PRODUCT-CENTERED REFLECTION-SYMMETRIC EQUILATERAL TRIANGLES IN FLORETION TRIANGULAR COORDINATES

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Abstract. Dement has recently studied the unordered triples of order-n floretion base vectors whose tile centroids form a nondegenerate equilateral triangle of the triangular lattice, and has determined in closed form the number of such triangles that are local, that are product-centered and local, and that are symmetric about a main axis of the tiling. Two columns of his table of reflection-symmetric counts are left open: no closed formula is known for the number $|A_n \cap P_n|$ of axis-symmetric product-centered triangles, nor for its nonlocal part $|A_n \cap X_n \cap P_n|$, and he obtains both only by exhaustive enumeration through order 6. We give a reduction of the per-axis version of that count from a search over triples, of size $\Theta(16^n)$, to a search over one word of $\{e, i, j, k\}^n$ and one binary word, and use it to push the open column four orders further. We obtain the new values $|A_n \cap P_n| = 16218$, 49930, 152346 at $n = 8, 9, 10$, and we find that on each single axis the count is $|A_n(\tau) \cap P_n| = 8 \cdot 3^{\,n-2} - 2^n$ at every order $n \geq 2$ we can reach, so that $|A_n \cap P_n| = 8 \cdot 3^{\,n-1} - 5 \cdot 2^n + 2$ for $2 \leq n \leq 10$. We emphasise that this closed form is verified, not proved: it is a conjecture with a ten-order audit. Two structural consequences are recorded. First, the Fibonacci numbers that appear in the conjectured nonlocal column are an artifact of subtracting Dement's local term: the per-axis product-centered count is Fibonacci-free, and it is therefore the object to attack. Second, the two integer equations that define the reduced count are read digit by digit with a bounded carry, which we prove; the reduced count is consequently recognised by a finite automaton, and an all-n proof of the closed form is reduced to a finite verification. The finite-order counts below are formally verified in Lean 4 — apart from the few statements we mark as lying outside the formal development — and we say exactly which of them rest on exhaustive computation and what was searched.

1. Introduction

The objects. Fix the alphabet $\{e, i, j, k\}$ and let $\Delta_n = \{e, i, j, k\}^n$ be the set of 4^n words of length n; these index the positive basis of the n-fold tensor power of the quaternions, and in the floretion coordinate model of Dement [1] they also index the 4^n tiles of an order-n recursive triangular tiling. The tile centroids are recorded by the *scaled integer centroid map* [1, Definition 2.3, eq. (2.3)]

$$\Lambda_n(b) \;=\; \sum_{r=1}^{n} \sigma_r(b)\, 2^{\,n-r} \lambda(b_r) \;\in\; \mathbb{Z}^2, \qquad \sigma_r(b) = (-1)^{\#\{s<r\,:\,b_s=e\}}, \tag{1}$$

where $\lambda(e) = (0, 0)$, $\lambda(i) = (1, 0)$, $\lambda(j) = (0, 1)$ and $\lambda(k) = (-1, -1)$. The map Λ_n is injective. Its target is an oblique coordinate system for the triangular lattice, whose induced quadratic form is $q(x, y) = x^2 - xy + y^2$; rotation by 60° is $(x, y) \mapsto (y, y - x)$. Let

$$E_n = \big\{\, T \subseteq \Delta_n \;:\; |T| = 3,\ \Lambda_n(T) \text{ a nondegenerate equilateral triangle} \,\big\}.$$

Three subfamilies of E_n concern us, all of them Dement's. Writing $\gamma\colon i \mapsto j \mapsto k \mapsto i$ with $\gamma(e) = e$, and γ_S for the application of γ in the coordinates of a set S, the *local* triangles are the orbits $T(b, S) = \{b, \gamma_S b, \gamma_S^2 b\}$ with $\varnothing \neq S \subseteq \operatorname{supp}(b)$; they form the class L_n, and $X_n = E_n \setminus L_n$ is the nonlocal class. Any two basis words commute or anticommute, so the product of the three vertices of a triangle is well defined up to a global sign; writing $p(T)$ for that unsigned product, T is *product-centered* when the centroid of the triangle is the point carried by $p(T)$, and P_n is the class of such triangles [1, Definition 1.1]. Finally, each transposition τ of $\{i, j, k\}$ acts digitwise on Δ_n and, through

Λ_n, as the reflection in one of the three main axes of the tiling; $A_n(\tau) = \{T \in E_n : \tau(T) = T\}$ and A_n is the union of the three sets $A_n(\tau)$.

The question. Dement determines $|E_n| = 4^n(4^n - 1)/12$, $|L_n| = (7^n - 4^n)/3$, and all eight of the reflection-symmetric counts

$$\begin{gathered} |A_n(\tau)| = 3 \cdot 4^{\,n-1} - 2^n, \qquad |A_n(\tau) \cap L_n| = 3^n - 2^n, \\ |A_n(\tau) \cap L_n \cap P_n| = F_{2n+2} - 2^n, \qquad |A_n \cap L_n \cap P_n| = 3F_{2n+2} - 5 \cdot 2^n + 2, \end{gathered} \tag{2}$$

together with $|A_n| = 9 \cdot 4^{\,n-1} - 5 \cdot 2^n + 2$ and the corresponding nonlocal counts [1, Corollary 4.4, Theorem 5.8, Theorem 7.4]; here $F_1 = F_2 = 1$. What he does not determine is the product-centered column. The relevant passage [1, Remark 7.5] reads, verbatim:

> The final two columns were obtained by exhaustive exact enumeration through order 6. No closed formula is presently known for either $|A_n \cap P_n|$ or $|A_n \cap X_n \cap P_n|$. … Determining such a formula is left for future work.

The same remark observes that the two unknown columns differ by the known term $|A_n \cap L_n \cap P_n|$ of (2), so that a closed formula for either gives the other: there is one open sequence here, not two. It also records the coincidence $|A_6 \cap L_6 \cap P_6| = |A_6 \cap X_6 \cap P_6| = 813$, of which it says "No structural explanation for this equality is presently known", and guesses that it is an isolated finite-order coincidence.

The obstruction to going further is arithmetic rather than conceptual. The published enumeration stops at order 6 because it runs over triples: $|E_7| = 22\,368\,256$ and $|E_9| = 5\,726\,601\,216$, and the natural algorithm — run over pairs and complete each pair by the two $60°$ rotations — costs $\Theta(16^n)$.

What is settled here. We reduce the per-axis count to a search of size $\Theta(8^n)$, and to $\Theta(4^n\, n)$ when one auxiliary table is precomputed. Fix τ to be the transposition $(j\,k)$, let $f_\tau = i$ be the noncentral digit it fixes, and let $W = \{e, i\}^n$ be the τ-fixed words. Writing $\Lambda_n(b) = (x(b), y(b))$ and $D(b) = \{r : b_r \in \{j, k\}\}$, the reduction (Remark 3.1) identifies $A_n(\tau)$ with the set of words b having $D(b) \neq \varnothing$ and $x(b) + y(b) \in x(W)$, and turns the product-centered condition into the single scalar equation $x(p) = x(b)$, where $p \in W$ is obtained from the τ-fixed vertex by exchanging e and i exactly on $D(b)$.

Running that reduction gives, for every order we can reach, a closed form on each single axis, and hence, through Dement's own inclusion–exclusion over the three axes, the open column itself:

$$|A_n(\tau) \cap P_n| = 8 \cdot 3^{\,n-2} - 2^n, \qquad |A_n \cap P_n| = 8 \cdot 3^{\,n-1} - 5 \cdot 2^n + 2 \qquad (n \geq 2). \tag{3}$$

Theorem 4.2 states the first of these at orders $2 \leq n \leq 7$ and Theorem 4.3 at orders $n = 8, 9, 10$; both are verified by exhaustive computation, and (3) for all n is Conjecture 4.5, which we do not prove. All three orders lie past the enumeration Dement reports, and the order-10 value is past what we were able to confirm by an independent triple search as well; the new entries of the open column are

$$|A_n \cap P_n| = 16218,\ 49930,\ 152346 \qquad \text{and} \qquad |A_n \cap X_n \cap P_n| = 9744,\ 32193,\ 104331$$

at $n = 8, 9, 10$. Two further points are worth isolating.

The Fibonacci numbers are an artifact. The conjectural nonlocal column $|A_n \cap X_n \cap P_n| = 8 \cdot 3^{\,n-1} - 3F_{2n+2}$ carries a Fibonacci term, but the per-axis product-centered count (3) does not; F_{2n+2} enters only through the subtraction of Dement's local term in (2). Compare his own $|A_n(\tau) \cap L_n| = 3^n - 2^n = 9 \cdot 3^{\,n-2} - 2^n$, which sits beside (3) with the same shape and a different leading constant. The per-axis identity is therefore the object to attack, and the two "unknown sequences" of Remark 7.5 are one statement. As a by-product, the order-6 coincidence that Dement flags becomes, conditionally on (3), the unique solution of an explicit exponential equation (Remark 5.1).

The reduced count is finite-state. Both defining equations are read coordinate by coordinate, most significant first, as a running difference δ obeying $\delta_r = 2\delta_{r-1} + c_r$ with $\delta_0 = 0$ and a bounded increment c_r. Lemma 6.1 shows that increments of size at most C cannot drive such a difference from outside $(-C, C)$ back to 0; since the two increment bounds here are 3 and 2 (Lemma 6.2), any accepting run keeps its two carry registers in $\{-2, \ldots, 2\}$ and $\{-1, 0, 1\}$. The reduced count is therefore recognised by a finite automaton, and proving (3) for all n becomes a finite verification plus a counting induction rather than an open problem in enumeration. We record the automaton we obtained in Remark 6.4 and are explicit that it is not part of the formal development.

What was searched. Since Remark 7.5 is a published open problem, the novelty of (3) is a claim about the literature and we state what supports it. The source is arXiv:2608.15925; an arXiv API query on its identifier, run on 2026-08-29 and again on 2026-08-30, returns exactly one entry, `v1`, with published = updated = 2026-08-16T20:39:16Z, so the columns are open in the version that exists. The Online Encyclopedia of Integer Sequences was queried on 2026-08-29 and re-queried on 2026-08-30 for the six sequences 1, 4, 16, 56, 184, 584, 1816, 5576 and 4, 16, 56, … (the per-axis column, with and without its exceptional first term), 1, 6, 34, 138, 490, 1626, 5194 and 6, 34, … (the open column), 0, 0, 9, 51, 216, 813, 2871 (the nonlocal open column) and 1, 8, 72, 598, 4622, 34638, 255922 (the unrestricted product-centered count); every one of the six returns no hit. Two positive controls run in the same batch do return entries: 1, 11, 93, 715, 5261, 37851 returns A016150, which lists $|L_{m+1}|$, and Dement's published 1, 8, 40, 176, 736, 3008, 12160 returns A165665, $\big(3\cdot 2^m - 2\big)2^m$, which is $|A_{m+1}(\tau)|$. So the null results are not an artifact of the query format. A full-text sweep of a local mirror of arXiv (367 shards) for the word *floretion* returns 56 occurrences resolving to three documents: the source, its companion [2], and one unrelated bibliography citing the author's web page.

That last fact is also the honest limit of the evidence. Floretions are this author's own construction, the literature on them is two papers and a web site, and absence from OEIS and from an arXiv mirror is weaker evidence here than it would be for a mainstream sequence. We read all of the above as *not found*, not as *new*. We add that the companion paper [2], whose Theorem 4.8 is a principal input to Sections 6–8 of the source, has not been read by us; nothing in this paper depends on it, because everything is derived from (1) directly.

2. Notation

Throughout, $n \geq 1$ and τ is the digitwise transposition $(j\,k)$ acting on Δ_n coordinate by coordinate, with $f_\tau = i$ its fixed noncentral digit and $W = \{e, i\}^n$ the set of τ-fixed words. We write $\Lambda_n(b) = (x(b), y(b))$ for the two coordinates of (1), reserving the letter X for the nonlocal class X_n, and we drop the subscript n from Λ when it is clear. We put

$$D(b) = \{r \in \{1, \ldots, n\} \;:\; b_r \in \{j, k\}\},$$

so that $D(b)$ is the set of coordinates moved by τ. Encoding $e = 00$, $i = 01$, $j = 10$, $k = 11$ in base 4, the unsigned product of words is the bitwise exclusive or of the codes, written $\oplus$: quaternion multiplication modulo the global sign is the Klein four-group. We identify a word $b_0 \in W$ with the binary word $v \in \{0,1\}^n$ that has $v_r = 1$ exactly where $(b_0)_r = i$, and write

$$g(v) = x(b_0), \qquad g(v) = \sum_{r\,:\,v_r = 1} \sigma_r\, 2^{\,n-r},$$

the sign σ_r flipping after each e. Finally $1_D \in \{0,1\}^n$ is the indicator of a set D of coordinates, and F_m denotes the Fibonacci numbers with $F_1 = F_2 = 1$.

3. The axis reduction

The reduction is elementary, and we state it in full because everything below runs through it. It is *not* part of the formal development: it is proved by hand, and what is machine-checked is that the two counts it equates agree at every order the unreduced enumeration reaches (Proposition 3.3).

Remark 3.1 (The axis reduction; proved by hand, outside the formal development). Let $n \geq 1$.

(A) $\Lambda \circ \tau = M \circ \Lambda$ where $M(x, y) = (x - y, -y)$ is a linear involution with fixed line $y = 0$, and $y(b) = 0$ if and only if $b \in W$, equivalently $D(b) = \varnothing$.

(B) Every $T \in A_n(\tau)$ has the form $\{b_0, b, \tau b\}$ with $b_0 \in W$ and $D(b) \neq \varnothing$, and the equilateral condition forces $x(b_0) \in \{x(b) + y(b),\ x(b) - 2y(b)\}$. The assignment $T \mapsto b$, taking b to be the vertex realising the first alternative, is a bijection

$$A_n(\tau) \;\longleftrightarrow\; \big\{\, b \in \Delta_n \;:\; D(b) \neq \varnothing,\ x(b) + y(b) \in x(W) \,\big\}.$$

(C) With T and b as in (B), the unsigned product $p = b_0 \oplus b \oplus \tau b$ lies in W and is obtained from b_0 by exchanging e and i exactly on $D(b)$; the product-centeredness of T is equivalent to the single equation $x(p) = x(b)$.

Consequently, with $v \in \{0,1\}^n$ coding b_0,

$$|A_n(\tau)| = \#\big\{\, b \in \Delta_n : D(b) \neq \varnothing,\ \exists v,\ g(v) = x(b) + y(b) \,\big\}, \tag{4}$$

$$|A_n(\tau) \cap P_n| = \#\big\{\, b \in \Delta_n : D(b) \neq \varnothing,\ \exists v,\ g(v) = x(b) + y(b) \text{ and } g(v \oplus 1_{D(b)}) = x(b) \,\big\}. \tag{5}$$

Proof of Remark 3.1. (A) Since τ fixes e and i and exchanges j and k, it fixes each σ_r, and $\lambda(k) = -\lambda(i) - \lambda(j)$ gives $\lambda(\tau d) = M\lambda(d)$ for each digit d; (1) is linear in the $\lambda(b_r)$, so $\Lambda(\tau b) = M\Lambda(b)$. For the second claim, $y(b) = \sum_{b_r = j} \sigma_r 2^{\,n-r} - \sum_{b_r = k} \sigma_r 2^{\,n-r}$ is a $\pm$-signed sum of *distinct* powers of two, and such a sum vanishes only when it is empty, i.e. only when $D(b) = \varnothing$.

(B) An involution of a three-element set fixes a point. It cannot fix all three, since τ-fixed vertices lie on the line $y = 0$ and three collinear points are not a nondegenerate triangle. So $T = \{b_0, b, \tau b\}$ with $b_0 \in W$ and $y(b) \neq 0$. Put $(x_1, y_1) = \Lambda(b)$ and $u = x(b_0) - x_1$. Then $\Lambda(b) - \Lambda(\tau b) = (y_1, 2y_1)$ and $\Lambda(b_0) - \Lambda(b) = (u, -y_1)$, so equating the two q-lengths gives $y_1^2 - 2y_1^2 + 4y_1^2 = u^2 + uy_1 + y_1^2$, that is $u^2 + uy_1 - 2y_1^2 = 0$, i.e. $(u + 2y_1)(u - y_1) = 0$. The two admissible values of $x(b_0)$ differ by $3y_1 \neq 0$, so for a given b_0 exactly one of $b, \tau b$ realises $u = y_1$; taking that one gives a well-defined map $T \mapsto b$, whose inverse recovers b_0 from the equation $x(b_0) = x(b) + y(b)$ by injectivity of Λ on W.

(C) Digitwise, $b \oplus \tau b$ is 00 at coordinates outside $D(b)$ and 01 at coordinates of $D(b)$, because $j \oplus k = 01$; hence $p = b_0 \oplus b \oplus \tau b$ agrees with b_0 off $D(b)$ and differs from it by the exchange $e \leftrightarrow i$ on $D(b)$, and in particular $p \in W$. Both the centroid of T and $\Lambda(p)$ lie on the axis $y = 0$: the centroid because $\Lambda(b) + \Lambda(\tau b)$ has zero second coordinate by (A), and $\Lambda(p)$ because $p \in W$. So product-centeredness is the single scalar equation $3x(p) = x(b_0) + x(b) + x(\tau b)$, and (A) with $x(\tau b) = x(b) - y(b)$ and $x(b_0) = x(b) + y(b)$ turns the right-hand side into $3x(b)$. □

The counts (4) and (5) are what the formal development computes. Their agreement with the unreduced enumeration is machine-checked at every order the latter reaches, and the unreduced enumeration is in turn checked against Dement's published tables. We state both facts, in that order.

Proposition 3.2 (The unreduced enumeration reproduces the published tables)**.** *Let $1 \leq n \leq 5$. Enumerating E_n from the definitions of Section 1 and classifying each triangle by locality, by product-centeredness, by the vanishing of its unsigned product, and by invariance under each of the three digitwise transpositions, reproduces at those orders the following published entries of* [1]*: the columns $|E_n|$ and $|L_n|$ of its Table 1, and hence also its $|X_n| = |E_n| - |L_n|$; all three columns $|P_n|, |P_n \cap L_n|, |P_n \cap X_n|$ of its Table 2; and all six columns $|A_n|, |A_n \cap L_n|, |A_n \cap X_n|, |A_n \cap L_n \cap P_n|, |A_n \cap P_n|, |A_n \cap X_n \cap P_n|$ of its Table 3, the reflection table. The remaining columns of Table 1 count pairs by their number of equilateral completions and are not computed here.*

Proposition 3.3 (The reduction agrees with the unreduced enumeration)**.** *For $1 \leq n \leq 5$, the right-hand sides of* (4) *and* (5) *equal, respectively, the numbers $|A_n(\tau)|$ and $|A_n(\tau) \cap P_n|$ produced by the enumeration of Proposition 3.2.*

Proposition 3.3 is a computation that anyone can redo once Remark 3.1 is written down, and we claim nothing more for it than that. Its role is that it removes the reduction from the critical path at the orders where both sides can be computed: at those orders the numbers below do not depend on the correctness of the hand proof above.

4. The reflection-symmetric product-centered counts

We first record the published control. Equation (2) gives $|A_n(\tau)| = 3 \cdot 4^{n-1} - 2^n$, and this is exactly what (4) returns.

Proposition 4.1 (Dement's per-axis count, recomputed through the reduction)**.** *For $1 \le n \le 7$, the right-hand side of* (4) *equals $3 \cdot 4^{n-1} - 2^n$.*

This is not new — it is [1, eq. (7.3), the first identity of Theorem 7.4], and the sequence 1, 8, 40, 176, 736, 3008, 12160 is A165665 — but it is the control that matters, since it exercises the same reduction, the same coordinate arithmetic and the same search as the results below, and it is checked against a formula proved by other means.

Theorem 4.2 (The per-axis product-centered count at orders ≤ 7)**.** *For $2 \le n \le 7$,*

$$|A_n(\tau) \cap P_n| \;=\; 8 \cdot 3^{n-2} - 2^n,$$

while $|A_1(\tau) \cap P_1| = 1$. Consequently, for $n = 6, 7$,

$$|A_n \cap P_n| \;=\; 3\,|A_n(\tau) \cap P_n| - 2\,(2^n - 1) \;=\; 8 \cdot 3^{n-1} - 5 \cdot 2^n + 2,$$

which is 1626 at $n = 6$ and 5194 at $n = 7$.

The passage from one axis to three is Dement's own inclusion–exclusion and we use it as given: a triangle symmetric about two distinct main axes is invariant under the whole digitwise S_3-action, the three pairwise intersections therefore coincide with the triple intersection, and that intersection consists of the $2^n - 1$ size-3 orbits, so $|A_n| = 3|A_n(\tau)| - 2(2^n - 1)$ and the same correction applies to any subfamily containing those orbits [1, proof of Theorem 7.4]. Those orbits are local and product-centered, so the correction applies to $A_n \cap P_n$.

The value 1626 at $n = 6$ agrees with the entry Dement prints; the value 5194 at $n = 7$ is the first entry past his enumeration. The reduction reaches three orders further inside the kernel.

Theorem 4.3 (The open column at orders $8, 9, 10$)**.** *We have*

$$|A_8(\tau) \cap P_8| = 5576, \qquad |A_9(\tau) \cap P_9| = 16984, \qquad |A_{10}(\tau) \cap P_{10}| = 51464,$$

these being $8 \cdot 3^{n-2} - 2^n$ at $n = 8, 9, 10$; hence

$$|A_8 \cap P_8| = 16218, \qquad |A_9 \cap P_9| = 49930, \qquad |A_{10} \cap P_{10}| = 152346.$$

Moreover $|A_n(\tau)| = 3 \cdot 4^{n-1} - 2^n$ at $n = 8$ and $n = 9$, so the published identity (2) *survives at these orders under the same computation.*

Corollary 4.4 (The nonlocal open column at orders $8, 9, 10$)**.** $|A_n \cap X_n \cap P_n| = 9744,\ 32193,\ 104331$ *at $n = 8, 9, 10$.*

Proof. By [1, Remark 7.5], $|A_n \cap X_n \cap P_n| = |A_n \cap P_n| - |A_n \cap L_n \cap P_n|$, and the subtrahend is $3F_{2n+2} - 5 \cdot 2^n + 2$ by (2), namely 6474, 17737 and 48015 at $n = 8, 9, 10$. Subtract from Theorem 4.3. □

Table 1 collects the columns as they now stand. Rows 1 through 6 of the last two columns are Dement's; rows 7 through 10 are new.

Conjecture 4.5. *The identities* (3) *hold for every $n \ge 2$.*

We stress what Conjecture 4.5 is and is not. It is verified at every order at which it has been tested — $2 \le n \le 10$ inside the formal development, and $2 \le n \le 12$ by the same reduction implemented outside it — and it is proved for no n beyond those. It is also not merely an interpolation through Dement's six published values: the entries beyond his table at $n = 7, 8, 9$ were produced a second time by a full triple enumeration written independently of the reduction, and the two computations agree in every cell.

TABLE 1. Reflection-symmetric product-centered counts. Rows 1–6 of the last two columns are [1, Table 3]; rows 7–10 are Theorems 4.2 and 4.3 with Corollary 4.4. The column $|A_n \cap L_n \cap P_n|$ is (2) and is proved for all n there.

n	$\lvert A_n(\tau) \cap P_n\rvert$	$\lvert A_n \cap L_n \cap P_n\rvert$	$\lvert A_n \cap P_n\rvert$	$\lvert A_n \cap X_n \cap P_n\rvert$
1	1	1	1	0
2	4	6	6	0
3	16	25	34	9
4	56	87	138	51
5	184	274	490	216
6	584	813	1626	813
7	1816	2323	5194	2871
8	5576	6474	16218	9744
9	16984	17737	49930	32193
10	51464	48015	152346	104331

5. The Fibonacci numbers, and the coincidence at order six

Substituting (3) into $|A_n \cap X_n \cap P_n| = |A_n \cap P_n| - |A_n \cap L_n \cap P_n|$ and using (2) gives, conditionally on Conjecture 4.5,

$$|A_n \cap X_n \cap P_n| = 8 \cdot 3^{n-1} - 3F_{2n+2}, \qquad |A_n(\tau) \cap X_n \cap P_n| = 8 \cdot 3^{n-2} - F_{2n+2}. \tag{6}$$

The Fibonacci term in (6) is inherited entirely from the subtracted local count of (2); the product-centered count itself, (3), has none. This is the practical content of the observation: the two sequences of Remark 7.5 are one statement about $A_n(\tau) \cap P_n$, and any attempt to prove a Fibonacci formula directly is attacking a difference of two terms rather than either term.

Remark 5.1 (The order-six coincidence; conditional, and outside the formal development). Assume Conjecture 4.5. Then, by (2) and (6), the equality $|A_n \cap L_n \cap P_n| = |A_n \cap X_n \cap P_n|$ that Dement observes at $n = 6$ is equivalent to

$$6F_{2n+2} = 8 \cdot 3^{n-1} + 5 \cdot 2^n - 2.$$

The right-hand side minus the left is $-2, -6, -16, -36, -58$ at $n = 1, \ldots, 5$, is 0 at $n = 6$, and is 548 at $n = 7$; using $F_{2n+4} = 3F_{2n+2} - F_{2n}$ one checks that it is positive and increasing for $n \geq 7$, so $n = 6$ is the only solution. Dement's guess that the coincidence is isolated is therefore correct, and for a reason: the left-hand side grows like φ^{2n} with $\varphi^2 = 2.618\ldots < 3$ and is outrun by the right-hand side exactly once. This argument is elementary and we have not formalised it; a machine-checked fragment of it is the statement that the coincidence does not already occur at order 5 (Proposition 7.1(iv)).

6. Bounded carry, and why the count is finite-state

Both conditions in (5) are linear equations between integers built from Δ_n-words most significant coordinate first: each of g, x and y satisfies value $\leftarrow 2 \cdot$ value $+$ (signed digit contribution). Reading such an equation as the running difference δ of its two sides therefore gives

$$\delta_r = 2\delta_{r-1} + c_r, \qquad \delta_0 = 0, \tag{7}$$

where c_r depends only on the r-th coordinates and on the two signs σ_r accumulated so far by the two sides. Whether δ stays in a bounded window is exactly what decides whether the count is finite-state. It does, and for the simplest possible reason.

Lemma 6.1 (Bounded carry)**.** *Let $C > 0$ be an integer and let $c_1, \ldots, c_m$ be integers with $|c_r| \leq C$ for every r. If $z \in \mathbb{Z}$ satisfies $|z| \geq C$, then the value obtained from z by iterating $\delta \mapsto 2\delta + c_r$ over*

$c_1, \dots, c_m$ again has absolute value at least C; in particular it is nonzero. Consequently, if a run of (7) *ends at $\delta_m = 0$, then $|\delta_r| < C$ for every $r \le m$.*

Proof. If $|\delta| \ge C \ge |c|$ then $|2\delta + c| \ge 2|\delta| - |c| \ge 2C - C = C$, so the set $\{|\delta| \ge C\}$ is forward-closed under the recursion and contains no zero; the last sentence is the contrapositive, applied to the suffix of the run after step r. ☐

Lemma 6.2 (The two increment bounds)**.** *Write $\ell = \lambda_x + \lambda_y$, so that ℓ takes the values $0, 1, 1, -2$ on e, i, j, k, and λ_x takes the values $0, 1, 0, -1$. Then, over all choices of the two relevant sign bits, of the digit $b_r \in \{e, i, j, k\}$ and of the relevant binary digit, the increment of the first equation of* (5) *is bounded in absolute value by* 3 *and that of the second by* 2*. Both bounds are attained.*

Corollary 6.3 (The carry window)**.** *On any run of* (7) *ending at* 0*, the carry register of the first equation of* (5) *lies in $\{-2, -1, 0, 1, 2\}$ and that of the second in $\{-1, 0, 1\}$. In particular the reduced count* (5) *is recognised by a finite automaton over the alphabet $\{e, i, j, k\}$.*

Lemma 6.1 is the standard bounded-carry argument for a linear equation read digit by digit, and we claim no novelty for it as mathematics; what it does here is convert an open enumeration problem into a finite verification. It is worth recording that it is three lines, because the working notes this paper grew out of had priced it as the only step with real content.

Remark 6.4 (The automaton; built by machine, outside the formal development)*.* With Corollary 6.3 in hand the carry automaton for (5) has states $(\sigma_b, \sigma_v, \sigma_p, \delta, \delta', h)$, where the three signs are those of (1) for b, for v and for p, the two carries range over $\{-2, \dots, 2\}$ and $\{-1, 0, 1\}$, and h records whether $D(b) \ne \varnothing$ has been witnessed. Quantifying v existentially and determinising reaches 14 subsets, which minimise to the six-state automaton with start A, accepting states C and F, dead state D, and transitions

	e	i	j	k
A	B	B	C	D
B	B	B	C	E
C	C	E	F	D
E	D	D	D	F
F	F	F	C	E

whose accepted-word counts are 1, 4, 16, 56, 184, 584, 1816, 5576, 16984, 51464 and so on. These match $8 \cdot 3^{n-2} - 2^n$ for $n \ge 2$ and give the true value 1 at $n = 1$, where the closed form is not an integer. Writing $c_m(s)$ for the number of words of length m carrying s to an accepting state, the five formulas $c_m(D) = 0$, $c_m(F) = 3^m$, $c_m(E) = 3^{m-1}$, $c_m(C) = 2 \cdot 3^{m-1}$ and $c_m(B) = 3^m - 2^m$ close under one simultaneous induction, and $c_n(A) = 2c_{n-1}(B) + c_{n-1}(C) = 8 \cdot 3^{n-2} - 2^n$. We present this as a route and not as a proof: neither the soundness and completeness of the carry automaton against (5) — an accumulator identity relating (1) to (7) — nor the subset construction is part of the formal development, and a five-state automaton guessed from the language at $n \le 4$ was found to reproduce the correct counts at $n \le 3$ and to be wrong from $n = 4$ on. What Corollary 6.3 supplies is the one ingredient that is not a finite check.

7. Negative controls

A count is evidence only if a wrong count would fail, and a reduction is evidence only if its hypotheses are not vacuous. The following are checked for that reason.

Proposition 7.1 (Controls)**.**
(i) $|A_5 \cap P_5| \notin \{489, 491\}$*; the reduced count at order* 7 *is neither* 1815 *nor* 1817*; the reduced count at order* 8 *is neither* 5575 *nor* 5577*. So an off-by-one in the open column is refuted at orders* 5*,* 7 *and* 8*, the last two being past the published enumeration.*
(ii) $|A_5 \cap P_5| \ne |P_5|$ *and the reduced count at order* 5 *is strictly smaller than* 4^5*: the counted family is a proper subfamily.*

(iii) *The product-centered condition strictly cuts the count — the right-hand side of* (5) *is strictly smaller than that of* (4) *at order* 5 *— and so does the condition* $D(b) \neq \varnothing$*, whose removal changes the count already at order* 2*.*

(iv) $|A_5 \cap L_5 \cap P_5| \neq |A_5 \cap X_5 \cap P_5|$*: the coincidence of Remark 5.1 does not already occur at order* 5*. Also* $|A_5 \cap X_5 \cap P_5| > 0$*, so the open column is not the trivially zero one.*

(v) *The two increment bounds of Lemma 6.2 are attained, so neither can be lowered; a run with increments in* $[-3, 3]$ *can pass through carry* 2 *and still end at* 0*, so the window of Corollary 6.3 is tight; and a carry that has left the window does fail to return even when the increments are as large as allowed.*

8. What rests on exhaustive computation

Every numerical statement above is exhaustive rather than sampled, and we say what was run.

The unreduced enumeration. For a given n it computes Λ_n on all 4^n words, stores them in a coordinate grid of side $2^{n+1} - 1$, runs over the $\binom{4^n}{2}$ unordered pairs, completes each pair by the two $\pm 60°$ lattice rotations [1, eq. (3.2)–(3.3)], and keeps a completion only when the third word exceeds both, so that each triangle is produced exactly once; it then classifies the triangle by locality (through the digitwise criterion of [1, Proposition 5.4], in both cyclic directions), by $3\Lambda(p(T)) = \sum_i \Lambda(b_i)$, by $p(T) = e_n$, and by invariance under each of the three digitwise transpositions. Inside the formal development this is run at $n \leq 5$, which is what Propositions 3.2 and 3.3 assert. Outside it, the same algorithm was implemented in C from Dement's definitions and run to $n = 9$: that is $4^9 = 262\,144$ words and about $3.4 \cdot 10^{10}$ pairs, some eleven CPU-minutes. As an external control the C implementation was cross-checked at $n \leq 3$ against a brute-force scan of all $\binom{4^n}{3}$ triples testing equilaterality by comparing the three q-distances directly, and it agrees with the published tables at $n \leq 6$. This is how the entries at $n = 7, 8, 9$ of Table 1 were confirmed a second time, without the reduction.

The reduced enumeration. For a given n it runs over all 4^n words b with $D(b) \neq \varnothing$ and, for each, scans the 2^n binary words v testing the two equations of (5); the cost is $4^n 2^n\, O(n)$. This is what Propositions 3.3 and 4.1 and Theorems 4.2 and 4.3 rest on, at $n \leq 5$, $n \leq 7$, $n \leq 7$ and $n \in \{8, 9, 10\}$ respectively; order 10 is $4^{10} \cdot 2^{10}$ word pairs and takes about five minutes in the kernel. Outside the formal development, g can be inverted by a table of its 2^n values, which brings the cost to $O(4^n n)$ and reaches $n = 12$ in seconds; that is the source of the claim in Section 4 that Conjecture 4.5 has been tested to order 12. Order 11 of the in-kernel version is estimated at about forty minutes and was not run; the full triple enumeration at $n = 10$ is estimated at about three CPU-hours and was not run either.

What is not computational. Lemma 6.1, Lemma 6.2 and Corollary 6.3 are proved outright; the first is an induction on the increment list and the other two are finite case distinctions over four digits and two sign bits. Remark 3.1 is a hand proof. The passage from one axis to three in Theorem 4.2 and Theorem 4.3 is Dement's inclusion–exclusion, quoted. Corollary 4.4 is a subtraction of (2) from a machine-checked value.

9. Formal verification

Propositions 3.2, 3.3, 4.1 and 7.1, Theorems 4.2 and 4.3, and Lemmas 6.1 and 6.2 with Corollary 6.3, are formalised in Lean 4 [5] and checked by its kernel; the development imports no library beyond a small internal attribute module, which is why the enumerations run in seconds to minutes rather than under a general-purpose algebraic library. Lemmas 6.1 and 6.2 and Corollary 6.3 are proved by `decide` and by linear-arithmetic automation and use no compiled evaluation at all: they depend at most on propositional extensionality, quotient soundness and the axiom of choice, and the increment bounds depend on no axioms whatever. The enumerations are discharged by compiled evaluation (`native_decide`), so each of them depends in addition on one axiom asserting that the compiled evaluation of the closed decidable proposition in question agrees with its value in the kernel. No declaration in the development depends on `sorryAx`. The objects the theorems are stated about are the definitions of Section 1 transcribed directly — Λ_n as in (1), the q-form, the local-cycle criterion, the unsigned product as bitwise exclusive or — and not the closed forms they are compared against, so the comparison in Proposition 3.2 is a genuine reproduction of Dement's tables rather than a restatement of them. Two things lie outside the guarantee and are labelled where they occur: the axis reduction of Remark 3.1, and the automaton of Remark 6.4; so do the independent triple enumeration at $n = 7, 8, 9$ described in Section 8, the verification of Conjecture 4.5 at orders 11 and 12, and the induction in Remark 5.1. The $|A_n \cap L_n \cap P_n|$ column of Table 1 is quoted from (2) and is not recomputed here beyond order 5. The development is held in a private repository; repository reference to be added at submission.

10. What remains

Conjecture 4.5 is the open statement, and by Remark 6.4 it is no longer open in the sense Remark 7.5 of [1] is: what is missing is an accumulator identity relating (1) to the carry recursion (7), the finitely many checks of a subset construction, and one counting induction. Two neighbouring questions are untouched here. The nonlocal product-centered column $|P_n \cap X_n|$ of [1, Table 2], of which the source says only that "No closed formula for the nonlocal column is asserted here", is not addressed by the axis reduction, which uses reflection symmetry essentially; the same carry method applied to the full equilateral condition would need four registers rather than two. And [1, Question 9.7], whether a nonlocal multiplication-generated triangle exists, remains open; the unreduced enumeration of Section 8, run outside the formal development, extends the negative evidence for it from the order 5 reported in the source to order 9, but we have no proof idea and record only the evidence.

References

[1] C. Dement, *Equilateral completion in floretion triangular coordinates: locality, product points, and reflection symmetry*, arXiv:2608.15925v1 [math.CO] (2026); dated 13 August 2026, announced 2026-08-16. All references here are to v1, the only version at the time of writing (arXiv API query on the identifier, 2026-08-30). Section, theorem, equation and table numbers were read from a compile of the e-print source of v1.

[2] C. Dement, *Bitwise triangular coordinates for central products of quaternion groups: floretion base vectors, digitwise S_3-actions, and centralizer tiles*, arXiv:2605.20265v5 [math.GM] (2026). Cited as `Dement2026` in [1]; not read by us, and nothing in this paper depends on it.

[3] G. A. Brouwer, J. Joe, A. A. Noble and M. Noble, *Problems on the triangular lattice*, arXiv:2405.12321 [math.CO] (2024). The finite triangular-lattice completion counts that [1] combines with its orientation obstruction to obtain $|E_n|$.

[4] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`. Entries A016150 (expansion of $1/((1-4x)(1-7x))$, with $a(m) = (7^{m+1}-4^{m+1})/3 = |L_{m+1}|$) and A165665 ($a(m) = (3{\cdot}2^m-2)2^m = |A_{m+1}(\tau)|$), both consulted 2026-08-30.

[5] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science, vol. 12699, Springer, 2021, pp. 625–635.