for every $k \leq 30$; a proof for all k is not given and is left open.

- **The census** (Theorem 7.1). $T_3(k)$, the number of three-fault sets of G_k with no re-rooted source, equals 50, 738, 4636, 18700, 57630, 148190, 334488, 684216 for $k = 3, \dots, 10$, out of $\binom{N}{3}$ three-fault sets; Table 2 extends this by enumeration to $k = 30$. The obstructed triples are the minority for $k \le 13$ and the majority from $k = 14$ on, and $k = 14$ is the first k with $|B_k|^3 < N^2$.
- **The published material, certified** (Section 8). The printed $B_3(0)$, the counterexample $\{0, 6, 12\}$, the count $|B_k| = 4k$ for $k \le 10$ and the two-fault theorem for $k \le 8$ are recomputed from the hop metric. We add (Remark 8.6) that G_3 has exactly two translation classes of obstructed triples, exchanged by an automorphism of G_3, and that the two published examples lie in the same class.

range of k	an obstructed three-fault set of G_k	how it is established
$k = 3$	$\{0, 6, 12\}$	the source's Proposition 1; recomputed here
$3 \le k \le 15$	$\{0,\ k+3,\ 3k+3\}$	thirteen exhaustive checks against R_k
$k \ge 16$	exists, no witness named	$\lvert B_k\rvert \le 4k+2$ and $\lvert B_k\rvert^3 + 3N < N^2$

TABLE 1. Theorem 5.2 in one line per range. The counting route with the exact value $|B_k| = 4k$ would start at $k = 14$; the two values $k = 14, 15$ are covered by the explicit triple instead (Remark 4.3).

What is new and what is not. The network, the re-rooting condition (1), Lemma 1 and Theorem 1, the example $\{0, 6, 12\}$ and the count $|B_k| = 4k$ are the source's; the representation of the nodes by the ℓ^1-ball $|x| + |y| \le k$ is classical [6, 8, 10]; the counting argument of Proposition 3.1 is elementary. What we claim is the answer to the first open question for every k (Theorem 5.2), the explicit obstructed triple over its certified range, the exact census over its certified range, and the machine-checked form of the hop-distance characterisation from which the sphere bound follows. The novelty claim is that nobody has taken these steps for this question, not that the steps are deep. Searches, all run on 2026-09-07: the author's fourteen June-2026 preprints (the arXiv API returns fourteen entries for the author, all in cs.DC), of which the five on this network and this question [1, 2, 3, 4, 5] were read — none computes a three-fault count and none claims existence beyond $k = 3$; a local full-text corpus of arXiv e-prints (376 shards, through the September 2026 announcements) searched for the phrase "dense Gaussian network" returns eleven e-prints, namely nine of the author's preprints, AlBdaiwi–Hussain–Cerny–Aldred [11] (arXiv:1601.06915; independent spanning trees, no re-rooting and no fault-set counting) and one false friend (arXiv:1605.03282, where the phrase names a wireless channel model); OpenAlex lists no work citing the source, neither under its preprint record nor under its journal record, and Semantic Scholar lists two, both the author's own [2, 4]; the 56 works citing [10] and the 13 citing [11] in OpenAlex were scanned by title and none concerns re-rooting or counts fault configurations. The nearest prior work on fault tolerance in this network is the spanning-tree line [11, 12], which builds redundant structures in advance and treats no re-rooting. The quantity $T_3(k)$ is not tabulated anywhere we could find. Read the negative as "not found".

Verification. Every theorem and proposition below that carries a formal counterpart in Appendix A is machine-checked in Lean 4 against *Mathlib*; Section 9 records the modules, the axioms and what is trusted. Two kinds of check occur and the text says which at each place: the counting theorem, the lattice characterisation of hop distance, the sphere bound and the case $k \ge 16$ of Theorem 5.2 are proved symbolically and checked by the Lean kernel with no compiled evaluation; the finite statements — the thirteen witnesses, the census, and the certifications of Section 8 — are decided by compiled evaluation (`native_decide`), which trusts the compiler on one auxiliary axiom per statement. What is outside the formal development (the enumerations beyond the certified ranges, the conversion between ordered pairs and three-element sets, the remarks on classes and automorphisms) is labelled as such where it occurs.

2. Preliminaries

2.1. **Circulants, hop distance and the boundary.** Let $N \geq 1$ and let J be a finite list of elements of $\mathbb{Z}_N$ closed under negation. In the circulant $\mathrm{Cay}(\mathbb{Z}_N, J)$ the node u is adjacent to $u + j$ for $j \in J$. The nodes within n hops of 0 are obtained by iterating "add one jump":

$$\mathrm{reach}_J(0) = \{0\}, \qquad \mathrm{reach}_J(n+1) = \mathrm{reach}_J(n) \cup \{u + j : u \in \mathrm{reach}_J(n),\ j \in J\}, \tag{2}$$

so that $d(0, v) = \min\{n : v \in \mathrm{reach}_J(n)\}$, and the nodes at hop distance exactly $n \geq 1$ form $\mathrm{sph}_J(n) = \mathrm{reach}_J(n) \setminus \mathrm{reach}_J(n-1)$. For G_k the jump list is $J_k = (k, -k, k+1, -(k+1))$ in $\mathbb{Z}_N$ with $N = 2k^2 + 2k + 1$, and

$$B_k := \mathrm{sph}_{J_k}(k). \tag{3}$$

This is the definition used throughout and the only one: B_k is the set of nodes at hop distance exactly k from 0, computed from the graph by (2), not a table. That k is the diameter of G_k, so that B_k is the outermost layer, is the source's and [10]'s; nothing below depends on it. Since translation by f is a graph automorphism, $\{v : d(f, v) = k\} = f + B_k$.

2.2. **Re-rooting and obstructed pairs.**

Definition 2.1. Let $S \subseteq \mathbb{Z}_N$ (the *boundary*) and $F \subseteq \mathbb{Z}_N$ (the *fault set*). F is *re-rootable for* S if there is a node $v \in \mathbb{Z}_N$ with $v - f \in S$ for every $f \in F$, that is, $v \in \bigcap_{f \in F}(f + S)$; otherwise F is *obstructed*. For $a, b \in \mathbb{Z}_N$ the pair (a, b) is *obstructed for* S if no $v \in S$ has both $v - a \in S$ and $v - b \in S$.

With $S = B_k$ this is exactly the source's condition: $v - f \in B_k$ means $d(f, v) = k$. The boundary is kept as a parameter because the counting theorem of Section 3 does not use anything about B_k beyond its size.

Lemma 2.2 (translation)**.** *For every $c \in \mathbb{Z}_N$, the set $F + c$ is re-rootable for S if and only if F is.*

Proof. If v serves F then $v + c$ serves $F + c$, since $(v + c) - (f + c) = v - f$; and conversely with $v - c$. □

Lemma 2.3. *(a, b) is obstructed for S if and only if $\{0, a, b\}$ is not re-rootable for S.*

Proof. A node v serving $\{0, a, b\}$ is a node with $v \in S$, $v - a \in S$ and $v - b \in S$. □

By Lemma 2.2 every three-fault set may be normalised to contain 0, and by Lemma 2.3 the normalised set $\{0, a, b\}$ is a question about the pair (a, b). The theorems below are stated for pairs with $a \neq 0$, $b \neq 0$, $a \neq b$, i.e. for genuine three-element sets; pairs violating one of these conditions are called *degenerate*. Both lemmas are checked by the kernel (Appendix A).

2.3. **Counting.**

Definition 2.4. For $S \subseteq \mathbb{Z}_N$ let $\mathrm{obs}(S)$ be the number of *ordered* pairs $(a, b) \in \mathbb{Z}_N^2$ with $a \neq 0$, $b \neq 0$, $a \neq b$ that are obstructed for S. For $k \geq 1$ let $T_3(k)$ be the number of three-element subsets $F \subseteq \mathbb{Z}_N$ that are obstructed for B_k in G_k.

Lemma 2.5. $6\,T_3(k) = N \cdot \mathrm{obs}(B_k)$.

Proof. Count the ordered triples (x, y, z) of distinct nodes whose underlying set is obstructed; there are $6\,T_3(k)$ of them. The map $(x, y, z) \mapsto (x, (y - x, z - x))$ is a bijection onto $\mathbb{Z}_N \times P$, where P is the set of ordered pairs (a, b) with $a, b \neq 0$, $a \neq b$ such that $\{0, a, b\}$ is obstructed: by Lemma 2.2 (translation by $-x$) the set $\{x, y, z\}$ is obstructed iff $\{0, y - x, z - x\}$ is, and the inverse map is $(x, (a, b)) \mapsto (x, x + a, x + b)$. By Lemma 2.3, $|P| = \mathrm{obs}(B_k)$. □

The lemma is elementary and is not part of the formal development; the formal statements carry the values of $\mathrm{obs}(B_k)$ (Theorem 7.1) together with a direct count of the three-element sets at $k = 3$, $T_3(3) = 50$, which is $25 \cdot 12/6$ as the lemma says.

2.4. **The source's results, as read.** Statement numbers and pages are those of the arXiv e-print of [1] compiled by us.

- *Lemma 1* (pp. 11–12): with $B = \{z : d(0,z) = k\}$, $\varphi(B) - \varphi(B) = \mathbb{Z}_N$. In our notation, $B_k - B_k = \mathbb{Z}_N$.
- *Theorem 1* (p. 13): for any two faulty nodes F_1, F_2 there is a node NS with $d(F_1, NS) = d(F_2, NS) = k$. The proof writes $\varphi(F_2) - \varphi(F_1) = \varphi(U) - \varphi(V)$ with $U, V \in B$ by Lemma 1 and takes $\varphi(NS) = \varphi(F_1) + \varphi(U)$. In our notation: every fault set with at most two elements is re-rootable for B_k, for every k.
- *Proposition 1* (statement p. 13, proof p. 14): at $k = 3$, the set $\{0, 6, 12\}$ is obstructed for B_3, with B_3 as printed in Section 1.

These are the source's results. Their finite instances are certified in Section 8; none of them is used in the proofs of Sections 3–7.

3. The counting theorem for circulants

Proposition 3.1 (counting theorem)**.** *Let $N \geq 1$ and $S \subseteq \mathbb{Z}_N$ with $|S|^3 + 3N < N^2$. Then there are $a, b \in \mathbb{Z}_N$ with $a \neq 0$, $b \neq 0$, $a \neq b$ such that no $v \in S$ has both $v - a \in S$ and $v - b \in S$; that is, (a, b) is obstructed for S and $\{0, a, b\}$ has no re-rooted source.*

Proof. Let $C \subseteq \mathbb{Z}_N^2$ be the image of $S \times S \times S$ under $(v, p, q) \mapsto (v - p,\, v - q)$, so $|C| \leq |S|^3$. If (a, b) is a pair that is *not* obstructed, with witness $v \in S$, $v - a \in S$, $v - b \in S$, then $(a, b) = (v - (v - a),\, v - (v - b))$ is the image of $(v,\, v - a,\, v - b) \in S^3$; so every non-obstructed pair lies in C. Let $D \subseteq \mathbb{Z}_N^2$ be the set of degenerate pairs, the union of $\{0\} \times \mathbb{Z}_N$, $\mathbb{Z}_N \times \{0\}$ and the diagonal $\{(x, x)\}$, so $|D| \leq 3N$. Since $|C \cup D| \leq |S|^3 + 3N < N^2 = |\mathbb{Z}_N^2|$, some pair (a, b) lies outside $C \cup D$: it is non-degenerate and obstructed. □

The statement is checked by the kernel for an arbitrary modulus N and an arbitrary finite S (`exists_obstructed_pair`, Appendix A); nothing about G_k enters.

Remark 3.2 (how crude the count is). The bound $|C| \leq |S|^3$ ignores every collision of the map, and $3N$ over-counts the degenerate pairs, which number exactly $3N - 2$; the non-degenerate pairs number $(N - 1)(N - 2)$. Both slacks are immaterial for Theorem 5.2, whose threshold is decided by the cubic-versus-quartic comparison of Proposition 5.1. The argument gives existence only: it names no obstructed pair, and it says nothing about how many there are beyond $\mathrm{obs}(S) \geq (N-1)(N-2) - |S|^3$, which is negative until $|S|^3 < (N - 1)(N - 2)$.

Remark 3.3 (more faults). Not formalised and not used. For a q-element fault set the same map $S^q \to \mathbb{Z}_N^{q-1}$, $(v, p_1, \ldots, p_{q-1}) \mapsto (v - p_1, \ldots, v - p_{q-1})$, covers every re-rootable normalised $(q-1)$-tuple, so an obstructed q-fault set exists as soon as $|S|^q$ plus the number of degenerate tuples is less than N^{q-1}. With $|S| = 4k$ and $N \approx 2k^2$ the comparison is $(4k)^q$ against $(2k^2)^{q-1}$, which favours obstruction for all large k once $q \geq 3$. Nothing here is claimed for $q \geq 4$.

4. Hop distance in G_k and the size of the boundary

For integers a, b write

$$\lambda_k(a, b) \;:=\; a \cdot k + b \cdot (k + 1) \;\in\; \mathbb{Z}_N,$$

so that $\lambda_k(x, y) = \varphi(x + yi)$ is the source's label of the lattice point (x, y). Each jump of G_k changes (a, b) by a unit vector: $\lambda_k(a \pm 1, b) = \lambda_k(a, b) \pm k$ and $\lambda_k(a, b \pm 1) = \lambda_k(a, b) \pm (k + 1)$.

Theorem 4.1 (hop distance is the lattice ℓ^1-norm)**.** *For every k, every $n \geq 0$ and every $v \in \mathbb{Z}_N$,*

$$v \in \mathrm{reach}_{J_k}(n) \quad\Longleftrightarrow\quad \exists\, a, b \in \mathbb{Z} :\ |a| + |b| \leq n \ \text{ and } \ \lambda_k(a, b) = v.$$

Equivalently, $d(0, v) = \min\{|a| + |b| : a, b \in \mathbb{Z},\ \lambda_k(a, b) = v\}$.

Proof. ($\Rightarrow$) By induction on n. For $n = 0$, $v = 0 = \lambda_k(0,0)$. If $v \in \mathrm{reach}(n+1)$ then either $v \in \mathrm{reach}(n)$, and the inductive hypothesis applies with the same (a,b), or $v = u + j$ with $u \in \mathrm{reach}(n)$ and $j \in J_k$; writing $u = \lambda_k(a,b)$ with $|a| + |b| \le n$, the four jumps give $v = \lambda_k(a \pm 1, b)$ or $\lambda_k(a, b \pm 1)$, of ℓ^1-norm at most $n+1$.

($\Leftarrow$) By induction on n, for all (a,b) at once. If $|a| + |b| = 0$ then $v = 0 \in \mathrm{reach}(0) \subseteq \mathrm{reach}(n)$. Otherwise $n \ge 1$ and one of a, b is nonzero; move it one step towards 0: if $a < 0$ then $\lambda_k(a,b) = \lambda_k(a+1,b) + (-k)$ with $|a+1| + |b| \le n-1$, so $\lambda_k(a+1,b) \in \mathrm{reach}(n-1)$ by induction and $v \in \mathrm{reach}(n)$ by (2); the cases $a > 0$, $b < 0$, $b > 0$ use the jumps k, $-(k+1)$, $k+1$. □

The proof is the one the kernel checks (`mem_reach_iff`); the forward direction reads the net counts of a word in the jumps, the backward direction builds a word of the right length.

Proposition 4.2 (the boundary lies on the lattice ring)**.** *Let*

$$R_k \;:=\; \{\lambda_k(a,b) : a,b \in \mathbb{Z},\ |a| + |b| = k\} \;\subseteq\; \mathbb{Z}_N.$$

Then $B_k \subseteq R_k$ and $|R_k| \le 4k+2$; in particular $|B_k| \le 4k+2$.

Proof. Let $v \in B_k = \mathrm{reach}(k) \setminus \mathrm{reach}(k-1)$. By Theorem 4.1, $v = \lambda_k(a,b)$ with $|a| + |b| \le k$; if $|a| + |b| \le k-1$ then $v \in \mathrm{reach}(k-1)$ by the same theorem, a contradiction. So $|a| + |b| = k$ and $v \in R_k$. A point with $|a| + |b| = k$ has $-k \le a \le k$ and $b = \pm(k - |a|)$, so R_k is the image of the index set $\{-k, \dots, k\} \times \{+, -\}$, of size $(2k+1) \cdot 2 = 4k+2$, under $(a, \pm) \mapsto \lambda_k(a, \pm(k - |a|))$. □

Remark 4.3 (the slack). The two corners $a = \pm k$ have $b = 0$ and are hit by both signs, so the index set already shows $|R_k| \le 4k$; the formal statement carries the bound $4k+2$ and that is what Proposition 5.1 uses. The true value is $|B_k| = 4k$ (the source's Section 3.1, certified for $k = 3, \dots, 10$ in Proposition 8.3), so $B_k \subseteq R_k$ with $|R_k| \le 4k = |B_k|$ forces $B_k = R_k$ whenever $|B_k| = 4k$ is known; by direct computation outside the formal development, $B_k = R_k$ for $k = 3, \dots, 30$. The slack of two costs two values of k: with $4k$ in place of $4k+2$ the inequality of Proposition 5.1 holds from $k = 14$ rather than $k = 16$ (its polynomial becomes $4k^4 - 56k^3 + 2k^2 - 2k - 2$, which is 362 at $k = 14$ and negative at $k = 13$). Nothing is lost, because $k = 14, 15$ are covered by Theorem 6.1.

5. An obstructed three-fault set for every $k \ge 3$

Proposition 5.1 (the case $k \ge 16$)**.** *For every $k \ge 16$ there are $a, b \in \mathbb{Z}_N$ with $a \ne 0$, $b \ne 0$, $a \ne b$ such that (a,b) is obstructed for B_k.*

Proof. By Proposition 4.2, $|B_k| \le 4k+2$, so $|B_k|^3 \le (4k+2)^3$, and it suffices by Proposition 3.1 to show $(4k+2)^3 + 3N < N^2$ for $k \ge 16$. With $N = 2k^2 + 2k + 1$,

$$N^2 - 3N - (4k+2)^3 \;=\; 4k^4 - 56k^3 - 94k^2 - 50k - 10 \;=:\; p(k).$$

For $k \ge 16$ one has $4k^4 - 56k^3 = 4k^3(k-14) \ge 8k^3$ and $8k^3 - 94k^2 - 50k - 10 = k^2(8k-94) - 50k - 10 \ge 34k^2 - 50k - 10 > 0$, so $p(k) > 0$. (The kernel proof substitutes $k = j + 16$ and closes the resulting polynomial inequality in $j \ge 0$ by `nlinarith`.) Since $p(15) = -8410 < 0$, the counting route with this input starts exactly at $k = 16$. □

Theorem 5.2 (main theorem)**.** *For every $k \ge 3$ there are $a, b \in \mathbb{Z}_N$, $N = 2k^2 + 2k + 1$, with $a \ne 0$, $b \ne 0$, $a \ne b$ such that the three-fault set $\{0, a, b\}$ is not re-rootable in G_k: no node of G_k is at hop distance k from all of 0, a and b.*

Proof. For $k \ge 16$, Proposition 5.1 gives an obstructed pair and Lemma 2.3 converts it. For $3 \le k \le 15$ take $(a,b) = (k+3,\ 3k+3)$: by Theorem 6.1 the pair is obstructed for R_k, hence for $B_k \subseteq R_k$ (an obstructed pair for a set is obstructed for every subset), and Lemma 2.3 converts it. The three side conditions hold since $3 \le k$ gives $0 < k+3 < 3k+3 < N$. □

The two halves have different characters and the formal development keeps them apart. The half $k \ge 16$ is a symbolic proof for infinitely many k, checked by the kernel alone; it produces no witness. The half $3 \le k \le 15$ is thirteen finite verifications by compiled evaluation, each of which names

its witness. The combined statement (`obstructed_triple_forall`) therefore depends on the thirteen compiled-evaluation axioms and on nothing else beyond the standard ones (Section 9).

6. The explicit triple $\{0,\, k+3,\, 3k+3\}$

Theorem 6.1. *For each $k = 3, 4, \ldots, 15$ the pair $(k+3,\, 3k+3)$ is obstructed for the lattice ring R_k of Proposition 4.2: no $v \in R_k$ has both $v-(k+3) \in R_k$ and $v-(3k+3) \in R_k$. Consequently $(k+3, 3k+3)$ is obstructed for B_k, and the three-fault set $\{0,\, k+3,\, 3k+3\}$ of G_k has no re-rooted source, for each of these thirteen k.*

Proof. For each k the statement is a finite check: R_k is listed from its $4k+2$ index pairs, and for each $v \in R_k$ the two memberships are decided. The thirteen checks are performed by compiled evaluation inside Lean (`obstructed_ring_3`, …, `obstructed_ring_15`), and each is reproduced independently in Remark 6.3. The consequence for B_k is Proposition 4.2 together with the monotonicity of obstruction under taking subsets, and the consequence for the fault set is Lemma 2.3. Checking against R_k rather than B_k makes each statement formally stronger and each check $O(k^2)$ instead of $O(N^2)$; by Remark 4.3 the two sets coincide, so nothing is gained or lost mathematically. □

Remark 6.2 (the triple in lattice coordinates). Since $\lambda_k(-2,3) = -2k+3(k+1) = k+3$ and $\lambda_k(0,3) = 3k+3$ for every k, the triple is the image of the fixed configuration $\{0,\, -2+3i,\, 3i\}$ of the Gaussian quotient, whatever k (for $k \geq 5$ these are the canonical representatives; for $k = 3, 4$ the node $-2+3i$ is represented inside the ball by another Gaussian integer, e.g. by 2 at $k = 3$, where $(-2+3i) - 2 = -4+3i = i(3+4i)$). At $k = 3$ the triple is $\{0, 6, 12\}$, the source's example. This is an identity of labels, not a theorem about obstruction.

Remark 6.3 (beyond the certified range; enumeration). Outside the formal development. A C program (breadth-first search from 0 in $C_N(k, k+1)$, the boundary B_k as a bitset, the triple $\{0, a, b\}$ tested as $B_k \wedge (B_k \ll a) \wedge (B_k \ll b) \neq 0$ with cyclic shifts) and, independently, a Python implementation written from the definitions of Section 2.1 (BFS, big-integer bitmasks) agree on the following.

- $\{0, k+3, 3k+3\}$ is obstructed for B_k for every $k = 3, \ldots, 60$.
- For every $k = 3, \ldots, 30$, where the full sweep over all pairs $1 \leq a < b < N$ was run, $(k+3, 3k+3)$ is the *lexicographically first* obstructed pair: the least a for which some (a, b) is obstructed is $k+3$, and the least such b is $3k+3$. At $k = 3$ the first hit is $(6, 12)$, the source's example; we take this as an indication of how the source's example may have been found, which the source does not say.
- For $k = 31, \ldots, 60$ only the obstruction of the named triple was checked, not its position in the lexicographic order.

A proof that $\{0, k+3, 3k+3\}$ is obstructed for every k is not given here and is open. By Theorem 6.1's formulation it would suffice to show, for every k, that $k+3$ and $3k+3$ are never simultaneously differences $v-v'$, $v-v''$ of three points v, v', v'' of the lattice ring R_k — a Diophantine statement about representations of $k+3$ and $3k+3$ modulo N as differences of two points of the ℓ^1-sphere $|a|+|b| = k$. Theorem 5.2 does not depend on it.

7. The census of obstructed three-fault sets

Theorem 7.1. *For $k = 3, \ldots, 10$ the number $\mathrm{obs}(B_k)$ of ordered pairs (a, b) of distinct nonzero nodes of G_k for which $\{0, a, b\}$ has no re-rooted source is*

$$12, \quad 108, \quad 456, \quad 1320, \quad 3060, \quad 6132, \quad 11088, \quad 18576,$$

respectively. Moreover, counted directly over all $\binom{25}{3} = 2300$ three-element subsets of $\mathbb{Z}_{25}$, exactly 50 three-fault sets of G_3 have no re-rooted source.

Proof. Each value is an exhaustive count: for $\mathrm{obs}(B_k)$, over all N^2 ordered pairs, each tested against all of B_k (an $O(N^2|B_k|)$ computation, $|B_k| = 4k$); for the direct count at $k = 3$, over all triples $x < y < z$ in $\mathbb{Z}_{25}$, each tested for a common node v with $v-x, v-y, v-z \in B_3$. The nine counts are performed

by compiled evaluation inside Lean (`obsOrd_3`, …, `obsOrd_10`, `obsTriples_3`) and reproduced by the two independent programs of Remark 6.3. □

Corollary 7.2. $T_3(k) = 50,\ 738,\ 4636,\ 18700,\ 57630,\ 148190,\ 334488,\ 684216$ *for* $k = 3, \dots, 10$.

Proof. Lemma 2.5: $T_3(k) = N \cdot \mathrm{obs}(B_k)/6$ with $N = 25, 41, 61, 85, 113, 145, 181, 221$. At $k = 3$ the direct count of Theorem 7.1 confirms the conversion. □

k	N	$\lvert B_k\rvert$	$\mathrm{obs}(B_k)$	$T_3(k)$	$\binom{N}{3}$	$T_3(k)/\binom{N}{3}$
3	25	12	12	50	2 300	0.022
4	41	16	108	738	10 660	0.069
5	61	20	456	4 636	35 990	0.129
6	85	24	1 320	18 700	98 770	0.189
7	113	28	3 060	57 630	234 136	0.246
8	145	32	6 132	148 190	497 640	0.298
9	181	36	11 088	334 488	971 970	0.344
10	221	40	18 576	684 216	1 774 630	0.386
11	265	44	29 340	1 295 850	3 066 580	0.423
12	313	48	44 220	2 306 810	5 061 836	0.456
13	365	52	64 152	3 902 580	8 038 030	0.486
14	421	56	90 168	6 326 788	12 347 930	0.512
15	481	60	123 396	9 892 246	18 431 920	0.537
20	841	80	443 916	62 222 226	98 783 860	0.630
25	1 301	100	1 169 136	253 507 656	366 166 450	0.692
30	1 861	120	2 548 056	790 322 036	1 072 475 690	0.737

TABLE 2. The census. Rows $k = 3, \dots, 10$: $\mathrm{obs}(B_k)$ and $T_3(3)$ certified in Lean (Theorem 7.1), $T_3(k)$ by Lemma 2.5. Rows $k \geq 11$: enumeration only (Remark 6.3), by two independent programs; not machine-checked. The last column is rounded.

Remark 7.3 (the transition). Outside the formal development. In the enumerated range the obstructed triples are a minority up to $k = 13$ (3 902 580 against 4 135 450 re-rootable) and a majority from $k = 14$ on (6 326 788 against 6 021 142). For a pair (a, b) the number of common boundary nodes is $|B_k \cap (a+B_k) \cap (b+B_k)|$, whose mean over all N^2 pairs is exactly $|B_k|^3/N^2 = 64k^3/N^2$; this mean is 1.055 at $k = 13$ and 0.991 at $k = 14$, so the majority transition occurs at the first k with $|B_k|^3 < N^2$ (indeed $N^2 - 64k^3 = 4k^4 - 56k^3 + 8k^2 + 4k + 1 = 4k^3(k-14) + 8k^2 + 4k + 1$). Whether this coincidence persists in any form for larger q or admits an explanation is not addressed; a closed form or an asymptotic for $T_3(k)/\binom{N}{3}$ is open, and the enumerated ratio (0.737 at $k = 30$) is still rising.

8. The published material, certified

The statements of this section are the source's; what is ours is their recomputation from the hop metric (2)–(3) rather than from the printed table, by compiled evaluation inside Lean.

Proposition 8.1 (the printed boundary)**.** $B_3 = \{2, 5, 9, 10, 11, 12, 13, 14, 15, 16, 20, 23\} \subseteq \mathbb{Z}_{25}$*, the set printed in the source's Proposition 1.*

Proposition 8.2 (the source's counterexample)**.** *The pair* $(6, 12)$ *is obstructed for* B_3*; the three-fault set* $\{0, 6, 12\}$ *of* G_3 *has no re-rooted source.*

Proposition 8.3 (the boundary count)**.** $|B_k| = 4k$ *for* $k = 3, 4, \dots, 10$.

Proposition 8.4 (the two-fault theorem at small k)**.** *For* $k = 3, \dots, 8$ *and every* $a \in \mathbb{Z}_N$*, the fault set* $\{0, a\}$ *is re-rootable for* B_k*; by translation, every fault set with at most two elements is re-rootable in* $G_3, \dots, G_8$.

Proposition 8.5 (controls)**.** *In G_3: the three-fault set $\{0,1,2\}$ is re-rootable; the pair $(1,2)$ is not obstructed; and $B_3 \neq \mathbb{Z}_{25}$.*

Proposition 8.4 is the source's Theorem 1 at six sizes; it is included as a control that the formal model is the source's object and that the two-fault half of the story is not vacuous, not as a proof of Theorem 1, which is the source's for all k. Proposition 8.5 refutes the too-large claim "every three-fault set of G_3 is obstructed" and the too-small reading in which the boundary is the whole network, so that neither quantifier in Definition 2.1 is vacuous.

Remark 8.6 (the two translation classes at $k = 3$). Outside the formal development, by direct computation in $\mathbb{Z}_{25}$. The six unordered obstructed pairs of G_3 are $(6,12)$, $(6,19)$, $(13,19)$ and $(8,16)$, $(8,17)$, $(9,17)$; the first three are the normalisations of one translation class of three-fault sets, represented by $\{0,6,12\}$, and the last three of another, represented by $\{0,8,16\}$, in agreement with $T_3(3) = 50 = 2\cdot 25$. Multiplication by $2k+1 = 7$ on $\mathbb{Z}_{25}$ sends the jumps $3,-3,4,-4$ to $21,4,3,22$, i.e. permutes $\pm 3, \pm 4$, so it is an automorphism of G_3 fixing 0 (in the lattice picture it is $z \mapsto iz$ up to sign; in general $(2k+1)k \equiv -(k+1)$ and $(2k+1)(k+1) \equiv k \pmod N$); it maps $\{0,6,12\}$ to $\{0,17,9\}$, which lies in the second class. So up to the automorphism group of G_3 there is exactly one obstructed three-fault set. The example of [4], the faults $(-3,0)$, $(-1,0)$, $(1,0)$, has labels $16,22,3$ and is $\{0,6,12\}+16$: the two published examples are the same fault set up to translation. In lattice terms both are three collinear nodes spaced two apart along an axis.

9. Verification

Theorems 4.1, 5.2, 6.1 and 7.1, Propositions 3.1, 4.2, 5.1 and 8.1–8.5, and Lemmas 2.2 and 2.3 rest on declarations formally verified in Lean 4 against *Mathlib* [15, 16] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. The development is three modules, 622 lines in all. The first (203 lines) is written for an arbitrary modulus and an arbitrary jump list: the iteration `reach` of (2), the sphere `sph`, the predicates `Rerootable` and `Obstructed`, the translation lemma, Lemma 2.3, the counters `obsOrd` and `obsTriples`, and the counting theorem `exists_obstructed_pair` (Proposition 3.1); it is shared with a parallel treatment of the Eisenstein–Jacobi networks of the same author, not reported here. The second (160 lines) is the symbolic layer for G_k: `gN`, `gJ`, `gB`, the label `latG`, the characterisation `mem_reach_iff` (Theorem 4.1), the ring `ringG` with `card_ringG_le` and `gB_subset_ringG` (Proposition 4.2), and `exists_obstructed_large` (Proposition 5.1); it contains no compiled evaluation. The third (259 lines) holds the finite statements, each decided by `native_decide`: the certifications of Section 8, the census (Theorem 7.1), the thirteen witnesses (Theorem 6.1), and the assembly `obstructed_triple_forall` (Theorem 5.2), whose proof is a case split at $k = 16$ with `interval_cases` below it.

The axiom print (`#print axioms`) of every declaration named in this paper was re-run read-only for it (49 declarations, the 45 bound to the results above and four supporting lemmas). Every declaration of the first two modules depends only on `propext`, `Classical.choice` and `Quot.sound` (Lemma 2.3 on `propext` and `Quot.sound` alone): in particular Proposition 3.1, Theorem 4.1, Proposition 4.2 and Proposition 5.1 are checked by the kernel with no compiled evaluation. Each `native_decide` statement of the third module depends, in addition, on exactly one auxiliary axiom of its own, generated for that statement and asserting the result of its compiled evaluation; the assembled Theorem 5.2 depends on the thirteen axioms of the witnesses `obstructed_ring_3`, …, `obstructed_ring_15` and on no other. No declaration depends on `sorryAx`, and no axiom is declared by hand. What compiled evaluation trusts is the Lean compiler and runtime on decidable finite statements; every such statement here was also computed by the two independent programs of Remark 6.3 (a C enumerator and a Python implementation written from the definitions), which agree with the Lean values in every cell of Table 2, reproduce B_3, the source's counterexample, the count $|B_k| = 4k$ for $k \le 60$, the source's Lemma 1 ($B_k - B_k = \mathbb{Z}_N$) for $k \le 30$, the two-fault theorem for $k \le 12$, the thirteen witnesses, and the thresholds 16 and 14 of Remark 4.3. As recorded when the development was written, the first module checks in about 11 seconds, the second in about 30 seconds, and the third in about four minutes, each with a peak resident set of 1.1–1.2 GB; the C enumeration to $k = 30$ with the full census takes under three processor-minutes and the triple

checks to $k = 60$ under one. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). `ZMod N` is $\mathbb{Z}_N$, `Finset` a finite set, `List` a list, `.card` the cardinality, and `Finset.univ` the set of all elements. In the first block `N` is an implicit variable with a `NeZero N` instance; the namespaces `LeanProblemSpec.Reroot` and `LeanProblemSpec.GaussReroot` are omitted.

Circulants, re-rooting and counting (Section 2, Proposition 3.1).

```
def step (J : List (ZMod N)) (s : Finset (ZMod N)) : Finset (ZMod N) :=
  s ∪ (s ×ˢ J.toFinset).image (fun p => p.1 + p.2)

def reach (J : List (ZMod N)) : ℕ → Finset (ZMod N)
  | 0 => {0}
  | n + 1 => step J (reach J n)

def sph (J : List (ZMod N)) (n : ℕ) : Finset (ZMod N) := reach J n \ reach J (n - 1)

def Rerootable (S : Finset (ZMod N)) (F : Finset (ZMod N)) : Prop :=
  ∃ v : ZMod N, ∀ f ∈ F, v - f ∈ S

theorem rerootable_translate (S : Finset (ZMod N)) (F : Finset (ZMod N)) (c : ZMod N) :
    Rerootable S (F.image (· + c)) ↔ Rerootable S F

theorem exists_obstructed_pair (S : Finset (ZMod N)) (h : S.card ^ 3 + 3 * N < N * N) :
    ∃ a b : ZMod N, a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧
      ∀ v : ZMod N, v ∈ S → v - a ∈ S → v - b ∉ S

def Obstructed (S : Finset (ZMod N)) (a b : ZMod N) : Prop :=
  ∀ v ∈ S, ¬(v - a ∈ S ∧ v - b ∈ S)

theorem obstructed_iff_not_rerootable {S : Finset (ZMod N)} {a b : ZMod N} :
    Obstructed S a b ↔ ¬ Rerootable S ({0, a, b} : Finset (ZMod N))

def obsOrd (S : Finset (ZMod N)) : ℕ :=
  ((Finset.univ : Finset (ZMod N × ZMod N)).filter
    (fun p => p.1 ≠ 0 ∧ p.2 ≠ 0 ∧ p.1 ≠ p.2 ∧ Obstructed S p.1 p.2)).card

def obsTriples (S : Finset (ZMod N)) : ℕ :=
  ((Finset.univ : Finset (ZMod N × ZMod N × ZMod N)).filter
    (fun p => p.1.val < p.2.1.val ∧ p.2.1.val < p.2.2.val ∧
      ¬ Rerootable S ({p.1, p.2.1, p.2.2} : Finset (ZMod N)))).card

def TwoFaultAlways (S : Finset (ZMod N)) : Prop :=
  ∀ a : ZMod N, Rerootable S ({0, a} : Finset (ZMod N))
```

The network G_k and the symbolic layer (Theorem 4.1, Propositions 4.2 and 5.1).

```
def gN (k : ℕ) : ℕ := 2 * k * k + 2 * k + 1

def gJ (k : ℕ) : List (ZMod (gN k)) :=
  [(k : ZMod (gN k)), -(k : ZMod (gN k)), (k : ZMod (gN k)) + 1, -((k : ZMod (gN k)) + 1)]
```

```
def gB (k : ℕ) : Finset (ZMod (gN k)) := sph (gJ k) k

def latG (k : ℕ) (a b : ℤ) : ZMod (gN k) :=
  (a : ZMod (gN k)) * (k : ZMod (gN k)) + (b : ZMod (gN k)) * ((k : ZMod (gN k)) + 1)

theorem mem_reach_iff (k : ℕ) (n : ℕ) (v : ZMod (gN k)) :
    v ∈ reach (gJ k) n ↔ ∃ a b : ℤ, a.natAbs + b.natAbs ≤ n ∧ latG k a b = v

def ringG (k : ℕ) : Finset (ZMod (gN k)) :=
  ((Finset.range (2 * k + 1)) ×ˢ ({0, 1} : Finset ℕ)).image
    (fun p => latG k ((p.1 : ℤ) - (k : ℤ))
      (if p.2 = 0 then (k : ℤ) - (((p.1 : ℤ) - (k : ℤ)).natAbs : ℤ)
       else -((k : ℤ) - (((p.1 : ℤ) - (k : ℤ)).natAbs : ℤ))))

theorem card_ringG_le (k : ℕ) : (ringG k).card ≤ 4 * k + 2

theorem gB_subset_ringG (k : ℕ) : gB k ⊆ ringG k

theorem exists_obstructed_large (k : ℕ) (hk : 16 ≤ k) :
    ∃ a b : ZMod (gN k), a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧ Obstructed (gB k) a b
```

The finite statements (Theorems 5.2, 6.1, 7.1, Section 8).

```
theorem gB3_eq :
    gB 3 = ({2, 5, 9, 10, 11, 12, 13, 14, 15, 16, 20, 23} : Finset (ZMod (gN 3)))

theorem source_counterexample : ¬ Rerootable (gB 3) ({0, 6, 12} : Finset (ZMod (gN 3)))

theorem card_gB_3 : (gB 3).card = 12          -- ... card_gB_10 : (gB 10).card = 40

theorem twoFault_3 : TwoFaultAlways (gB 3)   -- ... twoFault_8 : TwoFaultAlways (gB 8)

theorem rerootable_example : Rerootable (gB 3) ({0, 1, 2} : Finset (ZMod (gN 3)))
theorem not_all_obstructed : ¬ Obstructed (gB 3) 1 2
theorem gB3_ne_univ : gB 3 ≠ (Finset.univ : Finset (ZMod (gN 3)))

theorem obsTriples_3 : obsTriples (gB 3) = 50
theorem obsOrd_3 : obsOrd (gB 3) = 12        -- 108, 456, 1320, 3060, 6132, 11088,
                                             -- obsOrd_10 : obsOrd (gB 10) = 18576

theorem obstructed_ring_3 : Obstructed (ringG 3) 6 12      -- (k+3, 3k+3) for k = 3..15:
theorem obstructed_ring_15 : Obstructed (ringG 15) 18 48   -- 7 15, 8 18, ..., 17 45, 18 48

theorem obstructed_triple_forall (k : ℕ) (hk : 3 ≤ k) :
    ∃ a b : ZMod (gN k), a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧
      ¬ Rerootable (gB k) ({0, a, b} : Finset (ZMod (gN k)))
```

References

[1] B. Albader, M. R. Al-Mulla and G. Hassan, *Re-Rooting-Based Fault-Tolerant Broadcasting in Dense Gaussian Networks*, IEEE Access **14** (2026), 106148–106161 (doi:10.1109/ACCESS.2026.3712557, open access); preprint arXiv:2606.16954v1 [cs.DC], 15 June 2026, whose abstract page carries the comment "Submitted to IEEE Access" and announces a Zenodo copy. Read in full. The journal reference is from Crossref, and agrees with OpenAlex and dblp; the arXiv metadata does not yet carry it. *We worked from the e-print, not from the journal version, and every statement, section, equation and page number quoted in this paper is that of the e-print*, which (a single `main.tex`

and ten figures) was compiled by us with tectonic into a 28-page PDF on 2026-09-07 — twice, from two separately obtained copies of the source, with identical pagination: Section 3.1 (eq. (1) and the count of $4d$ nodes at distance d, p. 5), Section 3.2 (the definition of $G(\pi)$ and the label φ, pp. 5–6), Section 5 (the re-rooting condition, p. 9, in the section preamble and not in its subsection 5.1), Lemma 1 (statement p. 11, proof ending p. 12), Theorem 1 (p. 13, the first statement on that page), Proposition 1 (statement p. 13, proof p. 14; the only theorem-environment label in the `.aux`, `prop:threefault`, resolves to Proposition 1 on page 13), Tables 1–2 (simulation results, not used here). The quotations in Sections 1 and 2 are verbatim from that PDF. The numbering of the journal version was not checked and may differ. Version 1 is the only arXiv version as of 2026-09-07.

[2] B. Albader, *Closed-Form and Constant-Time New-Source Selection for Fault-Tolerant Broadcasting in Dense Gaussian Networks*, arXiv:2606.18715v1 [cs.DC], 17 June 2026. Read from the e-print source for its treatment of three faults: it has none (it treats one and two faults, with a constant-time selector).

[3] B. Albader, *Constant-Time Certificate Selection for Local Broadcast Repair in Dense Gaussian and Eisenstein–Jacobi Networks*, arXiv:2606.21132v1 [cs.DC], 19 June 2026. The sentence quoted in Section 1 is verbatim from the e-print source (its concluding discussion).

[4] B. Albader, *Re-Rooting-Assisted Edge-Minimum Runtime Repair for Node and Link Failures in Dense Gaussian Broadcast Networks*, arXiv:2606.21134v1 [cs.DC], 19 June 2026. The sentences quoted in Section 1 are verbatim from the e-print source, where the statement is typeset as "Theorem 4 (Re-rooting reliability threshold and triple-intersection criterion)" with the number written in the text rather than set by a counter. We compiled this e-print as well (32 pages): the statement so numbered stands on p. 18 of that PDF, its proof runs to p. 19, and the two quoted sentences are on pp. 18 and 19.

[5] B. Albader, *Multi-Orientation Edge-Minimum Repair for Non-Redundant Fault-Tolerant Broadcasting in Dense Gaussian Networks*, arXiv:2606.17528v1 [cs.DC], 16 June 2026. Read from a local text copy for its three-fault statements: a conjecture on the depth of an edge-minimum repair for three faults and a collinear three-fault repair certificate; a different question from the existence of a common distance-k node, about which it says nothing.

[6] C. Martínez, E. Vallejo, R. Beivide, C. Izu and M. Moretó, *Dense Gaussian networks: suitable topologies for on-chip multiprocessors*, International Journal of Parallel Programming **34** (2006), no. 3, 193–211. Bibliographic data checked against Crossref (doi:10.1007/s10766-006-0014-1); the source's reference [2]; not read for this work.

[7] C. Martínez, R. Beivide and E. Gabidulin, *Perfect codes for metrics induced by circulant graphs*, IEEE Transactions on Information Theory **53** (2007), no. 9, 3042–3052. Bibliographic data checked against Crossref (doi:10.1109/TIT.2007.903126); the metric of G_k as a graph metric on Gaussian integers; not read for this work.

[8] C. Martínez, R. Beivide, E. Stafford, M. Moretó and E. M. Gabidulin, *Modeling toroidal networks with the Gaussian integers*, IEEE Transactions on Computers **57** (2008), no. 8, 1046–1056. Bibliographic data checked against Crossref (doi:10.1109/TC.2008.57); not read for this work.

[9] C. Martínez, E. Stafford, R. Beivide and E. M. Gabidulin, *Modeling hexagonal constellations with Eisenstein–Jacobi graphs*, Problems of Information Transmission **44** (2008), no. 1, 1–11. Bibliographic data checked against Crossref (doi:10.1134/S0032946008010018); the Eisenstein–Jacobi counterpart of G_k; not read for this work.

[10] M. Flahive and B. Bose, *The topology of Gaussian and Eisenstein–Jacobi interconnection networks*, IEEE Transactions on Parallel and Distributed Systems **21** (2010), no. 8, 1132–1142. Bibliographic data checked against Crossref (doi:10.1109/TPDS.2009.132); the source's reference [1], cited as the source does for the topological description of the network (distance distribution and diameter); not read for this work. OpenAlex lists 56 citing works (2026-09-07), scanned by title for this paper.

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