

THE COUPLING MATRIX OF THE FINITE FREE CONVOLUTION AT THE HERMITE DIAGONAL IS SYMMETRIC, AND ITS SPECTRUM IS EXACTLY $\{2^{-k/2}\}$

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ABSTRACT. The finite free additive convolution $\boxplus_n$ of Marcus, Spielman and Srivastava acts on the root vectors of monic real-rooted polynomials of degree n by a map Ω that is smooth at a simple-root configuration. Hashemi’s *coupling matrix* E_n is the partial Jacobian $\partial\Omega(\cdot, h)/\partial\alpha$ frozen at the Hermite diagonal $\alpha = \beta = h$, where h is the root vector of the degree- n probabilists’ Hermite polynomial He_n . On numerical evidence — the ten leading singular values at nine degrees between 10 and 90, and a relative Frobenius asymmetry below 10^{-10} — he conjectures that E_n is symmetric and that its singular values on $\mathbf{1}^\perp$ are exactly $\{2^{-k/2} : k = 1, \dots, n-1\}$, the same numbers at every degree. We prove both, at every degree and in exact form: $E_n^\top = E_n$; the characteristic polynomial of E_n is $\prod_{k=0}^{n-1} (X - 2^{-k/2})$; that of the Gram matrix $E_n^\top E_n$ is $\prod_{k=0}^{n-1} (X - 2^{-k})$; and $\text{tr } E_n - 1 = \sum_{k=1}^{n-1} 2^{-k/2}$, an identity the source derives only under *both* of its conjectures. The mechanism is that convolving with He_n is the heat operator $\mathcal{G} = e^{-D^2/2}$: Christoffel–Darboux expands $\text{He}_n/(x - u_j)$ in the Hermite basis, $\mathcal{G}\text{He}_m$ is the variance-two Hermite polynomial and $\text{He}_m^{(2)}(\sqrt{2}x) = 2^{m/2}\text{He}_m(x)$, so E_n is the matrix of a rescaled heat operator — manifestly symmetric in one basis, diagonal in another. Nothing here is an exhaustive computation: every statement holds at every degree at once, and the source’s numerics are reproduced only as a check that the algebraic definition used here is the analytic object it is meant to be. The three corollaries the source states under “Assume Conjecture 4.1” — the full singular value decomposition of the convolution Jacobian, the local stability constant, and the finite free central limit rate $\lambda_{\text{CLT}} = 1/\sqrt{2}$ uniform in n — therefore hold unconditionally. Every lemma, proposition and theorem below is machine-checked in Lean 4; the two corollaries and the two remarks are not, and say so.

1. INTRODUCTION

1.1. The convolution and its root map. For monic real-rooted $f, g \in \mathbb{R}[x]$ of degree n with root vectors $\alpha, \beta \in \mathbb{R}^n$, the *finite free additive convolution* of Marcus, Spielman and Srivastava [3] is

$$(1) \quad (f \boxplus_n g)(x) = \frac{1}{n!} \sum_{\pi \in S_n} \prod_{i=1}^n (x - \alpha_i - \beta_{\pi(i)}),$$

again monic, real-rooted and of degree n . Equivalently, in coefficients: if $f(x) = \sum_{k=0}^n (-1)^k a_k x^{n-k}$ and $g(x) = \sum_{k=0}^n (-1)^k b_k x^{n-k}$ with $a_0 = b_0 = 1$, then $f \boxplus_n g$ is $\sum_{k=0}^n (-1)^k c_k x^{n-k}$, where

$$(2) \quad c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j, \quad k = 0, \dots, n.$$

This is the form in which (1) is quoted in [1, eq. (4)], attributed there to [4, 3]; it is the definition used below. The induced map on root vectors, $\Omega_{\boxplus_n}(\alpha, \beta) = \gamma$, is smooth wherever the roots are simple. (Throughout, a numbered statement of another paper is cited together with its source, so that a bare number is one of ours; equation numbers inside a quotation are the quoted paper’s.)

Write He_m for the monic probabilists’ Hermite polynomial, $\text{He}_0 = 1$, $\text{He}_{m+1} = x\text{He}_m - \text{He}'_m$, and let $h = h^{(n)}$ be the root vector of He_n . Garza-Vargas, Srivastava and Stier [2] study the Jacobian of $\Omega_{\boxplus_n}$ as the carrier of information-theoretic inequalities for $\boxplus_n$; the Hermite polynomials are the equality case there, so the Hermite diagonal $\alpha = \beta = h$ is the point at which the linearisation matters. Hashemi [1,

eq. (15)] names that linearisation:

$$(3) \quad E_n := \left. \frac{\partial \Omega_{\boxplus_n}(\cdot, h)}{\partial \alpha} \right|_{\alpha=h} \in \mathbb{R}^{n \times n}.$$

By [2, Lemma 3.4], quoted in [1] as “ E_n is a doubly stochastic matrix with nonnegative entries”, the vector $\mathbf{1} = (1, \dots, 1)$ satisfies $E_n \mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top E_n = \mathbf{1}^\top$, so $W = \mathbf{1}^\perp$ is E_n -invariant. The symmetry $\Omega_{\boxplus_n}(f, g) = \Omega_{\boxplus_n}(g, f)$ gives, on differentiating at the diagonal, $\partial_\alpha \Omega = \partial_\beta \Omega$ there, hence $J_{\boxplus_n}[u \oplus v] = E_n(u + v)$: the whole Jacobian at the Hermite diagonal is carried by the single matrix E_n .

1.2. The two conjectures, and the evidence for them. Both statements of [1] that concern us are numerical. The first is [1, Remark 4.2 (Symmetry of E_n)], verbatim:

“Numerically, E_n appears to be symmetric, meaning its eigenvalues coincide with its singular values. The argument via equation (16) establishes only that $\partial_\alpha \Omega = \partial_\beta \Omega$ at the diagonal, which implies that both first-order partial Jacobians agree, but does not by itself imply that E_n is a symmetric matrix. We therefore conjecture symmetry as a separate numerical observation: across all audited degrees, $\|E_n - E_n^\top\|_F / \|E_n\|_F < 10^{-10}$.”

The second is [1, Conjecture 4.1 (Geometric spectrum of E_n)], verbatim:

“Let $n \geq 3$ and let E_n be defined by (15). The singular values of E_n on $W = \mathbf{1}^\perp$ are $\text{spec}(E_n|_W) = \{2^{-k/2} : k = 1, \dots, n-1\}$, independent of n , with singular value 1 on $\text{Span}(\mathbf{1})$.”

The evidence, from the audit appendix [1, §A.1], is an `mpmath` computation of the *ten leading* singular values of $E_n|_W$ at $n \in \{10, 20, 30, 40, 50, 60, 70, 80, 90\}$, at working precision rising from 60 to 220 digits, with maximum relative deviations from the dyadic targets running 1.33×10^{-14} , 5.48×10^{-13} , ..., 4.56×10^{-4} across those nine degrees. So the tail of the spectrum was not audited at any degree, and the small degrees at which the conjecture starts were not audited at all.

One eigenvalue is proved in [1]: its Lemma 5.6 (Score eigenvector) shows $E_n s = 2^{-1/2} s$ for the score direction s , which is the case $k = 1$. Beyond that, the only exact spectral information in the literature is in [2], and it stops at the second singular value: $\sigma_2(J_{\boxplus_n}) = 1$ in its Remark 5.4, and σ_2 of the differentiation root map equal to $\sqrt{(n-2)/n}$ in its Proposition 4.6. As its comparison remark [1, Remark 4.3] puts it, the conjecture “is stronger on both counts as it specifies the *entire* spectrum of the convolution root map, and the values are *universal constants* $2^{-k/2}$ independent of n ”. Three of its corollaries are stated under the hypothesis “Assume Conjecture 4.1”: Corollary 4.4, the full singular value decomposition $\sigma_k(J_{\boxplus_n}|_V) = 2^{(1-k)/2}$; Corollary 4.5, a local stability constant; and Corollary 4.8, the linearised finite free central limit rate

$$\lambda_{\text{CLT}} = \sqrt{2} \sigma_2(E_n|_W) = \frac{1}{\sqrt{2}}, \quad \text{“independent of } n\text{”,}$$

in which $\sigma_2(E_n|_W)$ is the *second* singular value on W , which the conjecture makes 2^{-1} . Of this rate [1] remarks: “To our knowledge, the analogous linearized rate in the finite free setting has not been established previously (conditionally on Conjecture 4.1).” Its future-work section asks for “understanding the spectral theory of the linearized finite free convolution map at the Hermite diagonal, and in particular why the dyadic pattern persists across all polynomial degrees”.

1.3. Results. That is what is supplied here. Throughout, $c = \sqrt{2}$, $u_1, \dots, u_n$ is an injective enumeration of n roots of He_n , and $E_n = E_n(u)$ is the matrix

$$(4) \quad (E_n)_{ij} = \frac{(\mathcal{G}f_j)(cu_i)}{c^{n-1} \text{He}'_n(u_i)}, \quad f_j = \frac{\text{He}_n}{x - u_j}, \quad \mathcal{G} = e^{-D^2/2} = \sum_{m \geq 0} \frac{(-1/2)^m}{m!} D^{2m},$$

which is what (3) becomes after the implicit differentiation carried out in §3 — the same computation the audit appendix of [1] performs numerically. The sum defining $\mathcal{G}$ is finite on polynomials.

Theorem 1.1 (symmetry; Theorem 3.2). *For every $n \geq 1$ and every u as above, $E_n^\top = E_n$.*

Theorem 1.2 (the spectrum; Theorems 4.3 and 4.4). *For every $n \geq 1$ and every u as above,*

$$\det(X \cdot I - E_n) = \prod_{k=0}^{n-1} (X - 2^{-k/2}), \quad \det(X \cdot I - E_n^\top E_n) = \prod_{k=0}^{n-1} (X - 2^{-k}).$$

So the eigenvalues of E_n are exactly $2^{-k/2}$, $k = 0, \dots, n-1$, each simple, and the singular values of E_n are exactly the same numbers.

The second characteristic polynomial is the conjectured statement in the form that carries no square roots: the squared singular values are the roots of a polynomial with dyadic rational coefficients. Together with $E_n \mathbf{1} = \mathbf{1}$ (Proposition 4.5) and Theorem 1.1, it gives the conjecture in the source's own phrasing — the eigenvalue 1 is the one on $\text{Span}(\mathbf{1})$, and the remaining $n-1$ values $2^{-k/2}$, $k = 1, \dots, n-1$, are the singular values on $W = \mathbf{1}^\perp$ (Corollary 4.6).

Theorem 1.3 (the trace identity; Theorem 4.7). *For every $n \geq 1$ and every u as above, $\text{tr } E_n - 1 = \sum_{k=1}^{n-1} 2^{-k/2}$.*

The identity $\text{tr}(E_n|_W) = \sum_{k=1}^{n-1} 2^{-k/2} \rightarrow \sqrt{2} + 1$ is displayed in [1] under the hypothesis that *both* of its conjectures hold; here it is unconditional, and in a form — a trace, not a restricted trace — that does not presuppose the invariant splitting.

The proofs occupy §§2–4 and are short. The engine is that *convolving with He_n is the heat operator*: for every p of degree at most n ,

$$(5) \quad p \boxplus_n \text{He}_n = \mathcal{G}p,$$

which is (2) read in the right basis (Proposition 2.3), and is stated in [2] in almost these words. Christoffel–Darboux (Lemma 2.2) expands f_j in the Hermite basis, with coefficients

$$T_{mj} = \frac{(n-1)!}{m!} \cdot \frac{\text{He}_m(u_j)}{\text{He}_{n-1}(u_j)}, \quad 0 \leq m \leq n-1;$$

then $\mathcal{G}\text{He}_m = \text{He}_m^{(2)}$, the variance-two Hermite polynomial, and $\text{He}_m^{(2)}(cx) = c^m \text{He}_m(x)$ for $c^2 = 2$ (Lemma 2.4). Both facts together turn (4) into

$$(6) \quad (E_n)_{ij} = \frac{1}{c^{n-1} n!} \sum_{m=0}^{n-1} c^m m! T_{mi} T_{mj},$$

which is symmetric on its face — that is Theorem 1.1 — and simultaneously give $TE_n = \Lambda T$ with $\Lambda = \text{diag}(c^{m-n+1})_{m=0}^{n-1}$, from which Theorem 1.2 follows because T is invertible (a generalised Vandermonde matrix at n distinct nodes). In one sentence: E_n is the matrix, in the basis $\{-f_j\}$ of $\mathbb{R}[x]_{<n}$, of the operator $q \mapsto c^{-(n-1)}(\mathcal{G}q)(c \cdot)$, which is triangular with diagonal c^{m-n+1} in the monomial basis and *diagonal* in the Hermite basis. This is the exact finite-degree counterpart of the classical picture [6, 7] that [1] invokes for its rate $1/\sqrt{2}$: under the linearised classical central limit map at the Gaussian, the k -th Hermite mode decays by $2^{-k/2}$ per step.

Two consequences are worth recording. First, the three corollaries of [1] listed above lose their hypothesis (Corollary 4.8). Second, the dyadic law is not a large- n phenomenon that the audited range happened to reach: it holds at $n = 1$ and $n = 2$ as well, below the range $n \geq 3$ in which the conjecture was posed, and the eigenvalues are simple at every degree, so the matrix has no spectral degeneracy anywhere.

1.4. What is not claimed. Five limits, stated here rather than left to be inferred.

First, the passage from (3) to (4) is calculus done on paper. Everything proved below is a theorem about the matrix (4). The identification of that matrix with the partial Jacobian (3) is the implicit differentiation of §3: it is ordinary multivariable calculus, it is the same computation the audit appendix of [1] performs numerically, and it is *not* part of the machine-checked development. As a check on it,

(4) was compared against a high-precision central-difference Jacobian of the actual root map, computed from (2) and nothing else (Remark 6.1).

Second, the hypothesis on u . Every theorem below assumes that $u_1, \dots, u_n$ are n distinct roots of He_n . That He_n has n distinct real roots is classical [5], so the hypothesis is never vacuous; but it is a hypothesis in the verified development rather than a theorem there, because the library used has no simple-real-rootedness statement for Hermite polynomials. It is discharged explicitly at $n = 2$ and $n = 3$ in Proposition 5.1. Nothing depends on the enumeration being sorted: permuting u conjugates E_n by a permutation matrix, which changes neither symmetry nor spectrum, and the source's sorted convention is one case.

Third, the normalisation of h . The source's h is the Hermite root vector normalised to mean 0 and variance 1; the u above is the raw root vector. This is immaterial for (3), since $\Omega_{\boxplus_n}(t\alpha, t\beta) = t \Omega_{\boxplus_n}(\alpha, \beta)$ for $t > 0$ makes the partial Jacobian invariant under a common rescaling of both arguments. That observation, too, is on paper and not in the development.

Fourth, the restriction to W . What is verified is the pair of characteristic polynomials of Theorem 1.2 together with $E_n^\top = E_n$ and $E_n \mathbf{1} = \mathbf{1}$. The literal block statement of the conjecture — a singular value 1 on $\text{Span}(\mathbf{1})$ and the remaining values on $W = \mathbf{1}^\perp$ — is Corollary 4.6, one paragraph of standard linear algebra, and that paragraph is on paper.

Fifth, what is not touched. The nonnegativity of the entries of E_n , part of the double stochasticity statement of [2, Lemma 3.4], is not proved here; Proposition 4.5 proves only the two sum conditions. Neither the p -Stam inequalities of [1] for $p \neq 2$ nor its uniqueness statement for the Hermite equality case is addressed: both quantify over all real-rooted pairs of a given degree and are of a different shape from anything below.

2. THE HEAT OPERATOR AND THE HERMITE BASIS

Definition 2.1. $\text{He}_0 = 1$ and $\text{He}_{m+1} = x\text{He}_m - \text{He}'_m$; equivalently $\text{He}_{m+2} = x\text{He}_{m+1} - (m+1)\text{He}_m$ and $\text{He}'_{m+1} = (m+1)\text{He}_m$. Each He_m is monic of degree m . The variance-two Appell sequence is $\text{He}_0^{(2)} = 1$, $\text{He}_{m+1}^{(2)} = x\text{He}_m^{(2)} - 2(\text{He}_m^{(2)})'$. Finally $\mathcal{G}p = \sum_{m \geq 0} \frac{(-1/2)^m}{m!} p^{(2m)}$, a finite sum on polynomials.

Two classical facts are used, and both are proved from Definition 2.1 in the verified development, because the library used carries the Hermite polynomials but neither of these statements.

Lemma 2.2 (Christoffel–Darboux). *For every $m \geq 0$ and every $y \in \mathbb{R}$,*

$$(x - y) \sum_{r=0}^m \frac{\text{He}_r(x)\text{He}_r(y)}{r!} \cdot m! = \text{He}_{m+1}(x)\text{He}_m(y) - \text{He}_m(x)\text{He}_{m+1}(y).$$

Proof sketch. Induction on m using the three-term recurrence of Definition 2.1; the added term on the left telescopes against the difference of the two products on the right. Five lines. $\square$

Proposition 2.3 (convolving with He_n is $\mathcal{G}$). *For every $m \geq 0$, $\mathcal{G}x^m = \text{He}_m$. Consequently, for every $n \geq 0$ and every $p \in \mathbb{R}[x]$ of degree at most n , the coefficients (2) formed from p and He_n satisfy*

$$\sum_{k=0}^n (-1)^k c_k x^{n-k} = \mathcal{G}p, \quad \text{that is,} \quad p \boxplus_n \text{He}_n = \mathcal{G}p.$$

Proof sketch. $\mathcal{G}x^m = \text{He}_m$ is an induction on m from the commutation law $\mathcal{G}(xp) = (x - D)\mathcal{G}p$, which is itself the identity $\frac{(-1/2)^m}{m!} \cdot 2m = -\frac{(-1/2)^{m-1}}{(m-1)!}$ applied termwise to $D^{2m}(xp) = xD^{2m}p + 2mD^{2m-1}p$. For the second statement, insert $a_i = (-1)^i p_{n-i}$ and $b_j = (-1)^j (\text{He}_n)_{n-j}$ into (2); the signs $(-1)^i (-1)^{k-i} = (-1)^k$ come out, the sum is reflected to $\sum_j \frac{(n-k+j)! (n-j)!}{n! (n-k)!} p_{n-k+j} (\text{He}_n)_{n-j}$, the odd j drop because He_n has the parity of x^n , and the term $j = 2m$ reduces, through $(\text{He}_n)_{n-2m} = (-1)^m \frac{n!}{2^m m! (n-2m)!}$ and the falling factorial identity $(n - k + 2m)^{\underline{2m}} = (n - k + 2m)! / (n - k)!$, to the m -th term of $\mathcal{G}p$ in degree $n - k$. Both halves are verified. $\square$

Proposition 2.3 is not new: [2] says it in the same breath, “one can implement the heat flow by taking finite free convolutions with Hermite polynomials of different scales. Namely, if $p(x)$ is a polynomial of degree n and we write $p(x) = P(\partial_x)x^n$ for P a polynomial, then ...”, and it is the differential-operator description of $\boxplus_n$ from [3]. It is proved here from (2) for a specific reason: it is what makes the matrix (4) the object (3) of [1] and not a lookalike.

Lemma 2.4 (the eigen-relation). $\mathcal{G}\text{He}_m = \text{He}_m^{(2)}$ for every m , and $\text{He}_m^{(2)}(cx) = c^m \text{He}_m(x)$ for every c with $c^2 = 2$. Hence He_m is an eigenvector, with eigenvalue $2^{m/2}$, of the map $p \mapsto (\mathcal{G}p)(c \cdot)$ — convolve with He_n , then rescale by $\sqrt{2}$.

Proof sketch. Both are inductions. The first uses $\mathcal{G}(xp) = (x - D)\mathcal{G}p$ and $\mathcal{G}D = D\mathcal{G}$ against the two recurrences of Definition 2.1; the second uses $(\text{He}_m^{(2)})'(cx) \cdot c = (\text{He}_m^{(2)}(c \cdot))'(x)$ and $2c^{-1} = c$. Both are verified. $\square$

Lemma 2.4 is the whole mechanism, and it is folklore once Proposition 2.3 is in hand: convolving with He_n doubles the variance, rescaling by $\sqrt{2}$ restores it, and an Appell sequence is homogeneous under the rescaling. It is the finite-degree form of the classical statement [6, 7] that the k -th Hermite mode decays by $2^{-k/2}$ under the linearised central limit map.

3. THE COUPLING MATRIX

3.1. From (3) to (4). Fix $n \geq 1$ and let $u_1, \dots, u_n$ be distinct roots of He_n , so that $\text{He}_n = \prod_i (x - u_i)$ and $f_j := \text{He}_n/(x - u_j)$ is a polynomial of degree $n - 1$. Move the first argument of $\Omega_{\boxplus_n}(\cdot, h)$ away from u : a unit perturbation of the root α_j perturbs $f = \prod_i (x - \alpha_i)$ by $-f_j$, so by (5) it perturbs the convolution $f \boxplus_n \text{He}_n$ by $-\mathcal{G}f_j$. At $\alpha = u$ the convolution is $P := \text{He}_n \boxplus_n \text{He}_n = \mathcal{G}\text{He}_n = \text{He}_n^{(2)}$, and by Lemma 2.4 $P(y) = c^n \text{He}_n(y/c)$, so the roots of P are $\gamma_i = cu_i$ and $P'(cu_i) = c^{n-1} \text{He}_n'(u_i) \neq 0$. Differentiating $P(\gamma_i) = 0$ implicitly,

$$\frac{\partial \gamma_i}{\partial \alpha_j} = \frac{(\mathcal{G}f_j)(\gamma_i)}{P'(\gamma_i)} = \frac{(\mathcal{G}f_j)(cu_i)}{c^{n-1} \text{He}_n'(u_i)},$$

which is (4). As said in §1.4, this paragraph is the one step of the development that is done on paper; everything below is a theorem about the right-hand side of (4).

3.2. The closed form. Write, for $0 \leq m \leq n - 1$ and $1 \leq j \leq n$,

$$(7) \quad T_{mj} := \frac{(n-1)!}{m!} \cdot \frac{\text{He}_m(u_j)}{\text{He}_{n-1}(u_j)}.$$

This is well defined: $\text{He}_{n-1}(u_j) \neq 0$, since consecutive Hermite polynomials have no common root (immediate from the three-term recurrence), and $\text{He}_n(u_j) = 0$.

Proposition 3.1 (closed form). *For all i, j ,*

$$f_j = \sum_{m=0}^{n-1} T_{mj} \text{He}_m \quad \text{and} \quad (E_n)_{ij} = \frac{1}{c^{n-1}n!} \sum_{m=0}^{n-1} c^m m! T_{mi} T_{mj}.$$

Proof. Apply Lemma 2.2 at $m = n - 1$, $y = u_j$: the term $\text{He}_{n-1}(x)\text{He}_n(u_j)$ vanishes and the identity reads $(x - u_j) \sum_{r=0}^{n-1} \frac{\text{He}_r(x)\text{He}_r(u_j)}{r!} (n-1)! = \text{He}_n(x)\text{He}_{n-1}(u_j)$. Dividing by $(x - u_j)\text{He}_{n-1}(u_j)$ gives the first display, with T as in (7). Now apply $\mathcal{G}$ termwise and evaluate at cu_i : by Lemma 2.4, $(\mathcal{G}\text{He}_m)(cu_i) = \text{He}_m^{(2)}(cu_i) = c^m \text{He}_m(u_i)$, so the numerator of (4) is $\sum_m T_{mj} c^m \text{He}_m(u_i)$. Substituting $\text{He}_m(u_i) = \frac{m!}{(n-1)!} T_{mi} \text{He}_{n-1}(u_i)$ from (7) and $\text{He}_n'(u_i) = n \text{He}_{n-1}(u_i)$ from Definition 2.1, the factor $\text{He}_{n-1}(u_i)$ cancels and $(n-1)!n = n!$ remains in the denominator. $\square$

Theorem 3.2 (the symmetry conjecture). *For every $n \geq 1$ and every $u_1, \dots, u_n$ with $\text{He}_n(u_j) = 0$ for all j — injective or not — the matrix E_n of (4) satisfies $E_n^\top = E_n$.*

Proof. The right-hand side of the closed form in Proposition 3.1 is symmetric in (i, j) term by term, and that form is proved from $\text{He}_n(u_j) = 0$ alone. $\square$

Two remarks on Theorem 3.2. Its independence of injectivity is worth stating, because injectivity is genuinely needed for the spectrum (Proposition 5.3): symmetry survives the collision of two nodes and the dyadic law does not. And the theorem closes exactly the gap [1, Remark 4.2] identifies: the diagonal identity $\partial_\alpha \Omega = \partial_\beta \Omega$ is a statement about *which* Jacobian, not about the transpose of one, and the symmetry visible in (6) comes from somewhere else — from the fact that the Christoffel–Darboux kernel is symmetric and that $\mathcal{G}$ is diagonal in the Hermite basis.

4. THE SPECTRUM

Proposition 4.1 (eigen-relation). *Let $T = (T_{mj})_{0 \leq m \leq n-1, 1 \leq j \leq n}$ and $\Lambda = \text{diag}(c^{m-n+1})_{m=0}^{n-1}$. If u is injective then $TE_n = \Lambda T$, i.e. $\sum_i T_{mi}(E_n)_{ij} = c^{m-n+1} T_{mj}$ for all m, j .*

Proof. Both sides are the coefficient vectors, in the Hermite basis, of two polynomials of degree at most $n-1$:

$$P_1 = \sum_{m=0}^{n-1} c^{m-n+1} T_{mj} \text{He}_m, \quad P_2 = \sum_{m=0}^{n-1} \left(\sum_i T_{mi}(E_n)_{ij} \right) \text{He}_m = \sum_i (E_n)_{ij} f_i,$$

the last equality by the first display of Proposition 3.1. Evaluate both at u_k . On the left, $P_1(u_k) = c^{-(n-1)} \sum_m T_{mj} c^m \text{He}_m(u_k)$, which by the proof of Proposition 3.1 equals $c^{-(n-1)} (\mathcal{G} f_j)(c u_k)$, and so equals $(E_n)_{kj} \text{He}'_n(u_k)$ by the definition (4). On the right, $f_i(u_k) = 0$ for $i \neq k$ — here injectivity is used — and $f_k(u_k) = \text{He}'_n(u_k)$, so $P_2(u_k) = (E_n)_{kj} \text{He}'_n(u_k)$ as well. Two polynomials of degree $< n$ agreeing at the n distinct points $u_1, \dots, u_n$ are equal, and the He_m , $0 \leq m \leq n-1$, are linearly independent because their degrees are distinct, so the coefficient vectors agree. $\square$

Lemma 4.2. *If u is injective and $\text{He}_n(u_j) = 0$ for all j , then $\det T \neq 0$.*

Proof. Factor $T = D_1 (AV^\top) D_2$, where $D_1 = \text{diag}((n-1)!/m!)$, $D_2 = \text{diag}(\text{He}_{n-1}(u_j)^{-1})$, $A_{mk} = [x^k] \text{He}_m$ and V is the Vandermonde matrix of u . Both diagonal factors are invertible, A is triangular with unit diagonal because He_m is monic of degree m , so $\det A = 1$, and $\det V \neq 0$ because the u_j are distinct. $\square$

Theorem 4.3 (the eigenvalues). *For every $n \geq 1$ and every injective u with $\text{He}_n(u_j) = 0$ for all j ,*

$$\det(X \cdot I - E_n) = \prod_{k=0}^{n-1} (X - 2^{-k/2}).$$

Proof. By Proposition 4.1 and Lemma 4.2, $E_n = T^{-1} \Lambda T$, and similar matrices have equal characteristic polynomials; that of the diagonal matrix Λ is $\prod_{m=0}^{n-1} (X - c^{m-n+1})$, which is the asserted product after the substitution $k = n-1-m$ and $c = \sqrt{2}$. $\square$

Theorem 4.4 (the spectrum conjecture). *Under the same hypotheses,*

$$\det(X \cdot I - E_n^\top E_n) = \prod_{k=0}^{n-1} (X - 2^{-k}),$$

so the singular values of E_n are exactly $2^{-k/2}$, $k = 0, \dots, n-1$.

Proof. By Theorem 3.2, $E_n^\top E_n = E_n^2$, and $TE_n^2 = \Lambda^2 T$ by two applications of Proposition 4.1; conclude as in Theorem 4.3 from $\Lambda^2 = \text{diag}(2^{m-n+1})$. $\square$

Proposition 4.5 (double stochasticity). *Under the same hypotheses, every column of E_n sums to 1 and every row of E_n sums to 1; that is, $\mathbf{1}^\top E_n = \mathbf{1}^\top$ and $E_n \mathbf{1} = \mathbf{1}$.*

Proof. The row $m = n-1$ of T is constant, $T_{n-1,j} = 1$, so the case $m = n-1$ of Proposition 4.1 reads $\sum_i (E_n)_{ij} = c^0 \cdot 1 = 1$. The row sums follow by Theorem 3.2. $\square$

Proposition 4.5 is not new — it is the sum half of [2, Lemma 3.4], whose proof there is a coupling argument. It is reproved here in this form because the conjecture’s phrasing needs the eigenvalue 1 to be identified with the direction $\mathbf{1}$, and because the route through Proposition 4.1 costs one line. The nonnegativity of the entries, the other half of [2, Lemma 3.4], is not reproved.

Corollary 4.6 (the conjecture as stated). *Let $n \geq 2$ and let u be as above. Then $W = \mathbf{1}^\perp$ is invariant under E_n , the restriction $E_n|_W$ is symmetric, and*

$$\text{spec}(E_n|_W) = \{2^{-k/2} : k = 1, \dots, n-1\},$$

these being also its singular values, while E_n acts on $\text{Span}(\mathbf{1})$ as the identity. In particular [1, Conjecture 4.1] holds, for every $n \geq 2$ rather than only for $n \geq 3$.

Proof. By Theorem 4.3 the n eigenvalues are distinct, so each eigenspace is a line; by Theorem 3.2 the eigenvectors are pairwise orthogonal; by Proposition 4.5 the eigenvector for the eigenvalue 1 is $\mathbf{1}$. Hence W is the span of the remaining $n-1$ eigenvectors, is invariant, and $E_n|_W$ is symmetric with eigenvalues $2^{-k/2}$, $k = 1, \dots, n-1$; the singular values of a symmetric matrix with positive eigenvalues are its eigenvalues. (This paragraph is on paper, not in the verified development; see §1.4.) $\square$

Theorem 4.7 (the trace identity). *For every $n \geq 1$ and every injective u with $\text{He}_n(u_j) = 0$ for all j ,*

$$\text{tr } E_n - 1 = \sum_{k=1}^{n-1} 2^{-k/2} \quad \left(\xrightarrow[n \rightarrow \infty]{} \sqrt{2} + 1 \right).$$

Proof. $\text{tr } E_n = \text{tr}(T^{-1}\Lambda T) = \text{tr } \Lambda = \sum_{m=0}^{n-1} c^{m-n+1} = \sum_{k=0}^{n-1} 2^{-k/2}$, and the term $k = 0$ is the 1 subtracted on the left. $\square$

Corollary 4.8 (three conditional corollaries, unconditionally). *Corollaries 4.4, 4.5 and 4.8 of [1], each stated there under “Assume Conjecture 4.1”, hold with that hypothesis removed. In particular the linearised finite free central limit rate is $\lambda_{\text{CLT}} = \sqrt{2}\sigma_2(E_n|_W) = 1/\sqrt{2}$, the same value at every degree, and the local stability bound of [1, Cor. 4.5],*

$$\|w\|^2 - \|J_{\boxplus_n}(w)\|^2 \geq \frac{1}{2} \|P_\perp w\|^2$$

with e_1 the eigenvector of $E_n|_W$ for $2^{-1/2}$ and $P_\perp$ the orthogonal projection onto $\text{Span}\{(e_1, e_1)/\sqrt{2}\}^\perp$, holds unconditionally as well.

Proof. Their hypothesis is Corollary 4.6; their proofs are those of [1] and are not reproduced. In the source’s indexing $\sigma_k(E_n|_W) = 2^{-k/2}$, so the quantity entering [1, Cor. 4.8] is $\sigma_2(E_n|_W) = 2^{-1}$, the case $k = 2$ of Corollary 4.6, and $\lambda_{\text{CLT}} = \sqrt{2} \cdot \frac{1}{2} = 1/\sqrt{2}$; the case $k = 1$, $\sigma_1(E_n|_W) = 2^{-1/2}$, is the eigenvalue [1, Lemma 5.6] exhibits for the score direction, and was the only one of the $n-1$ previously known. $\square$

5. SMALL DEGREES, AND THREE CONTROLS

The theorems above are hypothesis-carrying, and their hypotheses deserve two kinds of scrutiny: that they are satisfiable at all, and that they are not so strong as to make the conclusion vacuous or automatic.

Proposition 5.1 (the hypotheses are satisfiable). $\text{He}_2 = x^2 - 1$ and $\text{He}_3 = x^3 - 3x$. Accordingly $(-1, 1)$ and $(-\sqrt{3}, 0, \sqrt{3})$ are injective enumerations of the roots of He_2 and He_3 , so the hypotheses of Theorems 3.2–4.7 hold at $n = 2$ and $n = 3$.

Proposition 5.2 (the anchor at $n = 2$). *With $u = (-1, 1)$ and $c = \sqrt{2}$, the definition (4) evaluates to*

$$E_2 = \begin{pmatrix} \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} \\ \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} \end{pmatrix}, \quad \det(X \cdot I - E_2) = (X-1)\left(X - \frac{1}{\sqrt{2}}\right),$$

the characteristic polynomial being computed from the trace and the determinant of a 2×2 matrix, a route that does not pass through Theorem 4.3.

Proposition 5.3 (controls). *In parts (i) and (ii), let $n \geq 1$ and let u be an injective enumeration of n roots of He_n ; part (iii) is about a node vector that is not injective.*

- (i) $\det(X \cdot I - E_n) \neq \prod_{k=1}^n (X - 2^{-k/2})$: the eigenvalue 1 really is present, and the spectrum is not the dyadic set shifted by one place.
- (ii) For $n \geq 2$, $\det(X \cdot I - E_n) \neq (X - 1)^n$: the matrix is not a perturbation of the identity with all eigenvalues equal to 1.
- (iii) Injectivity is load-bearing. The vector $u = (-1, -1)$ satisfies $\text{He}_2(u_j) = 0$ for both j but is not injective, and for it the same formula (4) produces the constant matrix $\frac{2+\sqrt{2}}{4} \mathbf{1}\mathbf{1}^\top$, whose characteristic polynomial is not $(X - 1)(X - 2^{-1/2})$.

Proof. (i) Evaluate both sides at $X = 1$: the left is 0 by Theorem 4.3, since the factor $k = 0$ vanishes, while the right is a product of the strictly positive numbers $1 - 2^{-k/2}$, $k \geq 1$. (ii) Evaluate at $X = 2^{-1/2}$: the left vanishes by the factor $k = 1$, which exists because $n \geq 2$, and the right is $(2^{-1/2} - 1)^n \neq 0$. (iii) The displayed matrix is what Proposition 3.1 gives at $u = (-1, -1)$ — the two columns of T coincide — and its determinant is 0, so its characteristic polynomial takes the value 0 at $X = 0$ while $(X - 1)(X - 2^{-1/2})$ takes the value $2^{-1/2}$. $\square$

Part (iii) is the perturbation control in the strict sense: it is not an ad hoc matrix but the construction (4) itself, run at a degenerate node vector, and it fails the conclusion. So the theorems of §4 are not consequences of the Christoffel–Darboux algebra alone.

6. WHAT IS FINITE HERE, AND WHAT IS NOT

No theorem above rests on an exhaustive computation. Each is a single statement quantified over all degrees n and all admissible node vectors, proved by the chain Lemma 2.2 $\rightarrow$ Proposition 2.3 $\rightarrow$ Lemma 2.4 $\rightarrow$ Proposition 3.1 $\rightarrow$ Proposition 4.1, in which no case distinction on n occurs and no search over a finite set is performed. The only finite computations in the paper are Propositions 5.1, 5.2 and 5.3(iii), each an explicit evaluation at $n = 2$ or $n = 3$ stated as such, and each proved exactly rather than numerically. In particular no floating-point or interval arithmetic enters any proof: $\sqrt{2}$ appears only through the identity $c^2 = 2$, which is all any of the arguments uses.

Remark 6.1 (numerical corroboration, outside the formal development). Before any claim above was made, the algebraic definition (4) was checked against the analytic object (3) it is supposed to compute. Two independent routines were run in 70-digit decimal arithmetic: a central-difference Jacobian ($\varepsilon = 10^{-25}$) of the actual root map — roots of He_n by interlacing bisection, elementary symmetric functions, the coefficients (2), then the roots of the convolution by Newton’s method from $\sqrt{2}u$, with no part of the theory entering — and the closed form (6). At the degrees $n = 2, \dots, 12$ and $n = 16, 20, 24, 28, 32, 34$ the two agree entrywise to at worst 7×10^{-37} , and the closed form is symmetric to the last digits of that arithmetic, the worst discrepancy at any of those degrees being of order 10^{-70} ; and at every degree $2 \leq n \leq 34$ the characteristic polynomial of $E_n^\top E_n$ restricted to $\mathbf{1}^\perp$ agrees coefficientwise with $\prod_{k=1}^{n-1} (x - 2^{-k})$, the worst deviation recorded being 3×10^{-63} . This is a stronger check than the source’s audit — the whole spectrum rather than the ten leading singular values, and every degree in the range rather than nine of them — but it is a check, not a proof, and it is not part of the verified development. It also reproduced, to the working precision, the exact small cases E_2 as in Proposition 5.2 and $E_3 = \frac{1}{12} \begin{pmatrix} 5+3\sqrt{2} & 2 & 5-3\sqrt{2} \\ 2 & 8 & 2 \\ 5-3\sqrt{2} & 2 & 5+3\sqrt{2} \end{pmatrix}$.

Remark 6.2 (E_n is not a rational matrix). Theorem 4.3 forces this, and it is worth recording because it rules out the obvious computational route to the conjecture. A matrix with rational entries has a Galois-stable spectrum, whereas the spectrum of E_n contains $2^{-1/2}$ and not $-2^{-1/2}$ as soon as $n \geq 2$. So no degree of the conjecture, however small, can be settled by exact rational arithmetic on an explicitly tabulated matrix; the entries live in a field that grows with n (already E_3 needs $\sqrt{2}$, and the nodes of He_4 are $\pm\sqrt{3 \pm \sqrt{6}}$). A general-degree argument was not a convenience here but the only route. (This remark is outside the formal development.)

7. VERIFICATION

Every statement above that is asserted as a lemma, proposition or theorem — Lemmas 2.2, 2.4 and 4.2, Propositions 2.3, 3.1, 4.1, 4.5, 5.1, 5.2 and 5.3, and Theorems 3.2, 4.3, 4.4 and 4.7 — is checked, in the form stated, by the kernel of the Lean 4 proof assistant [8] over the Mathlib library [9], which supplies the Hermite polynomials, the Vandermonde determinant and the characteristic polynomial of a matrix, but neither Christoffel–Darboux nor the finite free convolution. The development is six modules and about 1364 lines carrying 89 declarations: the Hermite recurrences and the operator $\mathcal{G}$ with its commutation laws, the variance-two sequence and the rescaling law, and Christoffel–Darboux; the coefficient formula (2) transcribed verbatim and the bridge (5); the matrix (4), the closed form (6), the symmetry theorem and the eigen-relation; a similarity lemma for characteristic polynomials, the generalised Vandermonde determinant, and the two characteristic polynomials; the headline statements at $c = \sqrt{2}$ in the dyadic indexing used above, with the trace identity; and the small-degree instances and controls. It contains no `sorry` and declares no axiom of its own; a print of the axiom set of each of the 24 headline declarations returns `propext`, `Classical.choice` and `Quot.sound` and nothing else — in particular no compiled-evaluation axiom, no external solver, no certificate file and no floating-point arithmetic anywhere, which is possible because, as §6 says, none of the theorems is a finite check. Elaboration of all six modules takes one to two minutes of processor time and about 7 GB of memory on the machines used, essentially all of both the library import. Five things stated above are deliberately outside that development and are labelled where they appear: the implicit differentiation of §3 that identifies (4) with (3); the scale-invariance remark of §1.4; the block decomposition in the proof of Corollary 4.6; Corollary 4.8, whose content is that the hypothesis is removed from three proofs of [1] which are not reproduced here; and Remarks 6.1 and 6.2. Additionally, the classical fact that He_n has n distinct real roots is used only to know that the hypotheses of the theorems are satisfiable at every degree, and is assumed rather than proved — verified only at $n = 2, 3$, in Proposition 5.1. The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. [1] and [2] were read from their arXiv e-prints, whose live listings were checked on 3 September 2026; every statement and equation number quoted from either was read off a compilation of that e-print’s own $\text{T}_{\text{E}}\text{X}$ source, so the numbering is pinned to the version cited. Journal data for [3], [4], [6] and [7] come from Crossref and zbMATH. [5] is cited without a section number, which could not be checked here.

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