FILL'S SEGMENT CONJECTURE FOR BIASED ADJACENT-TRANSPOSITION CHAINS: A PROOF AT $n = 3$, ITS SHARP FORM, AND A COUNTEREXAMPLE AT $n = 4$

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ABSTRACT. Fill's notes [1] study the Markov chain on the permutations of $[n]$ that picks one of the $n-1$ adjacent positions uniformly and swaps the two labels there with a probability depending only on the labels. Four conjectures are stated there; three are now theorems, by Greaves–Zhu [2] and by Jain–Mizgerd [3]. The fourth, stated for every n on numerical evidence, concerns the segment $p(t) = (1-t)p + tp_*$ from a regular parameter vector p to the uniform one p_*: it asserts that the spectral gap $\lambda_{K(t)}$ is nonincreasing *and convex* in $t \in [0,1]$. We settle it. At $n = 3$ we derive the spectrum from an annihilating identity and show that the whole conjecture collapses to the sign of one number, $M = (A+B)C - AB$ in the coordinates $A = p_{12} - \frac{1}{2}$, $B = p_{23} - \frac{1}{2}$, $C = p_{13} - \frac{1}{2}$: the gap along the segment is $\frac{1}{2}\big(1 - \sqrt{\frac{1}{4} - (1-t)^2 M}\big)$ exactly. Both halves of the conjecture then hold for every regular p; for an arbitrary parameter vector they hold if and only if $M \geq 0$, which is precisely the Gap-Problem inequality at that vector; and, for regular p, the gap is constant along the segment exactly when $p_{12} = p_{23} = \frac{1}{2}$, i.e. exactly at the vectors with a neutral label, which by [2, 3] are exactly the minimisers of the gap. We also show that the target p_* is essential and not merely the geometry of the regular polytope: two regular vectors at $n = 3$ whose joining segment carries a non-convex gap. At $n = 4$ the convexity half of the conjecture is *false*: for the regular vector $p_{12} = p_{13} = p_{23} = \frac{1}{2}$, $p_{14} = p_{24} = p_{34} = \frac{199}{200}$ and the equally spaced points $t = 0, \frac{5}{99}, \frac{10}{99}$ of the segment,

$$\lambda(\tfrac{5}{99}) - \tfrac{1}{2}\big(\lambda(0) + \lambda(\tfrac{10}{99})\big) \in \Big[\frac{1}{80000}, \frac{63}{2000000}\Big],$$

and for $p_{i4} = \frac{99}{100}$, $t = 0, \frac{9}{200}, \frac{9}{100}$ the same quantity is at least $177 \cdot 10^{-7}$. The proof exhibits the complete spectrum of the $n = 4$ chain on the one-parameter regular family $p_{ij} = \frac{1}{2}$ $(i < j \leq 3)$, $p_{i4} = q$: the relabelling symmetry of $\{1,2,3\}$ splits $\mathbb{R}^{24}$ into invariant blocks of dimensions $4+4+8+8$, and every eigenvalue is a root of $(\mu-1)(\mu-\frac{2}{3})((\mu-\frac{2}{3})^2 - \frac{2r}{9})$, of $\mu(\mu-\frac{1}{3})((\mu-\frac{1}{3})^2 - \frac{2r}{9})$ or of $P_r\big((6\mu-3)^2\big)$, where $r = q(1-q)$ and P_r is an explicit quartic; conversely each root w of P_r gives the eigenvalue $\frac{1}{2} + \sqrt{w}/6$ with an explicit eigenvector. The monotonicity half of the conjecture survived every search we ran. Every theorem of the paper, at $n = 3$ and at $n = 4$, is machine-checked in Lean 4 over Mathlib; the searches and the cross-checking computations are labelled as outside the formal development where they appear.

1. INTRODUCTION

1.1. **The chain.** Fix an integer $n \geq 2$ and real parameters $p_{ij} \in (0,1)$, one for each ordered pair of distinct labels $i, j \in [n]$, subject to $p_{ij} + p_{ji} = 1$. From a permutation $x = (x_1, \ldots, x_n) \in \mathfrak{S}_n$, written as the word listing the labels in positions $1, \ldots, n$, choose a position $r \in \{2, \ldots, n\}$ uniformly and interchange x_{r-1} with x_r with probability $p_{x_r, x_{r-1}}$; otherwise do nothing. The source describes the resulting transition matrix K formally, and we quote it verbatim [1]:

> "if x and y differ precisely by a transposition of adjacent items i and j, with i to the (immediate) left of j in x and j to the (immediate) left of i in y, then $K_{xy} = p_{ji}/(n-1)$";

off that pattern $K_{xy} = 0$ for $x \neq y$, and the diagonal takes up the rest. The chain is reversible with respect to

$$\pi_x = z^{-1} \prod_{1 \leq r < s \leq n} p_{x_r, x_s}, \tag{1}$$

z the normalising constant. Write β_K for the second-largest eigenvalue of K and $\lambda_K = 1 - \beta_K$ for the spectral gap. The uniform ("unweighted") parameter vector, all of whose entries are $\frac{1}{2}$, is denoted p_*; there K is the usual random adjacent-transposition walk.

A parameter vector is *regular* when, again verbatim [1],

$$(1)\ p_{i-1,i} \geq \tfrac{1}{2}\ (2 \leq i \leq n), \quad (2)\ p_{i-1,j} \geq p_{ij}\ (2 \leq i < j \leq n), \quad (3)\ p_{i,j+1} \geq p_{ij}\ (1 \leq i < j \leq n-1), \tag{2}$$

that is, when the upper-triangular array $(p_{ij})_{i<j}$ is $\geq \frac{1}{2}$ entrywise, nonincreasing along columns and nondecreasing along rows. The regular vectors form a convex set, as the source observes, and p_* is the point at which every inequality of (2) is an equality.

1.2. **The four conjectures, and the one that is left.** The notes [1] pose the *Gap Problem* — that over regular parameter vectors λ_K is minimised at p_* — and then, in a section headed "Stronger conjectures", four further items labelled (a)–(d). The source itself refutes (a) and (b) at $n = 4$, so what it leaves open is the Gap Problem, the two conjectures inside item (c) — a characterisation of the minimisers and a formula for the multiplicity of β_K — and item (d): four conjectures in all, of which three are now closed. Greaves and Zhu [2] prove the Gap Problem, as a corollary of a lower bound on λ_K valid for an arbitrary parameter vector. The characterisation of the minimisers — the gap is minimised if and only if p has a *neutral label*, in the words of [3] (v1 abstract) "the spectral gap is minimized if and only if $\vec{p}$ has a neutral label, i.e., there exists $c \in [n]$ such that $p_{c,i} = 1/2$ for all $i \neq c$" — and the multiplicity formula are proved in [3], which builds on the Greaves–Zhu argument, and also in the current version v2 of [2], whose v2 comment records them as added there. Nothing below depends on which of the two is cited.

The fourth is the subject of this paper. Verbatim [1]:

> "Observe that the regular vectors form a convex set. Let p_* denote p in the unweighted case. For given regular p and $0 \leq t \leq 1$, define $p(t) := (1-t)p + tp_*$, and let $K(t)$ denote the corresponding transition matrix. The Gap Problem formulates the conjecture that $K(1)$ has a smaller gap than does $K(0)$. Numerical evidence leads to the stronger conjecture that the gap $\lambda_{K(t)}$ is nonincreasing (and also convex) in $t \in [0, 1]$."

This is item (d) of that section; we refer to it as the *segment conjecture*, and write

$$\lambda(t) := \lambda_{K(t)}, \qquad t \in [0, 1], \tag{3}$$

suppressing the dependence on p. Two remarks on its status. First, the source proves nothing about it, at any n, and gives no range for the numerical evidence behind it. Second, neither [2] nor [3] touches it. We checked this against the current arXiv sources of both — v2 of [2] and v1 of [3], e-prints fetched on 3 September 2026: the segment $p(t)$ occurs in neither, the words "convex", "nonincreasing" and "monotone" do not occur in [2] at all, and in [3] "monotonicity" occurs only for the monotonicity built into the definition (2) of a regular vector. The same holds of the one further paper on these chains that our searches surfaced, Gheissari–Lee–Vigoda [4], which proves that the mixing time is $\Theta(n^2)$, with pre-cutoff, whenever $p_{ij} > \frac{1}{2} + \varepsilon$ for all $i < j$, and is not about the gap along the segment: in its current version (v2 of 18 August 2026, e-print fetched on 3 September 2026) neither the abstract nor the text contains "convex", "nonincreasing" or "interpolat", and "segment" occurs there only for intervals of positions, never for the parameter segment $p(t)$. It is also the only paper we found that cites [1]: it cites the notes for Fill's conjecture of polynomial mixing time under the bias and monotonicity conditions, and its v2 records the proof of the Gap Problem by [2] and of the minimiser characterisation by [3, 2], with no mention of item (d).

1.3. **What we prove.** The conjecture splits, and both halves are settled here in the sense made precise below.

(i) At $n = 3$ the whole conjecture is one number. Write $a = p_{12}$, $b = p_{23}$, $c = p_{13}$ and

$$D = p_{12}p_{23}p_{31} + p_{32}p_{21}p_{13} = ab(1-c) + (1-a)(1-b)c,$$

the quantity for which [1] reports $\lambda_K = \frac{1}{2}(1-\sqrt{D})$ at $n=3$. In the shifted coordinates $A = a - \frac{1}{2}$, $B = b - \frac{1}{2}$, $C = c - \frac{1}{2}$ one has $D = \frac{1}{4} - M$ with

$$M = M(p) := (A+B)C - AB, \tag{4}$$

and — because the segment scales A, B, C by the common factor $1-t$ — along $p(t)$

$$D\big(p(t)\big) = \tfrac{1}{4} - (1-t)^2 M \qquad \text{identically in } t \tag{5}$$

(Lemma 4.1). Hence $\lambda(t) = \frac{1}{2}\big(1 - \sqrt{\frac{1}{4} - (1-t)^2 M}\big)$, and the conjecture becomes a question about one quadratic.

(ii) The conjecture holds at $n=3$. Theorem 5.1: for every regular parameter vector at $n=3$ and every $t \in [0,1]$, the function (3) is nonincreasing and convex on $[0,1]$. The proof identifies $\lambda(t)$ with an explicit closed form at every t — not merely bounds it — and regularity enters exactly once, through $M = A(C-B) + BC \geq 0$.

(iii) It is sharp, and it is the Gap-Problem inequality. Theorem 5.2: for an arbitrary parameter vector (regular or not), the conclusion of the conjecture holds if and only if $M \geq 0$; when $M < 0$ the gap is *strictly increasing* and strictly concave on $[0,1]$, so both halves fail simultaneously. Since $D(p(0)) = \frac{1}{4} - M$ and $D(p(1)) = \frac{1}{4}$, the condition $M \geq 0$ is precisely $\lambda_{K(0)} \geq \lambda_{K(1)}$, the inequality the Gap Problem asserts. So at $n=3$ the segment conjecture is not stronger than the Gap Problem pointwise: it is equivalent to it, vector by vector.

(iv) The equality set is the set of minimisers. Theorem 5.3: for a regular vector at $n=3$ the gap is constant along the segment if and only if $p_{12} = p_{23} = \frac{1}{2}$, and otherwise it is strictly decreasing. Among regular vectors, $p_{12} = p_{23} = \frac{1}{2}$ is exactly the condition that label 2 be neutral (Remark 5.4), so the segment conjecture is an equality exactly at the vectors that [2, 3] characterise as the minimisers of the gap. This is obtained here in closed form and independently of [2, 3].

(v) The target p_* is essential. Theorem 6.1: $p = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ and $q = (\frac{1}{2}, \frac{1}{2}, \frac{7}{8})$ are both regular, and along the segment joining *them* the gap is not convex. So the conjecture is a statement about the direction p_*, not about the geometry of the regular polytope: the gap is not a convex function on that polytope.

(vi) At $n=4$ the convexity half is false. Theorem 7.10: for the regular vector $p_{12} = p_{13} = p_{23} = \frac{1}{2}$, $p_{14} = p_{24} = p_{34} = \frac{199}{200}$ and the equally spaced points $t = 0, \frac{5}{99}, \frac{10}{99}$ of the segment,

$$\frac{1}{80000} \;\leq\; \lambda(\tfrac{5}{99}) - \tfrac{1}{2}\big(\lambda(0) + \lambda(\tfrac{10}{99})\big) \;\leq\; \frac{63}{2000000},$$

and for $p_{i4} = \frac{99}{100}$ and $t = 0, \frac{9}{200}, \frac{9}{100}$,

$$\lambda(\tfrac{9}{200}) - \tfrac{1}{2}\big(\lambda(0) + \lambda(\tfrac{9}{100})\big) \geq \frac{177}{10^7} > 0.$$

Both are machine-checked. The monotonicity half of the conjecture was never violated in any of our searches, and at the three points of each witness the gap is still strictly decreasing (Proposition 7.11).

(vii) The complete spectrum at $n=4$ on the witness family. Theorem 7.4: for $p_{ij} = \frac{1}{2}$ $(i < j \leq 3)$ and $p_{i4} = q$, the relabelling symmetry of $\{1,2,3\}$ splits $\mathbb{R}^{24}$ into K-invariant blocks of dimensions $4+4+8+8$, the two 8-blocks carrying the same matrix, and every eigenvalue of K is a root of one of three explicit polynomials in μ whose coefficients are polynomials in $r = q(1-q)$; the one that carries the second-largest eigenvalue is $P_r\big((6\mu-3)^2\big)$ for the quartic

$$P_r(w) = w^4 - (6+16r)\,w^3 + (9+60r+64r^2)\,w^2 - (4+32r+160r^2)\,w + 192r^3. \tag{6}$$

Conversely (Theorem 7.5) every root w of P_r yields the eigenvalue $\frac{1}{2} + \sqrt{w}/6$ through an explicit eigenvector. This closed form is what makes (vi) provable: the second-largest eigenvalues in (vi) are bracketed by sign changes of P_r and excluded above the brackets by a five-coefficient sign certificate.

1.4. **What is not claimed.** The monotonicity half of the segment conjecture is *not* settled here for $n \geq 4$: we prove it only at $n = 3$, and Section 8 reports the searches in which it survived, which are evidence and not a proof. No statement below is uniform in n beyond $n = 3$. The counterexamples of Section 7 are two regular vectors at $n = 4$, both in the one-parameter family $p_{ij} = \frac{1}{2}$ $(i < j \leq 3)$, $p_{i4} = q$; we do not determine the set of regular vectors at which convexity fails, the spectrum of Theorem 7.4 is established for that family only, and the threshold values reported in Section 8 are floating-point measurements, not theorems. The Gap Problem itself, its equality case and the multiplicity conjecture are [2] and [3]; nothing here reproves them, and our $n = 3$ equality set is a statement about the segment, which is what makes its coincidence with their minimisers worth recording.

1.5. **Changes from version 1.** Version 1 of this paper contained Sections 2–6 and 8 as they stand, and stated the $n = 4$ counterexample — at $p_{i4} = \frac{99}{100}$, $t = 0, \frac{9}{200}, \frac{9}{100}$, with the bound $177 \cdot 10^{-7}$ — as a remark outside the formal development, proved in exact rational arithmetic by an LDL^{T} positive-semidefiniteness certificate and two Rayleigh-quotient witnesses. This version adds exactly the following. (1) The counterexample is now Theorem 7.10, machine-checked, at the version-1 vector with the same bound and at a second vector, $p_{i4} = \frac{199}{200}$ with equally spaced $t = 0, \frac{5}{99}, \frac{10}{99}$, where the convexity defect is bracketed on both sides. (2) The route is new: the block decomposition (Lemma 7.2), the complete spectrum on the family (Theorem 7.4) and its converse (Theorem 7.5) replace the LDL^{T} certificate and the Rayleigh witnesses by sign certificates on the quartic (6) (Theorem 7.9); the version-1 certificates survive as a cross-check, Remark 7.12, still outside the formal development. (3) Controls tying the $n = 4$ object to the source's transition rule (Proposition 7.1) and the negative and non-vacuity controls of Proposition 7.11. (4) Section 9 now covers the $n = 4$ module, and Appendix A lists every verified statement verbatim. The sentences of version 1 saying that the formal binding of the $n = 4$ counterexample is not claimed have been removed; nothing else in Sections 2–6 and 8 has changed beyond cross-references, the second parameter value named in Remark 8.1, and one added closing sentence in Remark 8.2. One sentence of the version-1 remark on the cross-checks called β a simple root of the characteristic polynomial at the three points; it is a double root, as the block structure now shows (Remark 7.12).

2. The chain at $n = 3$

At $n = 3$ we list $\mathfrak{S}_3$ in the order used by the source,

$$(1,2,3),\ (2,1,3),\ (1,3,2),\ (2,3,1),\ (3,1,2),\ (3,2,1), \tag{7}$$

and write $a = p_{12}$, $b = p_{23}$, $c = p_{13}$, so that $p_{21} = 1 - a$, $p_{32} = 1 - b$, $p_{31} = 1 - c$. The rule of §1.1, with $n - 1 = 2$, gives the transition matrix

$$K = \frac{1}{2}\begin{pmatrix} a+b & 1-a & 1-b & 0 & 0 & 0 \\ a & 1+c-a & 0 & 1-c & 0 & 0 \\ b & 0 & 1+c-b & 0 & 1-c & 0 \\ 0 & c & 0 & 1-c+b & 0 & 1-b \\ 0 & 0 & c & 0 & 1-c+a & 1-a \\ 0 & 0 & 0 & b & a & 2-b-a \end{pmatrix}, \tag{8}$$

which is the matrix printed in [1], entry for entry and in the source's own row and column order (7). The unnormalised stationary weights (1) in the order (7) are

$$w = \big(abc,\ (1-a)bc,\ ac(1-b),\ b(1-a)(1-c),\ a(1-b)(1-c),\ (1-a)(1-b)(1-c)\big). \tag{9}$$

Proposition 2.1. *For all real a, b, c, every row of* (8) *sums to* 1*, and K is self-adjoint for the w-weighted inner product:*

$$\sum_x w_x\, u_x\, (Kv)_x = \sum_x w_x\, (Ku)_x\, v_x \qquad \text{for all } u, v \in \mathbb{R}^6.$$

Proof. Both are finite identities in a, b, c and the coordinates of u, v, verified by expansion. The second is reversibility with respect to (1): the source states it without proof ("it is easy to see"), and it is proved here from (8) and (9) alone. (A third identity of the same kind, that the matrix (8) and the coordinatewise form of K used in the calculations below agree, is verified alongside them.) ☐

These three checks are not results; they are what makes the theorems below statements about Fill's chain rather than about an arbitrary 6×6 matrix. We record them for that reason. Reversibility also has a standard consequence used only for the reading of Theorem 3.1: since K is self-adjoint in a positive-definite inner product (the weights (9) are positive for $a, b, c \in (0,1)$), all its eigenvalues are real, so "largest real eigenvalue below 1" and "second-largest eigenvalue" name the same number. That deduction is classical and is not part of the formal development.

3. The spectrum at $n = 3$, re-derived

Put

$$D = D(a,b,c) = ab(1-c) + (1-a)(1-b)c = p_{12}p_{23}p_{31} + p_{32}p_{21}p_{13}. \tag{10}$$

Fill reports for $n = 3$ that $\lambda_K = \frac{1}{2}\big(1 - \sqrt{D}\big)$ and that β_K has multiplicity 2, obtained with a computer algebra system and stated with the remark that the results are easily verified. The following is that formula, with a proof. Call $\mu \in \mathbb{R}$ an *eigenvalue* of K if $Kv = \mu v$ for some nonzero $v \in \mathbb{R}^6$.

Theorem 3.1. *Let $Y = 4(K^2 - K)$. Then*

$$Y^2 = (D-1)\,Y, \tag{11}$$

identically in a, b, c; consequently every eigenvalue μ of K satisfies $\mu(\mu-1)\big(4\mu^2 - 4\mu + 1 - D\big) = 0$, so $\mu \in \{0, 1, \frac{1}{2}(1 \pm \sqrt{D})\}$. Moreover, if $s^2 = D$ then the explicit vector $v_\beta(a,b,c,s)$ of (12) below satisfies $Kv_\beta = \frac{1}{2}(1+s)v_\beta$, and its last coordinate is $2abc$. Hence for $a, b, c \in (0,1)$,

$$\tfrac{1}{2}\big(1 + \sqrt{D}\big) = \max\{\mu : \mu \textit{ an eigenvalue of } K,\ \mu < 1\}, \qquad \textit{i.e.} \qquad \lambda_K = \tfrac{1}{2}\big(1 - \sqrt{D}\big),$$

the maximum being attained.

The eigenvector is

$$v_\beta = \begin{pmatrix} (bc - b^2c + ac - 2ab + ab^2 - a^2c + a^2b) + (2ab - a - b)s \\ (-ab + abc - a^2c + a^2b) + (ac + ab - a)s \\ (-b^2c - ab + abc + ab^2) + (bc + ab - b)s \\ (-ac + 2abc) + (ac)s \\ (-bc + 2abc) + (bc)s \\ 2abc \end{pmatrix}. \tag{12}$$

Proof sketch. Identity (11) is six polynomial identities in a, b, c and the six coordinates of the argument vector, one per coordinate of the output; equivalently $K(K-1)\big(4K^2 - 4K + (1-D)\big) = 0$, so the minimal polynomial of K divides $x(x-1)\big(x^2 - x + \frac{1-D}{4}\big)$. If $Kv = \mu v$ with $v \neq 0$ then $Yv = 4(\mu^2 - \mu)v$, and applying (11) to v gives $\big(4(\mu^2 - \mu)\big)^2 = (D-1) \cdot 4(\mu^2 - \mu)$, which is the displayed quartic. For v_β: each of the six eigen-equations is a polynomial identity modulo $s^2 - D$, closed by subtracting the appropriate multiple of $s^2 = D$. (The vector was found as $\big(K - (1-\beta)\big)K(K-1)e_1$, which lies in $\ker(K - \beta)$ by (11), so no symbolic linear solve is involved; but the verification does not use its provenance.) For $a, b, c \in (0,1)$ one has $0 \le D < 1$ by inspection of (10), so $s = \sqrt{D} \in [0,1)$ and $\frac{1}{2}(1+s) < 1$; the last coordinate $2abc \neq 0$ makes $v_\beta \neq 0$, so $\frac{1}{2}(1+s)$ is an eigenvalue below 1; and any eigenvalue $\mu < 1$ satisfies either $\mu = 0 \le \frac{1}{2}(1+s)$ or $(2\mu - 1)^2 = s^2$, whence $2\mu - 1 \le s$. Nothing here is a numerical computation: all four steps are finite polynomial identities and elementary inequalities. ☐

Remark 3.2. Theorem 3.1 does *not* assert the multiplicity 2 of β_K that [1] reports; multiplicity plays no role below and is not proved here.

4. The segment reduction

Along $p(t) = (1-t)p + tp_*$ the three parameters become $(1-t)a + \frac{t}{2}$, $(1-t)b + \frac{t}{2}$, $(1-t)c + \frac{t}{2}$; in the shifted coordinates $A = a - \frac{1}{2}$, $B = b - \frac{1}{2}$, $C = c - \frac{1}{2}$ this is multiplication of all three by $1-t$. With M as in (4):

Lemma 4.1. *For all real a, b, c, t,*

$$D\big((1-t)a + \tfrac{t}{2},\ (1-t)b + \tfrac{t}{2},\ (1-t)c + \tfrac{t}{2}\big) = \tfrac{1}{4} - (1-t)^2 M(a,b,c).$$

In particular $D(a,b,c) = \frac{1}{4} - M$ at $t = 0$ and $D(p_*) = \frac{1}{4}$ at $t = 1$.

Proof. Expand both sides: it is a single polynomial identity in a, b, c, t. □

The lemma is elementary, and we state it separately only because it is the whole content of the $n = 3$ case. Two consequences are worth naming. First, combining it with Theorem 3.1, for a, b, c such that $p(t)$ stays in $(0,1)^3$,

$$\lambda(t) = g_M(t) := \tfrac{1}{2}\Big(1 - \sqrt{\tfrac{1}{4} - (1-t)^2 M}\Big), \tag{13}$$

so that the segment conjecture at $n = 3$ is a statement about the one-parameter family g_M. Second, the natural convexity discriminant of the radicand, $\big(\frac{d}{dt}D(p(t))\big)^2 - 2D(p(t))\frac{d^2}{dt^2}D(p(t))$, is identically equal to M; so a four-variable inequality collapses to the sign of a single number.

We also record the two elementary bounds on M that the analysis needs.

Lemma 4.2. *If p is regular at $n = 3$ — that is, $a, b \ge \frac{1}{2}$ and $c \ge a$, $c \ge b$, with $a, b, c < 1$ — then*

$$M = A(C - B) + BC \ \ge\ 0.$$

If merely $a, b, c \in [0,1]$ then $M \le \frac{1}{4}$, since $\frac{1}{4} - M = D \ge 0$.

Proof. For the first, $A, B \ge 0$, $C \ge B \ge 0$ and $C - B \ge 0$, and the displayed rewriting of M is a polynomial identity. For the second, $D = ab(1-c) + (1-a)(1-b)c \ge 0$ on $[0,1]^3$ and $D = \frac{1}{4} - M$ by Lemma 4.1 at $t = 0$. □

Note that at $n = 3$ the regularity conditions (2) are exactly $a \ge \frac{1}{2}$, $b \ge \frac{1}{2}$ (condition (1)), $c \ge b$ (condition (2) at $(i,j) = (2,3)$) and $c \ge a$ (condition (3) at $(i,j) = (1,2)$).

5. The conjecture at $n = 3$, and its sharp form

Theorem 5.1 (The segment conjecture at $n = 3$)**.** *Let $p = (a,b,c)$ be a regular parameter vector at $n = 3$ and let $M = M(p)$, so $0 \le M \le \frac{1}{4}$. Then:*

- (i) *for every $t \in [0,1]$ the number $\frac{1}{2}\big(1 + \sqrt{\frac{1}{4} - (1-t)^2 M}\big)$ is the largest eigenvalue of $K(t)$ below 1, and the maximum is attained; equivalently $\lambda(t) = g_M(t)$ as in* (13)*;*
- (ii) *$t \mapsto \lambda(t)$ is nonincreasing on* $[0,1]$*;*
- (iii) *$t \mapsto \lambda(t)$ is convex on* $[0,1]$.

Proof sketch. Regularity gives $a, b, c \in [\frac{1}{2}, 1)$, hence $p(t) \in (0,1)^3$ for every $t \in [0,1]$, and $0 \le M \le \frac{1}{4}$ by Lemma 4.2. Part (i) is Theorem 3.1 applied at the parameter vector $p(t)$, rewritten by Lemma 4.1. For (ii): with $0 \le M \le \frac{1}{4}$ the radicand $\frac{1}{4} - (1-t)^2 M$ is nonnegative on $[0,1]$ and nondecreasing in t there, because $(1-t)^2$ is nonincreasing; $\sqrt{\ }$ is nondecreasing, and $g_M = \frac{1}{2}(1 - \sqrt{\ })$ reverses the order. For (iii): the radicand $r(t) = \frac{1}{4} - (1-t)^2 M$ is concave, indeed $r\big(w_1 x + w_2 y\big) - w_1 r(x) - w_2 r(y) = M w_1 w_2 (x-y)^2 \ge 0$ for $w_1, w_2 \ge 0$ with $w_1 + w_2 = 1$; $\sqrt{\ }$ is concave and nondecreasing on $[0,\infty)$, so $\sqrt{r}$ is concave, and $g_M = \frac{1}{2}(1 - \sqrt{r})$ is convex. (The concavity of the square root is used in the two-point form $w_1\sqrt{u} + w_2\sqrt{v} \le \sqrt{w_1 u + w_2 v}$, which follows from $2\sqrt{u}\sqrt{v} \le u + v$.) No part of the argument is a computation over cases: it is a closed-form identity followed by two monotonicity steps. □

Theorem 5.2 (Sharpness, and the Gap-Problem inequality)**.** *Let $p = (a,b,c)$ be an arbitrary parameter vector at $n = 3$, $a, b, c \in (0,1)$, and $M = M(p)$ (so $M \le \frac{1}{4}$). Then:*

(i) *if* $M \geq 0$*, the gap is nonincreasing and convex on* $[0,1]$*;*
(ii) *if* $M < 0$*, the gap is* strictly increasing *on* $[0,1]$*, and strictly concave in the midpoint sense: for all* $0 \leq x < y \leq 1$*,*

$$\tfrac{1}{2}\big(\lambda(x) + \lambda(y)\big) < \lambda\big(\tfrac{x+y}{2}\big).$$

Consequently the conclusion of the segment conjecture holds at p *if and only if* $M \geq 0$*, and both of its halves fail simultaneously when it fails. Moreover*

$$M \geq 0 \iff \lambda_{K(0)} \geq \lambda_{K(1)},$$

so at $n = 3$ *the segment conjecture holds at* p *exactly when the Gap-Problem inequality does.*

Proof sketch. (i) is Theorem 5.1(ii),(iii), whose proof used only $0 \leq M \leq \frac{1}{4}$ and not regularity. For (ii), $M < 0$ makes $t \mapsto \frac{1}{4} - (1-t)^2 M$ strictly decreasing on $[0,1]$, hence g_M strictly increasing; the midpoint inequality follows from $\frac{1}{4} + (-M)(1-x)(1-y) < \sqrt{r(x)}\sqrt{r(y)}$, which after squaring is $(-M)\big((1-x)-(1-y)\big)^2 \cdot \frac{1}{4} > 0$ up to rearrangement, and then from $\big(2\sqrt{r(\frac{x+y}{2})}\big)^2 < \big(\sqrt{r(x)} + \sqrt{r(y)}\big)^2$. The last assertion is Lemma 4.1 at the two endpoints: $D(p(0)) = \frac{1}{4} - M$ and $D(p(1)) = \frac{1}{4}$, and $\lambda = \frac{1}{2}(1 - \sqrt{D})$ is a decreasing function of D, so $\lambda_{K(0)} \geq \lambda_{K(1)}$ if and only if $\frac{1}{4} - M \leq \frac{1}{4}$. ☐

By Lemma 4.2, $M \geq 0$ at every regular vector, which is Theorem 5.1 again; the content of Theorem 5.2 is that $M \geq 0$ is not merely sufficient. The witness $(\frac{9}{10}, \frac{9}{10}, \frac{1}{10})$, for which $M = -\frac{12}{25}$, is recorded in Proposition 6.2.

Theorem 5.3 (The equality set)**.** *Let* $p = (a, b, c)$ *be regular at* $n = 3$*. The following are equivalent:*
(i) $M(p) = 0$*;*
(ii) $p_{12} = p_{23} = \frac{1}{2}$*;*
(iii) λ *is constant on* $[0,1]$.

If they fail then $M(p) > 0$ *and* λ *is strictly decreasing on* $[0,1]$ *(and convex, by Theorem 5.1).*

Proof sketch. (i)⇔(ii): under regularity $M = A(C - B) + BC$ with both summands nonnegative, so $M = 0$ forces $A(C - B) = 0$ and $BC = 0$; the case $C = 0$ gives $a = b = \frac{1}{2}$ from $\frac{1}{2} \leq a \leq c$ and $\frac{1}{2} \leq b \leq c$, and the case $B = 0$ gives $b = \frac{1}{2}$ and then either $A = 0$ or $C = B = \frac{1}{2}$, which again forces $a = \frac{1}{2}$. Conversely $a = b = \frac{1}{2}$ gives $A = B = 0$, hence $M = 0$. (i)⇒(iii) is immediate from (13), since $g_0 \equiv \frac{1}{4}$. (iii)⇒(i): if $M \neq 0$ then $M > 0$ by Lemma 4.2, and then $t \mapsto \frac{1}{4} - (1-t)^2 M$ is strictly increasing on $[0,1]$, so g_M is strictly decreasing — in particular not constant. ☐

Remark 5.4. Among *regular* vectors at $n = 3$, condition (ii) is exactly the statement that p has a neutral label in the sense of [3]. Indeed label 2 is neutral iff $p_{21} = p_{23} = \frac{1}{2}$, i.e. iff $a = b = \frac{1}{2}$, which is (ii); label 1 neutral means $a = c = \frac{1}{2}$, and regularity ($\frac{1}{2} \leq b \leq c$) then forces $b = \frac{1}{2}$ as well, so $p = p_*$; symmetrically for label 3. So Theorem 5.3 says that the segment conjecture is an equality exactly at the vectors that [2, 3] identify as the minimisers of the gap — at $n = 3$, and proved here from (13) alone. At $n = 3$ that characterisation of the minimisers is in fact already asserted, without proof, in [1] itself: "one finds that λ_K is minimized by a given choice $\mathbf{p}$ if and only if $p_{12} = 1/2 = p_{23}$". What is new here is that the same set is the equality set of the segment conjecture. This also explains the aside that the source attaches to that sentence, namely "(with the value of p_{13} then being curiously irrelevant)": by (5), $M = 0$ erases c from the gap entirely.

6. The uniform endpoint is essential

The conjecture is about the specific direction p_*. It is not the assertion that the gap is a convex function on the regular polytope, and that assertion is false:

Theorem 6.1. *The vectors* $p = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ *and* $q = (\frac{1}{2}, \frac{1}{2}, \frac{7}{8})$ *are both regular at* $n = 3$*, their midpoint is* $m = (\frac{5}{8}, \frac{5}{8}, \frac{13}{16})$*, and*

$$\tfrac{1}{2}\big(\lambda_{K(p)} + \lambda_{K(q)}\big) < \lambda_{K(m)}.$$

Hence $p \mapsto \lambda_{K(p)}$ *is not convex on the polytope of regular vectors.*

Proof. Regularity of p and of q is immediate from (2). By (10), $D(p) = \frac{9}{16}\cdot\frac{1}{4} + \frac{1}{16}\cdot\frac{3}{4} = \frac{3}{16}$, $D(m) = \frac{25}{64}\cdot\frac{3}{16} + \frac{9}{64}\cdot\frac{13}{16} = \frac{3}{16}$ and $D(q) = \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{7}{8} = \frac{1}{4}$. By Theorem 3.1, $\lambda_{K(p)} = \lambda_{K(m)} = \frac{1}{2}(1-\sqrt{3/16})$ and $\lambda_{K(q)} = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$. Since $\sqrt{3/16} < \frac{1}{2}$ we get $\lambda_{K(q)} < \lambda_{K(m)} = \lambda_{K(p)}$, so the average of the two endpoint values is strictly below the midpoint value. Two of the three D-values are equal by an exact rational identity; no numerical evaluation enters. □

Proposition 6.2 (The hypotheses are not vacuous)**.** *At $n = 3$: $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ is regular with $M > 0$, so the strict conclusions of Theorem 5.3 are non-empty; $(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})$ is regular with $M = 0$, and its gap is constant along the segment; and $(\frac{9}{10}, \frac{9}{10}, \frac{1}{10})$ is* not *regular, has $M = -\frac{12}{25}$, and its gap is strictly increasing along the segment — so the regularity hypothesis in Theorem 5.1 cannot be dropped.*

Proof. Each item is the evaluation of (4) at an explicit rational point — the three values are $\frac{1}{16}$, 0 and $-\frac{12}{25}$ — together with the relevant part of Theorem 5.2 or Theorem 5.3; non-regularity of the third vector is the failure of $c \geq a$. (The verified statement at the first vector is $M > 0$; that its value is $\frac{1}{16}$ is the arithmetic just performed.) □

Remark 6.3. The failure in Theorem 6.1 is not exceptional. In a floating-point sweep, 922 of 4 000 random segments between two regular $n = 3$ vectors carried a non-convex gap, against 0 of 4 000 segments aimed at p_* — as Theorem 5.1 requires. This sweep is a measurement and no statement of this paper depends on it.

7. At $n = 4$ the convexity half is false

Everything stated in this section as a proposition, lemma or theorem is machine-checked in the same sense as Sections 2–6 (Section 9); the two remarks that report computations outside the formal development say so where they appear.

7.1. **A one-parameter regular family, and its segment.** Fix $q \in (0, 1)$ and take the $n = 4$ parameter vector

$$p_{12} = p_{13} = p_{23} = \tfrac{1}{2}, \qquad p_{14} = p_{24} = p_{34} = q, \tag{14}$$

with $p_{ji} = 1 - p_{ij}$. The array $(p_{ij})_{i<j}$ is then $\geq \frac{1}{2}$ entrywise, nonincreasing along columns and nondecreasing along rows exactly when $q \geq \frac{1}{2}$: conditions (2) and (3) of (2) are equalities except for $p_{14} \geq p_{13}$ and $p_{24} \geq p_{23}$, which read $q \geq \frac{1}{2}$. So (14) is regular if and only if $q \geq \frac{1}{2}$. Fill's segment $p(t) = (1-t)p + tp_*$ fixes the three entries $\frac{1}{2}$ and moves the three entries q together,

$$q(t) = (1-t)\, q_0 + \tfrac{t}{2}, \tag{15}$$

so along the segment the chain stays inside the family (14) and the segment conjecture becomes a question about one function of one real variable: with $\lambda_4(q)$ the spectral gap of the chain (14) at parameter q, $\lambda(t) = \lambda_4(q(t))$. For $q_0 \in [\frac{1}{2}, 1)$ one has $\frac{1}{2} \leq q(t) < 1$ for all $t \in [0, 1]$ (Lemma 7.8(ii)).

We list the 24 permutations of [4] as follows: state number $6(k-1)+j$, for $k \in \{1, 2, 3, 4\}$ and $j \in \{1, \ldots, 6\}$, is the permutation in which the label 4 sits in position k and the labels $1, 2, 3$ occupy the remaining positions, in order, as the j-th word of the list (7). Thus states 1–6 are $(4, 1, 2, 3)$, $(4, 2, 1, 3)$, $(4, 1, 3, 2)$, $(4, 2, 3, 1)$, $(4, 3, 1, 2)$, $(4, 3, 2, 1)$, and states 19–24 are the six permutations ending in 4. Write K_q for the transition operator of (14) on $\mathbb{R}^{24}$, an explicit 24×24 array of affine forms in q, and

$$w_x = q^{\,k-1}(1-q)^{4-k} \qquad (k \text{ the position of 4 in } x) \tag{16}$$

for the weights (1) with the common factor $(\frac{1}{2})^3$ from the pairs inside $\{1, 2, 3\}$ dropped. As before, $\mu \in \mathbb{R}$ is an *eigenvalue* of K_q if $K_q v = \mu v$ for some nonzero $v \in \mathbb{R}^{24}$.

Proposition 7.1 (K_q is Fill's chain)**.** *For every $q \in \mathbb{R}$:*

(i) *the list of 24 states consists of 24 distinct bijections of [4], and for every state x and every $r \in \{1, 2, 3\}$ the state $x^{(r)}$ obtained by interchanging the labels in positions r and $r+1$ is the one the transition table names; both facts are finite checks (288 cases for the second) carried out by the kernel;*

(ii) *for every* $v \in \mathbb{R}^{24}$ *and every state* x,

$$(K_q v)_x = \frac{1}{3}\sum_{r=1}^{3}\Big(p_{x_{r+1},x_r}\, v_{x^{(r)}} + \big(1 - p_{x_{r+1},x_r}\big)\, v_x\Big),$$

which is the source's rule of §1.1 with $n-1=3$ *(choose* r *uniformly, swap with probability* p_{x_{r+1},x_r}*, the diagonal takes up the rest);*

(iii) $K_q\mathbf{1} = \mathbf{1}$*;*

(iv) $\sum_x w_x\, u_x\, (K_q v)_x = \sum_x w_x\, (K_q u)_x\, v_x$ *for all* $u, v \in \mathbb{R}^{24}$*, and consequently* $\sum_x w_x (K_q v)_x = \sum_x w_x v_x$*.*

Proof. (i) is decided by exhaustive evaluation over the finite tables. (ii) is 24 identities between affine forms in q and the coordinates of v, one per state; (iii) is the special case $v = \mathbf{1}$; (iv) is one polynomial identity in q and the 48 coordinates of u, v, and stationarity is (iv) with $u = \mathbf{1}$ and (iii). ☐

As in Section 2, these are controls, recorded because they make the theorems below statements about the chain of [1]; and (iv) implies classically that the spectrum of K_q is real for $q \in (0,1)$, so that the supremum in Definition 7.7 below is the second-largest eigenvalue of K_q in the usual sense.

7.2. **The relabelling symmetry and the block decomposition.** The vector (14) is invariant under every relabelling of $\{1,2,3\}$, so K_q commutes with the induced action of $\mathfrak{S}_3$ on $\mathbb{R}^{24}$. That action is free with four orbits (indexed by the position k of the label 4), so $\mathbb{R}^{24}$ is four copies of the regular representation of $\mathfrak{S}_3$, which is triv ⊕ sgn ⊕ std ⊕ std; hence $\mathbb{R}^{24}$ splits into K_q-invariant subspaces of dimensions $4+4+8+8$, and the two 8-dimensional pieces carry the same matrix. This paragraph is the explanation; what is verified is the following explicit form of it. For $a \in \mathbb{R}^4$ and $a \in \mathbb{R}^8$ define $\iota_T(a), \iota_S(a), \iota_U(a), \iota_V(a) \in \mathbb{R}^{24}$ by giving, on the six states with 4 in position k (in the order (7) of the remaining labels),

$$\iota_T(a):\ (\alpha,\alpha,\alpha,\alpha,\alpha,\alpha), \qquad \iota_S(a):\ (\alpha,-\alpha,-\alpha,\alpha,\alpha,-\alpha),$$
$$\iota_U(a):\ (\alpha,\ \beta-\alpha,\ \alpha,\ -\beta,\ \beta-\alpha,\ -\beta), \qquad \iota_V(a):\ (\beta,\ \beta,\ \alpha-\beta,\ \alpha-\beta,\ -\alpha,\ -\alpha),$$

where $\alpha = a_k$ for ι_T, ι_S and $(\alpha,\beta) = (a_{2k-1}, a_{2k})$ for ι_U, ι_V; the sign pattern of ι_S is the sign of the permutation of $\{1,2,3\}$. Define the projections $\delta_T, \delta_S : \mathbb{R}^{24} \to \mathbb{R}^4$ and $\delta_U, \delta_V : \mathbb{R}^{24} \to \mathbb{R}^8$ by, on the same six coordinates $(v_1, \dots, v_6)$ of the k-th orbit,

$$\delta_T(v)_k = \tfrac{1}{6}(v_1+v_2+v_3+v_4+v_5+v_6), \qquad \delta_S(v)_k = \tfrac{1}{6}(v_1-v_2-v_3+v_4+v_5-v_6),$$
$$\big(\delta_U(v)_{2k-1}, \delta_U(v)_{2k}\big) = \big(\tfrac{1}{3}(v_1-v_2+v_3-v_4),\ \tfrac{1}{3}(v_3-v_4+v_5-v_6)\big),$$
$$\big(\delta_V(v)_{2k-1}, \delta_V(v)_{2k}\big) = \big(\tfrac{1}{3}(v_2+v_4-v_5-v_6),\ \tfrac{1}{3}(v_1+v_2-v_3-v_5)\big).$$

Finally put $\kappa = q/3$, $d = (1-q)/3$ and

$$B_T = \begin{pmatrix} \frac{3-q}{3} & \kappa & 0 & 0 \\ d & \frac{2}{3} & \kappa & 0 \\ 0 & d & \frac{2}{3} & \kappa \\ 0 & 0 & d & \frac{2+q}{3} \end{pmatrix}, \qquad B_S = \begin{pmatrix} d & \kappa & 0 & 0 \\ d & \frac{1}{3} & \kappa & 0 \\ 0 & d & \frac{1}{3} & \kappa \\ 0 & 0 & d & \kappa \end{pmatrix}, \tag{17}$$

$$B_U = \begin{pmatrix} C_1 & \kappa I & 0 & 0 \\ dI & C_2 & \kappa I & 0 \\ 0 & dI & C_3 & \kappa I \\ 0 & 0 & dI & C_4 \end{pmatrix}, \tag{18}$$

where I is the 2×2 identity and

$$C_1 = \begin{pmatrix} \frac{2-q}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2-q}{3} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \frac{2}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad C_3 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{2}{3} \end{pmatrix}, \quad C_4 = \begin{pmatrix} \frac{1+q}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1+q}{3} \end{pmatrix}.$$

(B_T is the birth-and-death chain followed by the position of the label 4.)

Lemma 7.2 (The block decomposition)**.** *For all* $q \in \mathbb{R}$ *and* $v \in \mathbb{R}^{24}$*,* $a \in \mathbb{R}^8$*:*

(i) $\delta_T(K_q v) = B_T\,\delta_T(v)$, $\delta_S(K_q v) = B_S\,\delta_S(v)$, $\delta_U(K_q v) = B_U\,\delta_U(v)$ *and* $\delta_V(K_q v) = B_U\,\delta_V(v)$*;*
(ii) $K_q\,\iota_U(a) = \iota_U(B_U a)$ *and* $\delta_U(\iota_U(a)) = a$*;*
(iii) $\iota_T\delta_T(v) + \iota_S\delta_S(v) + \iota_U\delta_U(v) + \iota_V\delta_V(v) = v$.

Proof. Each item is a finite list of linear identities in q and the coordinates of v or a (24 per identity of (i) and (ii), 24 for (iii)), verified by expansion. (Each projection also commutes with scalar multiplication, which is used below without comment.) □

Each block is (block-)tridiagonal with the invertible super-diagonal $\kappa = q/3$ (for $q \neq 0$), so forward substitution expresses every coordinate of an eigenvector in terms of the first one (B_T, B_S) or the first two (B_U), and the leftover rows are the characteristic equation. Put $r = q(1-q)$, let P_r be the quartic (6), and

$$\Delta_T(\mu) = (\mu-1)(\mu-\tfrac{2}{3})\big((\mu-\tfrac{2}{3})^2 - \tfrac{2r}{9}\big), \qquad \Delta_S(\mu) = \mu(\mu-\tfrac{1}{3})\big((\mu-\tfrac{1}{3})^2 - \tfrac{2r}{9}\big). \tag{19}$$

Lemma 7.3 (Characteristic equations of the blocks)**.** *Let $q \neq 0$ and $\mu \in \mathbb{R}$. If $B_T a = \mu a$ for $a \in \mathbb{R}^4$ then $\Delta_T(\mu)\,a = 0$; if $B_S a = \mu a$ then $\Delta_S(\mu)\,a = 0$; and if $B_U a = \mu a$ for $a \in \mathbb{R}^8$ then $P_r\big((6\mu-3)^2\big)\,a = 0$.*

Proof sketch. Forward substitution: from the first three rows of $B_T a = \mu a$ one obtains κa_2, $\kappa^2 a_3$, $\kappa^3 a_4$ as explicit polynomial multiples of a_1, and the fourth row then reads $\kappa^3\Delta_T(\mu)\,a_1 = 0$; dividing by $\kappa \neq 0$ gives $\Delta_T(\mu)a_1 = 0$, and multiplying the substitution formulas by $\Delta_T(\mu)$ gives the other three coordinates. B_S is identical in form. For B_U, the first six rows express $\kappa a_3, \kappa a_4, \kappa^2 a_5, \kappa^2 a_6, \kappa^3 a_7, \kappa^3 a_8$ as polynomial combinations of a_1, a_2, and the last two rows become a 2×2 linear system $m_{11}a_1 + m_{12}a_2 = 0$, $m_{21}a_1 + m_{22}a_2 = 0$ with the polynomial entries (20) below, whose determinant is $6^{-8}P_r\big((6\mu-3)^2\big)$; eliminating gives $P_r((6\mu-3)^2)\,a_1 = P_r((6\mu-3)^2)\,a_2 = 0$, and the remaining coordinates follow as before. Every step is a polynomial identity in q, μ and the coordinates. □

The notable feature is the last statement: the characteristic polynomial of B_U is even in $\mu - \frac{1}{2}$, so the spectrum of the standard block is $\frac{1}{2} \pm \sqrt{w}/6$ over the roots w of a *quartic*.

Theorem 7.4 (The complete spectrum on the family)**.** *Let $q \neq 0$. Every eigenvalue μ of K_q satisfies*

$$\Delta_T(\mu)\cdot\Delta_S(\mu)\cdot P_r\big((6\mu-3)^2\big) = 0, \qquad r = q(1-q).$$

Proof. Let $K_q v = \mu v$, $v \neq 0$. By Lemma 7.2(i) each of $\delta_T(v)$, $\delta_S(v)$, $\delta_U(v)$, $\delta_V(v)$ is an eigenvector (or zero) of the corresponding block for the eigenvalue μ, so by Lemma 7.3 each is annihilated by its own factor. The four inclusions are linear, so the product of the three factors annihilates each of the four summands in Lemma 7.2(iii), hence annihilates v; as $v \neq 0$ the product is 0. No case distinction on which block carries v is needed. □

7.3. **From a root of the quartic back to an eigenvector.** Forward substitution run backwards produces the eigenvector. Write

$$\begin{aligned}
m_{11} &= \tfrac{25}{324} - \tfrac{7}{108}q + \tfrac{2}{27}q^2 - \tfrac{1}{81}q^3 - \tfrac{5}{9}\mu + \tfrac{2}{9}\mu q - \tfrac{2}{9}\mu q^2 + \tfrac{14}{9}\mu^2 - \tfrac{2}{9}\mu^2 q + \tfrac{2}{9}\mu^2 q^2 - 2\mu^3 + \mu^4,\\
m_{12} &= \tfrac{35}{648} - \tfrac{1}{24}q + \tfrac{1}{27}q^2 + \tfrac{1}{162}q^3 - \tfrac{37}{108}\mu + \tfrac{2}{27}\mu q - \tfrac{2}{27}\mu q^2 + \tfrac{13}{18}\mu^2 - \tfrac{1}{2}\mu^3,\\
m_{21} &= \tfrac{43}{648} - \tfrac{7}{216}q + \tfrac{1}{27}q^2 - \tfrac{1}{162}q^3 - \tfrac{43}{108}\mu + \tfrac{2}{27}\mu q - \tfrac{2}{27}\mu q^2 + \tfrac{7}{9}\mu^2 - \tfrac{1}{2}\mu^3,\\
m_{22} &= \tfrac{89}{1296} - \tfrac{1}{18}q + \tfrac{5}{108}q^2 + \tfrac{1}{81}q^3 - \tfrac{19}{36}\mu + \tfrac{2}{9}\mu q - \tfrac{2}{9}\mu q^2 + \tfrac{55}{36}\mu^2 - \tfrac{2}{9}\mu^2 q + \tfrac{2}{9}\mu^2 q^2 - 2\mu^3 + \mu^4,
\end{aligned} \tag{20}$$

polynomials in q,μ, and for real x,y let $u(q,\mu,x,y)\in\mathbb{R}^8$ be the vector with coordinates

$$
\begin{aligned}
&u_1=\kappa^3x,\qquad u_2=\kappa^3y,\\
&u_3=\kappa^2\big((\mu-\tfrac23+\tfrac q3)x-\tfrac16y\big),\qquad u_4=\kappa^2\big(-\tfrac16x+(\mu-\tfrac23+\tfrac q3)y\big),\\
&u_5=\kappa\big((\tfrac49-\tfrac q3+\tfrac{q^2}{9}-\tfrac43\mu+\tfrac13\mu q+\mu^2)x+(\tfrac19-\tfrac16\mu)y\big),\\
&u_6=\kappa\big((\tfrac16-\tfrac{q}{18}-\tfrac13\mu)x+(\tfrac14-\tfrac29q+\tfrac19q^2-\mu+\tfrac13\mu q+\mu^2)y\big),\\
&u_7=\big(-\tfrac{19}{108}+\tfrac{7}{36}q-\tfrac{4}{27}q^2+\tfrac{1}{27}q^3+\tfrac{17}{18}\mu-\tfrac59\mu q+\tfrac29\mu q^2-\tfrac53\mu^2+\tfrac13\mu^2q+\mu^3\big)x\\
&\qquad+\big(-\tfrac{17}{216}+\tfrac{1}{18}q-\tfrac{1}{27}q^2+\tfrac13\mu-\tfrac{1}{18}\mu q-\tfrac13\mu^2\big)y,\\
&u_8=\big(-\tfrac19+\tfrac{1}{18}q-\tfrac{1}{54}q^2+\tfrac{7}{18}\mu-\tfrac{1}{18}\mu q-\tfrac13\mu^2\big)x\\
&\qquad+\big(-\tfrac16+\tfrac29q-\tfrac{5}{27}q^2+\tfrac{1}{27}q^3+\tfrac{11}{12}\mu-\tfrac59\mu q+\tfrac29\mu q^2-\tfrac53\mu^2+\tfrac13\mu^2q+\mu^3\big)y,
\end{aligned}
\tag{21}
$$

with $\kappa=q/3$. Put $u^\star(q,\mu)=u(q,\mu,\,m_{12},\,-m_{11})$.

Theorem 7.5 (Every root of the quartic is an eigenvalue)**.** (i) *For all real q,μ,x,y with $m_{11}x+m_{12}y=0$ and $m_{21}x+m_{22}y=0$, $B_U\,u(q,\mu,x,y)=\mu\,u(q,\mu,x,y)$.*

(ii) *If $P_r\big((6\mu-3)^2\big)=0$, $r=q(1-q)$, then $B_Uu^\star(q,\mu)=\mu\,u^\star(q,\mu)$.*

(iii) *If moreover $q\neq0$ and $m_{12}(q,\mu)\neq0$, then μ is an eigenvalue of K_q (the proof exhibits the eigenvector $\iota_U\big(u^\star(q,\mu)\big)$).*

In particular, if $w\geq0$ is a root of P_r, $q\neq0$, and $m_{12}(q,\frac12+\sqrt{w}/6)\neq0$, then $\frac12+\sqrt{w}/6$ is an eigenvalue of K_q.

Proof sketch. (i): the first six coordinates of $B_Uu-\mu u$ vanish identically in q,μ,x,y (six polynomial identities); the seventh and eighth are $-(m_{11}x+m_{12}y)$ and $-(m_{21}x+m_{22}y)$. (ii): with $(x,y)=(m_{12},-m_{11})$ the first condition of (i) is the identity $m_{11}m_{12}-m_{12}m_{11}=0$, and the second is $m_{21}m_{12}-m_{22}m_{11}=-6^{-8}P_r\big((6\mu-3)^2\big)$, a polynomial identity in q,μ once P_r is expanded; so it vanishes when the quartic does. (iii): $\iota_U(u^\star)$ is an eigenvector of K_q by Lemma 7.2(ii), and it is nonzero because $\delta_U\iota_U=\mathrm{id}$ and the first coordinate of $u^\star$ is $\kappa^3m_{12}\neq0$. The last sentence is (iii) with $(6\mu-3)^2=w$. ☐

The side condition $m_{12}\neq0$ is the only hypothesis of the converse; at each parameter value used below it is verified on the whole half-line of μ that matters, by writing m_{12} — a cubic in μ — in Taylor form about the left end of the bracket and checking that all four coefficients are negative.

Remark 7.6 (Two checks of the closed form; computations outside the formal development). (a) At the uniform vector $q=\frac12$, $r=\frac14$ and (6) factors as $w^4-10w^3+28w^2-22w+3=(w^2-6w+1)(w-1)(w-3)$; the largest root $w=3+2\sqrt2$ gives $\frac12+(1+\sqrt2)/6=\frac23+\sqrt2/6$, which is also the second-largest root of Δ_T, namely $\frac23+\sqrt{2r}/3$. So the gap of the unweighted $n=4$ chain is $\frac13-\sqrt2/6=(1-\cos(\pi/4))/(4-1)$, which is Theorem 1 of [1] at $n=4$ (the theorem, stated there for all n and attributed to [5], gives the unweighted gap as $(1-\cos(\pi/n))/(n-1)$); and this eigenvalue occurs once in the trivial block and once in each of the two standard blocks, multiplicity $3=n-1$, which is the multiplicity asserted for the unweighted case in item (c) of the source's stronger conjectures. Neither fact was assumed; both fall out of the decomposition. (b) At each of the six parameter values used in Theorems 7.9 and 7.10, P_r has no rational root and its resolvent cubic has no rational root (exact rational arithmetic), so P_r is irreducible over $\mathbb{Q}$ and $w=(6\beta-3)^2$ is an algebraic number of degree 4. Moreover β itself has degree 8 over $\mathbb{Q}$: the even polynomial $P_r(X^2)$ is irreducible over $\mathbb{Q}$ at each of the six values (checked by direct factorisation; it also follows from the irreducibility of P_r, since a factorisation would have the form $g(X)g(-X)$ with $g(0)^2=P_r(0)=192r^3$, and $192r^3=3r\,(8r)^2$ is not a rational square at any of the six values of $r=q(1-q)$). So there is no formula for $\beta_{K(t)}$ at $n=4$ as simple as $\frac12(1+\sqrt{D})$ at $n=3$. The closed form also reproduces, in floating point, the three eigenvalues of Remark 7.12 to 15 digits from a computation with nothing in common with the ones there.

7.4. **The second-largest eigenvalue at three points of the segment.**

Definition 7.7. For $q \in \mathbb{R}$ let $\beta_4(q) = \sup\{\mu : \mu \text{ an eigenvalue of } K_q,\ \mu < 1\}$ (the set is bounded above by 1, and is nonempty at every parameter value used below) and $\lambda_4(q) = 1 - \beta_4(q)$. For $q_0 = \frac{199}{200}$ and for $q_0 = \frac{99}{100}$ write $q(t)$ for (15).

Lemma 7.8. (i) *If the set $\{\mu : \mu \text{ an eigenvalue of } K_q,\ \mu < 1\}$ is nonempty, then $\beta_4(q)$ is its least upper bound.*

(ii) *For $q_0 = \frac{199}{200}$ and for $q_0 = \frac{99}{100}$, and every $t \in [0,1]$, $\frac{1}{2} \le q(t) < 1$; by §7.1 this is regularity of every point of either segment.*

Proof. (i) is the defining property of the supremum of a nonempty set bounded above. (ii): $q(t)$ is affine in t with values q_0 and $\frac{1}{2}$ at the endpoints. □

Theorem 7.9 (Three brackets)**.**

$$\frac{834441}{10^6} \le \beta_4\big(\tfrac{199}{200}\big) \le \frac{834451}{10^6}, \qquad \frac{8401}{10^4} \le \beta_4\big(\tfrac{97}{100}\big) \le \frac{840109}{10^6}, \qquad \frac{845802}{10^6} \le \beta_4\big(\tfrac{189}{200}\big) \le \frac{845812}{10^6}.$$

Proof sketch. Fix one of the three values of q and write $[b_-, b_+]$ for its bracket.

Lower bound. P_r changes sign between $(6b_- - 3)^2$ and $(6b_+ - 3)^2$: two exact rational evaluations of (6). By the intermediate value theorem there is a root w of P_r strictly between them, hence $\mu = \frac{1}{2} + \sqrt{w}/6 \in (b_-, b_+)$ with $(6\mu - 3)^2 = w$. The side condition $m_{12}(q,\mu) \ne 0$ holds for every $\mu \ge b_-$: the cubic $\mu \mapsto m_{12}(q,\mu)$ equals $A_0 + A_1(\mu - b_-) + A_2(\mu - b_-)^2 + A_3(\mu - b_-)^3$ for explicit rationals with $A_0 < 0$ and $A_1, A_2, A_3 \le 0$ (an identity in μ; here $A_3 = -\frac{1}{2}$ always). Theorem 7.5(iii) makes μ an eigenvalue of K_q; it lies below 1, so $\beta_4(q) \ge \mu > b_-$. The same eigenvalue shows the set in Lemma 7.8(i) nonempty.

Upper bound. Let $\mu < 1$ be an eigenvalue. By Theorem 7.4 one of three things holds. If $\Delta_T(\mu) = 0$ then $\mu \in \{1, \frac{2}{3}\}$ or $(\mu - \frac{2}{3})^2 = \frac{2r}{9}$; the first is excluded, and the others give $\mu \le b_+$ because $\frac{2r}{9} < (b_+ - \frac{2}{3})^2$ (one rational inequality). If $\Delta_S(\mu) = 0$, likewise, from $\frac{2r}{9} < (b_+ - \frac{1}{3})^2$. Otherwise $P_r\big((6\mu - 3)^2\big) = 0$. Put $W = (6b_+ - 3)^2$. There are explicit rationals $c_0 > 0$, $c_1, c_2, c_3 \ge 0$ with

$$P_r(w) = c_0 + c_1(w - W) + c_2(w - W)^2 + c_3(w - W)^3 + (w - W)^4 \qquad \text{identically in } w,$$

so $P_r(w) > 0$ for every $w \ge W$; if $\mu > b_+$ then $(6\mu - 3)^2 > W$ (as $b_+ > \frac{1}{2}$), a contradiction. The certificate is five rationals per bound — the largest denominator occurring in the whole development has 54 digits — and it exists because all four roots of P_r are real and at most W, a fact which is not needed and not proved: the identity is checked directly. There is no positive-semidefiniteness certificate, no Rayleigh quotient and no spectral theorem anywhere in the argument. □

Theorem 7.10 (The convexity half of the segment conjecture fails at $n = 4$)**.** (i) *Let $q_0 = \frac{199}{200}$, so that $t = 0, \frac{5}{99}, \frac{10}{99}$ — three equally spaced points of the segment — correspond to $q = \frac{199}{200}, \frac{97}{100}, \frac{189}{200}$. Then*

$$\frac{1}{80000} \ \le\ \lambda\big(\tfrac{5}{99}\big) - \tfrac{1}{2}\Big(\lambda(0) + \lambda\big(\tfrac{10}{99}\big)\Big) \ \le\ \frac{63}{2000000}.$$

(ii) *Let $q_0 = \frac{99}{100}$, so that $t = 0, \frac{9}{200}, \frac{9}{100}$ correspond to $q = \frac{99}{100}, \frac{19359}{20000}, \frac{9459}{10000}$. Then*

$$\lambda\big(\tfrac{9}{200}\big) - \tfrac{1}{2}\Big(\lambda(0) + \lambda\big(\tfrac{9}{100}\big)\Big) \ \ge\ \frac{177}{10^7}.$$

In both cases the middle point is the midpoint of the other two, so $t \mapsto \lambda(t)$ is not convex on $[0,1]$; both parameter vectors are regular, and so is every point of both segments. Hence the convexity half of the segment conjecture is false at $n = 4$.

Proof. (i): $\lambda(t) = 1 - \beta_4(q(t))$, so the displayed quantity equals $\frac{1}{2}\big(\beta_4(\frac{199}{200}) + \beta_4(\frac{189}{200})\big) - \beta_4(\frac{97}{100})$, and Theorem 7.9 bounds it below by $\frac{1}{2}(834441 + 845802) \cdot 10^{-6} - 840109 \cdot 10^{-6} = 12.5 \cdot 10^{-6}$ and above by $\frac{1}{2}(834451 + 845812) \cdot 10^{-6} - 840100 \cdot 10^{-6} = 31.5 \cdot 10^{-6}$. (ii): the same argument with the one-sided bounds $\beta_4(\frac{99}{100}) \ge \frac{8355691}{10^7}$, $\beta_4(\frac{9459}{10000}) \ge \frac{8456041}{10^7}$ and $\beta_4(\frac{19359}{20000}) \le \frac{8405689}{10^7}$, each obtained exactly as in Theorem 7.9 (the two lower bounds by a sign change of P_r and the side condition, the upper bound by a five-coefficient certificate at $W = (6 \cdot \frac{8405689}{10^7} - 3)^2$ together with the two rational inequalities for the trivial and sign blocks); then $\frac{1}{2}(8355691 + 8456041) - 8405689 = 177$. Regularity is Lemma 7.8(ii). □

Proposition 7.11 (Controls at $n = 4$)**.** (i) *The upper bound at the midpoint of Theorem 7.10(i) cannot be lowered to* $\frac{8401}{10^4}$*: at* $q = \frac{97}{100}$ *it is false that every eigenvalue of* K_q *below* 1 *is at most* $\frac{8401}{10^4}$.

(ii) *The convexity defect in Theorem 7.10(i) is bracketed on both sides, so it is genuinely of order* 10^{-5}*: neither an artefact of loose bounds nor larger than claimed.*

(iii) *The monotonicity half of the segment conjecture is untouched at the same three points:* $\lambda(\frac{10}{99}) < \lambda(\frac{5}{99}) < \lambda(0)$ *for* $q_0 = \frac{199}{200}$.

(iv) *At each of* $q = \frac{199}{200}, \frac{97}{100}, \frac{189}{200}$ *the set of eigenvalues of* K_q *below* 1 *is nonempty and* $\beta_4(q)$ *is its least upper bound.*

Proof. (i) is the eigenvalue exhibited in the lower bound of Theorem 7.9 at $q = \frac{97}{100}$, which exceeds $\frac{8401}{10^4}$. (ii) is Theorem 7.10(i) itself. (iii): from Theorem 7.9, $\beta_4(\frac{189}{200}) \geq 0.845802 > 0.840109 \geq \beta_4(\frac{97}{100})$ and $\beta_4(\frac{97}{100}) \geq 0.8401 > 0.834451 \geq \beta_4(\frac{199}{200})$. (iv) is the nonemptiness noted in the proof of Theorem 7.9 with Lemma 7.8(i). □

Item (i) is the counterpart of a negative control in the original exact-arithmetic proof (Remark 7.12), where the certificate for a bound 10^{-5} below the true eigenvalue correctly failed; item (iii) is the counterpart, at these three points, of the searches of Section 8 in which monotonicity was never violated.

Remark 7.12 (The original certificates, and the cross-checks; outside the formal development). Theorem 7.10(ii) was first established in exact rational arithmetic, before the closed form of Theorem 7.4 was available, by certificates on the 24×24 matrix itself. With $W = \mathrm{diag}(w)$, w as in (16) (there with the factor $(\frac{1}{2})^3$ kept), and $\sigma = \sum_x w_x$, reversibility makes $A = WK$ symmetric; every eigenvector for $\mu \neq 1$ satisfies $\sum_x w_x f_x = 0$, and on that subspace μ is the Rayleigh quotient $\langle f, Af\rangle / \langle f, Wf\rangle$. The upper bound $\beta(\frac{9}{200}) \leq \frac{8405689}{10^7}$ then follows from positive semidefiniteness of the rational matrix $\sigma(bW - A) + (1-b)ww^{\mathsf{T}}$, certified by an exact LDL^{T} factorisation over $\mathbb{Q}$ (nonnegative pivots), and the two lower bounds from explicit rational vectors f with $\sum_x w_x f_x = 0$ and Rayleigh quotients $0.835569182405987\ldots$ and $0.845604156388612\ldots$. Independently, the characteristic polynomial of the 24×24 matrix computed exactly over $\mathbb{Q}$ (Faddeev–LeVerrier) followed by Sturm-sequence root counting gives $\beta = 0.835569182405989, 0.840568823227253, 0.845604156388617$ at $q = \frac{99}{100}, \frac{19359}{20000}, \frac{9459}{10000}$ (a Sturm count counts distinct roots: β is in fact a *double* eigenvalue of K_q on the family, the two standard blocks of Lemma 7.2 carrying the same matrix, and an exact inertia count on the 24×24 matrix finds exactly two eigenvalues in each bracket of Theorem 7.9 and of part (ii), with only the eigenvalue 1 above them); two floating-point eigensolvers (cyclic Jacobi, and Lanczos with Sturm bisection) agree with these to $8 \cdot 10^{-15}$; and the quartic route of Theorem 7.4 returns the same three numbers to 15 digits, as well as 0.834447999626345, 0.840101397659255, 0.845809430782534 at the three points of Theorem 7.10(i), where the true defect is about $2.73 \cdot 10^{-5}$ — inside the certified bracket of Theorem 7.10(i), and about half as large again as the $177 \cdot 10^{-7}$ of part (ii). The matrix construction behind all of this was validated against the source before any claim was made: it reproduces entry for entry the 6×6 matrix (8) and the 2×2 matrix printed in [1], the reversibility relation (1) exactly at $n = 3$ and $n = 4$, the unweighted gap $(1 - \cos(\pi/n))/(n-1)$ of the source's Theorem 1 at $n = 3, 4, 5$, and the four decimal values that [1] prints in its discussion of the other stronger conjectures. None of the computations in this remark is machine-checked, and nothing in the paper depends on them: they are what Theorem 7.10 was checked against.

8. What the searches say

The following are floating-point measurements, reported to say where the failure lives and which half of the conjecture is worth attacking. They are not theorems and nothing above depends on them.

Remark 8.1 (Search output; floating point). *Monotonicity.* The monotonicity half of the segment conjecture was never violated: 400 random regular vectors at $n = 4$, 120 at $n = 5$, and adversarial optimisation of the forward difference, always gave a strictly positive minimum forward difference (worst $2.7 \cdot 10^{-5}$ at $n = 4$, $5.3 \cdot 10^{-5}$ at $n = 5$). As a control that the test is not vacuous, *non*-regular vectors violate monotonicity in 277 of 400 draws at $n = 4$ and 113 of 120 at $n = 5$.

Convexity at $n = 4$ is a boundary effect. In the one-parameter family (14), the minimum of the discrete second difference over an 81-point grid is $+0.0155$ at $q = 0.95$, $+0.0035$ at $q = 0.96$, -0.0039 at $q = 0.9655$ and -0.0437 at $q = 0.99$: convexity fails above a threshold $q^*(4) \approx 0.963$. A search over 400 random regular vectors with all $p_{ij} \le 0.95$ found no violation at all, while the same search with $p_{ij} \le 0.9999$ reaches $3.8 \cdot 10^{-5}$. This is why the counterexamples of Theorem 7.10 sit at $q = \frac{199}{200}$ and $q = \frac{99}{100}$. Every violation of the convexity half found at $n = 4$ was re-derived in exact arithmetic.

At $n = 5$ it is not a boundary effect. In the analogous family $p_{i5} = q$ for $i < 5$, $p_{ij} = \frac{1}{2}$ for $i < j \le 4$, the maximal midpoint violation over triples $(0, d, 2d)$ is absent at $q = 0.6, \ldots, 0.75$, turns positive at $q = 0.80$ $(+3.7 \cdot 10^{-7})$, and reaches $+3.8 \cdot 10^{-5}$ at $q = 0.85$, $+1.4 \cdot 10^{-4}$ at $q = 0.9$ and $+3.3 \cdot 10^{-4}$ at $q = 0.99$ — margins ten to twenty times the $n = 4$ ones, at vectors far from the boundary of the polytope. Random regular vectors at $n = 5$ violate convexity in 4 of 120 draws. So $q^*(5) \approx 0.80$, the threshold falls with n, and the failure of the convexity half is not an $n = 4$ accident. The $n = 5$ numbers were not re-derived in exact arithmetic.

Remark 8.2 (A strengthening that is not refuted). At $n = 3$, (5) says that $\nu(t) := \beta_{K(t)} - \frac{1}{2}$ satisfies $\nu(t)^2 = \frac{1}{4}\big(\frac{1}{4} - (1-t)^2 M\big)$, an affine function of $(1-t)^2$ and hence a concave quadratic in t; concavity of ν^2 together with $\nu \ge 0$ implies both halves of the segment conjecture. So "ν^2 is nondecreasing and concave in t" is the natural strengthening to ask about at general n. It survived 60 random regular vectors at $n = 4$; at $n = 5$ one apparent violation was seen at a near-boundary vector and has not been checked in exact arithmetic. We state this only as a question. On the family (14) at $n = 4$ it is now a question about one algebraic function, $\nu^2 = w_{\max}(r)/36$ with $w_{\max}$ the largest root of (6), and so is the exact value of the threshold $q^*(4)$ of Remark 8.1; neither is pursued here.

9. Verification

Every statement asserted above as a proposition, lemma or theorem — Propositions 2.1, 6.2, 7.1 and 7.11, Lemmas 4.1, 4.2, 7.2, 7.3 and 7.8, and Theorems 3.1, 5.1, 5.2, 5.3, 6.1, 7.4, 7.5, 7.9 and 7.10 — is checked, in the form stated, by the kernel of the Lean 4 proof assistant [6] over the Mathlib library [7]; Appendix A lists the verified statements verbatim. The development is four modules totalling 1 725 lines and 105 theorems, 56 of them tagged with the machine-readable pins that bind them to the statements above: three modules (691 lines, 37 theorems) for $n = 3$ — the matrix (8) and its operator form, the weights (9) and reversibility; the annihilating identity (11), the eigenvector (12) and the attained maximum of Theorem 3.1; the reduction (5), the bounds on M, and the monotonicity, convexity, strictness and $M < 0$ statements for the explicit function g_M; and the four control vectors — and one module (1 034 lines, 68 theorems) for $n = 4$: the operator K_q, the weights (16), the tables of Proposition 7.1, the maps and blocks of §7.2, the quartic (6), the eigenvector (21), the six parameter values with their sign changes, Taylor coefficients and five-coefficient certificates, and the definition of β_4 as a supremum. It contains no `sorry` and declares no axiom of its own: a print of the axiom set of each of the 105 theorems returns `propext`, `Classical.choice` and `Quot.sound` and nothing else (one of them, the check that the neighbour table of Proposition 7.1(i) is the adjacent transposition, needs only `propext` and `Quot.sound`) — in particular no `sorryAx` and no compiled-evaluation axiom. There is no use of `native_decide`, no external solver, no floating point and no interval arithmetic anywhere in the verified part; the three finite checks of Proposition 7.1(i) are decided by the kernel's own evaluation. This matters here because the objects are square roots of polynomials in the parameters at $n = 3$ and roots of a quartic at $n = 4$: what is verified is the closed form itself, not a numerical surrogate for it, and at $n = 4$ the numerical data of the proof are rational numbers (the largest with 54 digits) entering polynomial identities and rational inequalities. Total elaboration is a few CPU-minutes: the $n = 4$ module alone takes about 210 CPU-seconds (measured on 3 September 2026).

Three points of form. First, "eigenvalue" in Theorem 3.1, in Theorem 5.1(i) and throughout Section 7 means, in the verified statement, a real μ with $Kv = \mu v$ for a nonzero real v; Theorem 3.1 exhibits $\frac{1}{2}(1 + \sqrt{D})$ as the greatest element of the set of such μ below 1, and β_4 is defined as the supremum of that set at $n = 4$. That this is the second-largest eigenvalue of K in the usual sense uses the reality of the spectrum, which follows from Proposition 2.1, respectively Proposition 7.1(iv), by the classical argument recalled in §2 and is not part of the formal development. Second, the identification $M \geq 0 \iff \lambda_{K(0)} \geq \lambda_{K(1)}$ in Theorem 5.2, and the equivalence (i)⇔(iii) of Theorem 5.3, are the verified statements combined by one line of arithmetic each; the individual implications, not their combinations, are what the development names. So is the identification of $\lambda(t)$ with $g_M(t)$ at an arbitrary, possibly non-regular, vector in Theorem 5.2: it is Theorem 3.1 applied at $p(t)$, rewritten by Lemma 4.1, both of them verified. Likewise, at $n = 4$ the verified statements are about $\beta_4(q)$ at the six rational values of q and about the explicit maps (15); the identification $\lambda(t) = \lambda_4(q(t))$ is the definition of the segment on the family (14), and the arithmetic of Theorem 7.10 is verified. Third, Section 8, Remarks 6.3, 7.6 and 7.12, and the multiplicity claims recalled in §3 and in Remark 7.6 are deliberately outside the formal development and are labelled where they appear. (The line and theorem counts in this section are those of 3 September 2026.)

The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. All three of [1], [2] and [3] were read here from their arXiv e-prints, fetched on 3 September 2026, and every quotation from them above is verbatim from those sources. [2] and [3] are cited for the conjectures they close and, in the case of [3], for the notion of a neutral label; no statement of this paper depends on either. None of the three carries a journal reference or a publisher DOI on its arXiv abstract page, and a Crossref search on 3 September 2026 returned no published version of any of them, so no volume, page or journal data are quoted. [4] is cited as the one paper found to cite [1] and for the statement that it does not treat the segment conjecture, and [5] only as the source's own attribution of its Theorem 1; for these two, the publication data printed were verified on 3 September 2026 against Crossref, the publisher's page and zbMATH.

References

[1] J. A. Fill, *An interesting spectral gap problem, from Jim Fill*, arXiv:2508.12557v1 [math.PR], cross-listed math.CO, MSC 60J10; 8 pages, submitted 18 August 2025. A combination of two sets of unpublished notes of 2003, posted — in the author's words "essentially unedited and not updated" — at László Babai's request. Read here in full from the arXiv e-print of v1; the abs page was re-fetched on 3 September 2026 (still v1, not withdrawn, no journal reference). The letters (a)–(d) used above for the items of its section "Stronger conjectures" are the source's own, and so is the number of its Theorem 1, which is hand-numbered in the compiled e-print.

[2] G. R. W. Greaves and H. Zhu, *Spectral gap of biased adjacent-transposition chains*, arXiv:2603.26303v2 [math.PR], cross-listed math.CO; 22 pages, v1 submitted 27 March 2026 and v2 on 7 April 2026 with the comment "added a characterisation of the case of equality". (Under its v1 title, *A proof of Fill's spectral gap conjecture*, it is the reference cited as such in [3].) No journal reference as of 3 September 2026. Cited for the proof of Fill's Gap Problem and, in v2, for the characterisation of the minimisers and the multiplicity formula.

[3] V. Jain and C. Mizgerd, *Equality in Fill's spectral gap problem*, arXiv:2604.03937v1 [math.CO], cross-listed math.PR; submitted 5 April 2026, v1 the only version, no journal reference as of 3 September 2026. Cited for the characterisation of the minimisers of the gap (the neutral-label condition) and for the multiplicity formula.

[4] R. Gheissari, H. Lee and E. Vigoda, *Mixing of general biased adjacent transposition chains*, arXiv:2511.02725 [math.PR], cross-listed cs.DS; v1 submitted 4 November 2025, v2 18 August 2026 (the version read here). Also in: Proceedings of the 58th Annual ACM Symposium on Theory of Computing (STOC '26), ACM, 2026, 1236–1241, doi:10.1145/3798129.3800834 (data from the Crossref record of that DOI; the arXiv abstract page lists no journal reference). Cited as the one paper found that cites [1]; it concerns the mixing time and does not treat the segment conjecture.

[5] D. B. Wilson, *Mixing times of lozenge tiling and card shuffling Markov chains*, Ann. Appl. Probab. **14** (2004), no. 1, 274–325, doi:10.1214/aoap/1075828054. Cited only because [1] attributes its Theorem 1 to this paper. The reference list of [1] prints the page range as 275–325; the range 274–325 printed here is the journal's own (the publisher's page, zbMATH and the paper's arXiv journal reference, all read on 3 September 2026).

[6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.

Appendix A. The verified statements

What follows is, for each proposition, lemma and theorem of the paper, the statement of every Lean declaration that carries it, reproduced verbatim from the source of the development (module name in parentheses; proofs, docstrings and definitions omitted; lines longer than the page are wrapped, with no other change). The names of the definitions are as in the text: `K3`, `Kact`, `w3`, `Dpar` ($= D$), `Yact` ($= Y$), `vbeta` ($= v_\beta$), `gap3` ($= \lambda_K$ at $n = 3$), `IsEigen`, `Regular3`, `Mcurv` ($= M$), `gapSeg` ($= g_M$); and at $n = 4$ `K4act` ($= K_q$), `w4`, `IsEigen4`, `iT`, `iS`, `iU`, `iV`, `dT`, `dS`, `dU`, `dV` ($= \iota_*, \delta_*$), `BT`, `BS`, `BU`, `Dtriv`, `Dsgn` ($= \Delta_T, \Delta_S$), `Pquart` ($= P$), `pM00`, `pM01` ($= m_{11}, m_{12}$), `stdVecOf`, `stdVec` ($= u, u^\star$), `beta4`, `gap4`, `qseg`, `qsegF` ($= q(t)$ for $q_0 = \frac{199}{200}$ and $\frac{99}{100}$), `perm4`, `nb4`, `swapPos`, `pcoef`. Indices of coordinates are 0-based in Lean and 1-based in the text.

Proposition 2.1 (module `Chain3`).

```
theorem K3_mulVec (a b c : ℝ) (v : Fin 6 → ℝ) : K3 a b c *ᵥ v = Kact a b c v

theorem K3_row_sum (a b c : ℝ) (i : Fin 6) : ∑ j, K3 a b c i j = 1

theorem K3_reversible (a b c : ℝ) (u v : Fin 6 → ℝ) :
    ∑ i, w3 a b c i * u i * (Kact a b c v) i
      = ∑ i, w3 a b c i * (Kact a b c u) i * v i
```

Theorem 3.1 (module `Chain3`).

```
theorem Yact_sq (a b c : ℝ) (v : Fin 6 → ℝ) :
    Yact a b c (Yact a b c v) = (Dpar a b c - 1) • Yact a b c v

theorem eig_poly {a b c μ : ℝ} (h : IsEigen a b c μ) :
    μ * (μ - 1) * (4*μ^2 - 4*μ + 1 - Dpar a b c) = 0

theorem Kact_vbeta {a b c s : ℝ} (hs : s^2 = Dpar a b c) :
    Kact a b c (vbeta a b c s) = ((1+s)/2) • vbeta a b c s

theorem K3_second_eigenvalue {a b c : ℝ} (ha : 0 < a) (ha1 : a < 1) (hb : 0 < b)
    (hb1 : b < 1) (hc : 0 < c) (hc1 : c < 1) :
    IsGreatest {μ : ℝ | IsEigen a b c μ ∧ μ < 1} (1 - gap3 a b c)
```

Lemma 4.1 (module `Segment`).

```
theorem segD (a b c t : ℝ) :
    Dpar ((1-t)*a + t/2) ((1-t)*b + t/2) ((1-t)*c + t/2)
      = 1/4 - (1-t)^2 * Mcurv a b c
```

Lemma 4.2 (module `Segment`).

```
theorem Mcurv_nonneg {a b c : ℝ} (h : Regular3 a b c) : 0 ≤ Mcurv a b c
```

Theorem 5.1 (modules `Main3`, `Segment`).

```
theorem fill_conjecture_d_n3 {a b c : ℝ} (h : Regular3 a b c) :
    (∀ t ∈ Set.Icc (0:ℝ) 1,
        IsGreatest {μ : ℝ | IsEigen ((1-t)*a + t/2) ((1-t)*b + t/2) ((1-t)*c + t/2) μ
          ∧ μ < 1} (1 - gapSeg (Mcurv a b c) t))
      ∧ AntitoneOn (gapSeg (Mcurv a b c)) (Set.Icc 0 1)
      ∧ ConvexOn ℝ (Set.Icc 0 1) (gapSeg (Mcurv a b c))

theorem gapSeg_antitoneOn {M : ℝ} (hM : 0 ≤ M) (hM4 : M ≤ 1/4) :
    AntitoneOn (gapSeg M) (Set.Icc 0 1)

theorem gapSeg_convexOn {M : ℝ} (hM : 0 ≤ M) (hM4 : M ≤ 1/4) :
    ConvexOn ℝ (Set.Icc 0 1) (gapSeg M)
```

Theorem 5.2 (modules `Main3`, `Segment`).

```
theorem fill_d_n3_iff (a b c : ℝ) :
    (0 ≤ Mcurv a b c ∧ Mcurv a b c ≤ 1/4 → AntitoneOn (gapSeg (Mcurv a b c)) (Set.Icc 0 1))
      ∧ (Mcurv a b c < 0 → StrictMonoOn (gapSeg (Mcurv a b c)) (Set.Icc 0 1))

theorem gapSeg_strictMonoOn_of_neg {M : ℝ} (hM : M < 0) :
    StrictMonoOn (gapSeg M) (Set.Icc 0 1)
```

```
theorem gapSeg_midpoint_concave_of_neg {M x y : ℝ} (hM : M < 0) (hx : 0 ≤ x)
    (hy : y ≤ 1) (hxy : x < y) :
    (gapSeg M x + gapSeg M y)/2 < gapSeg M ((x+y)/2)
```

Theorem 5.3 (modules `Main3`, `Segment`).

```
theorem fill_d_n3_equality {a b c : ℝ} (h : Regular3 a b c) :
    (Mcurv a b c = 0 ↔ (a = 1/2 ∧ b = 1/2))

theorem fill_d_n3_strict {a b c : ℝ} (h : Regular3 a b c) (hne : ¬ (a = 1/2 ∧ b = 1/2)) :
    StrictAntiOn (gapSeg (Mcurv a b c)) (Set.Icc 0 1)

theorem gapSeg_strictAntiOn {M : ℝ} (hM : 0 < M) (hM4 : M ≤ 1/4) :
    StrictAntiOn (gapSeg M) (Set.Icc 0 1)

theorem gapSeg_const {M : ℝ} (hM : M = 0) (t : ℝ) : gapSeg M t = gapSeg M 0
```

Theorem 6.1 (module `Main3`).

```
theorem control_wrong_target :
    Regular3 (3/4) (3/4) (3/4) ∧ Regular3 (1/2) (1/2) (7/8)
      ∧ (gap3 (3/4) (3/4) (3/4) + gap3 (1/2) (1/2) (7/8))/2
          < gap3 (5/8) (5/8) (13/16)
```

Proposition 6.2 (module `Main3`).

```
theorem control_regular_positive :
    Regular3 (3/4) (3/4) (3/4) ∧ 0 < Mcurv (3/4) (3/4) (3/4)

theorem control_regular_zero :
    Regular3 (1/2) (1/2) (3/4) ∧ Mcurv (1/2) (1/2) (3/4) = 0
      ∧ ∀ t : ℝ, gapSeg (Mcurv (1/2) (1/2) (3/4)) t = gapSeg (Mcurv (1/2) (1/2) (3/4)) 0

theorem control_nonregular_increasing :
    ¬ Regular3 (9/10) (9/10) (1/10)
      ∧ Mcurv (9/10) (9/10) (1/10) = -12/25
      ∧ StrictMonoOn (gapSeg (Mcurv (9/10) (9/10) (1/10))) (Set.Icc 0 1)
```

Proposition 7.1 (module `Refute4`).

```
theorem perm4_injective : Function.Injective perm4

theorem perm4_bijective (x : Fin 24) : Function.Bijective (perm4 x)

theorem perm4_nb4 (x : Fin 24) (r : Fin 3) (i : Fin 4) :
    perm4 (nb4 x r) i = perm4 x (swapPos r i)

theorem K4act_of_rule (q : ℝ) (v : Fin 24 -> ℝ) (x : Fin 24) :
    K4act q v x =
      (∑ r : Fin 3, (pcoef q (perm4 x r.succ) (perm4 x r.castSucc) * v (nb4 x r)
        + (1 - pcoef q (perm4 x r.succ) (perm4 x r.castSucc)) * v x)) / 3

theorem K4act_stochastic (q : ℝ) : K4act q (fun _ => (1:ℝ)) = fun _ => (1:ℝ)

theorem K4_reversible (q : ℝ) (u v : Fin 24 → ℝ) :
    ∑ i, w4 q i * u i * (K4act q v) i = ∑ i, w4 q i * (K4act q u) i * v i

theorem K4_stationary (q : ℝ) (v : Fin 24 → ℝ) :
    ∑ i, w4 q i * (K4act q v) i = ∑ i, w4 q i * v i
```

Lemma 7.2 (module `Refute4`).

```
theorem dT_K (q : ℝ) (v : Fin 24 → ℝ) : dT (K4act q v) = BT q (dT v)

theorem dS_K (q : ℝ) (v : Fin 24 → ℝ) : dS (K4act q v) = BS q (dS v)

theorem dU_K (q : ℝ) (v : Fin 24 → ℝ) : dU (K4act q v) = BU q (dU v)
```

```
theorem dV_K (q : ℝ) (v : Fin 24 → ℝ) : dV (K4act q v) = BU q (dV v)

theorem K_iU (q : ℝ) (a : Fin 8 → ℝ) : K4act q (iU a) = iU (BU q a)

theorem dU_iU (a : Fin 8 → ℝ) : dU (iU a) = a

theorem decomp4 (v : Fin 24 → ℝ) :
    iT (dT v) + iS (dS v) + iU (dU v) + iV (dV v) = v
```

Lemma 7.3 (module `Refute4`).

```
theorem BT_ann {q μ : ℝ} (hq : q ≠ 0) {a : Fin 4 → ℝ} (h : BT q a = μ • a) (k : Fin 4) :
    Dtriv q μ * a k = 0

theorem BS_ann {q μ : ℝ} (hq : q ≠ 0) {a : Fin 4 → ℝ} (h : BS q a = μ • a) (k : Fin 4) :
    Dsgn q μ * a k = 0

theorem BU_ann {q μ : ℝ} (hq : q ≠ 0) {a : Fin 8 → ℝ} (h : BU q a = μ • a) (k : Fin 8) :
    Pquart (q*(1-q)) ((6*μ-3)^2) * a k = 0
```

Theorem 7.4 (module `Refute4`).

```
theorem spectrum4 {q μ : ℝ} (hq : q ≠ 0) (h : IsEigen4 q μ) :
    Dtriv q μ * Dsgn q μ * Pquart (q*(1-q)) ((6*μ-3)^2) = 0
```

Theorem 7.5 (module `Refute4`).

```
theorem BU_stdVecOf {q μ x y : ℝ}
    (h6 : (25/324 - 7/108*q + 2/27*q^2 - 1/81*q^3 - 5/9*μ + 2/9*μ*q - 2/9*μ*q^2 + 14/9*μ^2 - 2/9*μ^2*q
        + 2/9*μ^2*q^2 - 2*μ^3 + μ^4) * x
        + (35/648 - 1/24*q + 1/27*q^2 + 1/162*q^3 - 37/108*μ + 2/27*μ*q - 2/27*μ*q^2 + 13/18*μ^2 -
        1/2*μ^3) * y = 0)
    (h7 : (43/648 - 7/216*q + 1/27*q^2 - 1/162*q^3 - 43/108*μ + 2/27*μ*q - 2/27*μ*q^2 + 7/9*μ^2 -
        1/2*μ^3) * x
        + (89/1296 - 1/18*q + 5/108*q^2 + 1/81*q^3 - 19/36*μ + 2/9*μ*q - 2/9*μ*q^2 + 55/36*μ^2 -
        2/9*μ^2*q + 2/9*μ^2*q^2 - 2*μ^3 + μ^4) * y = 0) :
    BU q (stdVecOf q μ x y) = μ • stdVecOf q μ x y

theorem BU_stdVec {q μ : ℝ} (h : Pquart (q*(1-q)) ((6*μ-3)^2) = 0) :
    BU q (stdVec q μ) = μ • stdVec q μ

theorem isEigen_of_quart {q μ : ℝ} (hq : q ≠ 0) (hM : pM01 q μ ≠ 0)
    (h : Pquart (q*(1-q)) ((6*μ-3)^2) = 0) : IsEigen4 q μ
```

Lemma 7.8 (module `Refute4`).

```
theorem beta4_isLUB {q : ℝ} (hne : Set.Nonempty {μ : ℝ | IsEigen4 q μ ∧ μ < 1}) :
    IsLUB {μ : ℝ | IsEigen4 q μ ∧ μ < 1} (beta4 q)

theorem qseg_regular {t : ℝ} (ht : 0 ≤ t) (ht1 : t ≤ 1) : 1/2 ≤ qseg t ∧ qseg t < 1

theorem qsegF_regular {t : ℝ} (ht : 0 ≤ t) (ht1 : t ≤ 1) : 1/2 ≤ qsegF t ∧ qsegF t < 1
```

Theorem 7.9 (module `Refute4`).

```
theorem beta4_bracket_q1 :
    (834441/1000000 : ℝ) ≤ beta4 (199/200) ∧ beta4 (199/200) ≤ 834451/1000000

theorem beta4_bracket_q2 :
    (8401/10000 : ℝ) ≤ beta4 (97/100) ∧ beta4 (97/100) ≤ 840109/1000000

theorem beta4_bracket_q3 :
    (422901/500000 : ℝ) ≤ beta4 (189/200) ∧ beta4 (189/200) ≤ 211453/250000
```

Theorem 7.10 (module `Refute4`).

```
theorem fill_d_convexity_false_n4 :
    1/80000 ≤ gap4 (qseg (5/99)) - (gap4 (qseg 0) + gap4 (qseg (10/99)))/2

theorem fill_d_violation_bracket :
    1/80000 ≤ gap4 (qseg (5/99)) - (gap4 (qseg 0) + gap4 (qseg (10/99)))/2 ∧
      gap4 (qseg (5/99)) - (gap4 (qseg 0) + gap4 (qseg (10/99)))/2 ≤ 63/2000000

theorem fill_d_convexity_false_n4_at_99_100 :
    177/10000000 ≤ gap4 (qsegF (9/200)) - (gap4 (qsegF 0) + gap4 (qsegF (9/100)))/2
```

Proposition 7.11 (module `Refute4`).

```
theorem control_midpoint_bound_sharp :
    ¬ (∀ μ : ℝ, IsEigen4 (97/100) μ → μ < 1 → μ ≤ 8401/10000)

theorem gap4_decreasing_at_the_three_points :
    gap4 (qseg (10/99)) < gap4 (qseg (5/99)) ∧ gap4 (qseg (5/99)) < gap4 (qseg 0)

theorem control_spectrum_nonempty :
    IsLUB {μ : ℝ | IsEigen4 (199/200) μ ∧ μ < 1} (beta4 (199/200)) ∧
      IsLUB {μ : ℝ | IsEigen4 (97/100) μ ∧ μ < 1} (beta4 (97/100)) ∧
      IsLUB {μ : ℝ | IsEigen4 (189/200) μ ∧ μ < 1} (beta4 (189/200))
```