THE TRIANGLE OF FUNDAMENTAL ONE-DIMENSIONAL DISCRETE MODELS OF MAXIMUM LIKELIHOOD DEGREE ONE AND ITS REFLECTION-SYMMETRIC REFINEMENT

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. A one-dimensional discrete statistical model of maximum likelihood degree one is, after Bik and Marigliano, the image of a parameterisation $t \mapsto (c_i\, t^{\nu_i}(1-t)^{\mu_i})_{i=0}^{n}$ of the probability simplex Δ_n; it is *fundamental* when the exponent pairs (ν_i, μ_i) determine the scalings c_i. Améndola, Nguyen and Oldekop proved that the degree of such a model is at most $2n-1$, tabulated the number $T(n,d)$ of fundamental models in Δ_n of degree d for $n \le 7$, and conjectured that $T(n, 2n-2) = 2\sum_{k=1}^{n-1} a_k a_{n-k}$ with $a_k = T(k, 2k-1)$. We introduce the refinement $T^{\mathrm{sym}}(n,d)$, the number of those models whose support is invariant under the reflection $(\nu,\mu) \mapsto (\mu,\nu)$, i.e. under the reparameterisation $t \mapsto 1-t$; we prove that the reflection preserves fundamentality for every support, so that $T(n,d) \equiv T^{\mathrm{sym}}(n,d) \pmod 2$, and we compute $T^{\mathrm{sym}}(n,d)$ for $n \le 6$ and on the main diagonal up to $T^{\mathrm{sym}}(8,8) = 323$. The values are in no index we could find. On the almost sharp cells $d = 2n-2$ we find $T^{\mathrm{sym}}(n, 2n-2) = 0$ for $3 \le n \le 6$: no almost sharp fundamental model is its own reflection, which is the step left implicit in the published proof of the lower bound. Rows $n \le 4$ of both triangles, including the value $T(3,3) = 12$ that corrects an entry of Lebl and Lichtblau, are verified in the Lean 4 kernel; rows 5–7 and the value $T(8,8) = 1\,279\,349$, printed only in a figure of the source, are recomputed independently by an exact search, whose measured cost we use to price the remaining open cells.

1. INTRODUCTION

1.1. **The objects.** Let $\Delta_n = \{p \in \mathbb{R}^{n+1} : p_0 + \cdots + p_n = 1,\ p_i \ge 0\}$ be the probability simplex. By a result of Bik and Marigliano ([2, Proposition 2.2], as cited in [1, Section 2]), every one-dimensional discrete model in Δ_n whose maximum likelihood estimator is a rational function of the data — a model of *maximum likelihood degree one* — is the image of a parameterisation

$$p\colon [0,1] \to \Delta_n, \qquad t \longmapsto \big(c_i\, t^{\nu_i}(1-t)^{\mu_i}\big)_{i=0}^{n}, \qquad c_i > 0,\ \nu_i, \mu_i \in \mathbb{N}. \tag{1}$$

Following [1] such a model is *reduced* when the exponent pairs (ν_i, μ_i) are pairwise distinct and different from $(0,0)$, its *degree* is $\max_i(\nu_i + \mu_i)$, and a reduced model is *fundamental* [1, Definition 2.1] if, given the exponent pairs, the scalings c_i are uniquely determined by the constraint $p_0 + \cdots + p_n = 1$. A fundamental model is therefore the same thing as its *support* $S = \{(\nu_i, \mu_i)\}$, and one may count supports. Writing $x = t$ and $y = 1-t$, the constraint says that the bivariate polynomial $\sum_i c_i x^{\nu_i} y^{\mu_i}$, which has non-negative coefficients, equals 1 on the line $x + y = 1$. Such polynomials are the classical objects of the theory of proper monomial maps between balls in CR geometry; Lebl and Lichtblau [3] write $H(2,d)$ for the set of them of degree d, and [1] is built on this correspondence.

1.2. **The source and its tables.** Améndola, Nguyen and Oldekop [1] prove that every model (1) has degree at most $2n-1$ (their Theorem 2.2, stated without a number in their introduction; it settles [2, Conjecture 8.1], proved there for $n \le 4$ as [2, Theorem 1.2]), so that there are finitely many fundamental models in each Δ_n (their Corollary 2.3). They show that a fundamental model in Δ_n of degree d exists exactly when $n \le d \le 2n-1$ (their Propositions 6.1 and 6.2), call a reduced model of degree $2n-1$ *sharp* and of degree $2n-2$ *almost sharp* (their Definition 5.1), and define

$$T(n,d) = \#\{\text{fundamental models in } \Delta_n \text{ of degree } d\}.$$

Their Table 1 prints $T(n,d)$ for $1 \le n \le 7$ completely, and their Figure 9 repeats the table as a triangle and adds the two computed values $T(8,8) = 1\,279\,349$ and $T(8,9) = 285\,601$, the two values $T(8,15) = 4$ and $T(9,17) = 2$ taken from [3, Table 2], and the two conjectured values $T(8,14) = 96$ and $T(9,16) = 112$; every other cell of rows 8 and 9 is a question mark. Rows $n \le 5$ had already been computed by Bik and

Marigliano [2, Table 1], and the three top diagonals $d = 2n-1$, $2n-2$, $2n-3$ for small n by Lebl and Lichtblau [3, Table 2], who entered them in the OEIS as A143107, A143108 and A143109 [5]. Table 2 of [3] gives 11 for the cell $(n,d) = (3,3)$; [1, Proposition 6.3] ("Corrigendum") corrects this to 12, and the corrected diagonal is OEIS A387029. On the sharp diagonal two further values are known, $T(10,19) = 24$ and $T(11,21) = 2$, from Lebl's addendum [4], which reports that for degree 21 "the computation took over 8 months". The whole triangle is OEIS A386841, entered by the first author of [1]; its data stop one cell into row 8.

The recursion that Lebl and Lichtblau conjectured for the almost sharp diagonal is proved in [1] as a lower bound and left open as an equality: with $a_k = T(k, 2k-1)$,

$$T(n, 2n-2) \;\geq\; 2\,(a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1) \qquad (n \geq 3) \tag{2}$$

is [1, Proposition 6.4], and [1, Conjecture 6.5] asserts equality. The conjectured 96 and 112 of Figure 9 are the right-hand side of (2) at $n = 8, 9$. The proof of (2) composes a sharp model in Δ_k with one in Δ_{n-k}, obtaining $\sum_k a_k a_{n-k}$ models, and then doubles the count: "substituting $s := 1-t$ yields another fundamental model in Δ_n of degree $2n-2$. The substitution corresponds to a reflection on the main diagonal in the corresponding chip configuration." The doubling silently requires the reflected model to differ from the original.

1.3. **What this paper adds.** The substitution $t \mapsto 1-t$ acts on supports as the reflection $\sigma(\nu,\mu) = (\mu,\nu)$. We study the statistic

$$T^{\mathrm{sym}}(n,d) = \#\{\text{fundamental models in } \Delta_n \text{ of degree } d \text{ whose support } S \text{ satisfies } \sigma(S) = S\},$$

which is defined neither in [1] nor in any of the six OEIS entries it cites, and whose values match no OEIS entry (Remark 5.4). Our results are the following.

(i) For every support S, $\sigma(S)$ is fundamental if and only if S is (Theorem 3.1). Hence σ is an involution of the set of fundamental supports of each cell, $T^{\mathrm{sym}}(n,d)$ is its number of fixed points, and $T(n,d) \equiv T^{\mathrm{sym}}(n,d) \pmod 2$ (Remark 3.2). This explains the odd entries 3, 2421, 643 and 1 279 349 of the source's data.

(ii) The values of $T^{\mathrm{sym}}(n,d)$ for $n \leq 6$, together with $T^{\mathrm{sym}}(7,7) = 80$ and $T^{\mathrm{sym}}(8,8) = 323$ (Table 2); rows $n \leq 4$ are verified in the Lean 4 kernel (Theorem 4.4), the rest by an exact search (Remark 5.3).

(iii) $T^{\mathrm{sym}}(n, 2n-2) = 0$ for $n = 3, 4$ in the kernel and for $n = 5, 6$ by the exact search: no almost sharp fundamental model is its own reflection, so the reflection in the proof of (2) always produces a different model in these cases, and the cell splits into two-element orbits. This does not by itself show that the factor 2 in (2) is exact — that would also need the composed models to be disjoint from their reflections, which we do not settle (Section 6).

(iv) Rows $n \leq 4$ of $T(n,d)$, including $T(3,3) = 12 \neq 11$, are verified in the kernel from the definition (Theorems 4.1, 4.2, 4.3); rows 5–7 (as far as a fixed budget allows) and the value $T(8,8) = 1\,279\,349$, which appears only in Figure 9 of [1], are recomputed independently (Remarks 5.1, 5.2).

(v) The measured cost of that search prices the open cells of rows 8 and 9; the cell $T(8,14)$ that would test Conjecture 6.5 of [1] at $n = 8$ costs about 1400 CPU-hours with the present search (Remark 5.5).

Items (i)–(iv) in their kernel-verified form are machine-checked in Lean 4 over Mathlib (Section 7); everything else is a computation outside the formal development and is labelled as such.

2. Preliminaries

2.1. **Supports and the linear system.** A *support* is a finite set $S \subset \mathbb{N}^2 \setminus \{(0,0)\}$; its size is $|S|$ and its degree is $\deg S = \max_{(\nu,\mu)\in S}(\nu+\mu)$. For a field $K \supseteq \mathbb{Q}$ a *K-scaling* of S is a vector $c = (c_s)_{s\in S} \in K^S$ with

$$\sum_{(\nu,\mu)\in S} c_{(\nu,\mu)}\, t^{\nu}(1-t)^{\mu} \;=\; 1 \qquad \text{as polynomials in } t. \tag{3}$$

Equating coefficients of t^j, $0 \le j \le \deg S$, (3) is a linear system $A_S\, c = e_0$ with an integer matrix A_S of size $(\deg S + 1) \times |S|$; equivalently, after homogenisation [1, Lemma 3.2], $\sum_s c_s \binom{d-\nu_s-\mu_s}{j-\nu_s} = \binom{d}{j}$ for $0 \le j \le d = \deg S$, a system with non-negative entries. A support S with $|S| = n+1$ is *the support of a fundamental model* if a strictly positive real scaling exists and is the only positive real scaling; this is [1, Definition 2.1] read on supports.

Lemma 2.1. *For a support S the following are equivalent.*

(a) *S is the support of a fundamental model.*
(b) *The polynomials $t^\nu(1-t)^\mu$, $(\nu,\mu) \in S$, are linearly independent, and the unique solution c of* (3) *is strictly positive.*
(c) *A strictly positive rational scaling of S exists, and any two rational scalings of S coincide.*

In (b) linear independence may be taken over $\mathbb{Q}$ or over $\mathbb{R}$, and in (c) the identity (3) *may be required for all rational t only.*

Proof. The solution set of $A_S c = e_0$ over K is an affine subspace of K^S defined over $\mathbb{Q}$. Its dimension is $|S| - \operatorname{rank} A_S$, and the rank of an integer matrix is the same over $\mathbb{Q}$ and over $\mathbb{R}$; so the solution set is a single point over $\mathbb{R}$ if and only if it is over $\mathbb{Q}$, and in that case the point is rational. If the solution set has positive dimension and contains a strictly positive point c, then $c + \varepsilon k$ is a strictly positive solution for every kernel vector k and every sufficiently small ε, so the scalings are not uniquely determined; in the rational case a rational kernel vector gives a second rational scaling. This proves (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). Finally a polynomial in t that equals 1 at infinitely many rational t is the constant 1, so requiring (3) at all rational t is the same as requiring it identically. $\square$

Form (c) is the one formalised (Section 7); the argument above is written out here and is not part of the formal development. With this,

$$T(n,d) = \#\{S : |S| = n+1,\ \deg S = d,\ S \text{ fundamental}\},$$

and Proposition 6.1 of [1] says $T(n,d) = 0$ unless $n \le d \le 2n-1$.

2.2. **The reflection.** Let $\sigma\colon \mathbb{N}^2 \to \mathbb{N}^2$, $\sigma(\nu,\mu) = (\mu,\nu)$, and $\sigma(S) = \{\sigma(s) : s \in S\}$. Then $|\sigma(S)| = |S|$, $\deg \sigma(S) = \deg S$, and substituting $t \mapsto 1-t$ in (3) shows that c is a scaling of S if and only if it is a scaling of $\sigma(S)$ (with the entries indexed by σ). We call S *reflection-symmetric* when $\sigma(S) = S$ and set

$$T^{\mathrm{sym}}(n,d) = \#\{S : |S| = n+1,\ \deg S = d,\ S \text{ fundamental},\ \sigma(S) = S\}.$$

The reflection is not a special case of a lattice symmetry preserving fundamentality: Proposition 4.5 shows that the translation $(\nu,\mu) \mapsto (\nu+1,\mu)$ destroys it.

3. The reflection theorem

Theorem 3.1. *For every support S, $\sigma(S)$ is the support of a fundamental model if and only if S is.*

Proof. For any vector c and any t, $\sum_{s\in S} c_s\, t^{\mu_s}(1-t)^{\nu_s} = \sum_{s\in S} c_s\,(1-t)^{\nu_s}\big(1-(1-t)\big)^{\mu_s}$, so c is a scaling of $\sigma(S)$ exactly when it is a scaling of S (as $t \mapsto 1-t$ is a bijection of the parameter line). Existence of a positive scaling and uniqueness of the scaling are therefore the same statements for S and for $\sigma(S)$. $\square$

The theorem is proved in the kernel for every support, with no computation (Section 7). Since σ preserves size and degree, it restricts to an involution of the finite set $F(n,d)$ of fundamental supports of size $n+1$ and degree d, and $T^{\mathrm{sym}}(n,d)$ is the number of its fixed points.

Remark 3.2 (parity; not part of the formal development). The orbits of σ on $F(n,d)$ have size one or two, so

$$T(n,d) = T^{\mathrm{sym}}(n,d) + 2 \cdot \#\{\text{free orbits}\}, \qquad \text{hence} \qquad T(n,d) \equiv T^{\mathrm{sym}}(n,d) \pmod 2.$$

In particular $T(n,d)$ is odd exactly when an odd number of its models are reflection-symmetric. On the 23 cells where both counts are available (Tables 1 and 2) the congruence holds; besides the two one-element cells $T(1,1) = T(2,3) = 1$, the odd entries of the source's data on these cells, $T(2,2) = 3$, $T(6,7) = 2421$, $T(6,8) = 643$ and $T(8,8) = 1\,279\,349$, are odd because 1, 79, 5 and 323 of their models are reflection-symmetric. (The remaining odd value of Figure 9, $T(8,9) = 285\,601$, is on a cell whose symmetric count we did not compute.) The orbit-counting step is not formalised; only the ingredients (Theorem 3.1 and the individual counts) are.

4. The kernel-verified cells

4.1. **The computation.** All results of this section are exhaustive computations over the list $\mathrm{Supp}(n,d)$ of *all* supports of size $n+1$ and degree exactly d, i.e. all $(n+1)$-element subsets of the $N_d = \frac{1}{2}(d+1)(d+2)-1$ lattice points (ν,μ) with $1 \le \nu+\mu \le d$ that contain at least one point with $\nu+\mu = d$; for instance $|\mathrm{Supp}(3,3)| = 121$, and the four cells of row 4 have 1876, 13 502, 65 226 and 243 902 supports. For each S the decision is by a certificate checked against the definition:

- if every $(\nu,\mu) \in S$ has $\nu \neq 0$ (resp. $\mu \neq 0$), evaluation at $t = 0$ (resp. $t = 1$) shows that no scaling exists, so S is not fundamental;
- a non-zero vector k with $\sum_s k_s t^{\nu_s}(1-t)^{\mu_s} \equiv 0$ shows that the scaling, if any, is not unique, so S is not fundamental;
- a matrix M with $M \cdot E = I$, where E is the $(d+1)\times(n+1)$ evaluation matrix $E_{m,s} = m^{\nu_s}(1-m)^{\mu_s}$ at $m = 0,1,\ldots,d$, pins every scaling to $c = M \cdot \mathbf{1}$ (because $E\,c = \mathbf{1}$ for every scaling c). Then S is fundamental if this c satisfies (3) identically and is strictly positive, and not fundamental if c satisfies (3) but is not strictly positive, or if c fails (3) at some $m \in \{0,\ldots,d\}$.

The soundness of this calculus is a single theorem proved once in the kernel (the bridge `cert_sound` of Section 7). The certificates are found by Gauss–Jordan elimination over $\mathbb{Q}$ on E augmented by the identity: a pivot in every column yields M, a free column yields k. Since the monomials of S have degree at most d, evaluation at $d+1$ points is injective on their span, so a conclusive certificate always exists, and in every sweep reported below every certificate was conclusive. Each cell is one compiled sweep, accepted by the kernel through one `native_decide` axiom per cell; the counting principle that turns a verified sweep into a statement about the definition is itself kernel-clean.

4.2. **The counts.**

Theorem 4.1. *For $1 \le n \le 4$ the numbers $T(n,d)$ are*

$$T(1,1) = 1; \qquad T(2,2) = 3, \quad T(2,3) = 1;$$
$$T(3,3) = 12, \quad T(3,4) = 4, \quad T(3,5) = 2;$$
$$T(4,4) = 82, \quad T(4,5) = 38, \quad T(4,6) = 10, \quad T(4,7) = 4.$$

Proof. Ten exhaustive sweeps as in §4.1, one per cell. ☐

These ten values agree with Table 1 of [1], with Table 1 of [2] and with OEIS A386841; the contribution here is the verification from the definition, not the values. Before the kernel sweeps, the same ten cells (and $T(5,5) = 602$) were reproduced by two independent exact-rational implementations of the two linear systems of §2 and by the search of Section 5, all four implementations agreeing cell by cell.

Theorem 4.2 (the corrigendum)**.** $T(3,3) = 12$. *In particular* $T(3,3) \neq 11$ *and* $T(3,3) \neq 13$.

Proof. The sweep over the 121 supports of $\mathrm{Supp}(3,3)$. ☐

Table 2 of [3] prints 11 for this cell (as the number of polynomials in $H(2,3)$ with four terms); [1, Proposition 6.3] corrects it to 12, listing seven polynomials that represent the twelve up to swapping the variables (two of the seven are symmetric, in agreement with $T^{\mathrm{sym}}(3,3) = 2$ below), and OEIS A387029 records the same correction. Theorem 4.2 confirms the correction from the definition.

Theorem 4.3 (the ends of the degree window)**.** $T(3,2) = 0$ *and* $T(3,6) = 0$.

Proof. Two sweeps, over $\mathrm{Supp}(3,2)$ and $\mathrm{Supp}(3,6)$. ☐

These are the instances at $n = 3$ of [1, Proposition 6.1] ($d < n$) and of the degree bound [1, Theorem 2.2] ($d > 2n-1$), and they check that the enumeration is not vacuous at either end of the window $n \le d \le 2n-1$: the supports of degree 2 and 6 are enumerated and every one of them is rejected.

Theorem 4.4 (the reflection-symmetric triangle, rows 1–4)**.** *For $1 \le n \le 4$ the numbers $T^{\mathrm{sym}}(n,d)$ are*

$$T^{\mathrm{sym}}(1,1) = 1; \qquad T^{\mathrm{sym}}(2,2) = 1, \quad T^{\mathrm{sym}}(2,3) = 1;$$
$$T^{\mathrm{sym}}(3,3) = 2, \quad T^{\mathrm{sym}}(3,4) = 0, \quad T^{\mathrm{sym}}(3,5) = 0;$$
$$T^{\mathrm{sym}}(4,4) = 4, \quad T^{\mathrm{sym}}(4,5) = 8, \quad T^{\mathrm{sym}}(4,6) = 0, \quad T^{\mathrm{sym}}(4,7) = 2.$$

In particular $T^{\mathrm{sym}}(3,4) = T^{\mathrm{sym}}(4,6) = 0$: no almost sharp fundamental model in Δ_3 or Δ_4 is its own reflection. Moreover $T^{\mathrm{sym}}(4,5) = 8$ is neither 7 nor 9 nor the full count $T(4,5) = 38$.

Proof. The same ten sweeps as in Theorem 4.1: each sweep counts, besides the fundamental supports, those fundamental supports S for which $(\mu,\nu) \in S$ for every $(\nu,\mu) \in S$. ☐

The last sentence of the theorem is a control that the symmetric count is a strict sub-count and not an artefact of the predicate. Among the three fundamental models in Δ_2 of degree 2 the only reflection-symmetric one is the binomial model $t \mapsto (t^2, 2t(1-t), (1-t)^2)$; neither of the two sharp models in Δ_3 is reflection-symmetric.

Proposition 4.5 (non-vacuity and the wrong symmetry)**.** (a) *The binomial model B_2 with support $\{(2,0),(1,1),(0,2)\}$ has the scaling $(1,2,1)$, which is a scaling for every value of the parameter, and B_2 is fundamental.*

(b) *The translated support $\{(3,0),(2,1),(1,2)\}$ is not fundamental.*

Proof. (a) $t^2 + 2t(1-t) + (1-t)^2 = 1$ is a polynomial identity, checked in the kernel by normalisation, without computation over supports; fundamentality is the certificate check for this single support. (b) Every monomial t^3, $t^2(1-t)$, $t(1-t)^2$ vanishes at $t = 0$, so no scaling exists at all; this is the first certificate of §4.1. ☐

Part (a) is an instance of [1, Theorem 2.4] (the binomial model B_n is fundamental) at $n = 2$; part (b) shows that Theorem 3.1 is a property of the reflection and not of lattice symmetries in general.

$n \backslash d$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**														
2		**3**	**1**												
3			**12**	**4**	**2**										
4				**82**	**38**	**10**	**4**								
5					$602^{\dagger}$	$254^{\dagger}$	$88^{\dagger}$	$24^{\dagger}$	$2^{\dagger}$						
6						$6710^{\dagger}$	$2421^{\dagger}$	$643^{\dagger}$	$198^{\dagger}$	$32^{\dagger}$	$4^{\dagger}$				
7							$83906^{\dagger}$	$23285^{\dagger}$	$6445^{\dagger}$	$1442^{\dagger}$	332	56	8		
8								$1279349^{\dagger\ddagger}$	$285601^{\ddagger}$	?	?	?	?	96^{*}	$4^{\ddagger}$

TABLE 1. $T(n,d)$ as printed in Table 1 and Figure 9 of [1]. Bold: verified in the Lean 4 kernel here (Theorems 4.1, 4.2). †: recomputed here by the exact search of Section 5. ‡: printed in Figure 9 only (the 4 is taken there from [3, Table 2]). ∗: conjectured in Figure 9 (the right-hand side of (2)). ?: uncomputed in [1]. Blank cells are zero by [1, Proposition 6.1].

5. Beyond the kernel: the exact search

The results of this section are computations outside the formal development. They were produced by a depth-first search written in C over the homogenised system of §2, which we describe so that they can be reproduced. The exponent pairs are processed in increasing ν; the column of the pair (ν,μ) is supported on the row interval $[\nu, d-\mu]$, so each level of the search closes one equation. A branch is pruned when (i) the current row is not covered by any chosen column, (ii) the exact rational row-reduction of the closed rows is inconsistent, or leaves a column that can no longer receive a pivot, or determines a variable that is not positive, or (iii) a phase-1 simplex test in floating point on $\{c \ge 0 : A_{\mathrm{closed}} c = b_{\mathrm{closed}},\ A_{\mathrm{future}} c \le b_{\mathrm{future}}\}$ reports clear infeasibility — this last prune can only cost time, never lose a model. Rational

$n\backslash d$	1	2	3	4	5	6	7	8	9	10	11
1	**1**										
2		**1**	**1**								
3			**2**	**0**	**0**						
4				**4**	**8**	**0**	**2**				
5					$10^\dagger$	$4^\dagger$	$4^\dagger$	$0^\dagger$	$0^\dagger$		
6						$30^\dagger$	$79^\dagger$	$5^\dagger$	$14^\dagger$	$0^\dagger$	$0^\dagger$
7							$80^\dagger$				
8								$323^\dagger$			

TABLE 2. $T^{\mathrm{sym}}(n, d)$, the number of fundamental models in Δ_n of degree d whose support is reflection-symmetric. Bold: verified in the Lean 4 kernel (Theorem 4.4). †: computed by the exact search of Section 5, outside the formal development. Cells left blank in rows 7–8 were not computed.

arithmetic uses 64-bit numerators and denominators with 128-bit intermediates and an explicit overflow flag, which was zero on every run reported here. At each leaf the full system is solved exactly and the support is counted when its unique solution is strictly positive; the search also counts the supports with $\sigma(S) = S$. The search reproduces every cell of Theorem 4.1 and $T(5, 5) = 602$ against the two reference implementations. All timings are single-core wall times on a shared 20-core machine under load, so they are upper bounds by roughly a factor of two.

Remark 5.1 (rows 5–7). Rows 5 and 6 of Table 1 of [1] were recomputed exactly and agree cell by cell: $602, 254, 88, 24, 2$ and $6710, 2421, 643, 198, 32, 4$. Of row 7, the cells $T(7, 7) = 83\,906$, $T(7, 8) = 23\,285$, $T(7, 9) = 6445$ and $T(7, 10) = 1442$ were recomputed and agree; $T(7, 11)$, $T(7, 12)$, $T(7, 13)$ were not attempted within the budget of four CPU-hours per cell (see Remark 5.5). No printed cell disagreed with the search, so we have no erratum to report in Table 1.

Remark 5.2 (the cell $T(8, 8)$). The value $T(8, 8) = 1\,279\,349$ is printed in Figure 9 of [1] and, as far as we could find, nowhere else: it is absent from Table 1 of [1], from OEIS (the triangle A386841 stops at $\ldots, 83906, 4$, and a search for the number returns nothing), and from the tables of the authors' MathRepo page [6], which publishes the list of models for this cell only as a binary file. The search reproduces it exactly from the definition: $5.80 \cdot 10^8$ search nodes, 5573 s, overflow flag 0. This is, to our knowledge, the first independent recomputation of that cell. The companion value $T(8, 9) = 285\,601$ of Figure 9 was not recomputed (Remark 5.5).

Remark 5.3 (the reflection-symmetric counts beyond row 4). The same runs return the values of T^{sym} in rows 5 and 6 and on the main diagonal in Table 2: $10, 4, 4, 0, 0$; $30, 79, 5, 14, 0, 0$; $T^{\mathrm{sym}}(7, 7) = 80$; $T^{\mathrm{sym}}(8, 8) = 323$. In particular $T^{\mathrm{sym}}(5, 8) = T^{\mathrm{sym}}(6, 10) = 0$, extending the vanishing of Theorem 4.4 on the almost sharp diagonal to $n = 5, 6$, and every value is congruent to $T(n, d)$ modulo 2 as Remark 3.2 requires. The cost of the symmetric count is negligible against the search itself; the row-6 cells took between 10 s and 738 s each.

Remark 5.4 (indexes). On 2026-09-03 the OEIS was searched for the triangle of Table 2 read by rows, $1, 1, 1, 2, 0, 0, 4, 8, 0, 2, 10, 4, 4, 0, 0, 30, 79, 5, 14, 0, 0$, and for its main diagonal $1, 1, 2, 4, 10, 30, 80, 323$; neither matches any entry, and the short prefix $1, 1, 2, 4, 10, 30$ returns only entries unrelated to this object. The statistic is not defined in [1], in the six OEIS entries A143107, A143108, A143109, A386840, A386841, A387029 that [1] cites, or on the MathRepo page [6], whose code has not changed since July 2025. The main diagonal $T(n, n) = 1, 3, 12, 82, 602, 6710, 83906$ of the source's own table is likewise not in the OEIS.

Remark 5.5 (the price of the open cells). Measured single-core wall times of the search: $T(6, 6)$ 10 s, $T(6, 7)$ 17 s, $T(6, 8)$ 96 s, $T(6, 9)$ 197 s, $T(6, 10)$ 509 s, $T(6, 11)$ 738 s, $T(7, 7)$ 140 s, $T(7, 8)$ 736 s, $T(7, 9)$ 2092 s, $T(7, 10)$ 5015 s, $T(8, 8)$ 5573 s. The node count grows by a factor 2.3–3.9 per unit of d at fixed n, and by a factor 24.2, 23.4, 23.9, 26.7 per unit of n at fixed $d - n$; the latter law, fitted on rows 6 and 7, predicted the node count of $T(8, 8)$ to within 12% before that run. Extrapolating with it and with the throughput measured at $n = 8$ ($1.04 \cdot 10^5$ nodes per second) gives the following single-cell prices: $T(8,9) \approx 8$, $T(8,10) \approx 30$, $T(8,11) \approx 95$, $T(8,12) \approx 260$, $T(8,13) \approx 640$, $T(8,14) \approx 1400$ and $T(9,9) \approx 40$ CPU-hours. All exceed the budget of four CPU-hours per cell that we allowed ourselves, and none was run. Two improvements would change the verdict. First, Lemma 5.3 of [1] (from [3, Section 3]) restricts *sharp* supports severely — $(d,0)$ and $(0,d)$ belong to the support, no other point of the top diagonal or of the axes does, and no point $(j, d-1-j)$ with j even does; an analogue for almost sharp supports would cut $T(8,14)$, the cell that tests Conjecture 6.5 of [1] at $n=8$, by orders of magnitude. Second, the search prunes only on the closed rows; a relaxation using the future rows together with the bound on the number of columns still to be chosen is the obvious next prune.

6. Questions

Question 6.1. Is $T^{\mathrm{sym}}(n, 2n-2) = 0$ for every $n \geq 3$, i.e. is no almost sharp fundamental model its own reflection? It holds for $3 \leq n \leq 6$ (Theorem 4.4, Remark 5.3). By Remark 3.2 a positive answer would make $T(n, 2n-2)$ even for all $n \geq 3$, as the data of A143108 suggest.

Question 6.2. Is the family of composed models in the proof of [1, Proposition 6.4] disjoint from its own reflection? This is what the factor 2 in (2) needs, and it is stronger than Question 6.1; both concern sharp models composed pairwise, for which the support restrictions of [1, Lemma 5.3] are available.

Question 6.3. What is $T^{\mathrm{sym}}(n,d)$ on the sharp diagonal? The computed values $T^{\mathrm{sym}}(n, 2n-1)$ for $n = 1, \ldots, 6$ are $1, 1, 0, 2, 0, 0$. Table 1 of [3] lists the sharp polynomials of odd degree up to 17 up to swapping the variables, and the values above can be read off it: the symmetric ones are $x + y$ in degree 1, $x^3 + 3xy + y^3$ in degree 3, and $x^7 + 7x^3y + 7x^3y^3 + 7xy^3 + y^7$ and $x^7 + \frac{7}{2}x^5y + \frac{7}{2}xy + \frac{7}{2}xy^5 + y^7$, two of the three of degree 7; none of degree 5, 9 or 11 is symmetric. Read the same way, the table shows no symmetric sharp polynomial in degrees 13, 15, 17 either, while Lebl's addendum [4] reports that among the 13 sharp polynomials of degree 19 up to symmetry of the variables "Two of the polynomials are symmetric."

7. Verification

Every theorem and proposition of Sections 3 and 4 is formalised and checked in Lean 4 [7] (toolchain v4.32.0) over Mathlib [8]; the repository is private and a repository reference is to be added at submission. The development consists of five modules, in the namespace `LeanProblemSpec.FundamentalModelsTriangle`: `Core` (the predicate, the certificate calculus and its soundness theorem `cert_sound`), `Counts` (the enumeration `supports`, the certificate search `classify` and the two counting principles `count_of_sweep` and `count_and_of_sweep`), `Table` and `TableFour` (the sweeps of rows 1–3 and 4), and `Controls` (the reflection theorem and the controls). `cert_sound`, the two counting principles, `fundamental_swapS` and `binomial2_isScaling` depend only on the three standard axioms `propext`, `Classical.choice`, `Quot.sound`; each cell theorem depends in addition on exactly one axiom, generated by the compiled sweep of that cell and named after it, for the cell $(4,4)$

`sweep44._native.native_decide.ax_1_1`,

the same axiom serving $T(n,d)$ and $T^{\mathrm{sym}}(n,d)$; `binomial2_fundamental` and `shift_not_fundamental` each depend on one such axiom for their single support. These axiom lists were read off `#print axioms` on 2026-09-03.

The rational-versus-real bridge of Lemma 2.1 and the orbit count of Remark 3.2 are not formalised. The definitions the statements refer to are reproduced verbatim:

```
abbrev Pair := ℕ × ℕ
abbrev Support := List Pair
def dot (a b : List ℚ) : ℚ := ((a.zip b).map fun p => p.1 * p.2).sum
def evRow (S : Support) (t : ℚ) : List ℚ := S.map fun s => t ^ s.1 * (1 - t) ^ s.2
def paramSum (S : Support) (c : List ℚ) (t : ℚ) : ℚ := dot (evRow S t) c
def IsScaling (S : Support) (c : List ℚ) : Prop :=
  c.length = S.length ∧ ∀ t : ℚ, paramSum S c t = 1
def Fundamental (S : Support) : Prop :=
  (∃ c, IsScaling S c ∧ ∀ x ∈ c, 0 < x) ∧
    ∀ c₁ c₂, IsScaling S c₁ → IsScaling S c₂ → c₁ = c₂
def points (d : ℕ) : List Pair :=
```

```
  ((List.range (d + 1)).flatMap fun a =>
    (List.range (d + 1 - a)).map fun b => (a, b)).filter fun s => 1 ≤ s.1 + s.2
def degOf (S : Support) : ℕ := S.foldl (fun m s => max m (s.1 + s.2)) 0
def supports (n d : ℕ) : List Support :=
  ((points d).sublistsLen (n + 1)).filter fun S => degOf S == d
def symS (S : Support) : Bool := S.all (fun s => decide (((s.2, s.1) : Pair) ∈ S))
def swapS (S : Support) : Support := S.map Prod.swap
```

so `Fundamental` is form (c) of Lemma 2.1, `supports n d` is $\mathrm{Supp}(n, d)$ (each subset listed once, as a sublist of the duplicate-free list `points d`), `symS` is $\sigma(S) = S$ and `swapS` is σ. The bridge is

```
theorem cert_sound {S : Support} {r : ℕ} {ct : Cert} {b : Bool}
    (h : checkCert S r ct = some b) : (b = true ↔ Fundamental S)
```

The pinned statements follow verbatim (every count theorem is stated under `open Classical in`, which supplies the decidability of `Fundamental S`). Theorem 4.1:

```
theorem T_1_1 : (supports 1 1).countP (fun S => decide (Fundamental S)) = 1
theorem T_2_2 : (supports 2 2).countP (fun S => decide (Fundamental S)) = 3
theorem T_2_3 : (supports 2 3).countP (fun S => decide (Fundamental S)) = 1
theorem T_3_3 : (supports 3 3).countP (fun S => decide (Fundamental S)) = 12
theorem T_3_4 : (supports 3 4).countP (fun S => decide (Fundamental S)) = 4
theorem T_3_5 : (supports 3 5).countP (fun S => decide (Fundamental S)) = 2
theorem T_4_4 : (supports 4 4).countP (fun S => decide (Fundamental S)) = 82
theorem T_4_5 : (supports 4 5).countP (fun S => decide (Fundamental S)) = 38
theorem T_4_6 : (supports 4 6).countP (fun S => decide (Fundamental S)) = 10
theorem T_4_7 : (supports 4 7).countP (fun S => decide (Fundamental S)) = 4
```

Theorem 4.2 (`T_3_3` above, and):

```
theorem T_3_3_ne_11 : (supports 3 3).countP (fun S => decide (Fundamental S)) ≠ 11
theorem T_3_3_ne_13 : (supports 3 3).countP (fun S => decide (Fundamental S)) ≠ 13
```

Theorem 4.3:

```
theorem T_3_2_zero : (supports 3 2).countP (fun S => decide (Fundamental S)) = 0
theorem T_3_6_zero : (supports 3 6).countP (fun S => decide (Fundamental S)) = 0
```

Theorem 4.4:

```
theorem Tsym_1_1 :
    (supports 1 1).countP (fun S => decide (Fundamental S) && symS S) = 1
theorem Tsym_2_2 :
    (supports 2 2).countP (fun S => decide (Fundamental S) && symS S) = 1
theorem Tsym_2_3 :
    (supports 2 3).countP (fun S => decide (Fundamental S) && symS S) = 1
theorem Tsym_3_3 :
    (supports 3 3).countP (fun S => decide (Fundamental S) && symS S) = 2
theorem Tsym_3_4 :
    (supports 3 4).countP (fun S => decide (Fundamental S) && symS S) = 0
theorem Tsym_3_5 :
    (supports 3 5).countP (fun S => decide (Fundamental S) && symS S) = 0
theorem Tsym_4_4 :
    (supports 4 4).countP (fun S => decide (Fundamental S) && symS S) = 4
theorem Tsym_4_5 :
    (supports 4 5).countP (fun S => decide (Fundamental S) && symS S) = 8
theorem Tsym_4_6 :
    (supports 4 6).countP (fun S => decide (Fundamental S) && symS S) = 0
theorem Tsym_4_7 :
    (supports 4 7).countP (fun S => decide (Fundamental S) && symS S) = 2
theorem almost_sharp_no_reflection_fixed_point :
    (supports 3 4).countP (fun S => decide (Fundamental S) && symS S) = 0 ∧
    (supports 4 6).countP (fun S => decide (Fundamental S) && symS S) = 0
theorem Tsym_4_5_strict :
    (supports 4 5).countP (fun S => decide (Fundamental S) && symS S) ≠ 7 ∧
    (supports 4 5).countP (fun S => decide (Fundamental S) && symS S) ≠ 9 ∧
    (supports 4 5).countP (fun S => decide (Fundamental S) && symS S) ≠ 38
```

Theorem 3.1 (kernel-clean, every support):

```
theorem fundamental_swapS {S : Support} : Fundamental (swapS S) ↔ Fundamental S
```

Proposition 4.5:

```
theorem binomial2_isScaling : IsScaling [(2, 0), (1, 1), (0, 2)] [1, 2, 1]
theorem binomial2_fundamental : Fundamental [(2, 0), (1, 1), (0, 2)]
theorem shift_not_fundamental : ¬ Fundamental [(3, 0), (2, 1), (1, 2)]
```

The sweeps themselves are of the form `(supNM.all (okC d), supNM.countP (verdict d), supNM.countP (verdict d && symS)) = (true, T, Tsym)` by `native_decide`, where `okC d S` says that the certificate found for S verifies and `verdict d S` is its verdict; the row-4 sweeps took about 19 minutes in the Lean interpreter, the others under two minutes. The C search and the two Python reference implementations of Section 5 are outside the formal development and are supplied with the repository.

References

[1] C. Améndola, V. D. Nguyen and J. Oldekop, *One-dimensional discrete models of maximum likelihood degree one*, Advances in Applied Mathematics **178** (2026), 103095; `doi:10.1016/j.aam.2026.103095`. arXiv:2507.18686 (v2, 2 March 2026); statement numbers in this paper refer to the arXiv version.

[2] A. Bik and O. Marigliano, *Classifying one-dimensional discrete models with maximum likelihood degree one*, Advances in Applied Mathematics **170** (2025), 102928; `doi:10.1016/j.aam.2025.102928`. arXiv:2205.09547 (v3, 11 June 2025, "accepted version"); statement numbers, Table 1 and the spelling "chipsplitting" are quoted from that arXiv version.

[3] J. Lebl and D. Lichtblau, *Uniqueness of certain polynomials constant on a line*, Linear Algebra and its Applications **433** (2010), no. 4, 824–837; `doi:10.1016/j.laa.2010.04.020`. arXiv:0808.0284. Tables 1 and 2 quoted from the arXiv version (v3).

[4] J. Lebl, *Addendum to Uniqueness of certain polynomials constant on a line*, arXiv:1302.1441 (v2, 8 October 2013).

[5] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entries A143107, A143108, A143109, A386840, A386841, A387029; `https://oeis.org`, read 2026-09-03.

[6] C. Améndola, V. D. Nguyen and J. Oldekop, MathRepo page *FundamentalModels*, `https://mathrepo.mis.mpg.de/FundamentalModels/`, with code at `https://github.com/geniegeist/mathrepo-chipsplitting`; read 2026-09-03.

[7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: A. Platzer and G. Sutcliffe (eds.), Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635; `doi:10.1007/978-3-030-79876-5_37`.

[8] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; `doi:10.1145/3372885.3373824`.