THE EVENTUAL PERIOD OF $\Gamma(k, F_n)$: NINETY NEW VALUES AND A CLOSED FORM IN PISANO PERIODS

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Abstract. For relatively prime positive integers a, b exactly one of the two Diophantine equations $ax + by = (a-1)(b-1)/2$ and $1 + ax + by = (a-1)(b-1)/2$ has a nonnegative integral solution, and $\Gamma(a, b) \in \{0, 1\}$ records which one. Chu, Miller and Tresch (arXiv:2512.12681) write $T_k((a_n))$ for the eventual period of $(\Gamma(k, a_n))_{n\geq 1}$, prove that $T_k((F_n))$ divides the Pisano period $\pi(2k)$, print $T_k((F_n))$ for $k \leq 10$, and ask for its value for every k. We compute the ninety values $11 \leq k \leq 100$ and verify, for every $k \leq 200$ by machine-checked computation from the Diophantine definition and for every $k \leq 2000$ by an independent computation, that $T_k((F_n)) = \pi(P_k)$, where P_k ($= k$ for odd k, $2k$ for even k) is the period of $\Gamma(k, \cdot)$ on $\mathbb{N}$; that is, $T_k((F_n)) = \pi(k)$ for odd k and $\pi(2k)$ for even k. This value is strictly smaller than $\pi(2k)$ for 60 of the first 200 values of k, among them $k = 3$ of the printed table, and the set of k with $T_k((F_n)) = 2\pi(k)$ is not the set of powers of two ($k = 40$ is the least counterexample) but is described by a 2-adic condition on the Pisano period of the odd part of k. For the companion problem on arithmetic progressions $pn - r$ we note that the coprime case $\gcd(k, p) = 1$ follows for every k from the source's own periodicity theorem, and exhibit $T_{15}((5n-1)) = 1$ and $T_6((2n)) = 3 \neq 6 = T_6((2n-1))$, which rule out the natural candidate formulas in the non-coprime range. The closed form for all k is stated as a conjecture. Every computational statement is verified in Lean 4 by evaluated computation, without Mathlib.

1. Introduction

1.1. The two equations. Beiter [2] showed that for prime numbers a, b exactly one of the two Diophantine equations

$$(1) \qquad ax + by = \frac{(a-1)(b-1)}{2} \qquad \text{and} \qquad 1 + ax + by = \frac{(a-1)(b-1)}{2}$$

has a nonnegative integral solution (x, y), and that the solution is unique; Chu [4, Theorem 1.1] observed that the same holds for all relatively prime positive integers a, b. Arachchi, Chu, Liu, Luan, Marasinghe and Miller [1] turned "which equation" into a function. Following the normalisation of [6], for $a, b \in \mathbb{N} = \{1, 2, \dots\}$ with $g = \gcd(a, b)$,

$$(2) \qquad \Gamma(a, b) = \begin{cases} 0 & \text{if } \frac{a}{g}x + \frac{b}{g}y = \frac{(a/g-1)(b/g-1)}{2} \text{ has a nonnegative integral solution,} \\ 1 & \text{if } 1 + \frac{a}{g}x + \frac{b}{g}y = \frac{(a/g-1)(b/g-1)}{2} \text{ has a nonnegative integral solution.} \end{cases}$$

By the Beiter–Chu dichotomy exactly one case applies, so $\Gamma : \mathbb{N}^2 \to \{0, 1\}$ is well defined. (In [1] the function takes the values 1 and 2 instead, 1 when the first equation is the solvable one; the $\{0, 1\}$ normalisation, which is the value of [1] minus one, is that of [6], and it is used throughout.)

1.2. The source and its questions. The recent survey of Chu, Miller and Tresch [6] collects results on (1) and poses a list of problems. Its Section 2.3 is the starting point of this paper. For a fixed $k \in \mathbb{N}$ and a sequence of positive integers $(a_n)_{n\geq 1}$, [6] lets $T_k((a_n)_{n=1}^{\infty})$ denote the eventual period, if it exists, of the $\{0, 1\}$-word $(\Gamma(k, a_n))_{n\geq 1}$. Three facts from [6] frame everything that follows; theorem numbers refer to the arXiv PDF of version 1.

- [6, Theorem 2.8] (quoted there from [1, Theorem 1.8 of v1]): if k is odd then $T_k(\mathbb{N}) = k$ and in each period the number of 0's is one more than the number of 1's; if k is even then

$T_k(\mathbb{N}) = 2k$ and in each period the number of 0's is two more than the number of 1's. We write

$$P_k \;=\; \begin{cases} k & k \text{ odd}, \\ 2k & k \text{ even}, \end{cases} \tag{3}$$

for this period of $\Gamma(k,\cdot)$ on $\mathbb{N}$.

- [6, Theorem 2.14]: if (a_n) is eventually increasing and eventually satisfies a polynomial recurrence $a_n = c_1 a_{n-1}^{t_1} + \cdots + c_s a_{n-s}^{t_s}$, then $(\Gamma(k, a_n))_{n\geq 1}$ is eventually periodic for every k, and, writing $\pi(m)$ for the eventual period of $(a_n \bmod m)$, $T_k((a_n))$ divides $\pi(2k)$. For the Fibonacci numbers $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, this is [6, Corollary 2.16]: $T_k((F_n)_{n=1}^{\infty})$ divides the Pisano period $\pi(2k)$.
- [6, Table 1] prints $T_k((F_n)_{n=1}^{\infty})$ and $\pi(2k)$ for $1 \leq k \leq 10$ (reproduced as Table 1 below), and this is followed by the problem that this paper addresses, [6, Problem 2.17], quoted in full:

  For each k, compute $T_k((F_n)_{n=1}^{\infty})$.

The same section of [6] treats arithmetic progressions. Its Proposition 2.10 shows that for $a_n = pn - r$ with $0 \leq r < p$, odd k and $\gcd(k,p) = 1$ one has $T_k((a_n)) = k$, with one more 0 than 1 per period; its Proposition 2.11 shows $T_{2^j}((2n-1)) = 2^j$ with 2^{j-1} zeros and 2^{j-1} ones per period; and [6, Problem 2.12] asks:

> For a general arithmetic progression $a_n = pn - r$ with $0 \leq r < p$ and $k \geq 1$, compute $T_k((a_n)_{n=1}^{\infty})$.

That is the whole of what [6] says about these two periods: a ten-row table, one divisibility, two propositions on progressions, and two open problems. It states no formula and no conjecture for $T_k((F_n))$.

1.3. **Results.** Throughout, $T_k(F)$ abbreviates $T_k((F_n)_{n=1}^{\infty})$, and $\pi(m)$ is the Pisano period of m, the least period of $(F_n \bmod m)_{n\geq 1}$ (OEIS A001175 [8]).

- **Ninety new values.** Theorem 3.2 gives $T_k(F)$ for $11 \leq k \leq 100$; the source prints $k \leq 10$ and, to our knowledge, no other value has been published (see §1.5).
- **A closed form.** Theorem 3.3 verifies, for every $1 \leq k \leq 200$,

$$T_k(F) \;=\; \pi(P_k) \;=\; \begin{cases} \pi(k) & k \text{ odd}, \\ \pi(2k) & k \text{ even}. \end{cases} \tag{4}$$

  An independent computation confirms (4) for every $k \leq 2000$ (Remark 3.5). Identity (4) has exactly the shape of [6, Theorem 2.8], which says that $T_k(\mathbb{N}) = P_k$ is the period of $(n \bmod P_k)_{n\geq 1}$: along the Fibonacci numbers the period is that of $(F_n \bmod P_k)_{n\geq 1}$. We have not proved (4) for all k; it is stated as Conjecture 3.4, and the obstacle is described in Remark 3.6.
- **Two consequences a reader of the printed table would get wrong.** Theorem 3.7: the right-hand side of (4) is strictly smaller than the bound $\pi(2k)$ of [6, Corollary 2.16] for exactly 60 of the first 200 values of k, the smallest beyond the trivial $k = 1$ being $k = 3$, a row of the printed table ($8 < 24$). And the set of k with $T_k(F) = 2\pi(k)$, which the printed table exhibits only at $k = 2, 4, 8$, is not the set of powers of two: it begins $2, 4, 8, 16, 32, 40, 44, 48, \ldots$, it is described by a 2-adic condition on π of the odd part of k, and $k = 40$ is the least counterexample.
- **Arithmetic progressions.** Proposition 4.1 records two statements that are elementary consequences of [6, Theorem 2.8] but are not made in [6]: the parity hypothesis in its Proposition 2.10 is unnecessary ($\gcd(k,p) = 1$ gives $T_k((pn - r)) = P_k$ for every k), and without any hypothesis $T_k((pn - r))$ divides $P_k / \gcd(p, P_k)$. Theorem 4.2 then shows what

an answer to [6, Problem 2.12] cannot be in the range $\gcd(k,p) > 1$: $\Gamma(15, 5n-1)$ is the constant word 1, so $T_{15}((5n-1)) = 1$ while k, $k/\gcd(k,p)$ and the divisibility bound all equal 15 or 3; and $T_6((2n)) = 3 \neq 6 = T_6((2n-1))$, so the residue r matters.

1.4. **Method.** Every value of Γ used here is computed from the definition (2) by a bounded search for a solution of the Diophantine equation, not through the modular-inverse criterion [6, Theorem 2.1] that the proofs in [1, 6] use. The search is short even when $b = F_n$ has hundreds of digits: with $A = a/g$, $B = b/g$ and $R = (A-1)(B-1)/2$, any solution of $i + Ax + By = R$ has $By \le R < (A-1)B/2$, so $y \le (A-1)/2$, and x is read off the divisibility $A \mid R - i - By$. The infinite word $(\Gamma(k, F_n))_{n\ge1}$ is purely $\pi(2k)$-periodic (Lemma 2.1), so its least period is the least period of its prefix of length $2\pi(2k)$ (Lemma 2.2); that prefix is what is computed. For $k \le 200$ the prefix has at most 3000 terms and involves Fibonacci numbers of up to 627 decimal digits. The Beiter–Chu dichotomy, which makes (2) read Γ off the first equation alone, is itself machine-checked on every pair (k, F_n) with $k \le 120$ that the computation touches (Proposition 5.1).

Every theorem and proposition of Sections 3–5 has, in the finite form in which it is stated, a counterpart in a Lean 4 development that imports no library and closes every statement by evaluated computation; Section 6 and Appendix A give the details and the exact formal statements.

1.5. **Prior work.** The question is recent (December 2025) and, as far as we could determine on 3 September 2026, untouched. The four papers on "a pair of Diophantine equations" in this lineage are [1, 3, 5, 6]; only [6] poses the period question, and the other three contain no Pisano period and no T_k of any sequence (the Fibonacci content of [1] concerns $\Gamma(F_n, F_{n+1})$ for consecutive pairs, a different question). Version 2 of [1], dated 17 February 2026 and thus later than [6], was read for this purpose. The sequence $(T_k(F))_{k\ge1} = 1, 6, 8, 12, 20, 24, 16, 24, 24, 60, 10, \dots$ has no entry in the OEIS, nor does the doubling set $2, 4, 8, 16, 32, 40, 44, 48, \dots$, while the control query for the Pisano periods returns A001175. A full-text scan of an arXiv corpus of about 881,000 papers found 60 papers mentioning Beiter and 383 mentioning Pisano, and [6] is the only paper mentioning both. OpenAlex and Semantic Scholar list no citation of [6]. MathSciNet, zbMATH and the back issues of the *Fibonacci Quarterly* were not searched.

2. Preliminaries

2.1. **Notation.** $\mathbb{N} = \{1, 2, 3, \dots\}$. The Fibonacci numbers are $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$, the convention of [6]. For $m \in \mathbb{N}$ the Pisano period $\pi(m)$ is the least $t \ge 1$ with $F_{n+t} \equiv F_n \pmod{m}$ for all $n \ge 1$; the sequence $(F_n \bmod m)$ is purely periodic because the recurrence is invertible modulo m, and $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 8$, $\pi(2^j) = 3 \cdot 2^{j-1}$, $\pi(5) = 20$ [6, 8]. For a $\{0,1\}$-word $w = (w_n)_{n\ge1}$ a *period* is a $t \ge 1$ with $w_{n+t} = w_n$ for all $n \ge 1$, an *eventual period* is a $t \ge 1$ with $w_{n+t} = w_n$ for all sufficiently large n, and "the eventual period" means the least one. $v_2(n)$ is the exponent of 2 in n.

2.2. **Two lemmas that license the computation.** The computation determines least periods of finite prefixes. Two elementary observations turn these into the eventual periods that [6] asks for.

Lemma 2.1. *For every $k \in \mathbb{N}$ and every $n \ge 1$, $\Gamma(k, F_{n+\pi(2k)}) = \Gamma(k, F_n)$. In particular $(\Gamma(k, F_n))_{n\ge1}$ is purely periodic with period $\pi(2k)$.*

Proof. This is the argument of [6, proof of Theorem 2.14] specialised to (F_n). Since $(F_n \bmod 2k)_{n\ge1}$ is purely periodic with period $\pi(2k)$ and (F_n) is non-decreasing, for every $n \ge 1$ the difference $N = F_{n+\pi(2k)} - F_n$ is a nonnegative multiple of $2k$; with $d = \gcd(k, F_n) = \gcd(k, F_{n+\pi(2k)})$, [6, Lemma 2.15] applied to $a = k/d$, $b = F_n/d$ and N/d gives $\Gamma(k, F_{n+\pi(2k)}) = \Gamma(k, F_n)$, exactly as in [6], where the same step is carried out for all sufficiently large n. The hypotheses of that lemma hold here for every $n \ge 1$: N/d is a nonnegative integer because $F_{n+\pi(2k)} \ge F_n$ and $d \mid N$;

$\gcd(k/d, F_{n+\pi(2k)}/d) = 1$; and with $N = 2kj$ and y^* the y-coordinate of the solution for the pair $(k/d, F_n/d)$, $(N/d)\big((k/d-1)/2 - y^*\big) = (jk/d)(k/d - 1 - 2y^*)$ is an integer divisible by k/d. [6] restricts to large n only because a general sequence is assumed increasing and periodic modulo $2k$ only eventually; (F_n) is non-decreasing and purely periodic modulo $2k$ from $n = 1$. ☐

Lemma 2.2. *Let $w = (w_n)_{n\geq 1}$ be purely periodic with period P, let $L \geq 2P$, and let T be the least $t \geq 1$ with $w_{n+t} = w_n$ for all $1 \leq n \leq L - t$. Then T is the least period of w, and also its least eventual period.*

Proof. P itself satisfies the defining condition of T, so $T \leq P$ and $L - T \geq P$. Hence $w_{n+T} = w_n$ for $1 \leq n \leq P$, and by P-periodicity for all $n \geq 1$: T is a period of w. Every period of w satisfies the defining condition, so no period is smaller than T. Finally, if t is an eventual period of w, say $w_{n+t} = w_n$ for $n \geq n_0$, then for any $n \geq 1$ choose j with $n + jP \geq n_0$ and get $w_{n+t} = w_{n+t+jP} = w_{n+jP} = w_n$, so t is a period; thus least eventual period and least period coincide. ☐

By Lemmas 2.1 and 2.2, $T_k(F)$ is the least period of the finite word $(\Gamma(k, F_n))_{1\leq n\leq 2\pi(2k)}$, and that is the quantity the formal development computes and calls `TF` k. The same two lemmas apply to an arithmetic progression $a_n = pn - r$ once $(\Gamma(k, pn - r))_{n\geq 1}$ is known to be purely periodic with period $M = P_k/\gcd(p, P_k)$, which is Proposition 4.1(ii) below; there the prefix of length $2M$ is used.

3. The period along the Fibonacci numbers

Proposition 3.1. *For $1 \leq k \leq 10$ the triples $(k, T_k(F), \pi(2k))$ computed from the definition* (2) *are those of Table 1. Every entry agrees with* [6, Table 1].

k	1	2	3	4	5	6	7	8	9	10
$T_k(F)$	1	6	8	12	20	24	16	24	24	60
$\pi(2k)$	3	6	24	12	60	24	48	24	24	60

Table 1. The values printed in [6, Table 1], recomputed from (2) (Proposition 3.1).

This is the calibration of the method against the only data in [6]: the printed values are recovered from the raw Diophantine definition, not from the criterion the source's proofs rely on. The new values follow.

Theorem 3.2 (ninety new values)**.** *For $11 \leq k \leq 100$, $T_k(F)$ is as in Table 2.*

Theorem 3.3 (the closed form, $k \leq 200$)**.** (i) *For every $1 \leq k \leq 200$, $T_k(F) = \pi(P_k)$; explicitly, $T_k(F) = \pi(k)$ if k is odd and $T_k(F) = \pi(2k)$ if k is even.*
(ii) *For every odd $k \leq 399$, $\pi(2k) \neq 2\pi(k)$.*

Part (i) is checked in two forms, once through P_k and once spelled out by parity, so that no unfolding of (3) is left to the reader. Part (ii) is a statement about Pisano periods alone; it is the reason the two cases of (4) cannot be merged into "$\pi(2k)$ or $2\pi(k)$": for odd k the ratio $\pi(2k)/\pi(k)$ is never 2 (Remark 3.8 explains why it is 1 or 3).

Conjecture 3.4. *$T_k((F_n)_{n=1}^{\infty}) = \pi(P_k)$ for every $k \in \mathbb{N}$.*

Remark 3.5 (independent computation to $k \leq 2000$). Outside the formal development, (4) was recomputed in Python for every $1 \leq k \leq 2000$ with no failure, by two independently written implementations. Both use two shortcuts the formal development avoids: $\Gamma(k, b)$ is evaluated only for $1 \leq b \leq 2k$, using $\Gamma(k, F_n) = \Gamma(k, F_n \bmod 2k)$ (with $2k$ in place of the residue 0), which rests

k	11	12	13	14	15	16	17	18	19	20
$T_k(F)$	10	24	28	48	40	48*	36	24	18	60
k	21	22	23	24	25	26	27	28	29	30
$T_k(F)$	16	30	48	24	100	84	72	48	14	120
k	31	32	33	34	35	36	37	38	39	40
$T_k(F)$	30	96*	40	36	80	24	76	18	56	120*
k	41	42	43	44	45	46	47	48	49	50
$T_k(F)$	40	48	88	60*	120	48	32	48*	112	300
k	51	52	53	54	55	56	57	58	59	60
$T_k(F)$	72	84	108	72	20	48	72	42	58	120
k	61	62	63	64	65	66	67	68	69	70
$T_k(F)$	60	30	48	192*	140	120	136	36	48	240
k	71	72	73	74	75	76	77	78	79	80
$T_k(F)$	70	24	148	228	200	36*	80	168	78	240*
k	81	82	83	84	85	86	87	88	89	90
$T_k(F)$	216	120	168	48	180	264	56	120*	44	120
k	91	92	93	94	95	96	97	98	99	100
$T_k(F)$	112	48	120	96	180	96*	196	336	120	300

TABLE 2. $T_k(F) = T_k((F_n)_{n=1}^{\infty})$ for $11 \le k \le 100$ (Theorem 3.2). A star marks the values with $T_k(F) = 2\pi(k)$; all other entries equal $\pi(k)$ (Theorem 3.7(ii)).

on the P_k-periodicity of $\Gamma(k, \cdot)$ from [6, Theorem 2.8] and $P_k \mid 2k$; and solvability of $i + Ax + By = R$ is decided by the single residue $y_0 \equiv (R - i)B^{-1} \pmod{A}$, checking $By_0 \le R - i$. A third implementation that uses neither shortcut, the bounded search of §1.4 on the full Fibonacci numbers, agrees with them for every $k \le 300$. The same runs give the counts quoted in Theorem 3.7: $\pi(P_k) < \pi(2k)$ for 516 of the first 2000 values of k, and the doubling set of Theorem 3.7(ii) agrees with its 2-adic description for all $k \le 2000$.

Remark 3.6 (why it is not yet a theorem). Granting [6, Theorem 2.8], $\Gamma(k, \cdot)$ is P_k-periodic on $\mathbb{N}$, so $\Gamma(k, F_n)$ depends only on $F_n \bmod P_k$ and $T_k(F)$ divides $\pi(P_k)$. This already sharpens [6, Corollary 2.16] from $\pi(2k)$ to $\pi(P_k)$. Equality is the open half. For $a_n = n$ the source proves it by counting: each period of $(\Gamma(k, n))$ has one more 0 than 1 (odd k), which is incompatible with a proper divisor of P_k being a period. Along (F_n) the map $n \mapsto F_n \bmod P_k$ is not a bijection onto $\mathbb{Z}/P_k$, so that count does not transfer, and we do not know how to show that no proper divisor of $\pi(P_k)$ is a period. The ingredients one would expect to use are [6, Remark 2.9] ($\Gamma(k, s) \ne \Gamma(k, k-s)$ for odd k, $\Gamma(k, s) \ne \Gamma(k, 2k-s)$ for even k) and the structure of the orbit of (F_n, F_{n+1}) modulo P_k.

Theorem 3.7 (sharpness, and the doubling set)**.** (i) *Among* $1 \le k \le 200$ *there are exactly* 60 *values of* k *with* $\pi(P_k) < \pi(2k)$. *Beyond the trivial* $k = 1$ *(the constant word,* $1 < 3$*), the smallest is* $k = 3$*:* $\pi(P_3) = \pi(3) = 8$ *while* $\pi(6) = 24$. *Consequently, for these* 60 *values Theorem 3.3 is strictly stronger than the divisibility* $T_k(F) \mid \pi(2k)$ *of* [6, Corollary 2.16].

(ii) *For* $1 \le k \le 200$ *write* $k = 2^a k'$ *with* k' *odd. Then*

$$T_k(F) = \begin{cases} 2\pi(k) & \text{if } (a = 1 \text{ and } k' = 1) \text{ or } (a \ge 2 \text{ and } v_2(\pi(k')) < a), \\ \pi(k) & \text{otherwise.} \end{cases}$$

Part (ii) expresses the closed form through k alone, without $\pi(2k)$ or P_k, and it is the form that shows the doubling set is not what the printed table suggests. In [6, Table 1] the values with $T_k(F) \ne \pi(k)$ are $k = 2, 4, 8$ only, which invites the reading "$T_k(F) = 2\pi(k)$ exactly when

k is a power of two". That reading is false: $k = 40 = 2^3 \cdot 5$ has $v_2(\pi(5)) = v_2(20) = 2 < 3$, so $T_{40}(F) = 120 = 2\pi(40)$, and no $k < 40$ outside $\{2, 4, 8, 16, 32\}$ doubles (Proposition 5.5(iii)). The doubling set begins

$$2,\ 4,\ 8,\ 16,\ 32,\ 40,\ 44,\ 48,\ 64,\ 76,\ 80,\ 88,\ 96,\ 104,\ 116,\ 124,\ 128,$$
$$136,\ 144,\ 152,\ 160,\ 176,\ 192,\ 200,\ 208,\ 224,\ 232,\ 236,\ 240,\ 248,\ 256,\ \ldots$$

(the first 17 terms are those with $k \le 128$ in Theorem 3.7(ii); the rest are from the computation of Remark 3.5).

Remark 3.8 (Pisano arithmetic behind Theorem 3.7). Two classical facts about Pisano periods explain the shape of both parts, though neither is used in the formal development: $\pi(mn) = \operatorname{lcm}(\pi(m), \pi(n))$ for coprime m, n (by the Chinese remainder theorem; in the form $\pi(\prod p_i^{e_i}) = \operatorname{lcm}_i \pi(p_i^{e_i})$ it is in the formula section of [8]), and $\pi(2^a) = 3 \cdot 2^{a-1}$ for $a \ge 1$ [6, 8]; the classical study of $(F_n \bmod m)$ is [9]. For odd k they give $\pi(2k) = \operatorname{lcm}(3, \pi(k)) \in \{\pi(k), 3\pi(k)\}$, which is Theorem 3.3(ii), and $\pi(P_k) = \pi(k) < \pi(2k)$ exactly when $3 \nmid \pi(k)$; for even k, $\pi(P_k) = \pi(2k)$ trivially. So the 60 values of part (i) are the odd $k \le 200$ with $3 \nmid \pi(k)$, beginning $1, 3, 5, 7, 11, 13, 15, 21, 25, 29, \ldots$, and the sharpening is by a factor of exactly 3 at each of them. For even $k = 2^a k'$, $\pi(k) = \operatorname{lcm}(3 \cdot 2^{a-1}, \pi(k'))$ and $\pi(2k) = \operatorname{lcm}(3 \cdot 2^a, \pi(k'))$, so $\pi(2k) = 2\pi(k)$ exactly when $a > v_2(\pi(k'))$; since $\pi(m)$ is even for $m \ge 3$ (taking determinants in $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)^{\pi(m)} \equiv I \pmod m$ gives $(-1)^{\pi(m)} \equiv 1$), for $a = 1$ this reduces to $k' = 1$, which is the case split in part (ii). Hence, conditional on Conjecture 3.4, the description of the doubling set in part (ii) holds for every k, and (4) can equivalently be read as "$T_k(F) = \pi(2k)$ when $\pi(2k) = 2\pi(k)$, and $\pi(k)$ otherwise".

4. Arithmetic progressions

For $p \in \mathbb{N}$ and $0 \le r < p$ let $a_n = pn - r$, and write $T_k((pn - r))$ for $T_k((a_n)_{n=1}^{\infty})$; note $a_n \ge 1$ for all $n \ge 1$. [6, Theorem 2.8] gives $T_k(\mathbb{N}) = P_k$, and its Propositions 2.10 and 2.11 treat odd k coprime to p and $k = 2^j$, $p = 2$, $r = 1$. Two further statements follow from Theorem 2.8 alone.

Proposition 4.1. *Assume* [6, Theorem 2.8]*, in the form that* $\Gamma(k, m + P_k) = \Gamma(k, m)$ *for all* $m \ge 1$*.*

- (i) *If* $\gcd(k, p) = 1$ *then* $T_k((pn - r)) = P_k$ *for every* $k \in \mathbb{N}$ *and every* $0 \le r < p$*; that is, the hypothesis "k odd" in* [6, Proposition 2.10] *can be dropped, the value being k for odd k and $2k$ for even k.*
- (ii) *For all* $k, p \in \mathbb{N}$ *and* $0 \le r < p$*,* $(\Gamma(k, pn - r))_{n \ge 1}$ *is purely periodic with period* $M = P_k / \gcd(p, P_k)$*; hence* $T_k((pn - r))$ *divides* M*.*

Both statements are machine-checked, independently of Theorem 2.8, on the grid $1 \le k \le 40$*,* $1 \le p \le 24$*,* $0 \le r < p$ *(12,000 triples).*

Proof. (ii) $pM = \operatorname{lcm}(p, P_k)$ is a multiple of P_k, so $\Gamma(k, p(n + M) - r) = \Gamma(k, (pn - r) + pM) = \Gamma(k, pn - r)$. (i) If $\gcd(k, p) = 1$ then $\gcd(P_k, p) = 1$ as well (for even k, p is odd), so $n \mapsto pn - r$ induces a bijection of $\mathbb{Z}/P_k$. Let T be a period of $g(n) = \Gamma(k, pn - r)$. Then $\Gamma(k, m + pT) = \Gamma(k, m)$ for every $m \ge 1$: choose n with $pn - r \equiv m \pmod{P_k}$ and use P_k-periodicity. So pT is a period of $\Gamma(k, \cdot)$ on $\mathbb{N}$, whose least period is P_k by Theorem 2.8; thus $P_k \mid pT$, and $\gcd(p, P_k) = 1$ gives $P_k \mid T$. With (ii), $M = P_k$ is a period, so $T_k((pn - r)) = P_k$. ☐

The machine check of (i) and (ii) is a check of the periods of the prefixes of length $2M$, which by Lemma 2.2 are the eventual periods once (ii) is granted; the formal statements are those of Appendix A. Neither statement is deep, and we do not claim them as new theorems: they are recorded because [6] states only the odd half of (i), and because (ii) is the analogue for progressions of the bound $T_k(F) \mid \pi(2k)$ that [6, Theorem 2.14] provides for recurrence sequences.

What is not routine is the range $\gcd(k, p) > 1$, which is where [6, Problem 2.12] lives. There the period genuinely depends on r and can fall strictly below M.

Theorem 4.2 (what an answer to Problem 2.12 cannot be)**.** (i) $\Gamma(15, 5n-1) = 1$ *for* $1 \le n \le 40$*, and* $T_{15}((5n-1)) = 1$*, while* $k = 15$*,* $k/\gcd(k,p) = 3$ *and* $M = P_{15}/\gcd(5, P_{15}) = 3$*.*

(ii) $T_6((2n)) = 3$ *and* $T_6((2n-1)) = 6$*.*

By (i), in the range $\gcd(k,p) > 1$ the value of $T_k((pn-r))$ is in general none of k, $k/\gcd(k,p)$, or the divisibility bound M of Proposition 4.1(ii). By (ii), no formula in (k,p) alone can answer [6, Problem 2.12]: the residue r matters. In (i) the word is constant on the whole of $\mathbb{N}$, not only for $n \le 40$, since it is purely 3-periodic by Proposition 4.1(ii).

Remark 4.3 (a computation outside the formal development). On the 12,000-triple grid of Proposition 4.1, the ratio $M/T_k((pn-r))$ takes only the values $1, 2, 3, 6$, with frequencies 11544, 416, 38, 2 respectively, and by Proposition 4.1(i) every triple with ratio > 1 has $\gcd(k,p) > 1$. A criterion in terms of (k,p,r) for the ratio was not found and is not claimed. (The four frequencies were obtained by two independently written programs.)

5. Verification of published material, and controls

The statements of this section are not new. They are machine-checked instances of published theorems and data, and they exist so that the computations of Sections 3–4 rest on the definition (2) and on nothing that is merely cited.

Proposition 5.1 (faithfulness)**.** (i) *For every* $1 \le k \le 120$ *and every* n *with* $1 \le n \le 2\pi(2k)$*, exactly one of the two equations* $i + \frac{k}{g}x + \frac{F_n}{g}y = \frac{(k/g-1)(F_n/g-1)}{2}$*,* $i \in \{0,1\}$*,* $g = \gcd(k, F_n)$*, has a nonnegative integral solution. (These are the* 31,998 *pairs* (k, F_n) *that Proposition 3.1 and Theorem 3.2 touch, and more.)*

(ii) *For every* $1 \le k \le 200$*, the computed value* $T_k(F)$ *divides* $\pi(2k)$*, in agreement with* [6, Corollary 2.16].

Part (i) is the Beiter–Chu dichotomy [2, 4] on the pairs used: without it, reading Γ off the first equation alone, as (2) does, would not compute the function of [6]. For $121 \le k \le 200$, the range of Theorem 3.3 beyond that of part (i), faithfulness rests on the published dichotomy. Part (ii) is a consistency check, not an independent proof of the corollary, since the window of Lemma 2.2 is chosen using it.

Proposition 5.2 (calibration)**.** (i) $\Gamma(a, 2a) = 0$ *for* $1 \le a \le 200$ *and* $\Gamma(a, 2a-1) = 1$ *for* $2 \le a \le 200$ *(*[6, Proposition 2.5]*, quoted there from* [1, Corollary 2.1 of v1]*).*

(ii) $\Gamma(2^j, 2^j-1) = 0 = \Gamma(2^j, 1)$ *for* $1 \le j \le 12$ *(display (2.1) in the proof of* [6, Proposition 2.11]*).*

(iii) $\pi(m)$ *for* $1 \le m \le 20$ *is* $1, 3, 8, 6, 20, 24, 16, 12, 24, 60, 10, 24, 28, 48, 40, 24, 36, 24, 18, 60$*, in agreement with OEIS A001175* [8].

Proposition 5.3 (Theorem 2.8 of [6], $k \le 60$)**.** *For* $1 \le k \le 60$ *the least period of* $(\Gamma(k,n))_{1 \le n \le 2P_k}$ *is* P_k*, and among* $\Gamma(k,1), \dots, \Gamma(k, P_k)$ *the number of* 0*'s exceeds the number of* 1*'s by one if* k *is odd and by two if* k *is even.*

Proposition 5.4 (Propositions 2.10 and 2.11 of [6])**.** (i) *For odd* $k \le 25$*, every* $p \le 15$ *with* $\gcd(k,p) = 1$ *and every* $0 \le r < p$*,* $T_k((pn-r)) = k$*.*

(ii) *For* $1 \le j \le 8$*,* $T_{2^j}((2n-1)) = 2^j$*, and exactly* 2^{j-1} *of* $\Gamma(2^j, 1), \Gamma(2^j, 3), \dots, \Gamma(2^j, 2^{j+1}-1)$ *are* 0*.*

Proposition 5.5 (negative controls)**.** *Each of the following is machine-checked.*

(i) *(Not always* $\pi(2k)$*.)* $T_3(F) = 8 \ne 24 = \pi(6)$*; so* [6, Corollary 2.16] *cannot be strengthened to equality.*

(ii) *(Not always* $\pi(k)$*.)* $T_2(F) = 6 \ne 3 = \pi(2)$*; the doubling case is real.*

(iii) *(Not the powers of two.)* $T_{40}(F) = 2\pi(40)$*,* 40 *is not a power of two, and for* $1 \le k \le 39$ *one has* $T_k(F) = 2\pi(k)$ *exactly when* $k \in \{2, 4, 8, 16, 32\}$*.*

(iv) *(The period detector is sharp.) For* $k = 11$*, neither* 5 *nor* 9 *is a period of the prefix of length* $2\pi(22)$*, while* 10 *is; for* $k = 25$*,* $T_{25}(F) = 100$ *and the proper divisor* 50 *is not a period.*

(v) *(Window stability.) For* $k = 11$ *the least period of the prefix is* 10 *at two, three and four Pisano periods* $\pi(22)$ *alike.*

(vi) *(Non-degeneracy.) The word* $(\Gamma(7, F_n))$ *takes both values* 0 *and* 1*; for the pair* $(15, 4)$ *the first equation of* (1) *is unsolvable and the second solvable;* $\Gamma(2, 3) = 1$*; and the period detector returns* 1 *on the constant word* $(\Gamma(1, F_n))$*, so* $T_1(F) = 1$*.*

(vii) *(Both branches.)* $\pi(4) = 2\pi(2)$ *and* $\pi(6) \neq 2\pi(3)$*, so both cases of* (4) *occur inside the checked range.*

(viii) *(Hypothesis not vacuous, not removable.)* $\gcd(15, 5) = 5$*,* $T_{15}((5n-1)) = 1$ *and* $P_{15} = 15$*, so the coprimality hypothesis of Proposition 4.1(i) is needed.*

Every claim of this paper is a period, and a period claim is the kind that a misplaced quantifier satisfies for free: a detector that always returned the window length, or always 1, would make Theorem 3.2 look like a theorem. The controls refute each such degeneration inside the same formal development.

6. Verification

Every numbered statement of Sections 3–5 except Lemmas 2.1–2.2, Conjecture 3.4, the proof of Proposition 4.1 and the remarks is a theorem of a Lean 4 development (Lean `v4.32.0`) of four modules and about 540 lines, which imports no library: Mathlib is not used. Γ is defined on Lean's natural numbers by the bounded search of §1.4, the Pisano period by scanning for the first return of $(F_i, F_{i+1}) \equiv (0, 1)$, and the least period of a finite word by trying $T = 1, 2, \dots$ in turn; the relevant definitions are reproduced below. Each of the 28 theorems is closed by `native_decide`, that is, by running the compiled decision procedure; `#print axioms` shows every theorem depending only on the evaluation axioms that `native_decide` generates for that theorem (`<name>._native.native_decide.ax_1_j`) and on nothing else: no `sorryAx`, no `propext`, no `Classical.choice`, no `Quot.sound`. What is trusted is therefore the Lean compiler and runtime on these definitions. The whole development checks in well under four CPU-minutes (the slowest measured run took about 3.6 CPU-minutes from cold, the fastest about 45 CPU-seconds) with a peak resident set of about 1.6 GB. The exact statement of every theorem, in the order of the modules, is in Appendix A. Repository reference to be added at submission.

```
def rhs (A B : Nat) : Nat := (A - 1) * (B - 1) / 2

def solvable (A B i : Nat) : Bool :=
  let R := rhs A B
  if R < i then false
  else
    let R' := R - i
    (List.range (R' / B + 1)).any fun y =>
      decide (B * y ≤ R') && ((R' - B * y) % A == 0)

def Gam (a b : Nat) : Nat :=
  let g := Nat.gcd a b
  if solvable (a / g) (b / g) 0 then 0 else 1

def exclusive (a b : Nat) : Bool :=
  let g := Nat.gcd a b
  solvable (a / g) (b / g) 0 != solvable (a / g) (b / g) 1
```

```
def Pk (k : Nat) : Nat := if k % 2 == 1 then k else 2 * k

def win (k : Nat) : Nat := 2 * pisano (2 * k)
def TF (k : Nat) : Nat := TfibOn k (win k)

def TAP (k p r : Nat) : Nat :=
  let M := Pk k / Nat.gcd p (Pk k)
  minPeriod (gwordAP k p r (2 * M))
def TNat (k : Nat) : Nat := TAP k 1 0

def doubles (k : Nat) : Bool :=
  let a := v2 k
  let k' := oddPart k
  if a == 0 then false
  else if a == 1 then k' == 1
  else decide (v2 (pisano k') < a)
def Tpred (k : Nat) : Nat := if doubles k then 2 * pisano k else pisano k
```

Appendix A. The formal statements

The 28 theorems, verbatim from the four modules (`Basic`, `Table`, `Arith`, `Controls`), each with the attribute that binds it to the claim it verifies. `List.range' a n` is the list $a, a+1, \ldots, a+n-1$; `fibPrefix m` is $F_1, \ldots, F_m$; `gword k L` is the word $(\Gamma(k,b))_{b\in L}$; `TfibOn k m` is the least period of $(\Gamma(k,F_n))_{n\le m}$; `periodOk L T` decides whether T is a period of the finite word L; `zerosPerPeriod` and `onesPerPeriod` count $\Gamma(k,n)=0$, resp. 1, over $1 \le n \le P_k$.

```
@[dossier "gamma-fib-period" "GFP-06"]
theorem prop_2_5_upto_200 :
    ((List.range' 1 200).all fun a => Gam a (2 * a) == 0) = true ∧
    ((List.range' 2 199).all fun a => Gam a (2 * a - 1) == 1) = true := by
  constructor <;> native_decide

@[dossier "gamma-fib-period" "GFP-06"]
theorem source_er1 :
    ((List.range' 1 12).all fun j => Gam (2 ^ j) (2 ^ j - 1) == 0 && Gam (2 ^ j) 1 == 0) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-06"]
theorem pisano_a001175 :
    (List.range' 1 20).map pisano
      = [1, 3, 8, 6, 20, 24, 16, 12, 24, 60, 10, 24, 28, 48, 40, 24, 36, 24, 18, 60] := by
  native_decide

@[dossier "gamma-fib-period" "GFP-01"]
theorem source_table_1 :
    (List.range' 1 10).map (fun k => (k, TF k, pisano (2 * k)))
      = [(1, 1, 3), (2, 6, 6), (3, 8, 24), (4, 12, 12), (5, 20, 60),
         (6, 24, 24), (7, 16, 48), (8, 24, 24), (9, 24, 24), (10, 60, 60)] := by
  native_decide

@[dossier "gamma-fib-period" "GFP-02"]
theorem extension_11_to_100 :
    (List.range' 11 90).map TF
      = [10, 24, 28, 48, 40, 48, 36, 24, 18, 60, 16, 30, 48, 24, 100,
         84, 72, 48, 14, 120, 30, 96, 40, 36, 80, 24, 76, 18, 56, 120,
```

```
        40, 48, 88, 60, 120, 48, 32, 48, 112, 300, 72, 84, 108, 72, 20,
        48, 72, 42, 58, 120, 60, 30, 48, 192, 140, 120, 136, 36, 48, 240,
        70, 24, 148, 228, 200, 36, 80, 168, 78, 240, 216, 120, 168, 48, 180,
        264, 56, 120, 44, 120, 112, 48, 120, 96, 180, 96, 196, 336, 120, 300] := by
  native_decide

@[dossier "gamma-fib-period" "GFP-03"]
theorem closed_form_le_200 : ((List.range' 1 200).all fun k => TF k == pisano (Pk k)) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-03"]
theorem closed_form_by_parity_le_200 :
    ((List.range' 1 200).all fun k =>
      TF k == (if k % 2 == 1 then pisano k else pisano (2 * k))) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-04"]
theorem sharper_than_source :
    ((List.range' 1 200).filter fun k => decide (pisano (Pk k) < pisano (2 * k))).length = 60 ∧
      pisano (Pk 3) = 8 ∧ pisano 6 = 24 := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-04"]
theorem two_adic_le_200 : ((List.range' 1 200).all fun k => TF k == Tpred k) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-03"]
theorem odd_ratio_never_two :
    ((List.range' 1 200).all fun j => pisano (2 * (2 * j - 1)) != 2 * pisano (2 * j - 1)) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-05"]
theorem exclusive_le_120 :
    ((List.range' 1 120).all fun k =>
      (fibPrefix (win k)).all fun b => exclusive k b) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-05"]
theorem divides_pisano_two_k_le_200 :
    ((List.range' 1 200).all fun k => pisano (2 * k) % TF k == 0) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-07"]
theorem theorem_2_8_le_60 :
    ((List.range' 1 60).all fun k =>
      (TNat k == Pk k) &&
      (zerosPerPeriod k == onesPerPeriod k + (if k % 2 == 1 then 1 else 2))) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-08"]
theorem prop_2_10_grid :
    ((List.range' 1 13).all fun j =>
      let k := 2 * j - 1
      (List.range' 1 15).all fun p =>
        Nat.gcd k p != 1 || (List.range p).all fun r => TAP k p r == k) = true := by
  native_decide
```

```
@[dossier "gamma-fib-period" "GFP-08"]
theorem prop_2_11_upto_8 :
    ((List.range' 1 8).all fun j => TAP (2 ^ j) 2 1 == 2 ^ j) = true ∧
    ((List.range' 1 8).all fun j =>
      (((List.range' 1 (2 ^ j)).filter fun n => Gam (2 ^ j) (2 * n - 1) == 0).length
        == 2 ^ (j - 1))) = true := by
  constructor <;> native_decide

@[dossier "gamma-fib-period" "GFP-09"]
theorem coprime_gives_Pk :
    ((List.range' 1 40).all fun k =>
      (List.range' 1 24).all fun p =>
        Nat.gcd k p != 1 || (List.range p).all fun r => TAP k p r == Pk k) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-09"]
theorem ap_divides_bound :
    ((List.range' 1 40).all fun k =>
      (List.range' 1 24).all fun p =>
        (List.range p).all fun r =>
          (Pk k / Nat.gcd p (Pk k)) % TAP k p r == 0) = true := by
  native_decide

@[dossier "gamma-fib-period" "GFP-10"]
theorem prop_2_10_needs_coprimality :
    TAP 15 5 1 = 1 ∧ Pk 15 / Nat.gcd 5 (Pk 15) = 3 ∧
    ((List.range' 1 40).all fun n => Gam 15 (5 * n - 1) == 1) = true := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-10"]
theorem ap_depends_on_r : TAP 6 2 0 = 3 ∧ TAP 6 2 1 = 6 := by
  refine ⟨by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-11"]
theorem control_not_always_pi2k : TF 3 = 8 ∧ pisano 6 = 24 ∧ TF 3 ≠ pisano 6 := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-11"]
theorem control_not_always_pik : TF 2 = 6 ∧ pisano 2 = 3 ∧ TF 2 ≠ pisano 2 := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-11"]
theorem control_powers_of_two_is_wrong :
    TF 40 = 2 * pisano 40 ∧ ((List.range' 1 6).all fun j => 2 ^ j != 40) = true ∧
    ((List.range' 1 39).all fun k =>
      (TF k == 2 * pisano k) == ((List.range' 1 6).any fun j => 2 ^ j == k)) = true := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-11"]
theorem control_detector_sharp_11 :
    periodOk (gword 11 (fibPrefix (win 11))) 5 = false ∧
    periodOk (gword 11 (fibPrefix (win 11))) 9 = false ∧
    periodOk (gword 11 (fibPrefix (win 11))) 10 = true := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩

@[dossier "gamma-fib-period" "GFP-11"]
theorem control_proper_divisor_25 :
```

```
    TF 25 = 100 ∧ periodOk (gword 25 (fibPrefix (win 25))) 50 = false := by
  refine ⟨by native_decide, by native_decide⟩
@[dossier "gamma-fib-period" "GFP-11"]
theorem control_window_stable :
    TfibOn 11 (3 * pisano 22) = 10 ∧ TfibOn 11 (4 * pisano 22) = 10 ∧ TF 11 = 10 := by
  refine ⟨by native_decide, by native_decide, by native_decide⟩
@[dossier "gamma-fib-period" "GFP-11"]
theorem control_nondegenerate :
    ((gword 7 (fibPrefix (win 7))).toList.any (· == 0)) = true ∧
    ((gword 7 (fibPrefix (win 7))).toList.any (· == 1)) = true ∧
    solvable 15 4 0 = false ∧ solvable 15 4 1 = true ∧ Gam 2 3 = 1 ∧ TF 1 = 1 := by
  refine ⟨by native_decide, by native_decide, by native_decide, by native_decide,
          by native_decide, by native_decide⟩
@[dossier "gamma-fib-period" "GFP-11"]
theorem control_both_branches :
    (pisano 4 == 2 * pisano 2) = true ∧ (pisano 6 == 2 * pisano 3) = false := by
  refine ⟨by native_decide, by native_decide⟩
@[dossier "gamma-fib-period" "GFP-11"]
theorem control_coprimality_not_vacuous :
    Nat.gcd 15 5 = 5 ∧ TAP 15 5 1 = 1 ∧ Pk 15 = 15 ∧ TAP 15 5 1 ≠ Pk 15 := by
  refine ⟨by native_decide, by native_decide, by native_decide, by native_decide⟩
```

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