THE MAXIMAL GAP IN AN ELEMENTARY LOWER BOUND FOR THE GAUSSIAN ISOPERIMETRIC FUNCTION: A CERTIFIED BRACKET AND THE LOCATION OF THE MAXIMISER

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Abstract. Let φ and Φ be the density and the distribution function of the standard Gaussian law, and let $\mathcal{I}(p) = \varphi(\Phi^{-1}(p))$ be the Gaussian isoperimetric function. In a recent study of one-dimensional sub-Gaussian comparison in convex order, Zhang proves the elementary lower bound $\mathcal{I}(p) \geq L(p) := \frac{2}{\sqrt{\pi}}\, p(1-p)\sqrt{\log \frac{1-p}{p}/(1-2p)}$ and remarks, on the strength of a numerical evaluation only, that L is never below $\mathcal{I}$ by more than 0.00720 and that the maximal gap occurs at $p_0 \approx 0.10125$. Nothing about the size or the location of that gap is proved there. We prove $\mathcal{I}(p) - L(p) \leq 0.0072$ for every $p \in (0,1)$ with $p \neq \frac{1}{2}$; we exhibit an explicit point, $p_w = \Phi(-1.2745)$, at which the gap lies in $[0.00719656, 0.0071965623]$, so that $\Delta^* := \sup_p(\mathcal{I}(p) - L(p))$ lies in $[0.00719656, 0.0072]$; and we prove that every p at which the gap attains the certified level 0.00719656 satisfies $p \in [0.1006, 0.1019]$ or $1 - p \in [0.1006, 0.1019]$, which brackets the maximiser to three digits and confirms the value 0.10125 quoted in the source. We also exhibit a point at which the leading constant $2/\sqrt{\pi} = 1.128379\ldots$ already fails when raised to 1.12839, and identify $p = \frac{1}{2}$ as a removable singularity at which the bound is an equality. The upper bounds rest on exhaustive interval sweeps in exact rational arithmetic — 8070 intervals for the global bound, 19 245 + 17 917 more for the maximiser — over a self-contained rational enclosure of Φ and exp, built here because the library we verify in has no error function and nothing that evaluates or bounds the normal distribution function, and reusable elsewhere. All of this is machine-checked in Lean 4; Section 8 records exactly what the kernel checks and what it does not.

1. Introduction

1.1. The Gaussian isoperimetric function.

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$$\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy$$

for the density and the distribution function of the standard Gaussian law on $\mathbb{R}$, and let

$$\mathcal{I}(p) = \varphi\big(\Phi^{-1}(p)\big), \qquad p \in (0,1), \tag{1}$$

be the *Gaussian isoperimetric function.* It is the isoperimetric profile of Gauss space: among Borel sets of a given measure p in $\mathbb{R}^n$ with the standard Gaussian measure γ_n, half-spaces minimise the Gaussian perimeter, and the minimal value is $\mathcal{I}(p)$, independently of the dimension — the theorem of Borell [2] and of Sudakov and Tsirel'son [3]. The same function is the one that appears in Bobkov's functional form of Gaussian isoperimetry [4], and in the martingale proof Barthe and Maurey give of it [5]. It is concave, symmetric about $p = \frac{1}{2}$ with $\mathcal{I}(\frac{1}{2}) = \varphi(0) = 1/\sqrt{2\pi}$, and vanishes at the two endpoints.

Because Φ^{-1} is not elementary, quantitative work with $\mathcal{I}$ usually proceeds through elementary bounds. In a recent paper on sharp one-dimensional sub-Gaussian comparison in convex order, Zhang [1] establishes one such bound, which he introduces with the words *"En route to Theorem* `thm:cmgf`*, we establish a curious lower bound on $\mathcal{I}$ which may be of independent interest".*[1] Its statement (Lemma `lem:I` of [1]) is: for any $p \in [0,1]$,

$$\mathcal{I}(p) \;\geq\; \frac{2}{\sqrt{\pi}} \cdot p(1-p) \cdot \sqrt{\frac{\log \frac{1-p}{p}}{1-2p}} \;=:\; L(p). \tag{2}$$

[1]Quotations from [1] are from its v2 e-print; internal labels such as `thm:cmgf`, `lem:I` and `rk:I` are the source's own, and we write $\mathcal{I}$ where the source writes I.

The proof is deferred to an appendix of [1] and is, in that paper's own closing words, "purely analytical and offers no geometric insights".

Both sides of (2) are symmetric under $p \mapsto 1-p$, both vanish at the endpoints, and both equal $1/\sqrt{2\pi}$ at $p = \frac{1}{2}$ — the last after the removable singularity of the quotient $\log \frac{1-p}{p} / (1-2p)$ has been resolved (Section 3). The inequality (2) is therefore an equality at three points of $[0,1]$, and the interesting question is how large the gap

$$D(p) := \mathcal{I}(p) - L(p) \tag{3}$$

becomes in between. On this the source says exactly one thing, in the remark following (2):

> "In fact, as a consequence of the sharpness of c_{mgf} in Theorem `thm:cmgf`, the constant $2/\sqrt{\pi}$ on the RHS of (2) is also sharp in that it cannot be replaced with a larger constant. *Numerical evaluation suggests that the lower bound in* (2) *is never lower than* $\mathcal{I}$ *by more than* 0.00720 *uniformly on* $[0,1]$ *and the maximal gap occurs at* $p_0 \approx 0.10125$ *and* $1 - p_0 \approx 0.89875$." (emphasis added)

That sentence is the whole of the evidence offered there for the two constants: no range of the numerical evaluation is stated, no error is controlled, and neither $\Delta^* := \sup_p D(p)$ nor the location of the maximum is bounded anywhere in [1]. The sharpness of the leading constant is asserted, as a consequence of the source's main theorem, but no witness is exhibited.

1.2. **Results.** We settle the two constants of that remark, in the sense of two-sided rigorous bounds. Throughout, D is as in (3) and L is the right-hand side of (2), understood at $p = \frac{1}{2}$ through its limit (Section 3).

Theorem 1.1 (the global bound)**.** *For every $p \in (0,1)$ with $p \neq \frac{1}{2}$,*

$$\mathcal{I}(p) - L(p) \;\leq\; 0.0072.$$

Theorem 1.2 (the bracket, and a point where the gap is nearly extremal)**.** *Let $p_w = \Phi(-1.2745)$. Then*

$$0.00719656 \;\leq\; \mathcal{I}(p_w) - L(p_w) \;\leq\; 0.0071965623,$$

and consequently

$$\Delta^* \;=\; \sup_{p \in (0,1),\, p \neq 1/2} \big(\mathcal{I}(p) - L(p)\big) \;\in\; [0.00719656,\ 0.0072].$$

Theorem 1.3 (where the gap can be extremal)**.** *Let $p \in (0,1)$, $p \neq \frac{1}{2}$, and suppose $\mathcal{I}(p) - L(p) \geq 0.00719656$. Then*

$$p \in [0.1006,\ 0.1019] \qquad \text{or} \qquad 1 - p \in [0.1006,\ 0.1019].$$

Equivalently, in the coordinate $x = \Phi^{-1}(p)$: $|x| \in [1.270996\ldots, 1.278076\ldots]$, the endpoints being the dyadic rationals $2603/2048$ *and* $5235/4096$*.*

Theorem 1.2 confirms the source's 0.00720 at the precision at which it is printed — every number in the bracket $[0.00719656, 0.0072]$ rounds to 0.00720 — and pins the gap at the explicit point p_w to within $2.3 \cdot 10^{-9}$. Theorem 1.3 confirms the source's $p_0 \approx 0.10125$ to three digits: the value 0.10125 lies inside the interval $[0.1006, 0.1019]$, and by Theorem 1.2 the set appearing in Theorem 1.3 is not empty, since it contains p_w. Note that Theorem 1.3 does not presuppose that the supremum Δ^* is attained; it constrains every point at which the gap reaches the certified level 0.00719656, and every maximiser — if one exists — is such a point, by Theorem 1.2.

Two smaller statements complete the picture of the remark. The first makes the asserted sharpness of the leading constant checkable at a single point; the second identifies the one place where the formula (2) must be read as a limit.

Proposition 1.4 (the constant cannot be raised to 1.12839)**.** *Let $p_s = \Phi(-0.005)$ and write $L_\kappa(p) = \kappa\, p(1-p)\sqrt{\log \frac{1-p}{p} / (1-2p)}$, so that $L = L_\kappa$ for $\kappa = 2/\sqrt{\pi} = 1.1283791670\ldots$. Then*

$$\mathcal{I}(p_s) \;<\; L_{1.12839}(p_s).$$

In particular $\kappa = 1.12839$ *is already too large in* (2)*; granting* (2) *itself, which we do not reprove, the least constant for which it can hold lies in* $[2/\sqrt{\pi}, 1.12839)$.

Proposition 1.5 (the removable singularity at $p = \frac{1}{2}$)**.** $\lim_{p\to 1/2} \log \frac{1-p}{p}/(1-2p) = 2$, *and*

$$\frac{2}{\sqrt{\pi}} \cdot \frac{1}{2}\Big(1 - \frac{1}{2}\Big) \cdot \sqrt{2} \;=\; \varphi(0) \;=\; \mathcal{I}\Big(\frac{1}{2}\Big),$$

so with the limiting value inserted, (2) *is an* equality *at* $p = \frac{1}{2}$ *and* $D(\frac{1}{2}) = 0$.

1.3. **What is not claimed.** We do not reprove (2) itself. The inequality $\mathcal{I} \geq L$ is the source's Lemma, with a hand proof there; nothing below depends on it, and nothing below establishes it. All our statements bound $\mathcal{I} - L$ from above, or bound it from below at one explicit point; the sign of D away from that point is not part of what we prove.

We do not locate Δ^* or p_0 to more digits than stated. The bracket of Theorem 1.2 has width $2.3\cdot 10^{-9}$ at the witness but width $3.44 \cdot 10^{-6}$ for Δ^* itself, since its upper half is the global constant 0.0072; and Theorem 1.3 localises the maximiser to three digits, not five. Both limitations are consequences of the method rather than of the objects: see Remark 8.1.

We make no claim about the two endpoints $p = 0$ and $p = 1$, where $\mathcal{I}$ and L are defined only by continuous extension; our statements are about the open interval, which is the range of Φ. This is harmless for the supremum: for $|x| \geq 2.85$ one has $D(\Phi(x)) \leq \varphi(x) \leq 0.0069$ (Section 4), so no supremum is lost at the ends.

Finally, we have no proof that the source's remark is original in the sense of not appearing elsewhere; we searched for the bound (2) and for the two constants and found neither in the neighbouring literature, but a negative search is a negative search.

2. Preliminaries

2.1. **The gap in the coordinate** $x = \Phi^{-1}(p)$**.** Since $\Phi : \mathbb{R} \to (0, 1)$ is an increasing bijection and $\mathcal{I}(\Phi(x)) = \varphi(x)$ holds by the definition (1), it is convenient to transport everything to the x line. Put

$$g(x) := \varphi(x) - L(\Phi(x)), \qquad x \in \mathbb{R}, \tag{4}$$

so that $g(x) = D(\Phi(x))$ and, as p ranges over $(0, 1)$, $x = \Phi^{-1}(p)$ ranges over $\mathbb{R}$; the value $p = \frac{1}{2}$ corresponds to $x = 0$. In these coordinates the Φ^{-1} of (1) has disappeared: the only transcendental functions left are φ, Φ, log and exp, each at explicit rational arguments once a sweep grid is fixed.

Two elementary symmetries are used constantly.

Lemma 2.1. $\Phi(-x) = 1 - \Phi(x)$ *and* $L(1 - p) = L(p)$*; consequently* $g(-x) = g(x)$ *for all* x.

Proof. The first identity is the evenness of φ. For the second, replacing p by $1 - p$ replaces $\log \frac{1-p}{p}$ by its negative and $1 - 2p$ by its negative, so the quotient, and hence the square root, is unchanged, while $p(1 - p)$ is symmetric. The third follows from the first two together with $\varphi(-x) = \varphi(x)$. □

It therefore suffices to bound g on $(-\infty, 0)$.

2.2. **Two elementary inequalities.** The sweep uses one inequality for log and one for exp; both are classical, and both are used in the form of *rational* comparisons.

Lemma 2.2 (midpoint, or second Padé, bound)**.** *For* $w \geq 1$, $\log w \geq \dfrac{2(w-1)}{w+1}$. *Equivalently, for* $0 < q \leq \frac{1}{2}$,

$$\log \frac{1-q}{q} \;\geq\; 2(1 - 2q).$$

Proof. The difference $w \mapsto \log w - 2(w-1)/(w+1)$ vanishes at $w = 1$ and has derivative $(w-1)^2/\big(w(w+1)^2\big) \geq 0$. The second form is the first at $w = (1-q)/q$, where $2(w-1)/(w+1) = 2(1-2q)$. □

Lemma 2.2 is exactly tight as $q \to \frac{1}{2}$, and that is what makes a neighbourhood of $p = \frac{1}{2}$ coverable by finitely many intervals at all: it gives, for $0 < q < \frac{1}{2}$,

$$L(q) \;\geq\; \frac{2}{\sqrt{\pi}}\, q(1-q)\sqrt{2} \;=\; \frac{4}{\sqrt{2\pi}}\, q(1-q), \tag{5}$$

an inequality that becomes an equality in the limit $q \to \frac{1}{2}$ and whose right-hand side is a rational function. Without it, the margin available to a sweep would degenerate like $(1-2q)$ near $q = \frac{1}{2}$, the admissible step would shrink like $|x|$, and the number of intervals needed would diverge logarithmically at $x = 0$.

For the exponential we use the alternating Taylor sandwich on the negative axis, in the form recorded in Proposition 2.3 below.

2.3. **Rational enclosures for Φ, φ and** exp**.** The library we verify in has the Gaussian integral $\int e^{-bx^2} = \sqrt{\pi/b}$, the Gaussian density and the Gaussian measure, but no error function — the identifier occurs nowhere in it — and nothing that evaluates or bounds the normal distribution function: a distribution function is defined there generically, for an arbitrary measure on $\mathbb{R}$, and in the Gaussian case no property of it is proved. Everything numerical below therefore rests on a self-contained enclosure layer, which we describe because it is elementary, because it is the only place where analysis (as opposed to arithmetic) enters the certificates, and because it is reusable: any certified normal-tail constant — probit values, Gaussian concentration constants, Berry–Esseen numerics — can be produced from it.

Let $E_n(s) = \sum_{k<n} s^k/k!$ be the n-term Taylor sum of exp and $R_n(s) = e^s - E_n(s)$. Then $R'_{n+1} = R_n$, $R_n(0) = 0$ for $n \geq 1$, and hence, by induction on n using the sign of the derivative on $(-\infty, 0]$,

$$s \leq 0 \implies (-1)^n R_n(s) \geq 0, \tag{6}$$

that is: on the negative axis an even number of terms under-estimates e^s and an odd number over-estimates it. Substituting $s = -y^2/2$ and integrating the resulting polynomial sandwich over $[0, u]$ gives, with

$$G_n(u) = \sum_{k<n} \frac{(-1)^k u^{2k+1}}{2^k\, k!\, (2k+1)},$$

the two-sided estimate $G_n(u) \leq \int_0^u e^{-y^2/2}\, dy \leq G_m(u)$ for $u \geq 0$, n even, m odd. Together with $\Phi(0) = \frac{1}{2}$ (from the Gaussian integral) and

$$\Phi(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{-x} e^{-y^2/2}\, dy,$$

this yields rational two-sided bounds for Φ at every rational $x \leq 0$.

Proposition 2.3 (enclosures)**.** *There are explicitly computable maps $\mathbb{Q} \to \mathbb{Q}$ producing, for every rational $x \leq 0$, rationals $\underline{\Phi}(x) \leq \Phi(x) \leq \overline{\Phi}(x)$ and $\underline{\varphi}(x) \leq \varphi(x) \leq \overline{\varphi}(x)$, and, for every rational $r \in [0, 8]$, rationals $\underline{e}(r) \leq e^r \leq \overline{e}(r)$. All of these are obtained from* (6) *and its integrated form, together with the rational bounds $1/\sqrt{2\pi} \in [0.39894228040143267792, 0.39894228040143267795]$; no compiled evaluation is used in their proofs.*

The implementation is a running-term recursion over the rationals, with the outputs rounded outwards to multiples of 2^{-50} so that denominators do not compound along a sweep. At the term counts used here (10 to 34 terms, chosen by $|x|$), the truncation error contributes at most $1.7 \cdot 10^{-17}$ to the Φ bracket and at most $4 \cdot 10^{-16}$ to the φ bracket on all of $[-2.85, 0]$, so the delivered width is set by the rounding, $2^{-50} \approx 8.9 \cdot 10^{-16}$, and not by the series. For orientation,

$$\Phi(-1.2745) \in [0.101243136328517558,\ 0.101243136328518446],$$
$$\varphi(-1.2745) \in [0.177087085628484076,\ 0.177087085628484964].$$

3. The removable singularity at $p = \frac{1}{2}$

The expression on the right of (2) is a quotient whose numerator and denominator both vanish at $p = \frac{1}{2}$. Writing $p = \frac{1}{2} - t$ gives $\log \frac{1-p}{p} = \log \frac{1+2t}{1-2t} = 4t + O(t^3)$ and $1 - 2p = 2t$, so the quotient tends to 2; and $\sqrt{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{4} = \frac{1}{\sqrt{2\pi}} = \varphi(0)$.

Proposition 3.1. $\dfrac{2}{\sqrt{\pi}} \cdot \dfrac{1}{2}\Big(1 - \dfrac{1}{2}\Big) \cdot \sqrt{2} = \varphi(0)$. *Hence, with the limiting value* 2 *inserted for the quotient,* (2) *holds with equality at* $p = \frac{1}{2}$ *and* $D(\frac{1}{2}) = 0$. *If instead the formula is evaluated literally, under the convention* $y/0 = 0$ *used by our verification system, it returns* $L(\frac{1}{2}) = 0$*, which is not its limit; the hypothesis* $p \neq \frac{1}{2}$ *in Theorem 1.1 refers to this reading.*

Proof. $\sqrt{2} \cdot \sqrt{\pi} = \sqrt{2\pi}$ and $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{4} \cdot \sqrt{2} = \frac{\sqrt{2}}{2\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} = \varphi(0)$. The limit computation is the expansion above. The literal evaluation is immediate from $1 - 2p = 0$ at $p = \frac{1}{2}$. □

Proposition 3.1 is the reason the hypothesis $p \neq \frac{1}{2}$ appears in Theorem 1.1 and is not an artefact: under the literal reading the conclusion of Theorem 1.1 is *false* at $p = \frac{1}{2}$, since $D(\frac{1}{2})$ would be $\varphi(0) = 0.3989\ldots$; see Proposition 5.2(iii). Under the limiting reading it is true, but trivially so, both sides being equal.

4. The global bound

4.1. **The three regions.** By Lemma 2.1 it suffices to bound g on $(-\infty, 0)$. We split that half line at -2.85.

The tail. For $x \leq -2.85$ we use only $L \geq 0$ on $[0,1]$ and the monotonicity of φ on $(-\infty, 0]$:

$$g(x) \;\leq\; \varphi(x) \;\leq\; \varphi(-2.85) \;\leq\; 0.0068727667 \;<\; 0.0069 \;<\; 0.0072. \tag{7}$$

The last two comparisons are one evaluation of the enclosure of Proposition 2.3. (The sharper form 0.0069 is what Section 6 needs.)

The middle. On $[-397281/131072, 0) = [-3.0310134\ldots, 0)$ — an interval that covers $[-2.85, 0)$ with room to spare — we run a sweep.

The right half follows from $g(-x) = g(x)$.

4.2. **The per-interval test.** Fix a constant c (we take $c = 0.0072$ here) and an interval $[x_1 - h, x_1]$ with $x_1, h \in \mathbb{Q}$, $x_1 \leq 0$, $h \geq 0$. Let

$$\mathrm{ph} = \overline{\varphi}(x_1), \quad q_{\mathrm{lo}} = \underline{\Phi}(x_1), \quad b = \overline{\Phi}(x_1), \quad a = q_{\mathrm{lo}} - \mathrm{ph} \cdot h$$

be rationals produced by Proposition 2.3. For every x in the interval:

- $\varphi(x) \leq \varphi(x_1) \leq \mathrm{ph}$, by monotonicity of φ on the negative axis;
- $\Phi(x) \leq \Phi(x_1) \leq b$, by monotonicity of Φ;
- $\Phi(x) \geq \Phi(x_1) - \varphi(x_1)(x_1 - x) \geq a$, because $\Phi(x_1) - \Phi(x) = \int_x^{x_1} \varphi \leq \varphi(x_1)(x_1 - x)$, again by monotonicity.

The third item is a tangent bound rather than the naive $\Phi(x) \geq \Phi(x_1 - h)$; it makes the Φ-side error second order in h and needs Φ at one endpoint per interval instead of two.

Write $q = \Phi(x)$, so that $q \in [a, b]$ and, since $x \leq 0$, also $q \leq \frac{1}{2}$. Consequently $g(x) \leq c$ on the whole interval as soon as $A := \mathrm{ph} - c \leq L(q)$, and the test verifies that by one of two rational branches (a third, trivial, branch handles $A \leq 0$).

Branch 1 (near $p = \frac{1}{2}$*).* If $0 < a$, $2a \leq 1$ and $A \leq \frac{4}{\sqrt{2\pi}} a(1-a)$ — with $4/\sqrt{2\pi}$ replaced by a rational lower bound — then, since $q \mapsto q(1-q)$ increases on $[a, \frac{1}{2}]$ and $a \leq q \leq \frac{1}{2}$, (5) gives $L(q) \geq \frac{4}{\sqrt{2\pi}} q(1-q) \geq \frac{4}{\sqrt{2\pi}} a(1-a) \geq A$.

Branch 2 (elsewhere). Suppose $A > 0$, $0 < a$ and $2b < 1$. For $0 < q < \frac{1}{2}$ the inequality $A \leq L(q)$ is equivalent, after squaring, to

$$\log \frac{1-q}{q} \;\geq\; R(q) := \frac{A^2 \pi (1-2q)}{4\big(q(1-q)\big)^2},$$

and R decreases on $(0, \frac{1}{2})$, so $R(q) \le R(a)$. Since $q \le b < \frac{1}{2}$ we also have $\log \frac{1-q}{q} \ge \log \frac{1-b}{b}$, so it suffices that $e^{R(a)} \le (1-b)/b$; this is checked with the rational *upper* bound $\overline{e}$ of Proposition 2.3, applied to a rational upper bound for $R(a)$. No enclosure of log is needed anywhere.

4.3. **The sweep, and what was searched.** The middle region is covered by 8070 intervals arranged in 30 blocks of constant dyadic step, walking leftwards from 0 to the exact endpoint $-397281/131072$. The step is 2^{-3} at both ends of the range and falls to 2^{-17} on the block $[-1.28583\ldots, -1.26268\ldots]$, which is where the margin $L(q) - A$ available to the test is thinnest — that is, where the gap comes closest to the constant 0.0072 being certified. Every one of the 8070 intervals passes the test of the previous subsection with $c = 0.0072$: this is a single finite computation in exact rational arithmetic, and it is the only exhaustive search behind Theorem 1.1.

Theorem 4.1. *For every real $x \neq 0$, $g(x) \le 0.0072$. Equivalently, for every $p \in (0,1)$ with $p \neq \frac{1}{2}$, $\mathcal{I}(p) - L(p) \le 0.0072$.*

Proof sketch. By Lemma 2.1 we may take $x < 0$. If $x \le -2.85$, apply (7). Otherwise x lies in $[-397281/131072, 0)$, hence in one of the 8070 intervals of the sweep, and the per-interval test — verified exhaustively for all 8070 of them — gives $g(x) \le 0.0072$. The step from "the test holds at the interval" to "$g \le c$ on the interval" is the analytic content of Section 4 and is proved once, for a general interval; the computation supplies only the 8070 rational verifications. The second form is the first, read through $p = \Phi(x)$ and $\mathcal{I}(\Phi(x)) = \varphi(x)$. □

5. The value at a witness, and the bracket

The upper bound of Theorem 4.1 is complemented by a lower bound at one explicit point. Take $x_w = -1.2745$, so $p_w = \Phi(x_w) \in [0.101243136328517558, 0.101243136328518446]$ by Proposition 2.3.

Theorem 5.1. $0.00719656 \le g(x_w) \le 0.0071965623$*; that is,*

$$0.00719656 \ \le\ \mathcal{I}(p_w) - L(p_w) \ \le\ 0.0071965623.$$

Consequently

$$\Delta^* = \sup_{p\in(0,1),\, p\neq 1/2} \big(\mathcal{I}(p) - L(p)\big) \in [0.00719656,\ 0.0072].$$

Proof sketch. For the lower bound one runs Branch 2 of Section 4 in reverse at the single point x_w: with A' a rational lower bound for $\varphi(x_w) - 0.00719656$, the inequality $L(q) \le A'$ is equivalent, after squaring, to $\log \frac{1-q}{q} \le A'^2\pi(1-2q)/\big(4(q(1-q))^2\big)$, and that is certified by the rational *lower* bound $\underline{e}$ of Proposition 2.3 applied to the right-hand side, the enclosure of $\Phi(x_w)$ supplying q. This is one finite rational check. For the upper bound, the per-interval test of Section 4 at width $h = 0$ is a single-point upper bound; it passes with $c = 0.0071965623$, again one finite rational check. The bracket for Δ^* combines the lower bound here with Theorem 4.1. □

Theorem 5.1 also gives the negative statements that keep the bracket honest.

Proposition 5.2 (controls).
- (i) *The constant in Theorem 4.1 cannot be lowered to* 0.00719*: it is not true that $g(x) \le 0.00719$ for all $x \neq 0$.*
- (ii) *The constant* 0.0072 *is not attained at the witness: $g(x_w) < 0.0072$.*
- (iii) *The hypothesis $x \neq 0$ in Theorem 4.1 is not vacuous: under the literal reading of* (2) *discussed in Section 3, $g(0) = \varphi(0) = 0.3989\ldots > 0.0072$.*

Proof. (i) and (ii) are immediate from Theorem 5.1; (iii) is Proposition 3.1 together with one evaluation of $\underline{\varphi}(0)$. □

6. Where the gap can be extremal

The source's second constant, $p_0 \approx 0.10125$, is the location of the maximum. We bracket it by re-running the sweep of Section 4 with a constant *below* the value certified at the witness, on the complement of a small window.

Set $c_{\mathrm{loc}} = 0.0071965 < 0.00719656$, and let

$$x_L = -\frac{5235}{4096} = -1.278076171875, \qquad x_R = -\frac{2603}{2048} = -1.27099609375,$$

so that $x_w = -1.2745 \in [x_L, x_R]$. Two further sweeps establish $g \leq c_{\mathrm{loc}}$ off $[x_L, x_R]$ on the negative axis:

- sweep A: 19 245 intervals in 21 blocks, covering $[x_R,\, 0)$;
- sweep B: 17 917 intervals in 22 blocks, covering the interval $[-25435559/8388608,\, x_L)$, whose left endpoint is $-3.0321549\ldots$;
- the tail: $g(x) \leq 0.0069 < c_{\mathrm{loc}}$ for $x \leq -2.85$, by the sharper form of (7).

Together these cover $(-\infty, 0) \setminus [x_L, x_R]$ with a bound strictly below 0.00719656.

Theorem 6.1. *Let $x < 0$ with $g(x) \geq 0.00719656$. Then $x_L \leq x \leq x_R$, and consequently*

$$0.1006 \;\leq\; \Phi(x) \;\leq\; 0.1019.$$

In the source's variable: if $p \in (0,1)$, $p \neq \frac{1}{2}$, and $\mathcal{I}(p) - L(p) \geq 0.00719656$, then $p \in [0.1006, 0.1019]$ or $1 - p \in [0.1006, 0.1019]$.

Proof sketch. If $x < x_L$ then either $x \leq -2.85$, where the sharpened tail bound gives $g(x) \leq 0.0069 < 0.00719656$, or x lies in one of the 17 917 intervals of sweep B, where $g(x) \leq c_{\mathrm{loc}} < 0.00719656$; either way the hypothesis fails. If $x > x_R$ then x lies in one of the 19 245 intervals of sweep A, with the same conclusion. Hence $x \in [x_L, x_R]$. The bounds on $\Phi(x)$ then follow from the monotonicity of Φ and two evaluations of the enclosure, $\underline{\Phi}(x_L) \geq 0.1006$ and $\overline{\Phi}(x_R) \leq 0.1019$. The last sentence is the previous one together with Lemma 2.1, which sends $x > 0$ to $-x < 0$. The exhaustive part is the two sweeps, 37 162 intervals in all, each verified by the rational test of Section 4. □

The window is not an accident of the grid but of the level. With c the certified level and $g_{\max}$ the true maximum, the set $\{g \geq c\}$ has width about $\sqrt{(g_{\max} - c)/0.0085}$ in x, where 0.0085 is a numerical estimate of the local curvature of g; and the number of intervals a sweep needs near the edge of a window of half-width d was found to grow like $53/d$. (Both scalings are empirical, and neither is used in any proof.) Each further digit of p_0 therefore costs about a factor 10^2 in the size of the search. This is why the bracket is three digits wide and not five.

7. The leading constant

The source asserts that $2/\sqrt{\pi}$ in (2) is sharp, as a consequence of its main theorem, and exhibits no witness. The following exhibits one: a single point at which the inequality already fails when the constant is raised to 1.12839. It confirms a consequence of the source's assertion — which is about *every* larger constant — and does not reprove the assertion itself.

Proposition 7.1. *Let $x_s = -0.005$ and $p_s = \Phi(x_s) \in [0.498005296909258988,\, 0.498005296909259876]$. Then*

$$\mathcal{I}(p_s) \;=\; \varphi(x_s) \;<\; 1.12839 \cdot p_s(1-p_s) \cdot \sqrt{\frac{\log \frac{1-p_s}{p_s}}{1-2p_s}}.$$

Hence (2) fails with $2/\sqrt{\pi}$ replaced by 1.12839. Granting (2) itself — which is the source's lemma and is not reproved here — the least admissible constant lies in $[2/\sqrt{\pi},\, 1.12839)$, where $2/\sqrt{\pi} = 1.1283791670\ldots$

Proof. By Lemma 2.2, $\sqrt{\log \frac{1-q}{q} / (1-2q)} \geq \sqrt{2}$ for $0 < q < \frac{1}{2}$, so with $\kappa = 1.12839$ the right-hand side is at least $\kappa\, p_s(1-p_s)\sqrt{2}$; since $p_s < \frac{1}{2}$ and $q \mapsto q(1-q)$ increases below $\frac{1}{2}$, it remains to compare $\overline{\varphi}(x_s)$

with $\kappa\,\underline{\Phi}(x_s)(1-\underline{\Phi}(x_s))\sqrt{2}$, a rational comparison once $\sqrt{2}$ is replaced by a rational lower bound. That comparison is one finite evaluation. $\square$

8. Verification

Every proposition, lemma and theorem stated above — Lemmas 2.1 and 2.2, Propositions 2.3, 3.1, 5.2 and 7.1, and Theorems 4.1, 5.1 and 6.1, of which Theorems 1.1, 1.2, 1.3 and Propositions 1.4, 1.5 of the introduction are restatements — is machine-checked in Lean 4 [6] over the Mathlib library [7]; the development is seven modules and contains no `sorry`.

Three things above are stated in a form the development does not carry, and we say which, since none of them is mathematics. First, everything is formalised in the coordinate x: the statements in the source's variable are proved for $p = \Phi(x)$ with $x \neq 0$, and the identification of the open interval $(0,1)$ with the range of Φ is used as classical background rather than formalised. Second, the limit $\lim_{p\to 1/2} \log \frac{1-p}{p} \big/ (1-2p) = 2$ of Proposition 3.1 is proved by hand here; what is machine-checked at $p = \frac{1}{2}$ is the identity $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{4} \cdot \sqrt{2} = \varphi(0)$ and the junk-value evaluation $L(\frac{1}{2}) = 0$. Third, the development contains no supremum: the bracket for Δ^* in Theorem 5.1 is the one line of order theory that combines the two machine-checked inequalities $0.00719656 \le g(x_w)$ and $g \le 0.0072$ off $x = 0$. Everything else — every inequality, every enclosure, every sweep — is checked in the form printed.

The development is arranged so that the analysis is separated from the arithmetic: the enclosure layer of Proposition 2.3 and every reasoning step — the alternating Taylor sandwich, its integrated form, $\Phi(0) = \frac{1}{2}$, the monotonicity and tangent bounds for Φ, Lemma 2.2, the two branches of the per-interval test, and the passage from a verified sweep to a bound on g — are checked by the kernel alone, an axiom print returning `propext`, `Classical.choice` and `Quot.sound` and nothing else. Compiled evaluation is confined to ten finite computations in exact rational arithmetic, one for each of the ten invocations of compiled evaluation in the development: the three interval sweeps (8070, 19 245 and 17 917 intervals), the two tail comparisons for $\varphi(-2.85)$ (at the levels 0.0072 and 0.0069), the two checks at the witness x_w, the single check carrying both window comparisons $\underline{\Phi}(x_L) \ge 0.1006$ and $\overline{\Phi}(x_R) \le 0.1019$, the comparison of Proposition 7.1, and the evaluation of $\underline{\varphi}(0)$ used in Proposition 5.2(iii). The axiom print of a certified theorem names the three standard axioms above together with exactly those of these ten it uses, and nothing further; that is the precise and only sense in which the statements above rest on computation. Independently of the kernel, the two constants of the source's remark were recomputed here from its own definitions before anything was claimed, in 60-digit decimal arithmetic with an independently written implementation of the error function and a ternary search. This returns an approximate maximiser $x^* = -1.274478613636856963\ldots$, hence $\Phi(x^*) = 0.101246923628853767\ldots$, with value $g(x^*) = 0.007196562271631586\ldots$; the three agree with the source's printed 0.10125 and 0.00720 and with the brackets of Theorems 5.1 and 6.1, but they are numerical computations and are asserted here only as such. The source of the development is currently held in a private repository: repository reference to be added at submission.

Remark 8.1 (cost of the certificates; outside the formal development). The following are timings, not theorems, and nothing above depends on them. The 8070-interval sweep of Theorem 4.1 runs in one compiled evaluation taking 42.4 s wall and 15.5 s user time at 6.77 GB peak resident memory, the library import dominating; the two sweeps of Theorem 6.1 (37 162 intervals) take 41.7 s wall and 36.4 s user at 6.98 GB. The marginal cost is about 0.72 ms per interval. Replacing the naive envelope $\Phi(x) \ge \Phi(x_1 - h)$ by the tangent bound of Section 4 cuts the ideal interval count for the global sweep from about 8200 to about 6060 (and 8070 once the steps are rounded down to powers of two), and halves the number of Φ evaluations. Extrapolating the two scaling laws recorded at the end of Section 6, locating p_0 to five digits by the same method would need about $3.7 \cdot 10^6$ intervals, roughly 45 minutes of compiled evaluation and about 1.5 hours of grid generation — feasible, but not run here. A certified sign change of g' would be the cheaper route and would need the derivative of $L \circ \Phi$.

On the reference list

Reference [1] was read here from its arXiv LaTeX e-print, version 2, and every quotation above is from that file; the labels `lem:I`, `rk:I` and `thm:cmgf` are the source's own, and the numbering of the published version, if any, may differ. The abstract page was checked again on 3 September 2026: v2, submitted 14 June 2026, is the current version, and Lemma `lem:I` and Remark `rk:I` carry there the wording quoted above.

The four historical references are cited for the standard background on the Gaussian isoperimetric profile and are used in no proof above. Only [5] was read here, from the scan of the published article: the attributions in Section 1 follow its introduction, which credits [2] and [3] with the half-space property and [4] with the functional version, and its abstract, which calls its own proof a martingale proof. The bibliographic data of the other three were verified against Crossref and zbMATH and agree with its reference list. The statement in Section 2.3 about the verification library was checked against the version pinned by our development on 3 September 2026: the error function's identifier occurs nowhere in it, and it proves nothing about the distribution function of the Gaussian measure.

References

[1] Y. Zhang, *Sharp One-Dimensional Sub-Gaussian Comparison in Convex Order*, arXiv:2604.26819v2 [math.PR], 14 June 2026 (v1, 29 April 2026); cross-listed cs.IT, math.ST, stat.ML. Read here in the v2 e-print, which is what the labels above refer to.

[2] C. Borell, *The Brunn–Minkowski inequality in Gauss space*, Inventiones Mathematicae **30** (1975), no. 2, 207–216, doi:10.1007/BF01425510.

[3] V. N. Sudakov and B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures*, Journal of Soviet Mathematics **9** (1978), 9–18, doi:10.1007/BF01086099.

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[5] F. Barthe and B. Maurey, *Some remarks on isoperimetry of Gaussian type*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **36** (2000), no. 4, 419–434, doi:10.1016/S0246-0203(00)00131-X.

[6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.