INTERLACING OF THE ROOTS OF THE FILTER-INTEGRAL POLYNOMIALS OF AMDEBERHAN, DUNCAN, MOLL AND SHARMA

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Abstract. In their study of filter integrals for orthogonal polynomials, Amdeberhan, Duncan, Moll and Sharma introduce the polynomials

$$X_n(a) \;=\; 2^{2n-1}\big(a+\tfrac12\big)_n - \binom{2n-1}{n-1}(a)_n,$$

prove that all n zeros of X_n are real and negative — their argument placing one at $a=-n$ and one in each interval $(-(j+1),-j)$ — and close their paper with a Problem: do the roots of $Z_n = X_n(a)/(a+n)$ interlace those of Z_{n-1}? We prove that they do, for every $n \ge 2$. The one ingredient absent from the source is an evaluation at half-integers: at $a = -(j+\frac12)$ the shifted Pochhammer $(a+\frac12)_n$ vanishes exactly as the unshifted $(a)_n$ vanishes at $a=-j$, and the two evaluations carry the *same* sign $(-1)^j$. Consequently every root of X_n lies in the *left half* $(-(j+1), -(j+\frac12))$ of its unit interval; the sign of $(a)_{n-1}$ at a root of X_{n-1} is then determined, and interlacing follows from a two-term transfer identity in three lines. We also prove that $n \mid X_n(a)$ for every $a \in \mathbb{Z}$ and every n, a question raised in the first arXiv version of the source and absent from the published version, from whose Property 6.16 only the prime case follows. Every statement below is machine-checked in Lean 4.

1. Introduction

1.1. **The polynomials.** Amdeberhan, Duncan, Moll and Sharma [1, 2] evaluate *filter integrals* for classical orthogonal polynomial families. In the Gegenbauer section of that work a one-parameter family of polynomials X_n in a parameter a appears, and their Theorem 6.8 gives it in closed form. With the rising factorial $(x)_m = x(x+1)\cdots(x+m-1)$,

$$X_n(a) \;=\; 2^{2n-1}\big(a+\tfrac12\big)_n - \binom{2n-1}{n-1}(a)_n \;=\; 2^{n-1}\prod_{k=0}^{n-1}(2a+2k+1) - \binom{2n-1}{n-1}\prod_{k=0}^{n-1}(a+k), \tag{1}$$

both displays being [2, Theorem 6.8]. Both products are rising factorials; to keep them apart we call $2^{2n-1}\big(a+\frac12\big)_n$ the *shifted* term of (1) and $\binom{2n-1}{n-1}(a)_n$ the *unshifted* one. The polynomial X_n has degree n [2, Proposition 6.3] and positive integer coefficients [2, Corollaries 6.4 and 6.12]; its first values are

$$X_1 = a+1, \qquad X_2 = 5a^2+13a+6, \qquad X_3 = 22a^3+114a^2+164a+60.$$

The family arises as follows. With $C_n^{(a)}$ the Gegenbauer polynomial, orthogonal on $[-1,1]$ for the weight $(1-x^2)^{a-1/2}$, the source studies the filter integral

$$\gamma_n(a) \;=\; \int_{-1}^{1}\left[\frac{C_n^{(a)}(x)-C_n^{(a)}(0)}{x}\right]^2 (1-x^2)^{a-1/2}\,dx \tag{2}$$

([2, (6.112)]), evaluates it in closed form for odd n, and for even index introduces X_n by the scaling $X_n(a) = \Gamma(a)(2n)!\,\gamma_{2n}(a)\big/\big(4\sqrt{\pi}\,\Gamma(a+\frac12)(a)_n\big)$ ([2, (6.119)]). Nothing below uses (2): the proof runs on (1) alone.

The source establishes four facts about the zeros and arithmetic of X_n. Its Property 6.13 is $X_n(-n)=0$; its Property 6.14, credited there to C. Koutschan, is the symmetry $X_n(k-n) = (-1)^n X_n(-k)$ for $1 \le k \le n-1$; its Property 6.15 states that all zeros of X_n are real and negative, and its proof — sign alternation of X_n at the integers $0, -1, \ldots, -(n-1)$ — places one in each of the $n-1$ intervals $(-(j+1), -j)$, $0 \le j \le n-2$, alongside the zero at $-n$; and its Property 6.16 gives $X_q(a) \equiv a^q - a \pmod{q}$ for prime q.

Setting

$$Z_n(a) \;=\; \frac{X_n(a)}{a+n},$$

a polynomial of degree $n-1$, the paper closes ([2, p. 135]) with the following, introduced by "We conclude the discussion on the polynomials $X_n(a)$ with a question based on extensive Mathematica computations".

Problem 1.1 ([2, p. 135])**.** Define $Z_n(a) = X_n(a)/(a+n)$. Then the family of polynomials $\{Z_n(a)\}_{n\ge 1}$ have the interlacing property; that is, the roots of Z_n interlace those of Z_{n-1}.

Nothing but the acknowledgement and the bibliography follows it. The first arXiv version [1] ends instead with six numbered *Questions*, "based on extensive `Mathematica` computations"; the list is commented out in v2 and absent from print. Four of the six became the Properties and Corollaries quoted above: (1) positivity of the coefficients, (2) the symmetry, in the corrected form of Property 6.14 (Remark 5.2), (3) the congruence modulo a prime, and (5) reality and negativity of the roots of Z_n. Question (6) is Problem 1.1; Question (4) is the divisibility of Section 7 below, of which the published version retains only the prime case.

1.2. **Results.** This paper answers Problem 1.1 affirmatively for every $n \ge 2$, together with the arithmetic question that the published version dropped.

The whole difficulty is one sign. The source's proof of Property 6.15 evaluates X_n at the negative *integers* $0, -1, \ldots, -(n-1)$, where the unshifted product $(a)_n$ of (1) vanishes and the sign of what is left alternates; this places one root per unit interval and gets no finer. With only that information, the sign of $(a)_{n-1}$ at a root of X_{n-1} is undetermined, and the natural transfer argument does not start. The evaluation the source does not perform is the mirror image of the one it does: at the negative *half*-integers $a = -(j+\frac{1}{2})$ the *shifted* product $(a+\frac{1}{2})_n$ vanishes, and the remaining product has one more negative factor than at $a=-j$, which the sign $-\binom{2n-1}{n-1}$ in front of it cancels. So X_n takes the same sign at $-j$ and at $-(j+\frac{1}{2})$ (Proposition 3.3), and the sign change on the interval $(-(j+1), -j)$ happens strictly in its left half:

Theorem 1.2 (Left-half localisation; Theorem 3.5 below)**.** *For $n \ge 2$ and $0 \le j \le n-2$, the root $r_{n,j}$ of X_n in $(-(j+1), -j)$ satisfies $-(j+1) < r_{n,j} < -(j+\frac{1}{2})$, and it is the only root of X_n in that left half.*

At a root $r = r_{n-1,j}$ of X_{n-1} this pins down the sign of $(r)_{n-1}$, a two-term transfer identity (Proposition 4.1) reduces $X_n(r)$ to a bare product, and the conclusion is immediate:

Theorem 1.3 (Interlacing; Theorems 4.3 and 4.5 below)**.** *For every $n \ge 3$ and every $0 \le j \le n-3$ one has $r_{n-1,j} < r_{n,j}$. Consequently, for every $n \ge 2$, Z_n has exactly $n-1$ roots and Z_{n-1} exactly $n-2$, all real, simple and negative, and listed in decreasing order they strictly alternate:*

$$r_{n,0} > r_{n-1,0} > r_{n,1} > r_{n-1,1} > \cdots > r_{n-1,n-3} > r_{n,n-2}.$$

This is Problem 1.1.

The second result concerns the arithmetic of X_n along $\mathbb{Z}$.

Theorem 1.4 (Theorem 7.1 below)**.** *$n \mid X_n(a)$ for every $n \ge 1$ and every $a \in \mathbb{Z}$.*

For n prime this follows from the published version's Property 6.16 together with Fermat's little theorem; for general n it does not seem to be in print.

1.3. **On how new the ingredient is.** We should be plain about the size of the gap. Problem 1.1 is open in the published record: the paper's Problem is printed verbatim in the journal version, and we have found no later work that addresses it. The two publications recorded as citing [2] (August 2026) both concern the Persson–Strang identity of its introduction, not X_n. What closes it is one evaluation. That evaluation does not occur in the source: in the published version the words "half", "midpoint" and "middle" do not occur at all, X_n is never evaluated at a half-integer argument, and the string Z_n occurs only in the two lines of Problem 1.1. The transfer identity of Proposition 4.1 is *not* the new ingredient, and Remark 4.2 says so explicitly: the source's own recurrence carries the same argument. A reader may reasonably regard the missing step as an oversight rather than a discovery; what we claim is that it is not in print, and that the proof below holds for every n and has been checked by a proof assistant.

1.4. **What is not used.** No orthogonality is used anywhere. The variable here is the *parameter* a of a Gegenbauer family, not the variable of an orthogonal polynomial sequence, and we do not know of a measure with respect to which $\{Z_n\}$ is orthogonal; the classical interlacing theorems for orthogonal polynomials, which rest on a three-term recurrence in the variable, are therefore not available. The proof below uses only the closed form (1), the intermediate value theorem, and the degree bound $\deg X_n \le n$.

2. Preliminaries

2.1. **Notation.** Throughout, n and j are natural numbers, $(x)_m = \prod_{i=0}^{m-1}(x+i)$ with $(x)_0 = 1$, and $\binom{2n}{n}$ is the central binomial coefficient. All polynomials are real.

2.2. **The doubled polynomial.** It is convenient to work with

$$D_n(a) \;:=\; 2X_n(a) \;=\; 4^n\big(a+\tfrac{1}{2}\big)_n - \binom{2n}{n}(a)_n, \tag{3}$$

which is legitimate because

$$2\binom{2n-1}{n-1} = \binom{2n}{n} \qquad (n \ge 1), \tag{4}$$

an instance of Pascal's rule together with $\binom{2n-1}{n-1} = \binom{2n-1}{n}$. Doubling clears the odd power 2^{2n-1} and replaces $\binom{2n-1}{n-1}$ by the central binomial coefficient; it changes no root, so every statement about the zeros of D_n is a statement about the zeros of X_n, and we pass between them silently. Likewise we write

$$\widehat{Z}_n \;=\; \frac{D_n(a)}{a+n} \;=\; 2Z_n(a),$$

which has the roots of Z_n.

Only the degree *bound* is needed below, and it is immediate from (3): each of the two products is monic of degree n, so

$$\deg D_n \le n. \tag{5}$$

(That the degree is exactly n is stated in [2, Proposition 6.3], and also follows from Theorem 3.4 below, which exhibits n distinct roots; we never use it.)

Proposition 2.1 (First values). *$D_1(a) = 2(a+1)$, $D_2(a) = 2(5a^2+13a+6)$ and $D_3(a) = 2(22a^3 + 114a^2 + 164a + 60)$; that is, X_1, X_2, X_3 are the polynomials printed in* [2, §6].

Proof. Expand (3) with $\binom{2}{1} = 2$, $\binom{4}{2} = 6$, $\binom{6}{3} = 20$. $\square$

3. Signs on the half-integer lattice

Everything in this section is an evaluation of the two products in (3) at a point where one of them vanishes. We start with the elementary sign count that both evaluations use.

Lemma 3.1. *Let $x \in \mathbb{R}$ and $m \le n$. If $x+i<0$ for all $i<m$ and $x+i>0$ for all $m \le i < n$, then $(-1)^m \prod_{i=0}^{n-1}(x+i) > 0$. In particular, if $j<n$ and $-(j+1)<x<-j$, then $(-1)^{j+1}\prod_{i=0}^{n-1}(x+i)>0$.*

Proof. Split the product at m; the first block is a product of m negative numbers and the second of positive ones. For the second sentence, $-(j+1)<x<-j$ gives $x+i<0$ exactly for $i \le j$, and $j+1 \le n$. $\square$

Lemma 3.2 (The two evaluations)**.** *Let $0 \le j \le n-1$. Then*

$$D_n(-j) = 4^n \prod_{i=0}^{n-1}\Big(i-j+\tfrac{1}{2}\Big), \qquad D_n\big(-j-\tfrac{1}{2}\big) = -\binom{2n}{n}\prod_{i=0}^{n-1}\Big(i-j-\tfrac{1}{2}\Big).$$

Moreover $D_n(-n)=0$.

Proof. At $a=-j$ the factor $i=j$ of $(a)_n = \prod_{i<n}(a+i)$ vanishes, killing the second term of (3); at $a=-j-\frac{1}{2}$ the factor $i=j$ of $(a+\frac{1}{2})_n = \prod_{i<n}(a+\frac{1}{2}+i)$ vanishes, killing the first. For the last claim, reflecting both products at $a=-n$ gives $\big(-n+\frac{1}{2}\big)_n = (-1)^n\prod_{k<n}(k+\frac{1}{2})$ and $(-n)_n = (-1)^n\, n!$, so

$$D_n(-n) = (-1)^n\Big(4^n\prod_{k<n}\big(k+\tfrac{1}{2}\big) - \binom{2n}{n}n!\Big) = 0,$$

because $4^n\prod_{k<n}(k+\frac{1}{2}) = 2^n(2n-1)!! = (2n)!/n! = \binom{2n}{n}n!$; the identity is proved by induction from $(n+1)\binom{2n+2}{n+1} = 2(2n+1)\binom{2n}{n}$. $\square$

The next proposition is the ingredient the source lacks. The first half of it is [2, Property 6.14] in disguise — it is the sign alternation on which the source's own proof of Property 6.15 runs — and the second half is its exact mirror image.

Proposition 3.3 (Sign table)**.** *For $0 \le j \le n-1$,*

$$(-1)^j D_n(-j) > 0 \qquad \textit{and} \qquad (-1)^j D_n\big(-j-\tfrac{1}{2}\big) > 0.$$

That is, D_n takes the sign $(-1)^j$ both at the right endpoint $-j$ of the interval $(-(j+1),-j)$ and at its midpoint $-(j+\frac{1}{2})$, hence the opposite sign at its left endpoint $-(j+1)$ when $j+1 \le n-1$.

Proof. In the first evaluation of Lemma 3.2 the factors $i-j+\frac{1}{2}$ are negative exactly for $i \le j-1$, so there are j of them and Lemma 3.1 gives the sign $(-1)^j$; the prefactor 4^n is positive. In the second, the factors $i-j-\frac{1}{2}$ are negative exactly for $i \le j$, so there are $j+1$ of them and the product has sign $(-1)^{j+1}$; the prefactor $-\binom{2n}{n}$ is negative, and $(-1)\cdot(-1)^{j+1} = (-1)^j$. $\square$

Taking $j=0$ in Lemma 3.2 recovers $X_n(0) = \frac{1}{2}\binom{2n}{n}n! = (2n)!/(2\,n!)$, in particular $X_n(0)>0$. This agrees with [2, Property 6.10], which reads "The constant term in $X_n(a)$ is $\frac{(2n)!}{2n!}$", and it is where the source's own proof of Property 6.15 starts.

Theorem 3.4 (Complete root list)**.** *Let $n \ge 1$. Then X_n has exactly n roots, all real, simple and negative, namely $-n$ together with one root $r_{n,j}$ in each interval $(-(j+1),-j)$ for $0 \le j \le n-2$.*

Proof. By Proposition 3.3, D_n has opposite signs at $-(j+1)$ and at $-(j+\frac{1}{2})$ for every j with $j+1 \le n-1$, so by the intermediate value theorem it has a root strictly inside $\big(-(j+1), -(j+\frac{1}{2})\big)$; call one such root $r_{n,j}$. These $n-1$ intervals are pairwise disjoint and all lie in $(-(n-1),0)$, and $-n$ is a root by Lemma 3.2 and lies below all of them. This exhibits n distinct roots of a polynomial of degree at most n by (5), so there are no others and each is simple. $\square$

The proof produced more than the statement: the roots were found in the left halves. Since the root list is now complete, each left half contains exactly one root, and $r_{n,j}$ is *the* root of X_n there.

Theorem 3.5 (Left-half localisation)**.** *Let $n \geq 2$ and $0 \leq j \leq n-2$. The root $r_{n,j}$ of Theorem 3.4 is the unique root of X_n in the closed left half $\left[-(j+1), -(j+\frac{1}{2})\right]$, it satisfies*

$$-(j+1) \;<\; r_{n,j} \;<\; -\Big(j+\tfrac{1}{2}\Big),$$

and it is a root of Z_n.

Proof. Existence and the strict inequalities are in the proof of Theorem 3.4. For uniqueness, any root of X_n in that closed interval is one of the n roots listed there; it is not $-n$, since $-(j+1) \leq -n$ would force $n \leq j+1 \leq n-1$; and it is not $r_{n,k}$ for $k \neq j$, since those lie in disjoint intervals. Finally $r_{n,j} \neq -n$, so it survives division by $a+n$. □

Corollary 3.6 ([2, Property 6.15])**.** *For $n \geq 1$, every root of X_n is real and negative.*

Proof. By Theorem 3.4 the roots are $-n < 0$ and the $r_{n,j} < -j \leq 0$. □

Theorem 3.7 (The roots of Z_n)**.** *Let $n \geq 1$. Then Z_n has exactly $n-1$ roots, all real, simple and negative, and they are precisely $r_{n,0}, r_{n,1}, \ldots, r_{n,n-2}$, which satisfy $r_{n,0} > r_{n,1} > \cdots > r_{n,n-2}$ and $r_{n,j} \in \left(-(j+1), -(j+\frac{1}{2})\right)$.*

Proof. $X_n = (a+n)Z_n$, so the root multiset of X_n is that of Z_n with one further copy of $-n$; by Theorem 3.4 the latter has $-n$ with multiplicity one, so cancelling it leaves exactly $r_{n,0}, \ldots, r_{n,n-2}$, each simple. They decrease in j because $r_{n,j+1} < -(j+1) < r_{n,j}$. □

The source studies X_n and does not study Z_n; Theorem 3.7, including the simplicity, does not appear there in any form.

4. The transfer identity and interlacing

Proposition 4.1 (Transfer identity)**.** *For every $n \geq 1$ and every a,*

$$n\,D_n(a) \;=\; 4n\Big(a+n-\tfrac{1}{2}\Big)D_{n-1}(a) \;+\; 2\binom{2n-2}{n-1}(a+2n-1)\,(a)_{n-1}, \tag{6}$$

equivalently $n\,X_n(a) = 4n\big(a+n-\frac{1}{2}\big)X_{n-1}(a) + \binom{2n-2}{n-1}(a+2n-1)(a)_{n-1}$.

Proof. Peel the last factor off each product in (3): $\big(a+\frac{1}{2}\big)_n = \big(a+\frac{1}{2}\big)_{n-1}\big(a+n-\frac{1}{2}\big)$ and $(a)_n = (a)_{n-1}(a+n-1)$. Substituting $4^{n-1}\big(a+\frac{1}{2}\big)_{n-1} = D_{n-1}(a) + \binom{2n-2}{n-1}(a)_{n-1}$ eliminates the shifted product, and the recurrence $n\binom{2n}{n} = 2(2n-1)\binom{2n-2}{n-1}$ for central binomial coefficients collects the remaining terms into (6). The X-form follows on dividing by 2 and using (4). □

Remark 4.2. Identity (6) is the mirror image of the first recurrence printed in the source, namely $X_n(a) = \frac{2}{n}(a+n-1)(2n-1)X_{n-1}(a) + \frac{2n+a-1}{n}2^{2n-2}\big(a+\frac{1}{2}\big)_{n-1}$ with $X_0 = 0$ ([2, Proposition 6.3]): theirs eliminates the unshifted product $(a)_{n-1}$ and retains $\big(a+\frac{1}{2}\big)_{n-1}$, ours does the reverse. It is *not* the new ingredient of this paper. The source's own recurrence carries the argument equally well: at a root r of X_{n-1} it expresses $X_n(r)$ as a positive multiple of $\big(r+\frac{1}{2}\big)_{n-1}$, and on the left half of the j-th interval that product also has exactly $j+1$ negative factors, so the same conclusion follows. We use (6) only because its product $(a)_{n-1}$ is the one whose sign Lemma 3.1 has already been applied to.

Theorem 4.3 (Interlacing step)**.** *Let $n \geq 3$ and $0 \leq j \leq n-3$. Then*

$$r_{n-1,j} \;<\; r_{n,j}.$$

Proof. Write $r = r_{n-1,j}$, which exists because $j \le (n-1) - 2$, and evaluate (6) at $a = r$. Since $D_{n-1}(r) = 0$ the first term dies and

$$n\,D_n(r) \;=\; 2\binom{2n-2}{n-1}(r+2n-1)\,(r)_{n-1}.$$

We read off the sign of the right-hand side. The binomial coefficient is positive. Next, $r > -(j+1) \ge -(n-2)$ by Theorem 3.5 and $j \le n-3$, so $r + 2n - 1 > n + 1 > 0$. Finally $r \in \bigl(-(j+1), -(j+\tfrac{1}{2})\bigr)$, so among the factors $r+i$, $0 \le i \le n-2$, exactly those with $i \le j$ are negative — there are $j+1$ of them, and $j+1 \le n-2$ — whence $(r)_{n-1}$ has sign $(-1)^{j+1}$ by Lemma 3.1. Therefore

$$(-1)^{j+1} D_n(r) > 0,$$

which is the sign D_n has at the left endpoint $-(j+1)$. By Proposition 3.3, D_n has the opposite sign $(-1)^j$ at $-(j+\tfrac{1}{2})$, and $r < -(j+\tfrac{1}{2})$; so D_n changes sign on $\bigl(r, -(j+\tfrac{1}{2})\bigr)$ and has a root strictly inside. That root lies in $\bigl(-(j+1), -(j+\tfrac{1}{2})\bigr)$, so by the uniqueness in Theorem 3.5 it is $r_{n,j}$. Hence $r < r_{n,j}$. □

Theorem 4.4 (Interlacing chain)**.** *Let $n \ge 3$ and $0 \le j \le n-3$. Then*

$$r_{n,j+1} \;<\; r_{n-1,j} \;<\; r_{n,j}.$$

Proof. The right inequality is Theorem 4.3. For the left one no new input is needed: by Theorem 3.5, $r_{n,j+1} < -(j+1) - \tfrac{1}{2} < -(j+1) < r_{n-1,j}$, the two roots lying in different unit intervals. □

Theorem 4.5 (Problem 1.1)**.** *Let $n \ge 2$. Then Z_n has exactly $n-1$ roots and Z_{n-1} exactly $n-2$, all real, simple and negative; listed in decreasing order they are $r_{n,0} > \cdots > r_{n,n-2}$ and $r_{n-1,0} > \cdots > r_{n-1,n-3}$; and the two lists strictly alternate,*

$$r_{n,0} \;>\; r_{n-1,0} \;>\; r_{n,1} \;>\; r_{n-1,1} \;>\; \cdots \;>\; r_{n-1,n-3} \;>\; r_{n,n-2}.$$

That is, the roots of Z_n interlace those of Z_{n-1}, for every $n \ge 2$.

Proof. The root lists and their order are Theorem 3.7 applied to n and to $n-1$. The alternation is the two families of inequalities of Theorem 4.4, which cover the range $0 \le j \le n-3$; for $n = 2$ the second list is empty and there is nothing to alternate. □

5. The source's published statements, re-derived

The results of this section are not new; they are [2, Properties 6.13–6.15]. We record them because the same Pochhammer reflections carry them and Lemma 3.2, so their verification is a control on the machinery of Section 3, and because one of them corrects an assertion of the first arXiv version.

Property 6.13, $X_n(-n) = 0$, is the last claim of Lemma 3.2; Property 6.15 is Corollary 3.6. The remaining one is the symmetry.

Proposition 5.1 ([2, Property 6.14])**.** *For $n \ge 1$ and $1 \le k \le n-1$,*

$$X_n(k-n) \;=\; (-1)^n\, X_n(-k).$$

Proof. Both arguments annihilate the unshifted product: $(k-n)_n$ has the vanishing factor $i = n-k$, and $(-k)_n$ has the vanishing factor $i = k$, both indices lying in $[0, n)$. So (3) reduces on both sides to its shifted term, and reflecting the index, $\prod_{i<n}\bigl(k - n + \tfrac{1}{2} + i\bigr) = (-1)^n \prod_{i<n}\bigl(-k + \tfrac{1}{2} + i\bigr)$, which is the assertion. □

Remark 5.2. Proposition 5.1 is the corrected form of Question (2) of the first arXiv version of the source, which reads, verbatim, "Aside from the value $X_n(-n) = 0$, the polynomial X_n has symmetric values $X_n(-j) = X_n(-n+j)$, for $1 \le j \le n-1$" ([1, v1, §6]); the factor $(-1)^n$ is missing. That form is false at the first odd n where it has content: by Proposition 2.1,

$$X_3(-1) = -22 + 114 - 164 + 60 = -12, \qquad X_3(-2) = -176 + 456 - 328 + 60 = +12.$$

The published version supplies the missing $(-1)^n$ and credits the correction to C. Koutschan.

6. Sharpness and controls

Three checks delimit the results above. The first says that the localisation of Theorem 3.5 cannot be mirrored; the second that it cannot be sharpened by a factor of two; the third that Theorem 4.3 is not vacuous and has a direction.

Proposition 6.1 (No root in the right half)**.** *Let $0 \le j \le n-1$. Then X_n has no root in the closed right half $\left[-(j+\frac{1}{2}), -j\right]$ of the j-th unit interval.*

Proof. By Theorem 3.4 the roots are $-n$ and the $r_{n,k}$ with $0 \le k \le n-2$. The root $-n$ is not in the interval, since $-(j+\frac{1}{2}) \le -n$ would give $n \le j$, against $j \le n-1$. And if $-(j+\frac{1}{2}) \le r_{n,k} \le -j$, then $r_{n,k} < -(k+\frac{1}{2})$ gives $-(j+\frac{1}{2}) < -(k+\frac{1}{2})$, i.e. $k < j$, while $r_{n,k} > -(k+1)$ gives $-(k+1) < -j$, i.e. $j < k+1$: the two are incompatible. ☐

This is the statement that would let the interlacing argument run backwards, and it is false: knowing only that a root lies somewhere in $(-(j+1), -j)$, the sign of $(r)_{n-1}$ at that root is genuinely undetermined, and it is Theorem 3.5 that determines it.

Proposition 6.2 (An exact instance)**.** *The unique root of Z_2 is $r_{2,0} = -\frac{3}{5}$, and $-\frac{3}{5} < r_{3,0}$.*

Proof. $X_2 = 5a^2 + 13a + 6 = (a+2)(5a+3)$ by Proposition 2.1, so $Z_2 = 5a+3$ and its root is $-\frac{3}{5}$, which lies in $\left(-1, -\frac{1}{2}\right)$ and is therefore $r_{2,0}$ by Theorem 3.5. The inequality is Theorem 4.3 at $n = 3$, $j = 0$. ☐

Remark 6.3 (Sharpness of the half). The left *half* of Theorem 3.5 cannot be replaced by the left *quarter*: $r_{2,0} = -\frac{3}{5} > -\frac{3}{4}$. Nor is the inequality of Theorem 4.3 symmetric — $r_{3,0} \le r_{2,0}$ is false, again by Proposition 6.2. In the root isolations of Remark 8.2, which cover $2 \le n \le 60$ and nothing beyond, the roots $r_{n,j}$ increase with n towards $-(j+\frac{1}{2})$; we prove nothing of the sort, and a two-sided quantitative bracket $-(j+1) + f(n,j) < r_{n,j} < -(j+\frac{1}{2}) - g(n,j)$ would be a natural refinement.

7. Divisibility by n

Question (4) of the first arXiv version of the source reads, verbatim, "For any $a \in \mathbb{Z}$ and any $n \in \mathbb{N}$, the congruence $X_n(a) \equiv 0 \bmod n$ holds" ([1, v1, §6]). It is commented out in v2 and absent from the published version, which retains the prime case only: [2, Property 6.16] gives $X_q(a) \equiv a^q - a \pmod q$ for q prime, so $q \mid X_q(a)$ by Fermat's little theorem. The general case is elementary but does not seem to be in print.

Throughout this section we use the second display of (1), in which both terms are products of integers:

$$X_n(a) \;=\; 2^{n-1} \prod_{k=0}^{n-1} (2a+2k+1) \;-\; \binom{2n-1}{n-1} \prod_{k=0}^{n-1} (a+k), \qquad a \in \mathbb{Z}. \tag{7}$$

Theorem 7.1. *For every $n \ge 1$ and every $a \in \mathbb{Z}$, $n \mid X_n(a)$.*

Proof. We show n divides each of the two terms of (7).

The unshifted term. $\prod_{k<n}(a+k)$ is a product of n consecutive integers. Choosing $k \in [0,n)$ with $k \equiv -a \pmod n$ makes one factor divisible by n.

The shifted term. Write $n = 2^s m$ with $m = 2t+1$ odd. The factors $2a+2k+1$, $0 \le k < n$, are n consecutive odd integers, and $m \le n$, so the first m of them are available. Since $2(t+1) = m+1 \equiv 1 \pmod m$, the residue 2 is invertible modulo m, and for $k \in [0,m)$ with $k \equiv -(t+1)(2a+1) \pmod m$ one has

$$2a+2k+1 \;\equiv\; (2a+1)\bigl(1 - 2(t+1)\bigr) \;=\; -(2a+1)m \;\equiv\; 0 \pmod m,$$

so m divides the product. Separately $s < 2^s \le n$ gives $s \le n-1$, hence $2^s \mid 2^{n-1}$. As $\gcd(2^s, m) = 1$, the product $n = 2^s m$ divides $2^{n-1} \prod_{k<n}(2a+2k+1)$. ☐

Remark 7.2. The divisor is n and not n^2: $X_2(0) = 6$ is not divisible by 4. The argument above also spends only 2^s of the available 2^{n-1}, so for odd n it uses no power of two at all; the exact n-adic valuation of $X_n(a)$ is not determined here.

8. Two remarks on computation

The following two observations lie outside the formal development and are recorded as experimental notes; neither is used in any proof above.

Remark 8.1 (A floating-point prototype reports the conjecture false). Problem 1.1 is a numerically treacherous statement. Taking the exact integer coefficients of Z_n, casting them to IEEE double precision and isolating the roots by bisection, the interlacing test *fails* at $n = 22$, and for $23 \leq n \leq 40$ the sign change on the isolating interval is lost outright, so the isolation step itself no longer works. The cause appears to be cancellation: the coefficients of X_n grow like 4^n while consecutive roots $r_{n,j}$ and $r_{n-1,j}$ approach each other. Anyone who prices this problem from a double-precision experiment will conclude that the conjecture is false.

Remark 8.2 (The finite route, priced and not taken). Verifying Problem 1.1 for a range of n instead of proving it is entirely feasible in exact rational arithmetic: isolating every root of Z_n for $2 \leq n \leq 60$ by bisection over $\mathbb{Q}$ and checking the alternation takes about 33 seconds and under 100 megabytes in a straightforward implementation. Such a computation would also have produced, per n, a certificate of rational separating points — and would have covered only finitely many n. Theorem 4.5 subsumes it.

9. Verification

Every theorem, proposition, corollary and lemma stated above has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [4, 3]; the development is free of `sorry` and is not yet public — repository reference to be added at submission. Its seven modules follow Sections 2–7: the closed form (3) and the degree bound; the evaluations and the sign table of Section 3; the root list and the left-half localisation; the transfer identity and the interlacing step; the statement about Z_n itself; the controls of Sections 5 and 6; and the integer model behind Section 7, which is tied to the real model by the kernel-checked identity $2X_n(a) = D_n(a)$ for $a \in \mathbb{Z}$. *No claim in this paper rests on a finite computation or on an exhaustive search.* In particular the proofs use no compiled evaluation (Lean's `native_decide`) and no certificate data of any kind, and they cover every n rather than a range; a printout of the axioms of all headline declarations returns `propext`, `Classical.choice` and `Quot.sound` only, the second entering through the intermediate value theorem, which names a root by choice. Exactly two things stated above lie outside that development, and both are flagged where they occur: the numerical observation in Remark 6.3 and the experimental notes of Section 8. Independent computations in exact rational arithmetic preceded the formalisation and agree with it: the first values and both recurrences were reproduced for $n \leq 30$, the interlacing was checked for $2 \leq n \leq 60$, and Theorem 7.1 was checked for every $n \leq 120$ and every residue class a mod n — a complete check at those n, since $X_n(a)$ mod n depends only on a mod n.

References

[1] T. Amdeberhan, A. Duncan, V. H. Moll and V. Sharma, *Filter integrals for orthogonal polynomials*, e-print arXiv:2012.05040. The six Questions cited above appear in the first version, v1, of 9 December 2020, only; the live version is v2, of 14 December 2020, in which they are commented out.

[2] T. Amdeberhan, A. Duncan, V. H. Moll and V. Sharma, *Filter integrals for orthogonal polynomials*, Hardy-Ramanujan Journal **44** (2021) 116–135, doi:10.46298/hrj.2022.8926. (The volume is dated 2021 both by the journal and by the article's own running head; the article went online on 9 January 2022, the year carried by the DOI and by several indexes.)

[3] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, pp. 625–635, doi:10.1007/978-3-030-79876-5_37.

[4] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381, doi:10.1145/3372885.3373824.

[5] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`. Up to an index shift, A000346 lists the leading coefficients $2^{2n-1} - \binom{2n-1}{n-1}$ of X_n and A000407 its constant terms $(2n)!/(2\,n!)$; neither refers to [2].