EXACT CERTIFICATES FOR THE NONCOMMUTATIVE-MINKOWSKI UNITARY-CATEGORIFICATION CRITERION: TWO FUSION RINGS, TWO DEGENERATE CORNERS, AND THE EXCLUSION LIST RECOMPUTED

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Abstract. Lim (arXiv:2601.13490) derives from a noncommutative Minkowski integral inequality a necessary condition for a fusion ring to be the Grothendieck ring of a unitary fusion category: for every triple of basis elements, every exponent $1 \le p < \infty$ and every nonnegative vector a, an explicit inequality between two expressions in the structure constants and the quantum dimensions must hold. He searched for violations by floating-point gradient descent over the 28,450 fusion rings of the Vercleyen–Slingerland database, excluded 5,793 of them in 97 days of machine time, published no list, and reports that part of the data was lost. We observe that for a positive integer p and $a \in \{0,1\}^s$ the two sides of the inequality are algebraic in the dimensions with no root left to take on the left, so that a violation can be decided exactly. For the two rings the source prints — the rank-7 integral ring it names as the simplest of the 23 exclusions that the other criteria it tests do not reach, and the rank-4 ring with dimensions $(1, \frac{1+\sqrt{13}}{2}, 1, 1)$ — we give exact certificates at $p = 2$, valid for every positive character of the ring: the two sides are 2 and $\sqrt{2}$ for the first, and $\sqrt{4-\delta}$ and $\sqrt{\delta-1}$ with $\delta^2 = \delta + 3$ for the second, where the violation is exactly $\delta < 5/2$. We also record that the criterion is an identity at $p = 1$ and at $a \equiv 1$, two corners of the source's sampling window where its objective is identically 1, and two controls. These statements are verified in Lean 4 against *Mathlib*. Outside the formal development, an exact deterministic sweep of the current database (25,331 rings) certifies 6,205 exclusions with $p \le 10$; on the 11,739 rings of ranks $2, 3, 4, 6, 7, 8, 9$, where the database and the source's table agree cell for cell, it certifies 3,556 against the source's 3,119. The list is a lower bound on what the criterion excludes, not a classification.

1. Introduction

A *fusion ring* is a ring R that is free of finite rank as a $\mathbb{Z}$-module, with a distinguished basis $b_1 = 1, b_2, \ldots, b_s$ in which the structure constants $b_i b_j = \sum_k N_{ij}^k b_k$ are nonnegative integers, together with an involution $i \mapsto \bar{\imath}$ of the index set making $\sum n_i b_i \mapsto \sum n_i b_{\bar{\imath}}$ an anti-automorphism and satisfying $N_{ij}^1 = \delta_{j,\bar{\imath}}$ [1, Def. 3.5], after Lusztig [11] and [10, §3.1]. The Grothendieck ring of a fusion category is a fusion ring, and a fusion ring is *unitarily categorifiable* when it is the Grothendieck ring of a unitary fusion category. Deciding categorifiability means solving the pentagon equation, for which no systematic method is known, so the working tools are *exclusion criteria*: computable necessary conditions, each ruling out some rings. The primary criteria of Liu, Palcoux and Wu [5] and Huang, Liu, Palcoux and Wu [4], the d-number, extended cyclotomic, Lagrange and pseudo-unitarity Drinfeld criteria of Liu, Palcoux and Ren [6], the induction-matrix criterion of Alekseyev, Bruns, Dong and Palcoux [7, Rem. 3.2], and the formal-codegree proof of the primary criterion by Etingof, Nikshych and Ostrik [8] are the ones the source compares itself with.

Lim [1] adds a criterion of a different kind. From a noncommutative analogue of Minkowski's integral inequality for commuting squares of tracial von Neumann algebras ([1, Thm. A]) he derives an inequality for commuting squares of multi-matrix algebras ([1, Thm. B]) and, since endomorphism spaces of tensor products in a spherical unitary tensor category form such squares, the following statement, Corollary C of the introduction of [1] (Corollary 3.4 in its body; all numbers in this paper are read off the arXiv PDF of version 2): *if $\mathcal{C}$ is a spherical unitary (multi-)tensor category with simple objects $c_1, \ldots, c_s$, fusion rule $c_i \otimes c_j \cong \bigoplus_l N_{ij}^l c_l$ and quantum dimensions d_i, then for every triple i, j, k, every $1 \le p < \infty$*

and every $a \in \mathbb{R}^s_{\geq 0}$,

$$\Big(\sum_{l=1}^{s}\Big(\sum_{n=1}^{s}\frac{N_{lk}^{n}\,a_n d_n}{d_l}\Big)^p\frac{N_{ij}^{l}\,d_l}{d_j}\Big)^{1/p} \;\leq\; \sum_{m=1}^{s}\Big(\sum_{n=1}^{s}\frac{N_{im}^{n}\,a_n^p\,d_n}{d_m}\Big)^{1/p}\frac{N_{jk}^{m}\,d_m}{d_j}. \tag{1}$$

The source then draws the consequence it uses ([1, §3.2], verbatim): "If there is a triple of basis elements c_i, c_j, c_k of a unital based ring $\mathcal{R}$, nonnegative numbers $(a_n)_n$ and $1 \leq p < \infty$ that do not satisfy Corollary 3.4, then $\mathcal{R}$ does not admit a unitary categorification." A violating (a, p, i, j, k) is therefore a *certificate* that the ring has no unitary categorification.

The source's search. The source's own comment is that "difficulties lie in choosing the appropriate $(a_n)_n$ and p". Its answer is a randomised floating-point search: for each triple (i, j, k), draw $a \in [0, 1]^s$ and $1/p \in (0.1, 1]$ uniformly, run gradient descent on the ratio R/L of the two sides of (1) with difference-quotient derivatives ($\Delta a_n = \Delta(1/p) = 0.0001$), step size 0.05 and 1,000 iterations, repeated up to 20 times per ring, in MATLAB; a ring is excluded when the ratio falls below 1. Run over the 28,450 fusion rings of ranks 2 to 9 from the database of Vercleyen and Slingerland [2], this excluded 5,793 rings (20.36%), of which 540 pass the primary 3-criterion of [4], and 23 of those pass the four criteria of [6] as well. The caption of the source's Table 1 records the total running time: 97 d 15 h 22 m 12 s. Three things about this search motivate the present paper.

(1) Every exclusion rests on a floating-point minimisation reported to fifteen digits. The source prints two of them (its Example 3.6), with R/L = 0.923133094619110 and 0.767708161620070; we reproduce both to thirteen significant digits from the printed data (Section 3), so the numbers are right — but a numerical ratio below 1 is evidence, not a proof, of a violation of (1).
(2) The list of excluded rings is not published, so no reader can check any of the 5,793 exclusions, and the source reports that "some fusion rings are not excluded by our criterion, even though they should", with, in the footnote to that sentence: "We confirmed this by running the algorithm multiple times for certain subsets of fusion rings. At a certain trial run, the number of excluded fusion rings was higher than what is provided in Table 1. However, we lost the data and do not know which additional fusion rings were excluded."
(3) The search is expensive: 97 days of machine time.

What this paper does. The observation that makes all three cheap is that the search space contains an exact sub-family. Take p a positive integer and $a \in \{0, 1\}^s$. Then $a_n^p = a_n$, the p-th power of the left-hand side of (1) is a rational function of the dimensions d with rational coefficients, and the right-hand side is a sum of p-th roots of such functions with such coefficients; a violation is decided by comparing the p-th power of the left side with the p-th power of a rational upper bound of the right side, and no root is ever taken (Lemma 2.2). Since (1) is homogeneous of degree one in a, restricting a to $\{0, 1\}^s$ loses nothing that a rescaling would recover, and integer p lies inside the source's own window $1/p \in (0.1, 1]$ for $p \leq 9$, while $p = 10$ sits at the endpoint $1/p = 0.1$ that window excludes.

With this we prove, for the two rings the source prints, exact certificates at $p = 2$ that are simpler than the source's floating-point ones and give smaller ratios (Theorems 3.2 and 3.4). The rank-7 ring is the one the source names as "the simplest integral example among those that cannot be even ruled out by the criteria in [6] and [7]", i.e. one of the 23 rings its search excludes and that neither the primary 3-criterion of [4] nor the four criteria of [6] exclude; for it the two sides of (1) at $p = 2$, $a = (1, 1, 1, 1, 1, 0, 0)$, $(i, j, k) = (7, 6, 7)$ are exactly 2 and $\sqrt{2}$. The rank-4 ring is non-integral, with dimensions $(1, \delta, 1, 1)$, $\delta = \frac{1+\sqrt{13}}{2}$; for it the two sides at $p = 2$, $a = (1, 0, 1, 1)$, $(i, j, k) = (2, 2, 2)$ are $\sqrt{9/\delta^2} = \sqrt{4 - \delta}$ and $\sqrt{3/\delta} = \sqrt{\delta - 1}$, and the violation is exactly the statement $\delta < 5/2$, decided from $\delta^2 = \delta + 3$ with no root isolation. Both theorems are stated for *every positive character* of the ring, which is stronger than fixing the Frobenius–Perron dimension vector and, unlike it, needs no Perron–Frobenius theory: the character equations pin the dimensions down by elementary algebra (Lemmas 3.1 and 3.3). We explain in Section 2 why this is the right hypothesis.

We then record two identities that the source does not state (Propositions 4.1 and 4.2): at $p = 1$ the two sides of (1) are equal for every associative array of structure constants and every d without zero entries, and at $a \equiv 1$ they are equal for every fusion ring, every positive character and every $p > 0$. Both are corners of the source's sampling window $[0,1]^s \times (0.1, 1]$ at which its objective R/L is identically 1. We state this as an observation about the method, not as an explanation of any particular miss. Two controls follow (Propositions 5.1 and 5.2): the two rings do have positive characters, so the hypothesis of the theorems is not empty, and the group ring of $\mathbb{Z}/2$, which is unitarily categorifiable, is not excluded — at $p = 2$ the two sides of (1) agree on it for every a and every triple.

Finally, outside the formal development and labelled as such throughout Section 6, we recompute the list the source lost. A deterministic search over $p \in \{2, \dots, 10\}$, $a \in \{0,1\}^s$ and all s^3 triples, with every hit re-decided in exact rational arithmetic, certifies that 6,205 of the 25,331 rings of the current database admit no unitary categorification. The database has moved since the source ran (Section 6.1): 3,313 rank-5 rings are gone and 194 rank-10 rings are new. On the 11,739 rings of the other ranks, where the two datasets agree cell for cell in the (rank, multiplicity) grid, the exact search certifies 3,556 exclusions against the source's 3,119. The whole search takes under twenty processor-minutes. We stress that the list is a lower bound on what the criterion excludes and not a classification: a ring absent from it is not thereby categorifiable, and the two lists cannot be compared ring by ring, because the source's was never published.

What is new and what is not. The inequality (1) and the implication from a violation to non-categorifiability are the source's theorems; nothing here reproves them, and every exclusion below — the two theorems as much as the sweep — rests on them. That the positive character of a fusion ring is unique and equals the Frobenius–Perron dimension is standard [10, Prop. 3.3.6(3)], and that the quantum dimensions of a unitary categorification are positive is quoted from the source ([1, §3.2], citing [12]). What we claim is: the exact sub-family and its decision procedure (Lemma 2.2); the two certificates and their proofs for every positive character; the two identities, which are a few lines of algebra once stated but are not in the source and bear on its search design; and the recomputed list with its counts, which no published work contains. We found no work citing the source (OpenAlex, work `W7125226470`, cited-by count 0 on 2026-09-07; version 2 of 2026-01-21 is the latest on arXiv on that date). A full-text sweep of a corpus of 890,739 arXiv e-prints (edge 2609.04199) found "unitary categorification" (in three spellings) in 39 papers and "noncommutative Minkowski" (in three spellings) in 165, with the source as their only common member — the former are the criteria and classifications cited above and works on quantum Fourier analysis, the latter mostly noncommutative Minkowski space-times and a few noncommutative Minkowski-type inequalities; "Minkowski criterion" occurs in the source and three unrelated papers; the identifier $[7, 3, 2, 117]$ occurs nowhere; and the source's arXiv identifier occurs only in the source itself. So no paper in that corpus applies or lists exclusions by this criterion. The AnyonWiki database records no field for it. The nearest prior work is the exclusion data of [4] for the primary criterion (19,738 of 28,451 rings) and the categorifiability flags of [2], which we use as a control. For the rank-4 ring alone, near-group theory reaches the same conclusion by other means (Remark 3.6).

Verification. Theorems 3.2 and 3.4 and Propositions 4.1, 4.2, 5.1 and 5.2 are formalised in Lean 4 against *Mathlib*; Section 8 records the modules and the axioms, and Appendix A reproduces the formal statements. The sweep of Section 6 is exact rational arithmetic in ordinary code and is not formalised; Theorems 3.2 and 3.4 are the two of its 6,205 certificates that are machine-checked.

2. Preliminaries

2.1. **Fusion rings and characters.** Throughout, R is a fusion ring of rank s with basis $b_1 = 1, \dots, b_s$ and structure constants $N_{ij}^k \in \mathbb{Z}_{\geq 0}$, $b_i b_j = \sum_k N_{ij}^k b_k$. Associativity of R is the identity

$$\sum_l N_{ij}^l N_{lk}^n = \sum_m N_{im}^n N_{jk}^m \qquad (\text{all } i, j, k, n). \tag{2}$$

A *positive character* of R is a vector $d \in \mathbb{R}^s$ with

$$d_i > 0 \quad \text{and} \quad d_i d_j = \sum_n N_{ij}^n d_n \qquad (\text{all } i, j), \tag{3}$$

that is, a ring homomorphism $R \to \mathbb{R}$ positive on the basis. The Frobenius–Perron dimension $\mathrm{FPdim}(b_l)$ is the spectral radius of the nonnegative matrix $N_l = (N_{lm}^n)_{m,n}$ of left multiplication by b_l; $l \mapsto \mathrm{FPdim}(b_l)$ is a positive character, and it is the only one: FPdim is the unique character of a transitive unital $\mathbb{Z}_+$-ring of finite rank that takes non-negative values on the basis, and those values are strictly positive [10, Prop. 3.3.6(3)]; a unital based ring is transitive [10, Exercise 3.3.2]. So "every positive character" names a single vector. The reason for quantifying over all of them rather than naming that vector is that the quantum dimensions of a categorification are known to *be* a positive character before they are known to be the Frobenius–Perron one: the quantum dimension of a spherical structure is additive and multiplicative, hence a character, and for the spherical structure of a unitary tensor category it is positive on simple objects ([1, §3.2], after [12]). A statement proved for every positive character therefore applies to the quantum dimensions of any spherical unitary categorification with no appeal to Perron–Frobenius theory, and on the two rings treated here the equations (3) pin d down by elementary algebra (Lemmas 3.1 and 3.3).

2.2. **The criterion.** For $d \in \mathbb{R}^s_{>0}$, $a \in \mathbb{R}^s$, $p > 0$ and a triple (i, j, k) put

$$\mathrm{L}_p(d, a; i, j, k) = \Big(\sum_l \Big(\frac{\sum_n N_{lk}^n a_n d_n}{d_l}\Big)^p \frac{N_{ij}^l d_l}{d_j}\Big)^{1/p}, \tag{4}$$

$$\mathrm{R}_p(d, a; i, j, k) = \sum_m \Big(\frac{\sum_n N_{im}^n a_n^p d_n}{d_m}\Big)^{1/p} \frac{N_{jk}^m d_m}{d_j}, \tag{5}$$

the two sides of (1). (Real powers are the real-analytic ones, $x^t = \exp(t \log x)$ for $x > 0$, extended as in Mathlib by $0^t = 0$ for $t \neq 0$; every base occurring below is nonnegative.) Lim's Corollary C says $\mathrm{L}_p(d, a; i, j, k) \le \mathrm{R}_p(d, a; i, j, k)$ whenever d is the quantum-dimension vector of a spherical unitary categorification of R, $p \ge 1$ and $a \ge 0$.

Definition 2.1. Data (a, p, i, j, k) with $a \in \mathbb{R}^s_{\ge 0}$ and $p \ge 1$ is an *exact certificate of exclusion* for R if

$$\mathrm{R}_p(d, a; i, j, k) < \mathrm{L}_p(d, a; i, j, k) \qquad \text{for every positive character } d \text{ of } R.$$

By the source's Corollary C and the discussion above, a fusion ring with an exact certificate of exclusion admits no unitary categorification. This is the only place the operator-algebraic content of [1] enters, and it is used as a black box.

2.3. **Why integer p and $0/1$ vectors decide exactly.** Fix a positive integer p and $a \in \{0, 1\}^s$, so $a_n^p = a_n$, and write

$$A_l = \frac{\sum_n N_{lk}^n a_n d_n}{d_l}, \quad w_l = \frac{N_{ij}^l d_l}{d_j}, \quad B_m = \frac{\sum_n N_{im}^n a_n d_n}{d_m}, \quad u_m = \frac{N_{jk}^m d_m}{d_j},$$

so that $\mathrm{L}_p^p = \sum_l A_l^p w_l$ and $\mathrm{R}_p = \sum_m B_m^{1/p} u_m$, with all four families nonnegative and rational in d.

Lemma 2.2. *Let $q_m \ge 0$ be rationals with $q_m^p \ge B_m$ for every m with $u_m \neq 0$. If $\sum_l A_l^p w_l > \big(\sum_m q_m u_m\big)^p$, then $\mathrm{R}_p < \mathrm{L}_p$. If moreover d is enclosed in rational intervals $0 < \ell_n \le d_n \le h_n$, the same conclusion follows from the rational inequality*

$$\sum_l \Big(\frac{\sum_n N_{lk}^n a_n \ell_n}{h_l}\Big)^p \frac{N_{ij}^l \ell_l}{h_j} \;>\; \Big(\sum_m q_m \frac{N_{jk}^m h_m}{\ell_j}\Big)^p, \qquad q_m^p \ge \frac{\sum_n N_{im}^n a_n h_n}{\ell_m}.$$

Proof. $t \mapsto t^{1/p}$ is increasing on $[0, \infty)$, so $B_m^{1/p} \le q_m$ and $\mathrm{R}_p \le \sum_m q_m u_m$; then $\mathrm{R}_p^p \le (\sum_m q_m u_m)^p < \sum_l A_l^p w_l = \mathrm{L}_p^p$ and $\mathrm{R}_p < \mathrm{L}_p$ since both are nonnegative. For the second statement, every A_l, w_l is bounded below and every B_m, u_m above by the displayed rational expressions, by monotonicity in each d_n separately. ∎

For the two rings of Section 3 the dimensions are pinned exactly and the certificate is decided with $p = 2$ inside the formal development, where the shape lemma used is simply that $t \mapsto t^{1/2}$ is strictly increasing. For the sweep of Section 6 the intervals $[\ell_n, h_n]$ come from the Collatz–Wielandt bounds:

for a nonnegative matrix A and any positive vector v, $\min_m (Av)_m/v_m \le \rho(A) \le \max_m (Av)_m/v_m$ [13, Thm. 8.1.26], applied to $A = N_n$ with a rational v; the bounds are integer arithmetic whatever v is, so floating point only chooses v and never enters a decision.

3. The two printed rings

The source prints two fusion rings in its Example 3.6 as seven and four matrices, "the (n,m)-entry of the l-th matrix" being N_{lm}^n; we transcribe them from the version-2 e-print and give them here as multiplication tables. Both were located in the AnyonWiki database [3] by canonical form under relabelling of the non-unit basis elements, which is how the identifiers below are known; the database names a ring by [rank, multiplicity, #non-self-dual, index].

3.1. **The rank-7 ring** $[7,3,2,117]$**.** The basis is $b_1,\dots,b_7$ with $b_1 = 1$; the source names it "the simplest integral example among those that cannot be even ruled out by the criteria in [6] and [7]". The elements b_1,b_2,b_3,b_4 form the Klein four-group ($b_2^2 = b_3^2 = b_4^2 = 1$, $b_2b_3 = b_4$, $b_3b_4 = b_2$, $b_4b_2 = b_3$); each of b_2,b_3,b_4 fixes b_5,b_6,b_7 under multiplication on either side; and

$$b_5^2 = b_1 + b_2 + b_3 + b_4, \quad b_5b_6 = b_6b_5 = 2b_6, \quad b_5b_7 = b_7b_5 = 2b_7,$$

$$b_6^2 = b_6 + 3b_7, \quad b_7^2 = 3b_6 + b_7, \quad b_6b_7 = b_7b_6 = b_1 + b_2 + b_3 + b_4 + 2b_5 + b_6 + b_7.$$

The ring is commutative, $\bar{6} = 7$ and every other element is self-dual, and the largest structure constant is 3: rank 7, multiplicity 3, two non-self-dual elements, as the identifier says. Its Frobenius–Perron dimensions are $(1,1,1,1,2,4,4)$.

Lemma 3.1 (the dimensions are forced)**.** *Every positive character of* $[7,3,2,117]$ *is* $d = (1,1,1,1,2,4,4)$.

Proof. From (3): $d_1^2 = d_1$ and $d_1 > 0$ give $d_1 = 1$; $d_2^2 = d_3^2 = d_4^2 = d_1 = 1$ with positivity give $d_2 = d_3 = d_4 = 1$; $d_5d_6 = 2d_6$ with $d_6 > 0$ gives $d_5 = 2$; finally $d_6^2 = d_6 + 3d_7$ and $d_7^2 = 3d_6 + d_7$ subtract to $(d_6 - d_7)(d_6 + d_7 + 2) = 0$, so $d_6 = d_7$ by positivity, and then $d_6^2 = 4d_6$ gives $d_6 = d_7 = 4$. (Formally: `Ring7.dims`.) □

Theorem 3.2. *Let* $a = (1,1,1,1,1,0,0)$ *and* $(i,j,k) = (7,6,7)$*. For every positive character* d *of the fusion ring* $[7,3,2,117]$*,*

$$\mathrm{R}_2(d,a;7,6,7) < \mathrm{L}_2(d,a;7,6,7);$$

indeed $\mathrm{L}_2(d,a;7,6,7) = 2$ *and* $\mathrm{R}_2(d,a;7,6,7) = \sqrt{2}$*. Thus* $(a,2,7,6,7)$ *is an exact certificate of exclusion for* $[7,3,2,117]$.

Proof. By Lemma 3.1, $d = (1,1,1,1,2,4,4)$. With $k = 7$ and a supported on $\{1,\dots,5\}$, $A_l = \big(\sum_{n\le 5} N_{l7}^n d_n\big)/d_l$: for $l \le 4$ and $l = 5$ the product b_lb_7 is a multiple of b_7, and $b_7^2 = 3b_6 + b_7$, so $A_l = 0$ for $l \ne 6$, while $b_6b_7 = b_1 + b_2 + b_3 + b_4 + 2b_5 + b_6 + b_7$ gives $A_6 = (1+1+1+1+2\cdot 2)/4 = 2$. The weights are $w_l = N_{76}^l d_l/d_6$, and $w_6 = 1$. Hence $\mathrm{L}_2^2 = A_6^2 w_6 = 4$ and $\mathrm{L}_2 = 2$. On the right, $B_m = \big(\sum_{n\le 5} N_{7m}^n d_n\big)/d_m$ vanishes for $m \ne 6$ by the same products and $B_6 = 8/4 = 2$; the weights are $u_m = N_{67}^m d_m/d_6$ with $u_6 = 1$; so $\mathrm{R}_2 = \sqrt{2}$. Since $\sqrt{2} < 2$ the claim follows. Formally (`Ring7.excluded`), the two sides are rewritten to $4^{1/2}$ and $2^{1/2}$ by evaluating the finite sums with the pinned dimensions, and compared by the strict monotonicity of $t \mapsto t^{1/2}$. □

The source's certificate for this ring is $1/p = 0.393251890984615$, the seven-component vector a printed in its Example 3.6(2), the same triple $(7,6,7)$, and $\mathrm{R}/\mathrm{L} = 0.767708161620070$. Recomputing that ratio from the printed data and the source's own definitions, with the dimensions obtained by power iteration, reproduces the printed value to thirteen significant digits, $0.7677081616200\dots$ (the last digits of a double-precision ratio depend on the order of summation). The ratio of the exact certificate is $1/\sqrt{2} = 0.70710678\dots$, smaller, and every number in it is an integer.

3.2. **The rank-4 ring** $[4,1,2,3]$**.** The basis is $b_1 = 1, b_2, b_3, b_4$ with b_3, b_4 generating a copy of $\mathbb{Z}/3$ ($b_3^2 = b_4$, $b_4^2 = b_3$, $b_3b_4 = b_4b_3 = 1$), b_2 absorbing it ($b_2b_3 = b_3b_2 = b_2b_4 = b_4b_2 = b_2$), and

$$b_2^2 = b_1 + b_2 + b_3 + b_4.$$

Thus b_1, b_3, b_4 are the invertible basis elements, b_2 is the only non-invertible one, and $b_2^2 = \sum_{g\in\mathbb{Z}/3} g + 1\cdot b_2$: the ring carries the near-group fusion rule $(\mathbb{Z}/3, 1)$ in the sense of Siehler [14]. The source names it "the simplest such a fusion ring", the simplest ring that passes the primary 3-criterion of [4] and fails (1). It is not integral: its Frobenius–Perron dimensions are $(1, \delta, 1, 1)$ with $\delta = \frac{1+\sqrt{13}}{2}$, the positive root of $\delta^2 = \delta + 3$.

Lemma 3.3 (the dimensions are forced)**.** *Every positive character d of $[4,1,2,3]$ has $d_1 = d_3 = d_4 = 1$ and $d_2^2 = d_2 + 3$, hence $1 < d_2 < 5/2$.*

Proof. $d_1 = 1$ as before. From $b_3^2 = b_4$ and $b_3b_4 = b_1$, $d_3^3 = d_3d_4 = 1$, so $(d_3 - 1)(d_3^2 + d_3 + 1) = 0$ and $d_3 = 1$ by positivity, whence $d_4 = d_3^2 = 1$. Then $b_2^2 = b_1 + b_2 + b_3 + b_4$ gives $d_2^2 = d_2 + 3$. The bounds: if $0 < d_2 \le 1$ then $d_2^2 \le 1 < d_2 + 3$, and if $d_2 \ge 5/2$ then $d_2^2 - d_2 - 3 \ge (5/2)^2 - 5/2 - 3 = 3/4 > 0$ since $t \mapsto t^2 - t - 3$ is increasing for $t \ge 1/2$; both contradict $d_2^2 = d_2 + 3$. (Formally: `Ring4.dims`.) □

Theorem 3.4. *Let $a = (1,0,1,1)$ and $(i,j,k) = (2,2,2)$. For every positive character d of the fusion ring $[4,1,2,3]$,*

$$\mathrm{R}_2(d, a; 2, 2, 2) < \mathrm{L}_2(d, a; 2, 2, 2);$$

indeed, writing $\delta = d_2$, $\mathrm{L}_2(d, a; 2, 2, 2)^2 = 9/\delta^2 = 4 - \delta$ and $\mathrm{R}_2(d, a; 2, 2, 2)^2 = 3/\delta = \delta - 1$, and the inequality is equivalent to $\delta < 5/2$. Thus $(a, 2, 2, 2, 2)$ is an exact certificate of exclusion for $[4,1,2,3]$.

Proof. By Lemma 3.3, $d = (1, \delta, 1, 1)$ with $\delta^2 = \delta + 3$ and $\delta < 5/2$. With $k = 2$ and a supported on $\{1,3,4\}$, $A_l = \big(\sum_{n\in\{1,3,4\}} N_{l2}^n d_n\big)/d_l$ vanishes unless b_lb_2 contains one of b_1, b_3, b_4, which happens only for $l = 2$: $A_2 = (1 + 1 + 1)/\delta = 3/\delta$. The weight is $w_2 = N_{22}^2 d_2/d_2 = 1$, so $\mathrm{L}_2^2 = 9/\delta^2$. On the right, $B_m = \big(\sum_{n\in\{1,3,4\}} N_{2m}^n d_n\big)/d_m$ is nonzero only for $m = 2$, $B_2 = 3/\delta$, with weight $u_2 = 1$, so $\mathrm{R}_2^2 = 3/\delta$. Now $9/\delta^2 > 3/\delta$ if and only if $3 > \delta$, which holds since $\delta < 5/2$. The identities $9/\delta^2 = 4 - \delta$ and $3/\delta = \delta - 1$ follow from $\delta^2 = \delta + 3$ ($(4 - \delta)(\delta + 3) = 9$ and $(\delta - 1)\delta = 3$), and $4 - \delta > \delta - 1$ is $\delta < 5/2$. Formally (`Ring4.excluded`) the sides are rewritten to $(9/\delta^2)^{1/2}$ and $(3/\delta)^{1/2}$ and compared through $9/\delta^2 - 3/\delta = (9 - 3\delta)/\delta^2 > 0$. □

The source's certificate here is $1/p = 0.661975038859587$, $i = j = k = 2$,

$$a = (0.778280214647139,\ 0.006040665900893,\ 0.698943990142325,\ 0.856127138742457),$$

and $\mathrm{R}/\mathrm{L} = 0.923133094619110$, which we recompute to the same thirteen digits, $0.9231330946191\ldots$. The exact certificate has ratio $\sqrt{\delta/3} = 0.87612321\ldots$. The point of this ring is that the exact route does not need integrality: nothing about δ is used beyond its minimal relation.

Corollary 3.5. *By Corollary C of [1], neither $[7,3,2,117]$ nor $[4,1,2,3]$ admits a unitary categorification.*

The corollary is the source's implication applied to Theorems 3.2 and 3.4; the implication is not part of the formal development.

Remark 3.6 (the rank-4 conclusion is not new). For $[4,1,2,3]$ the conclusion of Corollary 3.5 is reached by older work on near-group fusion rules, and we do not claim it: for abelian G, a near-group fusion rule (G, k) with $k \ge 1$ admits a monoidal structure only if $|G| \le k + 1$ [14, Thm. 1.1], and the unitary near-group categories over a finite cyclic G obey the dichotomy $k = |G| - 1$ or $|G| \mid k$ [15]; the rule $(\mathbb{Z}/3, 1)$ of this ring obeys neither. What Theorem 3.4 adds is an exact certificate *by the criterion at hand*, on a non-integral ring, in place of the source's floating-point one. The rank-7 ring is not of this shape: it has three non-invertible basis elements.

4. Two identities

Proposition 4.1. *Let $N = (N_{ij}^k)$ be any array of nonnegative integers satisfying* (2)*, let $d \in \mathbb{R}^s$ have no zero entry, and let $a \in \mathbb{R}^s$. Then for every triple (i, j, k),*

$$\mathrm{L}_1(d, a; i, j, k) = \mathrm{R}_1(d, a; i, j, k) = \frac{1}{d_j} \sum_n \Bigl(\sum_l N_{ij}^l N_{lk}^n \Bigr) a_n d_n.$$

In particular, for each of the rings $[7, 3, 2, 117]$ *and* $[4, 1, 2, 3]$*, every positive character d, every $a \in \mathbb{R}^s$ and every triple,* $\mathrm{L}_1(d, a; i, j, k) = \mathrm{R}_1(d, a; i, j, k)$.

Proof. At $p = 1$ the powers disappear: $\mathrm{L}_1 = \sum_l \frac{\sum_n N_{lk}^n a_n d_n}{d_l} \cdot \frac{N_{ij}^l d_l}{d_j} = \frac{1}{d_j} \sum_n \bigl(\sum_l N_{ij}^l N_{lk}^n \bigr) a_n d_n$ and $\mathrm{R}_1 = \sum_m \frac{\sum_n N_{im}^n a_n d_n}{d_m} \cdot \frac{N_{jk}^m d_m}{d_j} = \frac{1}{d_j} \sum_n \bigl(\sum_m N_{im}^n N_{jk}^m \bigr) a_n d_n$, and the two inner sums agree by (2). The two rings are associative (`Ring7.assoc`, `Ring4.assoc`, checked by the kernel on all 7^4 and 4^4 instances of (2)), and a positive character has no zero entry. The general identity is `mink_eq_at_one`; the two instances for the rings are `Ring7.no_violation_at_one` and `Ring4.no_violation_at_one`. □

Proposition 4.2. *Let R be any fusion ring, d a positive character of R, $p > 0$ and (i, j, k) a triple. Then, for $\mathbf{1} = (1, \ldots, 1)$,*

$$\mathrm{L}_p(d, \mathbf{1}; i, j, k) = \mathrm{R}_p(d, \mathbf{1}; i, j, k).$$

Proof. By (3), $A_l = \bigl(\sum_n N_{lk}^n d_n \bigr) / d_l = d_l d_k / d_l = d_k$ for every l and $\sum_l w_l = \sum_l N_{ij}^l d_l / d_j = d_i d_j / d_j = d_i$, so $\mathrm{L}_p = (d_k^p d_i)^{1/p} = d_k d_i^{1/p}$. Likewise $B_m = d_i$ and $\sum_m u_m = d_k$, so $\mathrm{R}_p = d_i^{1/p} d_k$. (Formally `mink_eq_ones`; its statement is the equality, and the common value $d_k d_i^{1/p}$ appears in its proof. The hypothesis is only that d be a positive character; associativity and the unit are not used.) □

Observation 4.3. The source samples a uniformly from $[0, 1]^s$ and $1/p$ uniformly from $(0.1, 1]$ and descends on R/L. Proposition 4.2 says that at the corner $a = \mathbf{1}$ of that cube the objective equals 1 exactly, for every p; Proposition 4.1 says that on the whole face $1/p = 1$ it equals 1 exactly, for every a, on every associative ring. Neither corner can ever produce a certificate, and a descent that starts near either begins on a level set of the value it is trying to push below 1. We record this as a property of the objective; we do not know which, if any, of the source's misses it accounts for, and the sweep of Section 6 finds most of its certificates far from both corners ($p = 10$ and sparse a).

5. Controls

The hypothesis "for every positive character" in Theorems 3.2 and 3.4 would be vacuous on a ring with none.

Proposition 5.1. $(1, 1, 1, 1, 2, 4, 4)$ *is a positive character of* $[7, 3, 2, 117]$*, and* $\bigl(1, \frac{1+\sqrt{13}}{2}, 1, 1\bigr)$ *is a positive character of* $[4, 1, 2, 3]$.

Proof. Direct verification of (3): 49 equations for the first ring, all of them integer identities, and 16 for the second, where $b_2^2 = b_1 + b_2 + b_3 + b_4$ becomes $\delta^2 = \delta + 3$ and is checked from $\sqrt{13}^2 = 13$; positivity of δ uses $\sqrt{13} > 3$. (Formally `Ring7.dFP_isPosChar` and `Ring4.dFP_isPosChar`.) □

Together with Lemmas 3.1 and 3.3 this says each ring has exactly one positive character, as the general theory predicts. The second control is that the criterion does not fire on everything.

Proposition 5.2. *Let $R = \mathbb{Z}[\mathbb{Z}/2]$, the Grothendieck ring of* $\mathrm{Vec}_{\mathbb{Z}/2}$*, with basis $b_1 = 1$, $b_2 = g$, $g^2 = 1$, and $d = (1, 1)$. Then for every $a \in \mathbb{R}^2$ and every triple (i, j, k),* $\mathrm{L}_2(d, a; i, j, k) = \mathrm{R}_2(d, a; i, j, k)$.

Proof. Write the indices additively in $\mathbb{Z}/2$, so $N_{ij}^k = 1$ if $k = i + j$ and 0 otherwise. Then $\mathrm{L}_2^2 = \sum_l \bigl(\sum_n N_{lk}^n a_n \bigr)^2 N_{ij}^l = a_{i+j+k}^2$ and $\mathrm{R}_2 = \sum_m \bigl(\sum_n N_{im}^n a_n^2 \bigr)^{1/2} N_{jk}^m = (a_{i+j+k}^2)^{1/2}$, so both sides equal $|a_{i+j+k}|$. (Formally `Z2.no_violation`, by case analysis over the eight triples; `Z2.dOne_isPosChar` checks that $(1, 1)$ is a positive character.) □

$\mathrm{Vec}_{\mathbb{Z}/2}$ is a unitary fusion category, so the source's Corollary C requires exactly this, and a violation here would refute it.

Remark 5.3 (a control at scale, outside the formal development). The AnyonWiki database flags 60 of its rings as having a unitary categorification (`has_categories_with_props.Unitary`, and the same 60 under `Fusion`). None of the 60 appears among the 6,205 rings certified excluded in Section 6. A single overlap would have been either an error in our code or a counterexample to the source's theorem.

6. The exclusion list, recomputed

Everything in this section is exact rational arithmetic in ordinary code, not Lean, and is reported as computation. Theorems 3.2 and 3.4 are the two of its certificates that are machine-checked.

6.1. **The database and how it has moved.** The rings are the AnyonWiki fusion-ring database [3], file `AlgebraicStructures/FusionRings.tar.gz` at repository commit `41c1452c`, downloaded 2026-09-07 (SHA-256 `e8a67c1b…b6fe`), which is the database of Vercleyen and Slingerland [2] and the reference the source cites for its 28,450 rings. It holds 25,332 entries of ranks 1 to 10; we drop the trivial ring of rank 1 and work with 25,331 rings of ranks 2 to 10. Each entry carries the multiplication table, floating-point Frobenius–Perron dimensions, and the categorifiability flags used in Remark 5.3.

It is not the source's dataset. The source's Table 1 tabulates its 28,450 rings by rank (2–9) and multiplicity $\max N_{ij}^k$ (1–16). The current database agrees with that table cell for cell except at rank 5, multiplicities 9 to 12:

rank 5, multiplicity	9	10	11	12
source's Table 1	1463	1794	2283	3049
current database	863	1082	1383	1948
difference	600	712	900	1101

The four differences sum to 3,313, and $28{,}450 - (25{,}331 - 194) = 3{,}313$ exactly, where 194 is the number of rank-10 rings, which the current database has and the source did not test. Every other cell of the grid matches. We do not know the reason for the rank-5 change and do not speculate; the database is under active development. Two consequences: totals are comparable only rank by rank, and the honest comparison is on the 11,739 rings of ranks $2, 3, 4, 6, 7, 8, 9$, where the two datasets agree cell for cell. (Three counts want reconciling. [2] reports 28,451 fusion rings, and its own rank/multiplicity table — ranks 1 to 9, with a single ring of rank 1 — sums to that figure; the source's Table 1 reproduces that table cell for cell on ranks 2 to 9, so its 28,450 is 28,451 less the trivial ring. The introduction of [1] instead speaks of "28,541 fusion rings in [2]", a transposition of its own; and its "about 68.37%" for the 19,738 rings that fail the primary 3-criterion is carried over from the introduction of [4], whose own body gives 69.37% for the same two numbers — which is the ratio 19,738/28,451.)

6.2. **The search and its exactness.** *Search family.* $p \in \{2, \ldots, 10\}$, $a \in \{0,1\}^s \setminus \{0\}$, all s^3 triples (i, j, k); $p = 1$ is omitted by Proposition 4.1, $p \leq 9$ is the source's window $1/p \in (0.1, 1]$ restricted to integers, and $p = 10$ is the excluded endpoint $1/p = 0.1$ of that window, kept because it is where most certificates turn out to lie.

Screen. A C program evaluates R/L in double precision for every ring and every element of the family, using the database's floating-point dimensions, and reports for each ring the element with the smallest ratio if that ratio is below $1 - 10^{-9}$. The screen is a filter and asserts nothing.

Exact decision. For each screened ring the reported (a, p, i, j, k) is re-decided by Lemma 2.2 in exact rational arithmetic: a rational vector v is taken from the database's floating-point dimensions, the Collatz–Wielandt bounds of N_l at v give rational $\ell_l \leq \mathrm{FPdim}(b_l) \leq h_l$ for each l (a ring is dropped, not certified, if some $\ell_l \leq 0$), each q_m is a rational with $q_m^p \geq \left(\sum_n N_{im}^n a_n h_n\right)/\ell_m$ verified by one exact comparison, and the ring is certified excluded if and only if the displayed rational inequality of Lemma 2.2 holds. No floating-point number enters a decision. By Lemma 2.2, Lim's Corollary C, and the uniqueness of the positive character, a certified ring admits no unitary categorification.

6.3. **Results.** All 6,205 rings reported by the screen were certified; the exact decision rejected none. The counts by rank are:

rank	2	3	4	5	6	7	8	9	10	total
rings, source's Table 1	17	161	2787	16711	5502	2160	970	142	—	28450
excluded by the source (floating point)	0	21	469	2674	1389	770	419	51	—	5793
rings, current database	17	161	2787	13398	5502	2160	970	142	194	25331
certified here, $p \leq 10$	0	21	509	2563	1625	891	459	51	86	6205
certified here, $p = 2$ only	0	21	501	2399	1581	872	454	51	86	5965
certified here, $p \in \{2,3,4\}$	0	21	505	2465	1605	878	456	51	86	6067

Rank 5 is the only rank at which the two datasets differ. On the other seven ranks — 11,739 rings, identical cell for cell — the source's search excluded $0+21+469+1389+770+419+51 = 3{,}119$ rings and the exact search certifies $0+21+509+1625+891+459+51 = 3{,}556$, that is 437 more on the same rings, each with a certificate that can be re-checked in rational arithmetic. At ranks 3 and 9 the two counts coincide; at ranks $4, 6, 7, 8$ the exact search is strictly ahead. The family $p = 2$, $a \in \{0,1\}^s$ alone certifies 5,965 rings, more than the source's total.

Two further figures describe the certificates. Of the 6,205 best certificates (smallest screened ratio per ring), 6,136 have $p = 10$, the endpoint of the source's window — for those rings the best ratio at $p = 10$ is strictly below the best at any smaller p, so the search family is still improving where it stops; the smallest ratio found is 0.0232 (ring $[5, 16, 0, 1797]$, $p = 10$) and the largest that still certifies is 0.99987 (ring $[5, 14, 0, 442]$, $p = 2$), the latter decided exactly like the rest. For the two printed rings the sweep's best certificates also sit at $p = 10$, with screened ratios 0.536 for $[7, 3, 2, 117]$ and 0.788 for $[4, 1, 2, 3]$; the $p = 2$ certificates of Section 3 were chosen for the simplicity of their proofs, not for their ratios. The whole run was repeated twice for this paper from the committed code and the archived database and reproduced every count and every figure in this section, including the set of 6,205 identifiers and their certificates.

What the list is and is not. It is a lower bound on the set of rings the criterion excludes: the family searched is a thin slice of $\mathbb{R}^s_{\geq 0} \times [1, \infty)$, and a ring not on the list may well be excluded by other (a, p) — indeed the source excluded 111 more rank-5 rings than we do, on a rank-5 population that is 3,313 rings larger, so nothing can be concluded from that cell. A ring absent from the list is not thereby categorifiable. And the list cannot be compared with the source's ring by ring, because the source's list does not exist in print.

6.4. **Cost.** As measured (single core, at reduced scheduling priority, on a shared machine): exporting and canonically matching the database, 44–45 s; the screen over all 25,331 rings and $p \in \{2, \ldots, 10\}$, 9 wall-minutes as originally recorded and 8 m 46 s and 9 m 07 s in two re-runs; the screen at $p = 2$ alone, 45 s originally and 61–64 s in the re-runs; the exact re-decision of the 6,205 hits, 31–34 s. Each re-run reproduced the certified set line for line. The whole search is under twenty processor-minutes against the 97 d 15 h 22 m 12 s the source records. We do not draw a conclusion from the two figures beyond the obvious one, that a finite exact family is cheap to exhaust; the source's continuous family is larger and its search was not designed to be cheap.

7. Remarks and open questions

(1) *The other* 21 *of the* 23. The source reports 23 exclusions that survive the four criteria of [6] and prints one of them, the rank-7 ring of Theorem 3.2. Identifying the remaining 21 among the 6,205 certified rings needs those four criteria and the primary 3-criterion of [4], which for the 24,918 rings with non-integral dimensions is a positive-semidefiniteness test on a matrix of side s^3 over a number field; we have not done it. For the 413 rings whose Frobenius–Perron dimensions are integers (verified exactly) the matrix is rational and the test is cheap; 144 of the 6,205 certified rings are among them.

(2) *Wider exact families.* $a \in \{0,1,2\}^s$ and rational $p = u/v$ (raise the left side to the v-th power throughout) are exact in the same way and were not swept; since 6,136 of the best certificates sit at $p = 10$, the integer window is probably not the end of the criterion's reach.
(3) *More machine-checked certificates.* Every one of the 6,205 certificates has the shape of Theorem 3.2's proof; the only ring-specific work is Lemma 3.1, which for an integral ring is a few lines. For a non-integral ring the route of Theorem 3.4 — pin the dimension by its minimal relation, then decide a polynomial inequality — applies, and automating the pinning from the multiplication table is what would make the whole list formal.

8. Verification

Theorems 3.2 and 3.4, Propositions 4.1, 4.2, 5.1 and 5.2, and Lemmas 3.1 and 3.3 rest on declarations formally verified in Lean 4 against *Mathlib* [16, 17] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. The development is four modules, 546 lines in all, with 15 definitions and 18 theorems. The first (150 lines) defines positive characters, associativity, the two sides (4)–(5) of the criterion with real exponents, and exact exclusion (Definition 2.1 quantified over positive characters), and proves the $p = 1$ identity and two lemmas about real powers. The second and third (109 and 123 lines) transcribe the two printed rings, prove their associativity by kernel evaluation of all 4^4 and 7^4 instances of (2) (`decide`, not compiled evaluation), pin their dimensions (Lemmas 3.3 and 3.1) and prove the two certificates. The fourth (164 lines) holds the identity at $a \equiv 1$, the $p = 1$ identity on both rings, the two positive characters and $\mathrm{Vec}_{\mathbb{Z}/2}$. Where a printed statement says more than its Lean counterpart, the proof names the extra step: the common value $d_k d_i^{1/p}$ in Proposition 4.2, and the values 2, $\sqrt{2}$ and $\sqrt{4-\delta}$, $\sqrt{\delta - 1}$ in Theorems 3.2 and 3.4, which the formal proofs establish as $4^{1/2}$, $2^{1/2}$, $(9/\delta^2)^{1/2}$, $(3/\delta)^{1/2}$ on the way to the inequality. Corollary 3.5, Lemma 2.2, Remark 5.3 and all of Section 6 are outside the formal development. Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand; the axiom print (`#print axioms`) of each of the 18 theorems is `propext`, `Classical.choice`, `Quot.sound`. With *Mathlib* already loaded, the four modules elaborate in 11, 16, 31 and 12 seconds; the longest is the rank-7 module, whose associativity check is a `decide` over 7^4 instances. The exact sweep is a 58-line C screen and a 98-line Python re-decision in exact fractions, with a 73-line export from the database; the two ratios of the source's Example 3.6 were reproduced from its printed data before anything else was computed. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). Indices are 0-based, so the triple $(7,6,7)$ of Theorem 3.2 is `6 5 6` and $(2,2,2)$ of Theorem 3.4 is `1 1 1`; `Fin s → Fin s → Fin s → ℕ` is the array N, with `N i j k` $= N_{ij}^k$; `x ^ (p : ℝ)` is Mathlib's real power `Real.rpow`; `![...]` is a vector literal. The two rings' tables are the printed matrices, `mat l n m` $= N_{lm}^n$.

Definitions.

```
def IsPosChar (N : Fin s → Fin s → Fin s → ℕ) (d : Fin s → ℝ) : Prop :=
  (∀ i, 0 < d i) ∧ ∀ i j, d i * d j = ∑ n, (N i j n : ℝ) * d n

def IsAssoc (N : Fin s → Fin s → Fin s → ℕ) : Prop :=
  ∀ i j k n, ∑ l, N i j l * N l k n = ∑ m, N i m n * N j k m

noncomputable def minkLHS (N : Fin s → Fin s → Fin s → ℕ) (d a : Fin s → ℝ) (p : ℝ)
    (i j k : Fin s) : ℝ :=
  (∑ l, ((∑ n, (N l k n : ℝ) * a n * d n) / d l) ^ p * ((N i j l : ℝ) * d l / d j)) ^ (1 / p)

noncomputable def minkRHS (N : Fin s → Fin s → Fin s → ℕ) (d a : Fin s → ℝ) (p : ℝ)
    (i j k : Fin s) : ℝ :=
```

```
  ∑ m, ((∑ n, (N i m n : ℝ) * a n ^ p * d n) / d m) ^ (1 / p) * ((N j k m : ℝ) * d m / d j)

def MinkExcluded (N : Fin s → Fin s → Fin s → ℕ) (a : Fin s → ℝ) (p : ℝ) (i j k : Fin s) :
    Prop :=
  ∀ d : Fin s → ℝ, IsPosChar N d → minkRHS N d a p i j k < minkLHS N d a p i j k

-- Ring7 (AnyonWiki [7,3,2,117]) and Ring4 (AnyonWiki [4,1,2,3])
def Ring7.mat : Fin 7 → Fin 7 → Fin 7 → ℕ := ![ ... the seven printed matrices ... ]
def Ring7.N (l m n : Fin 7) : ℕ := Ring7.mat l n m
def Ring7.a : Fin 7 → ℝ := ![1, 1, 1, 1, 1, 0, 0]
def Ring7.dFP : Fin 7 → ℝ := ![1, 1, 1, 1, 2, 4, 4]

def Ring4.mat : Fin 4 → Fin 4 → Fin 4 → ℕ :=
  ![![![1,0,0,0],![0,1,0,0],![0,0,1,0],![0,0,0,1]],
    ![![0,1,0,0],![1,1,1,1],![0,1,0,0],![0,1,0,0]],
    ![![0,0,0,1],![0,1,0,0],![1,0,0,0],![0,0,1,0]],
    ![![0,0,1,0],![0,1,0,0],![0,0,0,1],![1,0,0,0]]]
def Ring4.N (l m n : Fin 4) : ℕ := Ring4.mat l n m
def Ring4.a : Fin 4 → ℝ := ![1, 0, 1, 1]
noncomputable def Ring4.dFP : Fin 4 → ℝ := ![1, (1 + Real.sqrt 13) / 2, 1, 1]

def Z2.N (l m n : Fin 2) : ℕ := if n = l + m then 1 else 0
def Z2.dOne : Fin 2 → ℝ := ![1, 1]
```

The seven 7×7 matrices of `Ring7.mat` are the source's, row by row, and are elided here for space; they are printed in Section 3.1 as a multiplication table.

The certificates (Theorems 3.2, 3.4) and the pinned dimensions (Lemmas 3.1, 3.3).

```
theorem Ring7.assoc : IsAssoc Ring7.N

theorem Ring7.dims {d : Fin 7 → ℝ} (h : IsPosChar Ring7.N d) :
    d 0 = 1 ∧ d 1 = 1 ∧ d 2 = 1 ∧ d 3 = 1 ∧ d 4 = 2 ∧ d 5 = 4 ∧ d 6 = 4

theorem Ring7.excluded : MinkExcluded Ring7.N Ring7.a 2 6 5 6

theorem Ring4.assoc : IsAssoc Ring4.N

theorem Ring4.dims {d : Fin 4 → ℝ} (h : IsPosChar Ring4.N d) :
    d 0 = 1 ∧ d 2 = 1 ∧ d 3 = 1 ∧ d 1 * d 1 = d 1 + 3 ∧ 1 < d 1 ∧ d 1 < 5 / 2

theorem Ring4.excluded : MinkExcluded Ring4.N Ring4.a 2 1 1 1
```

The identities (Propositions 4.1, 4.2).

```
theorem mink_eq_at_one {N : Fin s → Fin s → Fin s → ℕ} {d a : Fin s → ℝ}
    (hd : ∀ i, d i ≠ 0) (hassoc : IsAssoc N) (i j k : Fin s) :
    minkLHS N d a 1 i j k = minkRHS N d a 1 i j k

theorem Ring7.no_violation_at_one {d aa : Fin 7 → ℝ} (h : IsPosChar Ring7.N d)
    (i j k : Fin 7) :
    minkLHS Ring7.N d aa 1 i j k = minkRHS Ring7.N d aa 1 i j k

theorem Ring4.no_violation_at_one {d aa : Fin 4 → ℝ} (h : IsPosChar Ring4.N d)
    (i j k : Fin 4) :
    minkLHS Ring4.N d aa 1 i j k = minkRHS Ring4.N d aa 1 i j k
```

```
theorem mink_eq_ones {N : Fin s → Fin s → Fin s → ℕ} {d : Fin s → ℝ}
    (h : IsPosChar N d) {p : ℝ} (hp : 0 < p) (i j k : Fin s) :
    minkLHS N d (fun _ => 1) p i j k = minkRHS N d (fun _ => 1) p i j k
```

Controls (Propositions 5.1, 5.2).

```
theorem Ring7.dFP_isPosChar : IsPosChar Ring7.N Ring7.dFP

theorem Ring4.dFP_isPosChar : IsPosChar Ring4.N Ring4.dFP

theorem Z2.dOne_isPosChar : IsPosChar Z2.N Z2.dOne

theorem Z2.no_violation (aa : Fin 2 → ℝ) (i j k : Fin 2) :
    minkLHS Z2.N Z2.dOne aa 2 i j k = minkRHS Z2.N Z2.dOne aa 2 i j k
```

Shape lemmas.

```
theorem rpow_two_eq (x : ℝ) : x ^ (2 : ℝ) = x ^ (2 : ℕ)

theorem rpow_half_lt {X Y : ℝ} (hY : 0 ≤ Y) (h : Y < X) :
    Y ^ ((1 : ℝ) / 2) < X ^ ((1 : ℝ) / 2)

theorem Ring4.delta_sq : ((1 + Real.sqrt 13) / 2) * ((1 + Real.sqrt 13) / 2)
    = (1 + Real.sqrt 13) / 2 + 3

theorem Ring4.sqrt13_gt : (3 : ℝ) < Real.sqrt 13
```

References

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