SIGNED AND UNSIGNED GALLAI HOMOTHETY NUMBERS OF THREE-POINT SETS

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ABSTRACT. For a finite set $S \subset \mathbb{Z}$ and an integer $r \geq 1$, the *Gallai homothety number* $G_r(S)$ is the least N such that every r-colouring of N consecutive integers contains a monochromatic homothet $b + kS$ with $k \geq 1$ lying inside them. Letting the scale k range over all nonzero integers instead defines a second threshold $G_r^{\pm}(S)$, which for a three-point set $S = \{0, s, s+u\}$ is the Rado number of $ux+sz = (s+u)y$ in the convention that asks for solutions with distinct coordinates. We compute $G_3^{\pm}(\{0,2,5\}) = 48$, $G_3^{\pm}(\{0,3,7\}) \geq 70$ and $G_3^{\pm}(\{0,1,7\}) \geq 69$; the last of these is one row past the end of the published table for this family of equations, and an exhaustive search puts its value at 69. We have found no value of the signed threshold at three colours published for any three-point set other than $\{0,1,m\}$ with $m \leq 6$. We prove that the threshold $G_3^{\pm}(\{0,1,6\}) = 60$ has at least two extremal colourings that no permutation of the colours identifies, and that the window reflection carries one to the other with no relabelling at all: a statement about the extremal objects, which a refutation certificate cannot supply. The exhaustive halves proved here are discharged in one of two ways: by a refutation certificate re-checked inside the proof, or by a complete backtracking search that is proved sound in the one direction a refutation needs and stores nothing. The search closes two published cells the certificate route had priced out, $G_3^{\pm}(\{0,1,6\}) = 60$ and $G_3(\{0,1,5\}) = 70$, whose refutation certificates are 137 and 298 megabytes. We prove $G_3^{\pm}(\{0,2,5\}) < G_3(\{0,2,5\})$ as a comparison of the two thresholds rather than of two numerals, with a gap of at least 29. We give an elementary two-colouring proof of the uniform lower bound $G_2^{\pm}(\{0,s,s+u\}) \geq 4(s+u)+1$ for coprime s, u with $4 \nmid s$ and $4 \nmid u$, and we record exactly where the two readings separate at two colours: by one on the family $\{s,u\} = \{1,4m\}$ and by two at the sporadic pair $\{s,u\} = \{3,4\}$. Finally we correct the record. The two-colour signed values are a closed form published by Gupta, Thulasi Rangan and Tripathi in 2015; the recent literature that tabulates four of them, and that observes the drop at $\{1,4\}$ empirically, does not cite that theorem. Every theorem and proposition below is machine-checked in Lean 4; the numbers reported as measurements are labelled where they occur.

1. Introduction

1.1. **The object.** Let $S \subset \mathbb{Z}$ be finite. A *homothet* of S is a set $b + kS = \{b + ks : s \in S\}$ with $b \in \mathbb{Z}$ and $k \geq 1$. For $r \geq 1$, the *Gallai homothety number* $G_r(S)$ is the least N such that every colouring of N consecutive integers with r colours contains a monochromatic homothet of S lying inside them. Finiteness of $G_r(S)$ is the one-dimensional case of Gallai's theorem, proved by Gallai and first published by Rado, and independently by Witt; see [6] for a proof.

The number contains van der Waerden's as a special case. Homothets of an initial segment $\{0, 1, \ldots, t-1\}$ are exactly the t-term arithmetic progressions, so

$$G_r(\{0,1,\ldots,t-1\}) \;=\; W(t,r), \tag{1}$$

and the interesting cells of the problem are the ones where S is *not* an initial segment. Mouhib [13, 14] writes $G_r(S)$ for this quantity, normalises $\min S = 0$ throughout, and uses the numbers to drive lower bounds for Hales–Jewett numbers; the tables of [14] collect the two-colour values of eighteen four-point sets and a three-colour landscape for three-point sets. Both papers are by the same author, and [14] carries the comment *"supersedes v1 and arXiv:2607.02226"*, that is, it supersedes [13].

1.2. **The second reading.** The condition $k \geq 1$ is a choice, and it is not the only one in the literature. Allowing the scale to be any *nonzero* integer gives a different object. Write

$$G_r^{\pm}(S) \;=\; \begin{array}{l}\text{the least } N \text{ such that every } r\text{-colouring of } N \text{ consecutive integers con-}\\ \text{tains a monochromatic set } a+dS \text{ with } d\neq 0 \text{ lying inside them.}\end{array}$$

Since $a+dS$ and $a'+d'S$ can be equal as sets, $G_r^{\pm}(S)$ is really about *two* shapes at once: for $\min S = 0$ and $m = \max S$, avoiding every $a + dS$ with $d \neq 0$ is the same as avoiding every positive-scale homothet of S *and* every positive-scale homothet of the reflected pattern $m - S$ (Proposition 2.5). Consequently $G_r^{\pm}(S) \leq G_r(S)$ always, with equality whenever S is symmetric — in particular at every van der Waerden cell. The two-shape description and the inequality $G_r^{\pm}(S) \leq G_r(S)$ are [14, Proposition 26], there stated for a t-point set; what is added here is a proof from the definitions, with the normalisation $\min S = 0$, $\max S = m$ carried as a hypothesis rather than assumed.

The second reading is the one a *Rado number* computes. For $S = \{0, s, s+u\}$ the affine dependencies of $(0, s, s+u)$ are generated by $(u, -(s+u), s)$, so the integer solutions of

$$u\,x + s\,z \;=\; (s+u)\,y \tag{2}$$

are exactly the triples $(a, a+ds, a+d(s+u))$, and those with pairwise distinct coordinates are the ones with $d \neq 0$. Hence

$$G_r^{\pm}(\{0, s, s+u\}) \;=\; R_r\big(u\,x + s\,z = (s+u)\,y\big) \tag{3}$$

where R_r is the Rado number in the convention that requires a monochromatic solution with x, y, z distinct. Renaming variables, (2) is $ax + by = (a+b)z$ with $(a, b) = (s, u)$; the line $s = 1$, $u = k$ is the equation $z + kx = (k+1)y$ tabulated in [14, Table 5].

1.3. **What is open, and what is settled here.** Two frontiers meet in this problem. On the unsigned side, [14] closes $G_3(\{0,1,3\})$, $G_3(\{0,1,4\})$, $G_3(\{0,1,5\})$ and $G_3(\{0,2,5\})$ and leaves three rows of its three-colour table as lower bounds only — $G_3(\{0,1,6\}) \geq 82$, $G_3(\{0,1,7\}) \geq 87$, $G_3(\{0,2,7\}) \geq 87$ — saying that "their refutations lie beyond the present compute budget". On the signed side the published record is thinner than it looks: it consists of the full two-colour closed form for three-point sets, and of [14, Table 5], which is the single line $s = 1$, $u \leq 5$, at $r \leq 4$. We have found nothing published for $r \geq 3$ outside that line — neither for $s \geq 2$ nor for $s = 1$ with $u \geq 6$; the searches behind that statement are listed in Section 5.3.

This paper settles the following.

(A) $G_3^{\pm}(\{0,1,7\}) \geq 69$ (Theorem 6.9), with the value 69 settled by an exhaustive search that is reported here as a measurement and not as a theorem (Remark 6.10). In the notation of (3) this is the equation $z + 6x = 7y$, that is, $k = 6$ in the family tabulated in [14, Table 5], which stops at $k = 5$; it is off the two-colour closed form as well, so it lies off both published lines. We also prove $G_3^{\pm}(\{0,1,7\}) > G_3^{\pm}(\{0,1,6\}) = 60$ as a relation between thresholds: whatever the value at $k = 6$, it exceeds the last tabulated value at $k = 5$.

(B) At the threshold $G_3^{\pm}(\{0,1,6\}) = 60$ there are at least two extremal colourings, no permutation of the colours identifying them, and the window reflection carries one to the other with no relabelling at all (Theorem 6.7). This is a statement about the extremal objects rather than about the threshold, and it is available here because the exhaustive step is an enumeration and not a refutation certificate. Nothing of the kind is claimed in [14], [7], [1] or [8].

(C) $G_3^{\pm}(\{0,2,5\}) = 48$, both halves, and $G_3^{\pm}(\{0,3,7\}) \geq 70$ (Theorem 5.6). These are, as far as we have been able to determine, the first values of the signed threshold at three colours off both published lines. Moreover $G_3^{\pm}(\{0,2,5\}) < G_3(\{0,2,5\})$, proved as a statement about the two thresholds, with a gap of at least 29.

(D) Where the two readings separate at two colours (Theorem 3.4): by one at $\{0,1,5\}$, by two at $\{0,3,7\}$, and not at all at $\{0,2,5\}$. Those three cells are proved here. Placing the two published closed forms side by side then says, for every coprime pair, that the signed exceptional set is strictly larger than the unsigned one, that the signed exceptional value sits two below the generic value where the unsigned one sits one below, and that the value $4(s+u)$ never occurs on the signed side.

(E) A correction of the record (Section 4). The two-colour signed values are a theorem of Gupta, Thulasi Rangan and Tripathi from 2015 [7]. The remark in [14] that "at $k=4$ the reflected shape already bites" is exactly the exceptional family $(a,b)=(1,4k)$ of that theorem, and [14] does not cite it.

(F) An elementary proof of the uniform lower bound $G_2^{\pm}(\{0,s,s+u\}) \geq 4(s+u)+1$ whenever s,u are coprime with $4 \nmid s$ and $4 \nmid u$ (Theorem 3.5). The statement is contained in [7]; the proof is two explicit periodic colourings and a finite check on residues modulo 4, where the published argument is a long case analysis.

(G) Independent reproduction, from the definition and with machine-checked proofs, of the published values listed here: the thirteen unsigned two-colour values of diameter at most 9 (Theorem 3.1), the eighteen four-point values of [14, Table 4] (Proposition 5.2), the four values of [13, Table 2] (Proposition 5.1), and the three-colour entries of [14, Table 5] (Proposition 5.5). Two further published values are proved here rather than only bounded or quoted: $G_3^{\pm}(\{0,1,6\}) = 60$ (Proposition 6.5) and $G_3(\{0,1,5\}) = 70$ (Proposition 6.6), whose refutation certificates are 136 648 164 and 298 430 865 bytes respectively.

The exhaustive step behind (B) and behind the two cells just named — and, as a measurement, behind the value in (A) — is not a refutation certificate, and Section 6 is about it: a complete backtracking search over colourings, of which only one direction is proved — that its pruning test *under*-reports, so that a branch it cuts really is dead. Under-reporting is all a refutation consumes, and it keeps the part that has to be correct down to two short lemmas rather than a program.

1.4. **Method.** Each exact value is a pair of statements of very different cost. That a window of $N-1$ integers *can* be coloured without a monochromatic copy is settled by exhibiting one colouring and checking it against the definition, which is a polynomial enumeration of the copies inside the window. That a window of N integers cannot is a statement about all r^N colourings, and it is proved here in one of two ways. The first is a refutation certificate: the avoidance condition is encoded as a propositional formula, a SAT solver refutes the formula, and the resulting LRAT certificate is re-checked inside the proof. Section 2.4 states the encoding and the exact sense in which the symmetry-breaking clauses used to make the solver's task feasible are licensed rather than assumed. The second, which Section 6 sets out, is a complete backtracking search over colourings whose pruning is proved to be one-sided; it stores nothing, and it is what reaches the cells where the certificate would be hundreds of megabytes. Both routes are formally verified; see Section 7.

2. Preliminaries

2.1. **Windows, copies, thresholds.** Throughout, a *window* is the set $[0,N) = \{0,1,\ldots,N-1\}$ and an r-colouring of it is a map $[0,N) \to \{0,\ldots,r-1\}$. Since "$N$ consecutive integers" is translation invariant, working in $[0,N)$ rather than in an arbitrary interval is a normal form and not a convention that could hide an off-by-one.

Definition 2.1. For a finite list S of non-negative integers and $b,k \geq 0$, the *copy* $b+kS$ is the list $(b+ks)_{s\in S}$. A colouring c of $[0,N)$ *avoids* S if there is no pair (b,k) with $k \geq 1$ such that every point of $b+kS$ lies in $[0,N)$ and all of them receive the same colour.

Definition 2.2. Write $F_S^r(N)$ for the assertion that some r-colouring of $[0,N)$ avoids S. Restricting an avoiding colouring to a smaller window leaves it avoiding, so F_S^r is downward closed, and

$$G_r(S) = N \quad :\Longleftrightarrow \quad F_S^r(N-1) \text{ and } \neg F_S^r(N).$$

This is the "least N that forces" convention; both halves are asserted, so a claimed value is refuted by either half alone.

The one convention that does carry a risk is the range of the base. The definition in [13, 14] lets b run over $\mathbb{Z}$; Definition 2.1 lets it run over $\mathbb{N}$.

Proposition 2.3. *Let S be a finite set of non-negative integers with $0 \in S$. Then a colouring of $[0, N)$ avoids every copy $b + kS$ with $b \in \mathbb{N}$, $k \geq 1$ inside the window if and only if it avoids every copy with $b \in \mathbb{Z}$, $k \geq 1$ inside the window. The hypothesis $0 \in S$ cannot be dropped: for $S = \{1\}$ and $N = 1$, the integer base $b = -1$ and scale $k = 1$ produce the copy $\{0\}$, which lies in the window, while no base in $\mathbb{N}$ does.*

Proof sketch. One direction is trivial. For the other, if $0 \in S$ then $b = b + k \cdot 0$ is itself a point of the copy, so a copy inside $[0, N)$ has $b \geq 0$ automatically. The counterexample is checked by inspection. ☐

Every set below satisfies $0 \in S$, which is the normalisation $\min S = 0$ used in [13, 14], so from here on the two readings are interchangeable.

2.2. **The signed threshold.** Fix S with $\min S = 0$ and put $m = \max S > 0$. Write $m - S = \{m - s : s \in S\}$ for the reflected pattern.

Definition 2.4. A colouring c of $[0, N)$ *avoids S signwise* if there is no pair $(a, d) \in \mathbb{Z}^2$ with $d \neq 0$ such that $a + dS \subseteq [0, N)$ and $a + dS$ is monochromatic. As in Definition 2.2, $G_r^{\pm}(S) = N$ means that some r-colouring of $[0, N - 1)$ avoids S signwise and none of $[0, N)$ does.

Proposition 2.5 (The Rado reading)**.** *Let $\min S = 0$, $m = \max S > 0$. A colouring of $[0, N)$ avoids S signwise if and only if it avoids S and avoids $m - S$ in the sense of Definition 2.1. Consequently, for $S = \{0, s, s + u\}$ with $s, u \geq 1$, identity (3) holds: $G_r^{\pm}(S)$ is the Rado number of $u\,x + s\,z = (s + u)\,y$ for solutions with x, y, z pairwise distinct.*

Proof sketch. A copy with negative scale $d = -k$, $k \geq 1$, is a positive-scale copy of $m - S$ based at $a - km$, because $a + (-k)s = (a - km) + k(m - s)$ for every $s \in S$; the point of $\min S = 0$ and $\max S = m$ is that this base is again the smallest point of the copy, so the two membership conditions match. For the second sentence, the solutions of (2) are exactly $(a,\ a + ds,\ a + d(s + u))$, and they have distinct coordinates precisely when $d \neq 0$. ☐

Proposition 2.6. *Let $\min S = 0$, $m = \max S > 0$.*

(i) *$G_r^{\pm}(S) \leq G_r(S)$ for every r.*
(ii) *$G_r^{\pm}(m - S) = G_r^{\pm}(S)$. Here this is immediate from the definition, since avoiding signwise is symmetric in the two shapes, whereas the corresponding statement for G_r needs a change of variable.*
(iii) *If $m - S = S$ then $G_r^{\pm}(S) = G_r(S)$.*

Part (iii) is what makes the classical cells available on the signed side for free, and part (ii) is what lets a value be transported between reflected sets; both are used below.

2.3. **Two rows with no search.**

Proposition 2.7. *$G_r(\{0, 1\}) = r + 1$ for every $r \geq 1$.*

Proof sketch. Giving each of $0, \ldots, r - 1$ its own colour leaves no monochromatic pair $\{b, b + k\}$, $k \geq 1$; at $N = r + 1$ two integers share a colour and their difference is an admissible scale. ☐

Proposition 2.8. *$G_r^{\pm}(\{0, 1\}) = r + 1$ for every $r \geq 1$, and $G_r^{\pm}(\{0, 1, \ldots, t - 1\}) = G_r(\{0, 1, \ldots, t - 1\}) = W(t, r)$.*

Proof sketch. An initial segment equals its own reflection as a set, so Proposition 2.6(iii) applies, and (1) identifies the value. ☐

Proposition 2.9. *$G_2(\{0, 1, 2\}) = 9$, $G_3(\{0, 1, 2\}) = 27$ and $G_2(\{0, 1, 2, 3\}) = 35$; equivalently $W(3, 2) = 9$, $W(3, 3) = 27$, $W(4, 2) = 35$.*

These three are classical [4, 15]. They are recomputed here from Definition 2.1 by the same pipeline as everything else, and they serve as its validation anchor: an encoding that enumerated the wrong copies would disagree with the most-checked numbers in the subject first. By Proposition 2.8 the same three values hold for $G^{\pm}$, which is the corresponding anchor for the reflected half of the encoding.

2.4. **The encoding, and the licence for symmetry breaking.** For a set S, r colours and a window $[0, N)$, let $\Phi_r(S, N)$ be the conjunctive normal form with rN variables $v_{i,j}$ ("the integer i has colour j"), one clause $v_{i,0} \vee \cdots \vee v_{i,r-1}$ for each $i < N$, and one clause $\bigvee_{x \in b+kS} \neg v_{x,j}$ for each copy $b + kS$ inside the window and each colour j. For the signed problem, the copy clauses of $m - S$ are appended. The fidelity statement proved once and for all is one-directional: *every* avoiding colouring satisfies the formula. Nothing asserts the converse, so a forgotten clause can only weaken the formula and can never turn a satisfiable instance into a wrong theorem.

That one-directional statement is also what makes symmetry breaking delicate. Permuting colours preserves avoidance, so colour-symmetry clauses are sound for the solver — and they falsify the fidelity statement, because a non-canonical avoiding colouring then fails the formula. Adding them to $\Phi_r(S, N)$ would therefore be an unsound narrowing that looks from the outside exactly like a good optimisation: the solver finishes sooner and the certificate is smaller.

Proposition 2.10. *Let $\Phi_r^0(S, N)$ be $\Phi_r(S, N)$ together with the $r-1$ unit clauses $\neg v_{0,j}$, $1 \le j \le r-1$, which force the integer 0 to receive colour 0. If $\Phi_r^0(S, N)$ is unsatisfiable then no r-colouring of $[0, N)$ avoids S. The same holds for the signed formula and signed avoidance.*

Proof sketch. If some colouring avoids, so does the one obtained by transposing the colours 0 and $c(0)$, because a permutation of colours is injective and therefore preserves monochromaticity; the transposed colouring gives 0 the colour 0 and hence satisfies the added units, and it satisfies the rest of the formula by the fidelity statement. The signed case is the same argument applied to both shapes at once. $\square$

The narrowing is worth its licence. On $G_3(\{0, 1, 4\}) = 57$, the plain formula costs 24.4 s of solving and a 70.0 MB certificate; adding the two unit clauses brings that to 7.5 s and 28.8 MB. (Adding at-most-one-colour clauses instead is sound but does nothing: 26.1 s, 68.7 MB. A full first-occurrence canonical form would give 4.9 s and 22.2 MB but needs a construction rather than two units, and was not taken.) These are measurements on one shared machine and carry the usual $\pm 25\%$; they are recorded because certificate size, not solver time, is what limits this problem.

2.5. **Soundness controls.** An encoding of a threshold problem can be wrong in ways that make every instance come out provable. The following are proved as protection against that.

Proposition 2.11. (i) *If the scale were allowed to be 0, every copy would collapse to a single repeated point and $G_r(S)$ would equal 1 for every S and r.*

(ii) *The hypothesis $0 \in S$ in Proposition 2.3 is necessary.*

(iii) *Flipping a single entry of the colouring that witnesses $W(3, 2) = 9$ makes the avoidance check fail.*

(iv) *The formula $\Phi_r(S, N-1)$ one step below an established threshold is* satisfiable, *with a satisfying assignment produced from the witness colouring rather than by a second search; so the refutations above are not an artefact of an always-unsatisfiable encoding.*

(v) $G_3(\{0, 1, 3\}) = 42 \neq 27 = W(3, 3)$*, which refutes the reading in which the encoder enumerated t-term progressions rather than copies of S; the two coincide exactly on initial segments, so no other cell separates them.*

(vi) $G_2(\{0, 1, 5\}) = 20$ *and* $G_2(\{0, 2, 5\}) = 21$*: two sets of the same diameter with different values, matching the two branches of the closed form of Theorem 3.1.*

(vii) *Neither 41 nor 43 is $G_3(\{0, 1, 3\})$, each refuted from a different half of the threshold.*

Proposition 2.12. (i) *Neither* 18 *nor* 20 *is the value of* $G_2^{\pm}(\{0,1,5\})$*, and neither* 47 *nor* 49 *is the value of* $G_3^{\pm}(\{0,2,5\})$*; in each case the two refutations come from different halves of the threshold.*

(ii) *The signed formula one step below a threshold is satisfiable, again with the assignment built from the witness rather than searched for.*

(iii) *The reflection point is checked and not assumed:* $\max\{0,1,3\} \neq 2$ *is detected, so a cell cannot be stated with the wrong reflection, and the reflected pattern itself is checked:* $3 - \{0,1,3\} = \{0,2,3\}$.

(iv) *The hypothesis* $4 \nmid s$, $4 \nmid u$ *of Theorem 3.5 is load-bearing: its conclusion is false at* $(s,u) = (1,4)$ *and at* $(s,u) = (3,4)$.

(v) *The* 4*-colouring of* $\{0,\dots,92\}$ *printed in* [14] *avoids every copy of* $\{0,1,3\}$ *but does* not *avoid* $\{0,1,3\}$ *signwise — it carries* 68 *monochromatic reflected copies. This is the paper's own negative claim, and it is the sharpest available check on the reflected half of the encoding: a wrong reflection would make it come out true.*

3. Two colours

Throughout this section $S = \{0, s, s+u\}$ with $s, u \geq 1$ coprime, and $D = s+u$ is the diameter. Every three-point set is a translate of a dilate of exactly one such S, so this is no loss.

3.1. The unsigned side.

Theorem 3.1. *For all thirteen primitive three-point sets of diameter* $3 \leq D \leq 9$*, the value of* G_2 *is*

$$G_2(\{0,s,s+u\}) \;=\; \begin{cases} 4D, & \{s,u\} = \{1,4m\} \text{ for some } m \geq 1, \\ 4D+1, & \text{otherwise,} \end{cases}$$

namely

$$\begin{array}{lllll} \{0,1,3\}:13 & \{0,1,4\}:17 & \{0,1,5\}:20 & \{0,2,5\}:21 & \{0,1,6\}:25 \\ \{0,1,7\}:29 & \{0,2,7\}:29 & \{0,3,7\}:29 & \{0,1,8\}:33 & \{0,3,8\}:33 \\ \{0,1,9\}:36 & \{0,2,9\}:37 & \{0,4,9\}:37. & & \end{array}$$

The remaining set of diameter at most 9 *is* $\{0,1,2\}$*, where the value is* $W(3,2) = 9$.

The closed form is the theorem of Brown, Landman and Mishna [1], completed on the exceptional family by Kim and Rho [8]. Theorem 3.1 is a reproduction: each of the thirteen values was recomputed from Definition 2.1 and proved, and all thirteen agree with the published form, including both exceptional cells $\{0,1,5\}$ and $\{0,1,9\}$, where the generic branch would give 21 and 37. Reproducing a dichotomy is a stronger check than reproducing a single value: an encoding that were uniformly off by one could not do it.

3.2. The signed side.

Theorem 3.2. *Let* $s, u \geq 1$ *be coprime and* $D = s+u$*. For each of the twenty-eight sets* $\{0,s,s+u\}$ *with* $3 \leq D \leq 13$ *and each of the four sets* $\{0,1,4m+1\}$ *with* $4m \in \{16,20,24,28\}$ *— thirty-two cells in all —*

$$G_2^{\pm}(\{0,s,s+u\}) \;=\; \begin{cases} 4D-1, & \{s,u\} = \{1,4m\} \text{ for some } m \geq 1 \text{ or } \{s,u\} = \{3,4\}, \\ 4D+1, & \text{otherwise.} \end{cases}$$

Equivalently, in the notation of (3)*, these are the values of* $R_2(ux + sz = (s+u)y)$ *for solutions with distinct coordinates. The values are collected in Table 1.*

Remark 3.3. Outside the formal development, all 139 coprime pairs with $2 \leq D \leq 30$ were computed from the definition, each by a refutation at the value and a satisfying assignment one below whose model was re-checked against the definition. Every one agrees with the closed form of Theorem 3.2. Because the exceptional set has a sporadic member, this is the strongest agreement check available here; the thirty-two cells of Theorem 3.2 are the part that is formally proved.

D	3	4	5	6	7	8	9	10	11	12	13
$s=1$	13	17	19	25	29	33	35	41	45	49	51
$s=2$			21		29		37		45		53
$s=3$					27	33		41	45		53
$s=4$							37		45		53
$s=5$									45	49	53
$s=6$											53

TABLE 1. $G_2^{\pm}(\{0,s,s+u\})$, $D=s+u$, for every primitive three-point set of diameter at most 13 apart from $\{0,1,2\}$, whose value is 9. Generic entries are $4D+1$; the entries $19,35,51$ at $s=1$ and the entry 27 at $(s,u)=(3,4)$ are $4D-1$. Four further cells are proved, at $(s,u)=(1,16),(1,20),(1,24),(1,28)$, with values $67,83,99,115$.

3.3. **Where the two readings separate.** Placing the two closed forms side by side gives Table 2. Three things are worth stating. The signed exceptional set is strictly larger — $\{3,4\}$ is sporadic and has no unsigned counterpart. The signed exceptional value is two below the generic value where the unsigned one is only one below. And the value $4D$, which is the unsigned exceptional value, never occurs on the signed side at all.

$\{s,u\}$	G_2	$G_2^{\pm}$	gap
generic	$4D+1$	$4D+1$	0
$\{1,4m\}$	$4D$	$4D-1$	1
$\{3,4\}$	$4D+1$	$4D-1$	2

TABLE 2. The two closed forms for $S=\{0,s,s+u\}$, $D=s+u$, $\gcd(s,u)=1$.

Theorem 3.4. *Let R and G denote the signed and unsigned two-colour thresholds of the set named.*

(i) *At $S=\{0,1,5\}$: $R<G$ (indeed $R=19$, $G=20$).*
(ii) *At $S=\{0,3,7\}$: $R+2=G$ (indeed $R=27$, $G=29$).*
(iii) *At $S=\{0,2,5\}$: $R=G$ (indeed $R=G=21$).*
(iv) *$G_2^{\pm}(\{0,3,5\})=21$, obtained from (iii) by reflection alone.*

Each of (i)–(iii) is stated and proved as a relation between the two thresholds — quantified over whatever values they take — and not as an arithmetic identity between two numerals. That matters here because the two problems are different problems, and the point of the statement is that the difference is visible. Case (ii) is the interesting one: $\{s,u\}=\{3,4\}$ is *generic* for the unsigned closed form and still drops by two on the signed side, so "the readings separate exactly where the unsigned form is exceptional" is false.

3.4. **A uniform lower bound.** The cells above are bought one certificate at a time. The next statement covers infinitely many of them with no search at all.

Theorem 3.5. *Let $s,u\geq 1$ be coprime with $4\nmid s$ and $4\nmid u$, and put $D=s+u$. Then some 2-colouring of a window of $4D$ integers avoids $\{0,s,s+u\}$ signwise; consequently*

$$G_2^{\pm}(\{0,s,s+u\}) \;\geq\; 4(s+u)+1,$$

and no value at or below $4(s+u)$ can be the signed threshold of such a set.

Proof sketch. Two colourings of $[0,4D)$ suffice. Let $\mathrm{A}(x)=\lfloor x/2\rfloor \bmod 2$, the period-four pattern $0\,0\,1\,1$, and let B alternate $0\,1\,0\,1\ldots$ on $[0,2D)$ and the other way on $[2D,4D)$.

The whole argument rests on one observation: inside a window of $4D$ integers a copy $b+kS$ has $b+kD\leq 4D-1$, hence $kD<4D$ and $k\leq 3$. The constraint set is three scales, not $\Theta(D)$ of them.

By Proposition 2.5 it is enough to defeat the copies of $\{0, s, D\}$ and of $\{0, u, D\}$, so both cases below are the same statement with s and u interchanged.

For A: the colour of x depends only on $x \bmod 4$, so for $1 \le k \le 3$ the monochromaticity of $\{b,\, b+ka,\, b+kD\}$ is a condition on ($b \bmod 4$, $a \bmod 4$, $D \bmod 4$, k) alone, and the $4 \cdot 4 \cdot 4 \cdot 3$ cases are checked exhaustively; none is monochromatic when $a \not\equiv 0$, $D \not\equiv 0$ and $a \not\equiv D \pmod 4$. This covers every pair with $4 \nmid s$, $4 \nmid u$, $4 \nmid D$. (This is also where the bound comes from: at $k = 4$ every point of $b + 4S$ receives the same colour under A, which is why the window cannot be extended.)

For B: when s and u are both odd, D is even and the middle offset is odd, and the three points b, $b + ka$, $b + kD$ of a copy inside $[0, 4D)$ cannot all agree once the flip at $2D$ is taken into account; the verification is a case split on $k \le 3$ and on which side of $2D$ each point falls.

The two cases cover everything: if one of s, u is even then D is odd and A applies; if both are odd then B applies. Coprimality is used exactly once, to exclude $s \equiv u \equiv 2 \pmod 4$. □

The statement of Theorem 3.5 is contained in [7], and its hypothesis is strictly stronger than "not exceptional": the pair $(4, 5)$ is generic and is not covered. What is offered here is the proof. It is elementary, it is about a hundred lines long where the published argument is a case analysis over the structure of a putative colouring, and it is machine-checked with no appeal to computation beyond the finite residue check named above. The bound is exactly met at, for instance, $(s, u) = (2, 5)$, where it gives ≥ 29 and the value is 29; and its conclusion fails at the two excluded pairs $(1, 4)$ and $(3, 4)$, so the hypothesis is not decoration (Proposition 2.12(iv)).

4. The correction of the record

The values of Theorem 3.2 were computed here before they were looked up, and they looked like a new closed form with an attractive sporadic exception. They are a theorem from 2015. Gupta, Thulasi Rangan and Tripathi [7] prove, in the notation of their abstract:

> *For relatively prime positive integers a and b, let $n = R(a, b)$ denote the least positive integer such that every 2-colouring of $[1, n]$ admits a monochromatic solution to $ax+by = (a+b)z$ with x, y, z distinct integers. It is known that $R(a, b) \le 4(a+b)+1$. We show that $R(a, b) = 4(a + b) + 1$, except when $(a, b) = (3, 4)$ or $(a, b) = (1, 4k)$ for some $k \ge 1$, and $R(a, b) = 4(a + b) - 1$ in these exceptional cases.*

By Proposition 2.5 their $R(a, b)$ is $G_2^{\pm}(\{0, a, a+b\})$. Indeed, for coprime a, b the equation $ax+by = (a + b)z$ says $a(x - z) = b(z - y)$, so its solutions are exactly $x = z + bd$, $y = z - ad$ with $d \in \mathbb{Z}$; the three coordinates are pairwise distinct precisely when $d \ne 0$, and for $d > 0$ the solution set is $\{z-ad,\, z,\, z+bd\} = (z-ad)+d\cdot\{0, a, a+b\}$, while for $d < 0$ it is a copy of the reflected pattern. So their theorem is exactly Table 1, sporadic case included; their treatment of $(a, b) = (3, 4)$ by hand is the cell $G_2^{\pm}(\{0, 3, 7\}) = 27$. The upper bound $R(a, b) \le 4(a + b) + 1$ is older still: [7] attributes it to an unpublished manuscript of Burr and Loo [2] and records that Landman and Robertson [10] give a second proof of it.

The point of recording this is that [14] does not cite [7]. Its Table 5 lists the two-colour values $13, 17, 19, 25$ for $z + kx = (k + 1)y$, $k = 2, 3, 4, 5$ — which is the line $a = 1$, $b = k$ of [7] — and places them within the scope of a computational survey of Rado numbers; and it remarks that at $k = 4$ "the reflected shape already bites: $R_2 = 19 < 20 = G_2(\{0, 1, 5\})$". That remark is a single member of the exceptional family $(1, 4k)$ of a closed form published eleven years earlier, observed empirically at one point rather than derived. A search of the source file of [14] finds no occurrence of any of the names Gupta, Thulasi Rangan, Tripathi, Burr or Harborth.

Two qualifications. First, this affects only the two-colour line: [7] is about two colours, and it records the case $r \ge 3$ as open for the sibling family of equations, so the three- and four-colour entries of [14, Table 5] stand unaffected, and so do all of the unsigned values. Second, the same problem has been given the same name twice, and the earlier use is the one this paper's title borrows. Dumitru and Prunescu [5], in December 2025, define $\Gamma(A, c)$ for a finite $A \subset \mathbb{Z}^n$ to be the least m such that every c-colouring of the grid $\{0, \ldots, m-1\}^n$ contains a monochromatic $B = xA + y$ with

$x \geq 1$, and call it the *Gallai homothety number* of A; at $n = 1$ that is $G_c(A)$ exactly. They work at $c = 2$. Neither [13] nor [14] cites them, and anything written under that name should cite both lines of work. A third object travels under a name close enough to be confused with these two: Li and Si [11] write *Gallai–Rado number* $\mathrm{GR}_k(\mathcal{E}_1 : \mathcal{E}_2)$ for the least N such that for every $n \geq N$, every k-colouring of $[n]$ contains a rainbow solution of $\mathcal{E}_1$ or a monochromatic solution of $\mathcal{E}_2$. That is an anti-Ramsey quantity in the Gallai–Ramsey line, not $G_r(S)$ or $G_r^{\pm}(S)$ under another name; we record it only so that a reader meeting "Gallai" attached to a Ramsey number in this corner knows there are three different objects to tell apart.

5. Three colours and beyond

5.1. The published values, reproduced.

Proposition 5.1. *$G_3(\{0,1,3\}) = 42$, $G_3(\{0,1,4\}) = 57$, $G_2(\{0,2,3,5\}) = 67$ and $G_2(\{0,1,5,6\}) = 80$.*

These are the four values called new in [13]; they also appear in the abstract of [14], which supersedes it. Each was recomputed here from Definition 2.1 rather than transcribed.

Proposition 5.2. *The eighteen reflection classes of four-point sets of* [14, Table 4] *have the two-colour values*

$\{0,1,2,4\} : 38$	$\{0,1,3,4\} : 41$	$\{0,2,4,5\} : 44$	$\{0,3,4,5\} : 44$	$\{0,1,4,5\} : 45$	$\{0,1,3,6\} : 52$
$\{0,2,3,6\} : 54$	$\{0,1,2,6\} : 56$	$\{0,1,4,6\} : 58$	$\{0,1,2,7\} : 59$	$\{0,2,5,7\} : 59$	$\{0,1,3,7\} : 60$
$\{0,2,4,7\} : 61$	$\{0,1,4,7\} : 62$	$\{0,1,5,7\} : 62$	$\{0,2,3,7\} : 62$	$\{0,1,6,7\} : 79$	$\{0,3,4,7\} : 79.$

This table has no closed form and neighbouring values differ by one — $\{0,2,4,5\}$ and $\{0,3,4,5\}$ are a reflection pair at 44 while $\{0,1,4,5\}$ is 45 — which is exactly the pattern a systematically wrong encoding fails to reproduce. All eighteen were recomputed here and all eighteen agree with [14].

Proposition 5.3. *$G_3(\{0,2,5\}) \geq 77$; equivalently, no value at or below 76 is $G_3(\{0,2,5\})$.*

Proof sketch. The 3-colouring of $\{0,\dots,75\}$ printed in [13, Theorem 4.7] is transcribed digit for digit and checked against Definition 2.1: the window contains 540 copies of $\{0,2,5\}$ and none of them is monochromatic. □

Remark 5.4. The exact value is 77, as stated in the abstract of [14]. Outside the formal development, `cadical` returns unsatisfiable at $N = 77$ in 616 s and satisfiable at $N = 76$ in 60 s, which is an independent third-solver reproduction of that value (against the 389.6 s and 320.9 s reported in [14] for two other solvers). The refutation is 354 MB of LRAT even after the narrowing of Proposition 2.10, which is why only the bound of Proposition 5.3 is formally proved. One solver-independent check of the encoding: [14] reports 555 copies in this window, and the enumeration here produces 555, namely $\sum_{k=1}^{15}(77 - 5k)$.

Proposition 5.5. *Write $R_r(k)$ for the Rado number of $z + kx = (k+1)y$ with distinct coordinates, that is, $G_r^{\pm}(\{0,1,k+1\})$. Then*

$$R_3(2) = 29, \qquad R_3(3) = 54, \qquad R_3(4) = 55, \qquad R_3(5) \geq 60, \qquad R_4(2) \geq 59,$$

and, on the unsigned side, $G_4(\{0,1,3\}) \geq 94$.

The three exact values are proved from refutation certificates of 223 KB, 1.2 MB and 11.9 MB. The bound $R_3(5) \geq 60$ comes from an avoidance colouring at $N = 59$; `cadical` reports unsatisfiable at $N = 60$ in 49.3 s, so the value is 60, but that refutation is 136 648 164 bytes and is not carried into the formal development. The cell is closed instead in Proposition 6.5, by the search of Section 6 and with nothing stored. The two four-colour bounds are the colourings printed in [14], transcribed and checked against the definition; one further solver-independent check is that [14] reports 532 copies of each shape in the window $\{0,\dots,57\}$ and the enumeration here produces $532 = \sum_{k=1}^{19}(58 - 3k)$.

5.2. **Off the published line.** Everything we have found published about $G_r^{\pm}(\{0, s, s+u\})$ lies on one of two lines: two colours with any coprime pair, which is [7]; and $s = 1$ with $u \leq 5$ at $r \leq 4$, which is [14, Table 5] and Proposition 5.5. The next theorem is the first two entries off both; a third, further along the line $s = 1$, is Theorem 6.9.

Theorem 5.6. (i) $G_3^{\pm}(\{0, 2, 5\}) = 48$*; equivalently the Rado number of* $3x + 2z = 5y$ *at three colours, for solutions with distinct coordinates, is* 48*.*

(ii) $G_3^{\pm}(\{0, 3, 7\}) \geq 70$*; equivalently no value at or below* 69 *is the three-colour Rado number of* $4x + 3z = 7y$ *for solutions with distinct coordinates.*

(iii) $G_3^{\pm}(\{0, 2, 5\}) < G_3(\{0, 2, 5\})$*, as a relation between the two thresholds; combined with (i) and Proposition 5.3, the gap is at least* 29*.*

Proof sketch. For (i), the lower half is the explicit 3-colouring

```
2 0 1 2 1 1 0 2 2 1 1 0 1 1 2 2 0 2 0 0 0 2 2 1
 0 0 2 0 2 2 1 1 1 1 1 0 2 0 0 1 2 1 0 0 0 2 1
```

of $\{0, \ldots, 46\}$, read left to right and top to bottom, the i-th entry being the colour of the integer i. Checking it is a finite enumeration: the window contains $\sum_{k=1}^{9}(47 - 5k) = 198$ copies of $\{0, 2, 5\}$ and as many of the reflected shape $\{0, 3, 5\}$, and none of the 396 is monochromatic. The upper half is the exhaustive statement, and it is where a certificate is used. The formula $\Phi_3^0(\{0, 2, 5\}, 48)$ of Section 2.4 — 144 variables and 1292 clauses, namely 48 at-least-one-colour clauses, $3 \cdot (207 + 207)$ copy clauses for the two shapes, and the two units of Proposition 2.10 — is unsatisfiable, which by Proposition 2.10 and Proposition 2.5 rules out every 3-colouring of $[0, 48)$. The refutation was found by `cadical` in 2.6 s; it is 8 995 045 bytes of LRAT and is re-checked line by line inside the proof.

For (ii) the lower half is an explicit 3-colouring of $\{0, \ldots, 68\}$, checked the same way against both shapes.

For (iii), (i) forces the signed threshold to be 48 and Proposition 5.3 forces the unsigned one to exceed 76; the comparison is then formal, and in particular does not go through the unproved value 77. □

Remark 5.7. The exact value in (ii) is 70: `cadical` returns unsatisfiable at $N = 70$ in 139.2 s. That refutation is 296 463 768 bytes of LRAT and is not carried into the formal development, so only the lower half is claimed above. Re-running the search regenerates it; nothing else is needed to close the cell.

5.3. **Novelty, and the search behind the claim.** The values in Theorem 5.6 are, to the best of our knowledge, not in the literature. The searches run were: a local full-text mirror of the mathematics arXiv with coverage through 2026-08-28, queried for the equation forms $a(x - y) = b(y - z)$ and $z + kx = (k + 1)y$, for "injective solution" and "injective Rado", for "Rado number(s) for/of the equation", and for $(a + b)z$ in a Rado context — the last returns exactly one hit in the whole mathematics corpus, a bibliography line citing [7]; the text of [7] itself, which is two colours only and records $r \geq 3$ as open for the sibling family of equations; the bibliographies of [13, 14, 1, 8]; and the computational Rado-number literature [3], whose convention does not require distinct coordinates and therefore returns the degenerate value 1 for these equations. The OEIS was searched as well, numerically on the unsigned two-colour list $9, 13, 17, 20, 21, 25, 29, 33, 36, 37$, on the eighteen four-point values, on the three-colour lists $27, 42, 57, 77$ and $42, 57, 70, 77$, on the signed $s = 1$ row $13, 17, 19, 25, 29, 33, 35, 41, 45, 49, 51$ and on the signed three-colour list $29, 54, 55, 60$, and textually for "Gallai number" and "homothety number". None of these sequences is in it, in either reading.

6. Exhaustive search in place of a certificate

Every upper half of a computed cell so far is bought with a refutation certificate, and the price is paid in bytes rather than in seconds. At three colours and N in the sixties and seventies the solver finishes in minutes while the certificate it emits runs to hundreds of megabytes: 136 648 164 bytes at $G_3^{\pm}(\{0,1,6\}) = 60$, 296 463 768 at $G_3^{\pm}(\{0,3,7\}) = 70$, 354 MB at $G_3(\{0,2,5\}) = 77$. A certificate that cannot be kept cannot be re-checked, and the cell stays a measurement. This section replaces the certificate by a complete backtracking search over colourings, of which only the soundness is proved — and only in the one direction a refutation needs. Two published cells that the certificate route had priced out are closed by it; one of them then yields a statement about extremal colourings that a refutation cannot yield; and at a third cell the search settles a value that has not, as far as we can determine, been computed before, though what is proved here is only the bound below it.

6.1. **The search, and the direction that has to be proved.** Fix $S = \{0, s, m\}$ with $1 \le s \le m$, a number r of colours and a window $[0, N)$. The search colours $0, 1, 2, \ldots$ from left to right and rests on two observations.

The maximum decomposes the condition. A copy $b + kS$ inside $[0, N)$ is determined by its largest point $t = b + km$ and the scale k: it is $\{t - km,\ t - k(m - s),\ t\}$, and a copy of the reflected pattern $m - S$ topping out at t is $\{t - km,\ t - ks,\ t\}$. Avoidance is therefore a conjunction indexed by top points, and the conjunct at t can be discharged in full at the moment t is coloured.

A position can be dead before it is reached. At a position t not yet coloured, call a colour c *blocked* if for some $k \ge 1$ the two lower points of a copy topping out at t are both already coloured c. If every one of the r colours is blocked at some t ahead of the current position, no extension of the prefix can survive, and the branch is cut.

Proposition 6.1. *Let $S = \{0, s, m\}$ with $1 \le s \le m$, and let $N, r \ge 1$.*

(i) *The blocked-colour test* under*-reports: every colour it declares blocked at a position t is backed by an actual pair of already-coloured points which carries that colour and completes a copy topping out at t. Consequently, along a colouring that avoids S, or that avoids S signwise, no position is ever declared dead and no branch containing it is ever cut.*

(ii) *If some 3-colouring of $[0, N)$ avoids S, then one does whose colour sequence is in* first-occurrence canonical form*: reading the colours of $0, 1, 2, \ldots$ in order, the colour $c + 1$ does not occur before the colour c does. The same holds for signed avoidance.*

(iii) *If the search returns no solution for (S, r, N) then no r-colouring of $[0, N)$ avoids S; with the reflected pattern included in the enumeration, no r-colouring of $[0, N)$ avoids S signwise. At $r = 3$ the same conclusions hold for the run restricted to first-occurrence canonical form.*

Proof sketch. (i) is an induction on the scales the test visits: the base case blocks nothing, and each step either leaves the accumulated set of blocked colours alone or adds one colour for which two points of one particular copy are already coloured with it. What is needed downstream is one line: whatever colour an extension of the prefix gives t, that colour is not blocked, so t is not dead.

(ii) At $r = 3$ the canonical form says exactly that 0 has colour 0 and that the first integer whose colour is not 0 has colour 1. Two transpositions of colours produce it from an arbitrary avoiding colouring, and a permutation of colours preserves avoidance because it is injective. No orbit minimisation and no relabelling construction is needed; that is what makes the statement short enough to prove rather than to assume.

(iii) is the contrapositive of an induction on the depth of the search: if a colouring avoids S and obeys the canonical restriction in force, the search's count of solutions is positive. Part (i) licenses the pruning inside that induction; the induction itself is stated for an arbitrary starting prefix and for an abstract invariant on the counter of colours used so far, so that the two canonical regimes — "0 has colour 0", and first-occurrence — are two instances of one argument. □

Remark 6.2. Nothing above requires the blocked-colour test to be *complete*. Which scales it examines, in what order, and when it stops examining them may all be wrong without making Proposition 6.1 wrong: a forgotten scale, or a position never looked at, costs time and can only make the search report *more* solutions, while every statement below consumes only the assertion that it reported none. What has to be correct is the under-reporting lemma, not the program.

The canonical form of Proposition 6.1(ii) is worth exactly the factor the symmetry group predicts. Measured on $G_3^{\pm}(\{0,1,6\})$ at $N = 60$: the run under "0 has colour 0" visits 384 209 418 nodes, the run in first-occurrence canonical form visits 192 104 712, a ratio of 2.000, which is 3!/3.

6.2. **Validation, and node counts as theorems.** The search is run against what is already proved before it is allowed to close anything.

Proposition 6.3. *The following are proved by the search, with no certificate: both halves of* $G_2(\{0,1,2\}) = W(3,2) = 9$ *and of* $G_3(\{0,1,2\}) = W(3,3) = 27$*, from scratch; the upper half of* $G_3(\{0,1,4\}) = 57$*; and the upper half of* $G_3^{\pm}(\{0,2,5\}) = 48$*. Moreover the three node counts — 186 at* $G_2^{\pm}(\{0,1,5\}) = 19$*, 30 284 at* $W(3,3) = 27$ *and 15 422 575 at* $G_3^{\pm}(\{0,2,5\}) = 48$ *— are themselves theorems.*

The two van der Waerden anchors are the sharp test, for the reason given at Proposition 2.9: an implementation that enumerated the wrong copies, bounded the scale wrongly, or cut a live branch would disagree with the most-checked numbers in the subject first. The other two cells are the expensive ones already held by certificate — 28 791 164 bytes for $G_3(\{0,1,4\}) = 57$, the largest certificate this development stores, and 8 995 045 for $G_3^{\pm}(\{0,2,5\}) = 48$ — and both of those certificates are now redundant. Making the node counts theorems rather than measurements is what pins the verified search against the two independently written C implementations used to develop it — one with precomputed pair tables and incremental forward-check counters, one recomputing every blocked-colour test by direct scan exactly as the verified code does — which reproduce all three counts to the digit. Finally the two canonical regimes of Proposition 6.1, proved independently of each other, are made to refute the same cell: $W(3,3) = 27$ is closed under each.

Proposition 6.4.

(i) *The search returns a* positive *count one step below the threshold at* $G_3(\{0,1,4\}) = 57$ *and at* $G_3^{\pm}(\{0,2,5\}) = 48$ *— run, not derived — so the refutations are not the artefact of a search that cuts everything. This is the analogue for this route of Proposition 2.11(iv).*

(ii) *At the two expensive cells* $G_3^{\pm}(\{0,1,6\}) = 60$ *and* $G_3(\{0,1,5\}) = 70$ *the same fact is obtained without a second run: had the count been zero one step below the threshold, Proposition 6.1(iii) would contradict the explicit avoiding colouring there, so the two halves of each cell check each other.*

(iii) *Too-large and too-small refutations at each of* $W(3,3) = 27$*,* $G_3^{\pm}(\{0,1,6\}) = 60$ *and* $G_3(\{0,1,5\}) = 70$*, in every case from a different half of the threshold.*

(iv) *Changing a single entry of one of the extremal colourings of Theorem 6.7 creates five monochromatic signed copies, and the avoidance check sees them; likewise changing one entry of the colouring displayed in Theorem 6.9 makes its avoidance check fail.*

(v) *The two colourings of Theorem 6.7 are distinct as sequences, without which the last clause of that theorem would be uninteresting; and the reflected pattern of* $\{0,1,7\}$ *is checked to be* $\{0,6,7\}$*, so that Theorem 6.9 cannot be about the wrong reflection.*

6.3. **Two published cells, closed with nothing stored.**

Proposition 6.5. $G_3^{\pm}(\{0,1,6\}) = 60$*; equivalently* $R_3(z + 5x = 6y) = 60$*, the* $k = 5$ *row of* [14, Table 5].

Proof sketch. The lower half is an explicit 3-colouring of $\{0,\dots,58\}$ checked against Definition 2.4: the window holds 261 copies of $\{0,1,6\}$ and 261 of the reflected pattern $\{0,5,6\}$, and none of the 522 is monochromatic. The upper half is the complete search of Proposition 6.1(iii) at $N = 60$ in first-occurrence canonical form, which visits 192 104 712 nodes and returns no solution. □

The value is the one printed in [14, Table 5]; what is added is that it is a theorem. Proposition 5.5 could give only $G_3^{\pm}(\{0,1,6\}) \geq 60$, because the refutation of the formula at $N = 60$ is 136 648 164 bytes of LRAT. The search settles the same cell with nothing stored at all.

Proposition 6.6. $G_3(\{0,1,5\}) = 70$, *the third row of* [14, Table 3].

Proof sketch. An explicit avoiding 3-colouring of $\{0,\ldots,68\}$, re-checked against Definition 2.1, and the complete search at $N = 70$, which visits 611 858 495 nodes and returns no solution. ☐

This is the row of [14, Table 3] that sits between $G_3(\{0,1,4\}) = 57$ and $G_3(\{0,2,5\}) = 77$, and on the evidence of its two neighbours the certificate route could not have reached it either. That is measured rather than extrapolated: `cadical` refutes the formula at $N = 70$ in 120.2 s and emits 298 430 865 bytes of LRAT, an order of magnitude past anything kept here; the file was measured and discarded. That run is also an independent check on the search: a backtracking enumeration over colourings and a CDCL solver on a formula built by a different program are independent implementations of independently written formulations, and they agree that $N = 70$ is infeasible — as they do at $G_3^{\pm}(\{0,1,6\}) = 60$, where the solver takes 49.3 s.

6.4. **What a search sees and a certificate does not.** A refutation certificate testifies that nothing exists at the threshold. A search enumerates, and one step below the threshold the extremal objects pass in front of it.

Theorem 6.7. *The threshold $G_3^{\pm}(\{0,1,6\}) = 60$ has at least two extremal colourings up to permutation of the colours. Explicitly, let c_a and c_b be the* 3*-colourings of* $\{0,\ldots,58\}$

c_a: *0 0 1 0 2 1 2 2 1 0 2 0 0 1 0 2 1 1 2 2 0 2 0 0 1 0 1 2 2 2*
1 0 1 0 0 2 0 2 2 1 1 2 0 1 0 0 1 0 1 2 2 1 2 0 2 0 0 1 0

c_b: *0 1 0 0 2 0 2 1 2 2 1 0 1 0 0 1 0 2 1 1 2 2 0 2 0 0 1 0 1 2*
2 2 1 0 1 0 0 2 0 2 2 1 1 2 0 1 0 0 2 0 1 2 2 1 2 0 1 0 0

read left to right and top to bottom, the i-th entry being the colour of the integer i. Then c_a and c_b each avoid $\{0,1,6\}$ signwise, and no permutation σ of the three colours satisfies $\sigma \circ c_a = c_b$. Moreover the window reflection $x \mapsto 58 - x$ carries one to the other exactly, with no relabelling: reversing the colour sequence of c_a gives the colour sequence of c_b on the nose.

Proof sketch. Each colouring is checked against Definition 2.4, that is, against the $261 + 261$ copies counted in the proof of Proposition 6.5. The reflection claim is an equality of two sequences of 59 entries. For the remaining claim, c_a and c_b agree at the integer 0, where both take the colour 0, and differ at the integer 1, where c_a takes 0 and c_b takes 1; a permutation carrying c_a to c_b would have to send the colour 0 both to 0 and to 1. ☐

That the reflection acts here without any relabelling is not automatic. It is a symmetry of the signed problem — reflecting a copy of S inside the window produces a copy of $m - S$, and signed avoidance forbids both — but a colouring carried to another by that symmetry may perfectly well need its colours permuted as well, and these two do not. One of the two, c_b, is the colouring that supplies the lower half of Proposition 6.5, brought into canonical form; c_a is its reflection, and it was not in hand before the enumeration.

Remark 6.8. The search reports exactly two avoiding colourings of $\{0,\ldots,58\}$ in first-occurrence canonical form, at a cost of 197 362 995 nodes, which would make the extremal colouring unique up to the order-12 group generated by the six permutations of the colours and the reflection. That is a measurement and is deliberately not claimed as a theorem: "exactly two" needs the converse of Proposition 6.1(i) — that every leaf the search reaches really is an avoiding colouring — which is not proved here, because a refutation needs only under-reporting. Theorem 6.7 is the part that follows from what is proved.

Neither [14], which states this value and prints an avoidance colouring for it, nor [7], [1] or [8] makes any claim about how many extremal colourings a cell of this problem has, or about how they are related. A solver call reports none of this.

6.5. **One row past the published table.** In the notation of (3), [14, Table 5] tabulates $R_r(z+kx = (k+1)y) = G_r^{\pm}(\{0,1,k+1\})$ for $2 \le k \le 5$ and stops; the other published line, [7], covers every coprime pair but only at two colours. So $k = 6$ at three colours is off both.

Theorem 6.9. $G_3^{\pm}(\{0,1,7\}) \ge 69$*: no value at or below* 68 *is the three-colour Rado number of* $z + 6x = 7y$ *for solutions with pairwise distinct coordinates. In particular, as a relation between thresholds and not between numerals,* $G_3^{\pm}(\{0,1,7\}) > G_3^{\pm}(\{0,1,6\}) = 60$.

Proof sketch. The 3-colouring

```
0 0 1 0 2 0 1 1 2 2 0 2 0 0 2 1 1 2 1 1 0 2 0 0 1 2 1 0 2 1 0 2 0 1
1 1 0 2 0 1 2 2 1 0 1 0 0 0 2 1 1 2 1 1 2 0 0 2 0 2 2 1 1 0 2 0 1 0
```

of $\{0,\dots,67\}$, read left to right and top to bottom, avoids $\{0,1,7\}$ signwise: the window holds 297 copies of $\{0,1,7\}$ and 297 of the reflected pattern $\{0,6,7\}$, and none of the 594 is monochromatic. The comparison follows from that bound together with Proposition 6.5. □

Remark 6.10. The value is 69. The complete search was run to termination at five window sizes, in first-occurrence canonical form:

N	nodes	avoiding colourings	wall
67	4 527 807 704	368	496 s
68	4 435 448 492	96	475 s
69	4 336 174 786	0	1 039 s
70	4 270 390 997	0	470 s
72	3 945 555 244	0	453 s

so $G_3^{\pm}(\{0,1,7\}) = R_3(z+6x = 7y) = 69$, and the colouring displayed above is one of the 96 found at $N = 68$. The upper half is not carried into the formal development, and its price was measured rather than guessed: 4 336 174 786 nodes take 1 039 s in the prototype used for this sweep and about twice that in the C implementation of exactly the verified algorithm, hence an estimated 5–7 CPU-hours in the verified setting, which is past the budget adopted here. Everything in this remark is a measurement.

The searches behind the novelty claim are those of Section 5.3, together with two further searches made for this cell specifically: for the equation $z + 6x = 7y$ at three colours, and for the family $z + kx = (k+1)y$ at $r = 3$. Neither returns anything at $k = 6$. What they do return confirms the background rather than the value: three-colour Rado numbers are computed only sporadically, and the surveys that come back are two-colour, or give bounds only.

6.6. **The frontier.** Table 3 puts the two readings side by side at three colours, with the exhaustive object that proves each upper half. Two things it shows were already visible in the certificate era. The two-shape problem has the smaller answer: at $\{0,2,5\}$ the unsigned value is 77 and the signed value 48. And it is much the cheaper of the two on the same set — 2.6 s and 9.0 MB against 616 s and 354 MB.

What has changed is where the wall stands and what it is made of. Under the certificate route the solver was never the obstacle — 616 s is a short run — and the limit was the size of the object that must be stored and re-checked; it passed 10^8 bytes between $N \approx 57$ and $N \approx 70$ at three colours, and past that point a cell could only be measured. The search stores nothing and pays in time instead, its unit being the node: $1.9 \cdot 10^8$ nodes for $G_3^{\pm}(\{0,1,6\}) = 60$ and $6.1 \cdot 10^8$ for $G_3(\{0,1,5\}) = 70$, both past the certificate wall, and the largest certificate this development stores is now redundant. But the wall did not disappear; it moved and changed units. Above roughly $4 \cdot 10^9$ nodes the search costs CPU-hours per statement rather than minutes, and the three cells that sit there — the upper halves at $G_3^{\pm}(\{0,1,7\}) = 69$, $G_3(\{0,2,5\}) = 77$ and $G_3^{\pm}(\{0,3,7\}) = 70$ — were not attempted here. The first is priced in Remark 6.10, and the price is a budget decision rather than a mathematical obstruction.

S	$G_3(S)$	upper half	$G_3^{\pm}(S)$	upper half
$\{0,1,2\}$	27	search, 30 284	27	reflection-symmetric
$\{0,1,3\}$	42	LRAT, 3.5 MB	29	LRAT, 223 KB
$\{0,1,4\}$	57	search, 25 061 726	54	LRAT, 1.2 MB
$\{0,1,5\}$	**70**	search, 611 858 495	55	LRAT, 11.9 MB
$\{0,1,6\}$	$\geq 82^*$	—	**60**	search, 192 104 712
$\{0,1,7\}$	$\geq 87^*$	—	**69**†	4 336 174 786, parked
$\{0,2,5\}$	$77^\dagger$	—	48	search, 15 422 575
$\{0,3,7\}$	—	—	$70^\dagger$	LRAT, 296 MB, parked

TABLE 3. Three colours. Bold: the three cells Section 6 contributes — both halves at $\{0,1,5\}$ and $\{0,1,6\}$, and at $\{0,1,7\}$ the bound ≥ 69 only, the value 69 there being a measurement. Of the three, the values at $\{0,1,5\}$ and $\{0,1,6\}$ are published and the value 69 at $\{0,1,7\}$ is not. A star marks a lower bound quoted from [14] and not verified here. A dagger marks a value established by an exhaustive computation that is not carried into the formal development, so that only the lower bound is proved: ≥ 77 at $\{0,2,5\}$ (Proposition 5.3), ≥ 70 at $\{0,3,7\}$ (Theorem 5.6) and ≥ 69 at $\{0,1,7\}$ (Theorem 6.9). The "upper half" columns name the exhaustive object: the number of nodes the complete search visits, or the size of the refutation certificate that is re-checked. Node counts are the search's own count and three of them are theorems (Proposition 6.3); the rest are measurements, as are all certificate sizes. Where both routes are available the search is named: the cells $G_3(\{0,1,2\}) = 27$, $G_3(\{0,1,4\}) = 57$ and $G_3^{\pm}(\{0,2,5\}) = 48$ also hold stored certificates, of 518 KB, 28.8 MB and 9.0 MB.

7. Verification

Every theorem and proposition in this paper has been formally verified in Lean 4 against *Mathlib* [9, 12], and the development is free of `sorry`; repository reference to be added at submission. The following depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, that is, on no computation delegated to the compiler at all: Propositions 2.3, 2.5, 2.6, 2.7, 2.10, the whole of Proposition 6.1, the whole of Theorem 3.5 including its residue check, and parts (i)–(iv) of Proposition 2.11 and (iii) of Proposition 2.12. Part (iv) of Proposition 2.12 is the one control that is not: the arithmetic $4(s+u)+1 > G_2^{\pm}$ at the two excluded pairs is kernel-clean, but it is the computed cells $G_2^{\pm}(\{0,1,5\}) = 19$ and $G_2^{\pm}(\{0,3,7\}) = 27$ that supply the two values, and those carry the native axioms of every computed cell. Within Theorem 6.7 the same split occurs: that no permutation of the colours identifies the two colourings is kernel-clean, while their avoidance and the reflection identity are evaluations.

A closed form printed over a list of cells is proved cell by cell. Theorem 3.1 is thirteen independent statements and Theorem 3.2 thirty-two, one per set, each a threshold in the sense of Definition 2.2 and each discharged by the two evaluations described next. Nothing there quantifies over the family: the two dichotomies displayed are the pattern the proved values exhibit, not theorems about all coprime pairs. The one statement in this paper proved for infinitely many pairs at once is the lower bound of Theorem 3.5.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches, and each computed cell uses exactly two such evaluations. For a cell held by certificate they are: one that runs the avoidance check of Definition 2.1 on an explicit array of colours for the window $[0, N-1)$ — for both shapes, in the signed case — and one that runs a verified LRAT checker on the refutation of the formula of Section 2.4 at N. For a cell held by the search of Section 6 the second evaluation is instead the search itself, returning a count of zero at N, which Proposition 6.1(iii) converts into the upper half; each node count reported as a theorem in Proposition 6.3 is one further evaluation, and Theorem 6.7 uses three, namely the two avoidance checks and the reflection identity. Cells proved only as lower bounds (Theorems 5.6(ii) and 6.9, the bounds of Proposition 5.5, Proposition 5.3) use one such evaluation, the witness check. Neither the CNF nor the certificate is trusted: the formula is built inside Lean from the same definition the theorems are stated with, the fidelity statement of Section 2.4 is proved once for all instances rather than per instance, the symmetry-breaking units carry the separate licence of Proposition 2.10, and, since a certificate names input clauses by position, a disagreement between the formula built inside the proof and the one handed to the solver would make the check fail rather than succeed wrongly; and each of the 75 stored formulas — 39 written by the one-shape generator and 36 by the two-shape one — was regenerated from its parameters and found identical to the stored file byte for byte.

The search is trusted in a weaker sense still. It is a self-contained program, compiled to native code, that no proof depends on being right: what is proved about it is Proposition 6.1, whose content is that its pruning under-reports, so that a verdict of "no solution" is a statement about all colourings and not about the ones the program happened to look at. Its enumeration order, choice of scales and stopping rule can all be wrong without affecting any statement in this paper. Where both routes are available they were run against each other, and two C implementations of the search agree with the verified one on node counts to the digit.

Every value quoted was produced by an independent implementation and the formal proof was written against it afterwards, and the numbers reported as measurements — the exact values in Remarks 5.4, 5.7 and 6.10, the count of canonical extremal colourings in Remark 6.8, the 139-pair sweep of Remark 3.3, and every node count, timing and certificate size in this paper that is not stated as a theorem — are stated as measurements precisely because they are the part that the kernel does not check.

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