SWAP DIFFERENCES OF POWERS: A CONJECTURE OF FILL AND JANSON ON NONNEGATIVE COEFFICIENTS, AND TWO CASES OF THEIR ROOT CONJECTURE

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ABSTRACT. In an appendix to their study of positive autocorrelation at unit lag for stationary random walk Metropolis–Hastings, Fill and Janson introduce the two-variable polynomial

$$h(t,a;r) \;=\; [1+(3+a)t]^r\big([1+(2+a)t]^{r+1}-[1+at]^{r+1}\big) \\ -\;[1+(1+a)t]^r\big([1+(4+a)t]^{r+1}-[1+(2+a)t]^{r+1}\big),$$

prove that it is nonnegative for real $r \ge 1$ and real $a, t \ge 0$ by a Jensen argument, and then conjecture, "in passing", a considerable strengthening: that for fixed $r \in \mathbb{N}$ the polynomial $h(t,a;r)$ in (t,a) has *nonnegative coefficients*. We prove that conjecture, for every $r \in \mathbb{N}$. The proof rests on the observation that with $P = 1+(1+a)t$ and $Q = 1+(3+a)t$ the four inner brackets of h are exactly $P \pm t$ and $Q \pm t$, so that h is the *swap difference* $Q^r\big((P+t)^{r+1}-(P-t)^{r+1}\big) - P^r\big((Q+t)^{r+1}-(Q-t)^{r+1}\big)$, together with a polynomial identity, valid over any commutative ring, that writes such a swap difference as a sum of products of P, Q, y and $Q-P$ with nonnegative integer scalars. Since $Q-P = 2t$ here, positivity is termwise: there is no inequality and no analysis in the argument. What we actually prove is the abstract statement — if P, Q, y and $Q-P$ all have nonnegative coefficients then so does the swap difference — of which Fill and Janson's conjecture is one instance, and we show by explicit counterexamples that neither the exponent balance $(r, r+1)$ nor the hypothesis on $Q-P$ can be dropped. We also settle the first two cases of Fill and Janson's second conjecture, on the roots of $g(k;r) = g(k+1;r)$ with $g(k;r) = (2k-1)^r/\sum_{i\le k} i^r$: the case $k=1$ has $r=1$ as its unique real root, and the case $k=2$ has a unique real root, which lies strictly between 2 and 3. Finally we close six of the thirteen inequalities on which Fill and Janson's own proof of $s(r) = \max_{k \le k_0(r)} g(k;r)$ rests through what they describe as "the clear indication from a plot", and we do so on all of $(1,\infty)$. Every theorem below has been formally verified in Lean 4 against *Mathlib*.

1. INTRODUCTION

1.1. **The polynomial and the conjecture.** Fill and Janson [1] study the autocorrelation at unit lag of a stationary random walk Metropolis–Hastings chain. The last appendix of their paper concerns the quantity

$$s(r) \;:=\; \sup\; \frac{\mathbb{E}|X_1 - X_0|^r}{\mathbb{E}|X_0|^r}, \tag{1}$$

the supremum being taken over nondegenerate symmetric discrete unimodal distributions π for $X_0 \in L^r$ and symmetric proposal step-distributions concentrated on the odd integers. Their route to $s(r)$ passes through a two-variable polynomial. We quote its definition, and the inequality they prove about it, verbatim from [1, Lemma `L:helpful2`(a)]:

> Suppose that $r \in [1,\infty)$.
> (a) For real $a, t \in [0,\infty)$ we have
>
> $$h(t,a;r) := [1+(3+a)t]^r\Big([1+(2+a)t]^{r+1}-[1+at]^{r+1}\Big) \\ -\,[1+(1+a)t]^r\Big([1+(4+a)t]^{r+1}-[1+(2+a)t]^{r+1}\Big) \;\ge\; 0.$$

Their proof of this is an application of Jensen's inequality: the function $h_1(t,a;r) := [1+(1+a)t]^{-r}\big([1+(2+a)t]^{r+1}-[1+at]^{r+1}\big)$ equals $(r+1)t\int_{-1}^{1}(1+x\tau)^r\,dx$ with $\tau = t/[1+(1+a)t]$ nonincreasing in a, and

the integral is nondecreasing in τ; the displayed inequality is $h_1(t, a+2; r) \le h_1(t, a; r)$. Immediately after that lemma [1] write, again verbatim:

> In passing, we conjecture a considerable strengthening of [the displayed inequality] when $r \in \mathbb{N}$:
>
> **Conjecture** (`Conj:strong!`). "If $r \in \mathbb{N}$ is arbitrary but fixed, then the polynomial $h(t, a; r)$ in (t, a) has nonnegative coefficients."

The strengthening is considerable in a precise sense. What the appendix of [1] actually consumes is a single one-parameter slice: the proof of part (b) of the same lemma, that $f_1(k; r)$ is nonincreasing in the integer $k \ge 1$, reduces — in the authors' words, "omitting some calculations" — to $h\big((2k+1)^{-1}, 0; r\big) \ge 0$ for $k \ge 1$. So the lemma is used only along $a = 0$, at the countably many points $t = (2k+1)^{-1}$, whereas the conjecture asks for a statement about every monomial of h. It is also strictly stronger than the lemma as stated: nonnegativity on the closed positive quadrant does not imply nonnegative coefficients, and we record an explicit witness for that in Proposition 5.3.

1.2. **What is proved here.** The first contribution of this paper is a proof of the conjecture, for every $r \in \mathbb{N}$. We prove it in the form: for every integer $r \ge 0$ and every monomial $t^i a^j$, the coefficient of $t^i a^j$ in $h(t, a; r) \in \mathbb{Z}[t, a]$ is nonnegative. Since we allow $r = 0$, the result covers the conjecture on either reading of $\mathbb{N}$.

The mechanism is a reformulation that removes the analysis entirely. Put

$$P \;:=\; 1 + (1+a)t, \qquad Q \;:=\; 1 + (3+a)t. \tag{2}$$

Then the four inner brackets of h are exactly $P \pm t$ and $Q \pm t$:

$$1 + (2+a)t = P + t, \quad 1 + at = P - t, \quad 1 + (4+a)t = Q + t, \quad 1 + (2+a)t = Q - t,$$

so that

$$h(t, a; r) \;=\; Q^r\big((P+t)^{r+1} - (P-t)^{r+1}\big) - P^r\big((Q+t)^{r+1} - (Q-t)^{r+1}\big), \qquad Q - P = 2t. \tag{3}$$

We call the right-hand side of (3) a *swap difference*: P and Q have been exchanged between the outer power and the inner odd part. It is the numerator of the difference $h_1(t, a; r) - h_1(t, a+2; r)$ that [1] bound by Jensen's inequality, and the conjecture is exactly the assertion that this numerator is coefficientwise nonnegative.

Section 3 proves an identity for swap differences over an arbitrary commutative ring (Proposition 3.2), which expresses the swap difference as a sum of terms each of which is a nonnegative integer times a product of powers of P, Q and y with a single factor $Q - P$. Section 4 draws the consequence:

> *If P, Q, y and $Q - P$ all have nonnegative coefficients, then so does the swap difference* (Theorem 4.1),

and Fill and Janson's conjecture (Theorem 4.2) is the instance $P = 1 + (1+a)t$, $Q = 1 + (3+a)t$, $y = t$, where $Q - P = 2t$. We regard Theorem 4.1 as the heart of the paper: it is more general than the conjecture on two independent axes. The constant 3 in Q may be replaced by any $1 + \delta$ with δ coefficientwise nonnegative, and the base polynomials P, Q, y are arbitrary. What the theorem isolates is one hypothesis, that $Q - P$ has nonnegative coefficients, and Proposition 5.3 shows that this hypothesis cannot be dropped.

Section 5 records what h actually is: $h(\cdot, \cdot; 0) = h(\cdot, \cdot; 1) = 0$, $h(t, a; 2) = 8t^4(1 + 2t + at)$, and $t^4 \mid h(\cdot, \cdot; r)$ for every r. The last of these follows from the identity, because the two lowest terms of the sum vanish, one for having an empty geometric factor and one for having scalar 0. Section 5 also collects the sharpness statements: with the inner exponent pushed from $r+1$ to $r+2$ the swap difference acquires negative coefficients, and so does the variant with $Q = 1 + at$, for which $Q - P = -t$.

Sections 6 and 7 turn to two further open ends of the same appendix. With

$$S(k; r) := \sum_{i=1}^{k} i^r, \qquad g(k; r) := \frac{(2k-1)^r}{S(k; r)}, \tag{4}$$

Fill and Janson conjecture [1, `Conj:Kr`], verbatim:

> "For each $k \geq 1$ there is a unique real root $r = r_k$ to the equation $g(k;r) = g(k+1;r)$ (with $r_1 = 1$), and $r_k \in [1,\infty)$ increases strictly monotonically to ∞ as $k \to \infty$; indeed $r_k \sim k \ln 3$ as $k \to \infty$. If $r \in (r_{k-1}, r_k)$ with $k \geq 2$, then $s(r) = g(j;r)$ uniquely for $j = k$. If $r = r_k$, then $s(r) = g(j;r)$ precisely for $j \in \{k, k+1\}$."

They prove no case of it. (A third conjecture of the same appendix, `Conj:s g`, asserts that $s(r) = \sup_{k\in\mathbb{N}} g(k;r) = \max_{k\leq\lceil r\rceil} g(k;r)$ for all $r \in (0,\infty)$; we do not treat it here.) Section 6 settles the uniqueness-and-existence half for $k = 1$ (Theorem 6.1) and for $k = 2$ (Theorem 6.5), the latter with an exact bracket $r_2 \in (2,3)$. The case $k = 1$ is a standard exercise — $3^r = 1+2^r$ becomes $(1/3)^r+(2/3)^r = 1$, whose left side is strictly decreasing — and we include it because two of the six inequalities closed in Section 7 reduce to it. The case $k = 2$ is not: after dividing $3^r(1+2^r+3^r) = 5^r(1+2^r)$ by 10^r one is left with

$$\chi(r) \;=\; (3/10)^r + (3/5)^r + (9/10)^r - (1/2)^r - 1 \;=\; 0,$$

which is not a sum of monotone pieces. Nor does a sign count settle it: writing $g(k;r) = g(k+1;r)$ as the exponential sum $\sum_{i=1}^{k}\big(\frac{i}{2k-1}\big)^r - \sum_{i=1}^{k+1}\big(\frac{i}{2k+1}\big)^r = 0$ and sorting its $2k+1$ bases, the coefficients $+1$ and -1 interleave with $2k-1$ sign changes, so the classical extension of Descartes' rule of signs to exponential sums [3] bounds the number of real roots only by $2k-1$, hence by 3 at $k = 2$. What replaces it is a global derivative bound: $\chi' < 0$ on all of $\mathbb{R}$, obtained from four explicit logarithm bounds and three crude ranges.

Section 7 concerns a gap that [1] flag themselves. Their Remark `R:Isymmunimodalr`(c) says that the proof of

$$s(r) \;=\; \max_{k\in\mathbb{N}:\, k\leq k_0(r)} g(k;r), \qquad k_0(r) := \Big\lfloor \tfrac{1}{2} + (1-2^{-1/r})^{-1} \Big\rfloor, \tag{5}$$

on $[1.043,\infty)$ is "slightly incomplete", because on $[2.198,\infty)$ it rests on asymptotics with an undetermined threshold together with "the clear indication from a plot that a certain piecewise smooth function is nonnegative", and "similarly for $r \in [1.043, 2.198)$ we rely on the clear indication from a plot that a certain function is strictly positive". On $[1.043, 2.198)$ the plots are asked to establish thirteen explicit inequalities $f(k,y;r) < \max\{g(2;r), g(3;r)\}$, over $(k,y) \in \mathbb{Z}^2 \setminus \{(2,1),(3,2)\}$ with $0 \leq y \leq k-1 \leq 4$, together with a fourteenth, $f_1(6;r) < \max\{g(2;r), g(3;r)\}$. Proposition 7.1 proves six of the thirteen, and on all of $(1,\infty)$ rather than only on the interval needed. The remaining eight statements are not closed here; Remark 7.2 records how tight they are.

1.3. **Status of the conjectures.** The source [1] was posted on 27 January 2026 and is still at v1; we checked the arXiv record on 3 September 2026, when its journal-reference field was empty, no journal version of it was indexed by Crossref or zbMATH, and its citation count was zero in both OpenAlex and Semantic Scholar. The entry for [1] in the second author's own publication list [2] carries neither a journal reference nor the "proved by" annotation that the list gives to conjectures of his that have since been settled. We are therefore not aware of any proof of `Conj:strong!`, or of any case of `Conj:Kr`, in the literature; a preprint outside arXiv, a thesis or a talk would be invisible to every channel just named, so a referee should re-run the check at submission.

2. Notation and preliminaries

Throughout, $\mathbb{N} = \{0,1,2,\dots\}$, r denotes an element of $\mathbb{N}$ in Sections 3–5 and a real number in Sections 6–7, and $\binom{n}{m}$ is the usual binomial coefficient, taken to be 0 when $m > n$.

Definition 2.1 (swap difference)**.** Let R be a commutative ring, let $P, Q, y \in R$ and let $r \in \mathbb{N}$. The *swap difference* of P and Q at y and order r is

$$D_r(P,Q;y) \;:=\; Q^r\big((P+y)^{r+1} - (P-y)^{r+1}\big) \;-\; P^r\big((Q+y)^{r+1} - (Q-y)^{r+1}\big) \;\in\; R.$$

The name records the shape: the same expression appears twice with P and Q interchanged between the outer power and the odd part. Equivalently, up to the factor $(PQ)^{-r}$, $D_r(P,Q;y)$ measures how far the map $X \mapsto \big((X+y)^{r+1} - (X-y)^{r+1}\big)/X^r$ is from being constant in X; the Jensen argument of [1] shows that it is monotone. We use no property of that map below, only Definition 2.1.

Definition 2.2 (coefficientwise nonnegativity)**.** Write $\mathbb{Z}[t,a]$ for the ring of polynomials in two commuting indeterminates over $\mathbb{Z}$ and $\mathbb{N}[t,a]$ for the corresponding semiring over $\mathbb{N}$. A polynomial $p \in \mathbb{Z}[t,a]$ *has nonnegative coefficients*, written $p \succeq 0$, if p lies in the image of the coefficient extension $\mathbb{N}[t,a] \to \mathbb{Z}[t,a]$; equivalently, if $[t^i a^j]\, p \ \geq 0$ for every $(i,j) \in \mathbb{N}^2$.

The following is the only structural fact we need about $\succeq 0$, and it is the reason the proof of Theorem 4.1 is short.

Lemma 2.3. *$\{p \in \mathbb{Z}[t,a] : p \succeq 0\}$ is a subsemiring of $\mathbb{Z}[t,a]$. In particular it contains 0, 1, t, a and every natural number, and it is closed under addition, multiplication, powers and finite sums.*

Proof. It is the set-theoretic image of a semiring homomorphism. □

For Sections 6 and 7 we use the notation of [1]: $S(k;r)$ and $g(k;r)$ as in (4), $k_0(r)$ as in (5), and

$$f(k,y;r) := \frac{(2y+1)^r (k-y)}{S(k;r)}, \qquad f_1(k;r) := \frac{1}{2} \cdot \frac{r^r}{(r+1)^{r+1}} \cdot \frac{(2k+1)^{r+1}}{S(k;r)}, \tag{6}$$

for $k \in \mathbb{N}$, $k \geq 1$, $0 \leq y \leq k-1$ and $r \in (0,\infty)$; here $f_1(k;r)$ is the value of $f(k,\cdot\,;r)$ at its real maximiser, so $f(k,y;r) \leq f_1(k;r)$. All powers with real exponent are the usual real powers of positive reals.

3. The swap identity

Everything in this section holds over an arbitrary commutative ring. We start from the odd part of a binomial expansion, written so that no case split on the parity of the summation index is ever needed: the scalar $1-(-1)^m$ is 0 for even m and 2 for odd m, and carries the parity by itself.

Lemma 3.1. *Let R be a commutative ring, $x, y \in R$ and $n \in \mathbb{N}$. Then*

$$(x+y)^n - (x-y)^n \ = \ \sum_{m=0}^{n} \bigl(1-(-1)^m\bigr) \binom{n}{m} y^m x^{n-m}.$$

Proof. Expand both powers by the binomial theorem, writing $x-y = (-y)+x$, and subtract termwise: the m-th terms are $\binom{n}{m} y^m x^{n-m}$ and $(-1)^m \binom{n}{m} y^m x^{n-m}$. □

Proposition 3.2 (swap identity)**.** *Let R be a commutative ring, let $P, Q, y \in R$ and let $r \in \mathbb{N}$. Then*

$$D_r(P,Q;y) \ = \ \sum_{m=0}^{r} \bigl(1-(-1)^{m+1}\bigr) \binom{r+1}{m+1} y^{m+1} \, (PQ)^{r-m} \, (Q-P) \sum_{i=0}^{m-1} Q^i P^{\,m-1-i}, \tag{7}$$

where the inner sum is empty (hence 0) when $m = 0$.

Proof. Apply Lemma 3.1 to both odd parts in Definition 2.1 with $n = r+1$, and collect the two resulting sums into one:

$$D_r(P,Q;y) = \sum_{k=0}^{r+1} \bigl(1-(-1)^k\bigr) \binom{r+1}{k} y^k \Bigl(Q^r P^{\,r+1-k} - P^r Q^{\,r+1-k} \Bigr).$$

The term $k=0$ vanishes because its scalar $1-(-1)^0$ is 0. Reindex the rest by $k = m+1$ with $0 \leq m \leq r$. Since $m \leq r$ we may write $Q^r = Q^{\,r-m} Q^m$ and $P^r = P^{\,r-m} P^m$, whence

$$Q^r P^{\,r-m} - P^r Q^{\,r-m} = (PQ)^{\,r-m} \bigl(Q^m - P^m \bigr) = (PQ)^{\,r-m} (Q-P) \sum_{i=0}^{m-1} Q^i P^{\,m-1-i},$$

the last step being the standard factorisation of $Q^m - P^m$. Substituting gives (7). □

Two features of (7) do the work later. First, the scalar $1-(-1)^{m+1} \in \{0,2\}$ is a nonnegative integer for every m, so no cancellation between terms is needed and no parity of r is distinguished. Second, the single occurrence of $Q-P$ is the only place where a difference survives; everything else on the right-hand side is built from P, Q and y by products, powers and sums.

Remark 3.3. The identity uses the exponent balance of Definition 2.1 — outer exponent r, inner exponent $r+1$ — through the step $Q^r = Q^{r-m}Q^m$, which needs $m \le r$. With the inner exponent replaced by $r+2$ the analogous sum has a surviving term at $m=0$, namely $2(r+2)\,y\,(PQ)^r(P-Q)$, of the wrong sign; Proposition 5.3(i) turns this into an explicit refutation.

4. Positivity

Theorem 4.1 (abstract positivity)**.** *Let $P, Q, y \in \mathbb{Z}[t,a]$ satisfy $P \succeq 0$, $Q \succeq 0$, $y \succeq 0$ and $Q - P \succeq 0$. Then $D_r(P,Q;y) \succeq 0$ for every $r \in \mathbb{N}$.*

Proof. By Proposition 3.2 it suffices to show that each summand of (7) has nonnegative coefficients. Fix m. If $m+1$ is even the scalar $1-(-1)^{m+1}$ is 0 and the summand is 0. If $m+1$ is odd the scalar is the natural number 2, and the summand is a product of: the natural numbers 2 and $\binom{r+1}{m+1}$; the power y^{m+1}; the power $(PQ)^{r-m}$; the factor $Q-P$; and the finite sum $\sum_{i<m} Q^i P^{m-1-i}$. Each factor lies in the subsemiring of Lemma 2.3 by hypothesis, and the subsemiring is closed under the operations used, so the summand does too. A finite sum of such summands again lies in the subsemiring. □

Theorem 4.2 (Fill and Janson's conjecture)**.** *For every $r \in \mathbb{N}$ and every $(i,j) \in \mathbb{N}^2$,*

$$[t^i a^j]\; h(t,a;r) \;\ge\; 0.$$

That is, for every fixed $r \in \mathbb{N}$ the polynomial $h(t,a;r)$ in (t,a) has nonnegative coefficients.

Proof. With P and Q as in (2) and $y = t$, expanding both sides shows that $h(t,a;r) = D_r(P,Q;t)$, which is (3). Now $P = 1 + (1+a)t \succeq 0$, $Q = 1 + (3+a)t \succeq 0$ and $t \succeq 0$ by Lemma 2.3, and $Q - P = 2t \succeq 0$. Theorem 4.1 applies. □

Remark 4.3. Proposition 3.2 holds over an arbitrary commutative ring, and the proof of Theorem 4.1 uses nothing about $\mathbb{Z}[t,a]$ beyond Lemma 2.3. The same argument therefore proves the corresponding statement in any commutative ring equipped with a subsemiring containing P, Q, y and $Q-P$; we state and verify only the bivariate polynomial case, which is what Theorem 4.2 needs. The wider statement is an observation about the proof, not a separately verified theorem.

Remark 4.4. Neither proof uses an inequality, an integral, a division, or any property of $\mathbb{R}$: the whole argument is the identity (7) together with the fact that a homomorphic image of a semiring is a semiring. In particular nothing here is a strengthening of the Jensen argument of [1]; it is a different argument, which happens to prove the stronger statement and does not reprove the weaker one along the way. What it does reprove is the pointwise inequality on the closed positive quadrant, since a polynomial with nonnegative coefficients is nonnegative wherever its variables are — though only for integer $r \ge 0$, whereas [1] prove it for all real $r \ge 1$.

Remark 4.5 (the constant 3 is not special). Theorem 4.1 says nothing about the particular polynomials (2). Taking $Q = 1 + (1+\delta+a)t$ for any $\delta \in \mathbb{Z}[t,a]$ with $\delta \succeq 0$ gives $Q - P = \delta t \succeq 0$, so the swap difference has nonnegative coefficients for every such δ; Fill and Janson's h is $\delta = 2$. Outside the formal development we verified this by exact expansion for the integer values $\delta = 0, 1, \dots, 6$ and $r \le 12$.

5. Structure of h, and sharpness

Proposition 5.1 (structure)**.** *In $\mathbb{Z}[t,a]$:*

(i) $h(t,a;0) = h(t,a;1) = 0$;
(ii) $h(t,a;2) = 8t^4\,(1 + 2t + at)$;
(iii) $t^4 \mid h(t,a;r)$ *for every* $r \in \mathbb{N}$.

Proof. (i) and (ii) are expansions. For (iii), rewrite h by (3) and apply Proposition 3.2. The summand at $m=0$ vanishes because its geometric factor is an empty sum, and the summand at $m=1$ vanishes because its scalar is $1-(-1)^2 = 0$. Every remaining summand, $m \ge 2$, carries the factor $y^{m+1}(Q-P) = t^{m+1} \cdot 2t = 2t^{m+2}$, which is divisible by t^4. □

So the conjecture has content exactly from $r = 2$ on, and every monomial of $h(\cdot, \cdot; r)$ has t-degree at least 4.

Remark 5.2 (computations outside the formal development). We record, without proof and outside the formally verified material, three numerical observations made while setting up the problem. Exact expansion of $h(t, a; r)$ over $\mathbb{Z}$ for $r = 0, 1, \ldots, 14$ produces

$$0,\ 0,\ 3,\ 10,\ 21,\ 36,\ 55,\ 78,\ 105,\ 136,\ 171,\ 210,\ 253,\ 300,\ 351$$

nonzero monomials respectively, all with positive coefficients. Substituting $u := 1 + at$ makes

$$H(u,t) \;=\; (u+3t)^r\big((u+2t)^{r+1} - u^{r+1}\big) - (u+t)^r\big((u+4t)^{r+1} - (u+2t)^{r+1}\big)$$

homogeneous of degree $2r + 1$, with $[t^n u^{2r+1-n}]H$ equal to

$$c_n \;=\; \sum_{j=1}^{n} \binom{r}{n-j}\binom{r+1}{j}\big(3^{\,n-j}2^{\,j} + 2^{\,j} - 4^{\,j}\big);$$

this closed form agreed with direct expansion at every n for $r \le 40$, and $c_n \ge 0$ with $c_n = 0$ for $n \le 3$ for every $r \le 200$. Finally, the coefficient of t^4 in $h(\cdot, \cdot; r)$ appears to be $8\binom{r+1}{3}$, matching (ii) at $r = 2$; this was checked for $2 \le r \le 14$ and is not proved here. Both the closed form for c_n and the $8\binom{r+1}{3}$ pattern are numerical observations; no result below depends on either.

The next proposition collects the sharpness statements. Each is a refutation at an explicit point, so each is an ordinary computation rather than a search.

Proposition 5.3 (sharpness)**.** *Let $P = 1 + (1 + a)t$ and $Q = 1 + (3 + a)t$ as in* (2)*, and write*

$$D_r^{+}(P, Q; y) \;:=\; Q^r\big((P+y)^{r+2} - (P-y)^{r+2}\big) - P^r\big((Q+y)^{r+2} - (Q-y)^{r+2}\big)$$

for the variant of Definition 2.1 with the inner exponent raised by one. Then:

- (i) *$D_1^{+}(P, Q; t) \not\succeq 0$. Indeed its value at $(t, a) = (1, 0)$ is -92.*
- (ii) *$D_2\big(P,\, 1 + at;\, t\big) \not\succeq 0$, where now $Q - P = -t$. Indeed its value at $(t, a) = (1, 0)$ is -6. So the hypothesis $Q - P \succeq 0$ of Theorem 4.1 cannot be dropped.*
- (iii) *$h(\cdot, \cdot; 2) \neq 0$; its value at $(t, a) = (1, 0)$ is 24.*
- (iv) *$(t-1)^2$ is nonnegative at every point of $\mathbb{Z}^2$, and $(t-1)^2 \not\succeq 0$. So nonnegativity on the positive quadrant, which is what* [1] *prove, does not imply Theorem 4.2.*

Proof. A polynomial with nonnegative coefficients takes nonnegative values at every point with nonnegative coordinates, so a negative value at such a point refutes $\succeq 0$. For (i), at $(t, a) = (1, 0)$ we have $P = 2$, $Q = 4$, $y = 1$, $r = 1$ and inner exponent 3, giving $4(3^3 - 1^3) - 2(5^3 - 3^3) = 104 - 196 = -92$. For (ii), at the same point $P = 2$, $Q = 1$, $y = 1$, $r = 2$ and inner exponent 3, giving $1 \cdot (3^3 - 1^3) - 4 \cdot (2^3 - 0^3) = 26 - 32 = -6$. For (iii), $h(1, 0; 2) = 8 \cdot 1 \cdot (1 + 2 + 0) = 24$ by Proposition 5.1(ii). For (iv), $(t-1)^2$ is a square, hence nonnegative everywhere; but a polynomial with nonnegative coefficients has an evaluation that is monotone nondecreasing in each variable on the nonnegative orthant, whereas the evaluation of $(t-1)^2$ drops from 1 at $(t, a) = (0, 0)$ to 0 at $(1, 0)$. □

Remark 5.4. Part (i) is not an artefact of the chosen point or of $r = 1$: exact expansion finds $6, 15, 28, 42, 60, 76$ negative coefficients in $D_r^{+}(P, Q; t)$ for $r = 1, \ldots, 6$, and $3, 9, 21, 35$ negative coefficients in the variant of (ii) for $r = 2, 3, 4, 5$. In the other direction, replacing the inner exponent $r + 1$ by r or by $r - 1$ leaves the swap difference coefficientwise nonnegative for $r \le 9$, so $r + 1$ appears to be the largest inner exponent that works rather than the only one. These counts are exact integer computations outside the formal development; only the two single-point refutations of Proposition 5.3 are part of it.

6. Two cases of the root conjecture

Throughout this section r is real and S, g are as in (4). Explicitly $g(1;r) = 1$, $g(2;r) = 3^r/(1+2^r)$ and $g(3;r) = 5^r/(1+2^r+3^r)$.

Theorem 6.1 (`Conj:Kr` at $k=1$)**.** *For real r, $g(1;r) = g(2;r)$ if and only if $r = 1$.*

Proof. Since $1 + 2^r > 0$, the equation is $1 + 2^r = 3^r$, and dividing by $3^r > 0$ it is $F(r) = 1$ with $F(r) := (1/3)^r + (2/3)^r$. Both summands are strictly decreasing in r because their bases lie in $(0,1)$, so F is strictly decreasing, hence injective; and $F(1) = \frac{1}{3} + \frac{2}{3} = 1$. □

This is the standard argument, and we claim no novelty for it; it is stated because Proposition 7.1 uses its strict form twice.

For $k = 2$ the same trick fails. Clearing denominators, $g(2;r) = g(3;r)$ reads $3^r(1+2^r+3^r) = 5^r(1+2^r)$, that is $3^r + 6^r + 9^r = 5^r + 10^r$; dividing by 10^r gives $\chi(r) = 0$ with

$$\chi(r) \;:=\; (3/10)^r + (3/5)^r + (9/10)^r - (1/2)^r - 1. \tag{8}$$

The term $-(1/2)^r$ has the wrong sign for a monotonicity argument on the individual summands, and the classical Descartes rule for exponential sums [3] gives only a bound of $2k - 1 = 3$ on the number of real roots. What we use instead is that $\chi' < 0$ everywhere.

Lemma 6.2. $\log(10/3) > 1$, $\log(5/3) > \frac{1}{2}$, $\log(10/9) > \frac{1}{10}$, *and* $\log 2 < \frac{7}{10}$.

Proof. Each is a comparison of e with an explicit rational, obtained by exponentiating and raising to a power: $e < 10/3$; $e = (e^{1/2})^2 < (5/3)^2 = 25/9$; $e = (e^{1/10})^{10} < (10/9)^{10}$; and $2^{10} = 1024 < e^7$. All four follow from the two-sided rational bounds on e to nine decimal places. □

Lemma 6.3. *For every real r, $(3/5)^r + \frac{1}{2}(6/5)^r + \frac{1}{10}(9/5)^r > \frac{7}{10}$.*

Proof. Three ranges, each crude. If $r \le 0$ then $(3/5)^r \ge 1 > \frac{7}{10}$. If $0 \le r \le 4$ then $(3/5)^r \ge (3/5)^4 = 81/625$ and $(6/5)^r, (9/5)^r \ge 1$, so the left side is at least $81/625 + \frac{1}{2} + \frac{1}{10} = 0.7296 > \frac{7}{10}$. If $r \ge 4$ then $\frac{1}{10}(9/5)^r \ge \frac{1}{10}(9/5)^4 = 6561/6250 > 1$. □

Lemma 6.4. *χ is differentiable on $\mathbb{R}$ with $\chi' < 0$ everywhere; in particular χ is strictly decreasing, hence injective.*

Proof. Differentiating (8) termwise,

$$\chi'(r) = (3/10)^r \log\tfrac{3}{10} + (3/5)^r \log\tfrac{3}{5} + (9/10)^r \log\tfrac{9}{10} - (1/2)^r \log\tfrac{1}{2}.$$

By Lemma 6.2, $\log\frac{3}{10} < -1$, $\log\frac{3}{5} < -\frac{1}{2}$, $\log\frac{9}{10} < -\frac{1}{10}$ and $\log\frac{1}{2} > -\frac{7}{10}$. Since all four powers are positive, replacing each logarithm by its bound gives

$$\chi'(r) \;<\; -(3/10)^r - \tfrac{1}{2}(3/5)^r - \tfrac{1}{10}(9/10)^r + \tfrac{7}{10}(1/2)^r.$$

Now $(3/10)^r = (3/5)^r(1/2)^r$, $(3/5)^r = (6/5)^r(1/2)^r$ and $(9/10)^r = (9/5)^r(1/2)^r$, so the right side equals

$$(1/2)^r\Big(\tfrac{7}{10} - \big[(3/5)^r + \tfrac{1}{2}(6/5)^r + \tfrac{1}{10}(9/5)^r\big]\Big),$$

which is negative by Lemma 6.3 and $(1/2)^r > 0$. □

Theorem 6.5 (`Conj:Kr` at $k=2$)**.** *There is exactly one real r with $g(2;r) = g(3;r)$, and it satisfies $2 < r < 3$.*

Proof. For every real r the denominators $1+2^r$, $1+2^r+3^r$ and 10^r are positive, so $g(2;r) = g(3;r)$ is equivalent to $\chi(r) = 0$. The values

$$\chi(2) = \tfrac{9}{100} + \tfrac{9}{25} + \tfrac{81}{100} - \tfrac{1}{4} - 1 = \tfrac{1}{100}, \qquad \chi(3) = \tfrac{27}{1000} + \tfrac{27}{125} + \tfrac{729}{1000} - \tfrac{1}{8} - 1 = -\tfrac{153}{1000}$$

are exact rationals of opposite sign, and χ is continuous by Lemma 6.4, so the intermediate value theorem gives a root in $[2,3]$. Uniqueness is the injectivity of χ from Lemma 6.4. Finally, if r is the root then $r > 2$, since $r \le 2$ would give $\chi(r) \ge \chi(2) = \frac{1}{100} > 0$, and $r < 3$, since $r \ge 3$ would give $\chi(r) \le \chi(3) < 0$. □

Remark 6.6. Bisection places the root at $r_2 = 2.049616\ldots$, consistent with the asymptotic $r_k \sim k \ln 3$ conjectured in [1] ($2\ln 3 = 2.1972\ldots$); the numerical value is outside the formal development and nothing above uses it. Theorems 6.1 and 6.5 settle only the uniqueness-and-existence half of `Conj:Kr` at $k = 1, 2$. The monotonicity of (r_k) and the asymptotic $r_k \sim k \ln 3$ are untouched; $r_1 = 1 < 2 < r_2$ is the only instance of the monotonicity that follows. The argument for $k = 2$ is a template rather than a general method: for each k one divides by the largest base, bounds the derivative globally, and brackets the root between consecutive integers where the value is an exact rational. It is the global derivative bound that would have to be made uniform in k.

7. Six of the thirteen plot-based inequalities

We turn to the gap that [1] flag in their Remark `R:Isymmunimodalr`(c) and quote in Section 1. On $r \in [1.043, 2.198)$ the plot is asked to establish

$$\max_{\substack{(k,y)\in\mathbb{Z}^2\setminus\{(2,1),(3,2)\}\\ 0\le y\le k-1\le 4}} f(k,y;r) \;<\; \max\{g(2;r), g(3;r)\},$$

which is thirteen inequalities, one per admissible pair (k, y), together with a fourteenth inequality $f_1(6;r) < \max\{g(2;r), g(3;r)\}$. The following proposition proves six of the thirteen, and does so on all of $(1, \infty)$. In each case we prove the stronger statement that $f(k, y; r)$ is smaller than one designated branch of the right-hand maximum, which implies the inequality against the maximum.

Proposition 7.1. *For every real $r > 1$:*

(i)	$f(1,0;r) < g(2;r)$,	*(iv)*	$f(3,1;r) < g(2;r)$,
(ii)	$f(2,0;r) < g(2;r)$,	*(v)*	$f(4,0;r) < g(3;r)$,
(iii)	$f(3,0;r) < g(3;r)$,	*(vi)*	$f(5,0;r) < g(3;r)$.

Proof. Throughout, $S(k;r) > 0$ and all powers are positive. By Theorem 6.1 and the strict monotonicity of $F(r) = (1/3)^r + (2/3)^r$ used in its proof, $F(r) < F(1) = 1$ for $r > 1$, i.e.

$$1 + 2^r < 3^r \qquad (r > 1); \tag{9}$$

and $3 < 3^r$, $5 < 5^r$ for $r > 1$ since b^r is strictly increasing in r for $b > 1$.

(i) $f(1,0;r) = 1/S(1;r) = 1$ and $g(2;r) = 3^r/(1+2^r)$, so the claim is $1 + 2^r < 3^r$, which is (9).

(ii) $f(2,0;r) = 2/S(2;r)$ and $g(2;r) = 3^r/S(2;r)$ have the same positive denominator, so the claim is $2 < 3^r$, which follows from $3 < 3^r$.

(iii) $f(3,0;r) = 3/S(3;r)$ and $g(3;r) = 5^r/S(3;r)$ have the same positive denominator, so the claim is $3 < 5^r$, which follows from $5 < 5^r$.

(iv) $f(3,1;r) = 2\cdot 3^r/S(3;r)$ and $g(2;r) = 3^r/S(2;r)$, so after clearing the two positive denominators and cancelling $3^r > 0$ the claim is $2(1 + 2^r) < 1 + 2^r + 3^r$, i.e. (9) again.

(v) $f(4,0;r) = 4/S(4;r)$ and $g(3;r) = 5^r/S(3;r)$, so the claim is $4\,S(3;r) < 5^r\,S(4;r)$. This holds because $4 < 5^r$ and $S(3;r) < S(4;r)$, the latter since $S(4;r) - S(3;r) = 4^r > 0$.

(vi) $f(5,0;r) = 5/S(5;r)$ and $g(3;r) = 5^r/S(3;r)$, so the claim is $5\,S(3;r) < 5^r\,S(5;r)$, which holds because $5 < 5^r$ and $S(5;r) - S(3;r) = 4^r + 5^r > 0$. □

Note that (i) and (iv) are both (9), the strict form of the $k = 1$ case of `Conj:Kr`; so Theorem 6.1, although a known exercise, is what closes two of the six. Of the fifteen pairs with $0 \le y \le k - 1 \le 4$, two are excluded in [1], namely $(2, 1)$ and $(3, 2)$; for those two, $f(2,1;r) = 3^r/S(2;r) = g(2;r)$ and $f(3,2;r) = 5^r/S(3;r) = g(3;r)$, so a strict inequality could not hold. The seven pairs not covered above are $(4,1)$, $(4,2)$, $(4,3)$, $(5,1)$, $(5,2)$, $(5,3)$ and $(5,4)$.

Remark 7.2 (measured margins; not part of the formal development). The seven remaining pairs and the fourteenth inequality behave differently from the six above, and we record how tight they are, as measurements rather than as theorems. On a 4001-point grid of $[1.043, 2.198)$ their minimum absolute margins against $\max\{g(2;r), g(3;r)\}$ are 0.1249, 0.00227, 0.1454, 0.2319, 0.01087, 0.01540 and 0.3145 for $(4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4)$ respectively, so $(4, 2)$ is the tightest. Since $10^r\chi(r)$ is the numerator of $g(2;r) - g(3;r)$, Lemma 6.4 makes the right-hand maximum $g(2;r)$ for $r < r_2$ and

$g(3;r)$ for $r > r_2$, with r_2 the root of Theorem 6.5; [1] record only that it is $g(2;r)$ when $k_0(r) = 2$, that is for $r < \ln 2/\ln(5/3) \approx 1.3569$. On the grid each of the seven in fact stays below the single branch $g(2;r)$ over the whole interval, with margins down to 0.00227, so what separates them from the six of Proposition 7.1 is not a branch split but that each compares sums of rth powers of several bases rather than two terms; $f(4,2;r)$ and $f(5,2;r)$ meet $g(2;r)$ at $r = 1$, which is where their small margins come from. The fourteenth inequality does need both branches: $f_1(6;r)$ carries the additional factor $r^r/(r+1)^{r+1}$, exceeds $g(2;r)$ once $r > 2.075$, and has minimum margin against the maximum 2.014×10^{-5} absolute (1.96×10^{-5} relative), attained at the left endpoint $r = 1.043$, where the two sides are 1.027627 and 1.027647. For the other flagged interval, $r \in [2.198, \infty)$, the statement in question is $f_1(k_0(r)+1;r) \le g(\lceil r/\ln 3 \rceil; r)$; on the grid 2.198, 2.200, 2.202, ..., 202 the ratio of the two sides has minimum 1.004227 at $r = 3.294$ and drifts up to between 1.0254 and 1.0256 over the last quarter of that range, consistent with the quotient $\big(2/(3\sqrt{3})\big)\big/\big(1/(e\sqrt{2}\ln 2)\big) = 1.02561$ of the two asymptotic constants computed in [1]. Every figure here is a double-precision grid measurement at the resolution stated, reported to indicate difficulty and not as a bound; nothing in this remark is proved, and no statement above depends on it.

8. Verification

Every theorem, proposition and lemma stated above — Lemmas 2.3, 3.1, 6.2, 6.3 and 6.4, Propositions 3.2, 5.1, 5.3 and 7.1, and Theorems 4.1, 4.2, 6.1 and 6.5 — has been formally verified in Lean 4 [4] against *Mathlib* [5]. The development is four modules of 948 lines and contains no `sorry`: a print of the axiom set of each of the 26 declarations corresponding to the statements above and their principal auxiliaries returns `propext`, `Classical.choice` and `Quot.sound` and nothing else. In particular there is no compiled-evaluation axiom and no appeal to a decision procedure over a finite search space: every statement listed in the previous sentence is proved by ordinary kernel-checked reasoning, and the four modules check in under five minutes of elapsed time in total. The two design choices worth recording are that coefficientwise nonnegativity is carried as membership in the image subsemiring of the coefficient extension $\mathbb{N}[t,a] \to \mathbb{Z}[t,a]$, so that Lemma 2.3 is a definition rather than a computation and the proof of Theorem 4.1 never touches an individual coefficient; and that the four logarithm bounds of Lemma 6.2 are obtained inside exact rational arithmetic: three from the library's nine-decimal rational bounds on e, raised to the second and tenth powers, and $\log 2 < 7/10$ from its nine-decimal bound on $\log 2$. No floating-point number enters Section 6.

No result in this paper rests on an exhaustive computation. The finite computations that were run are of two kinds, both reported above as such and neither load-bearing. First, first-value checks made before the proofs were attempted: exact integer expansion of $h(t,a;r)$ for $0 \le r \le 14$; agreement of the closed form for c_n in Remark 5.2 with direct expansion at every n for $r \le 40$, and $c_n \ge 0$ for $r \le 200$; the coefficientwise nonnegativity of the δ-variants of Remark 4.5 for $\delta \le 6$ and $r \le 12$, and of the inner exponents r and $r-1$ for $r \le 9$; the negative-coefficient counts of Remark 5.4 for $r \le 6$; and reproduction of seven of the numerical values printed in [1]. Second, the grid measurements of Remark 7.2, at the resolutions stated there. The repository containing the Lean development is private; repository reference to be added at submission.

References

[1] J. A. Fill and S. Janson, *Positive autocorrelation at unit lag for stationary random walk Metropolis–Hastings in* $\mathbb{R}^d$, arXiv:2601.19323v1 [math.PR], 27 January 2026, 52 pages (math.ST cross-list). All quotations above are cut from the arXiv LaTeX e-print of v1, in which the conjecture quoted in Section 1 is labelled `Conj:strong!`, the root conjecture `Conj:Kr` and the remark on the incomplete proof `R:Isymmunimodalr`(c); $S(k;r)$, $g(k;r)$, $k_0(r)$, $f(k,y;r)$ and $f_1(k;r)$ are the displayed definitions of that appendix. arXiv record checked live on 3 September 2026: still one version, journal-reference field empty.

[2] S. Janson, publication list, `www2.math.uu.se/~svantejs/papers/`, entry [400], consulted 3 September 2026; the entry for [1] carries no journal reference and no note recording a proof of either conjecture, where entry [372] of the same list ends "(The conjecture made here was quickly proved by Zipei Nie arXiv:2301.05704.)"

[3] G. Pólya and G. Szegő, *Problems and Theorems in Analysis II: Theory of Functions, Zeros, Polynomials, Determinants, Number Theory, Geometry*, translated from the German by C. E. Billigheimer, Springer, 1976; reprinted in Classics in Mathematics, Springer, 1998. The generalisation of Descartes' rule of signs to exponential sums $\sum_j c_j\, b_j^r$

— the number of real zeros is at most the number of sign changes in (c_j) when the bases b_j are put in increasing order — is classical, and is used above only to say what the rule does *not* give at $k = 2$.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: A. Platzer and G. Sutcliffe (eds.), Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, pp. 625–635. DOI 10.1007/978-3-030-79876-5_37.

[5] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381. DOI 10.1145/3372885.3373824; preprint arXiv:1910.09336.