EIGENVALUES OF THE FIBONACCI–REDHEFFER MATRIX: THE SHARP ENCLOSURE $F_i < \lambda_i < F_i + 1$ FOR $3 \le n \le 64$

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ABSTRACT. For $n \ge 1$ let $F_R(n)$ be the $n \times n$ integer matrix with (i,j) entry 1 if $j = 1$, F_i if $i \mid j$, and 0 otherwise, where F_i is the i-th Fibonacci number: the Fibonacci-weighted analogue of Redheffer's matrix introduced by Doumas and Psarrakos. They prove that the eigenvalues $\lambda_1 < \cdots < \lambda_n$ of $F_R(n)$ are real and simple, that $-1 < \lambda_1 < 0$, and that $F_i < \lambda_i < F_{i+1}$ for $2 \le i \le n-1$; the *sharp* upper bound $\lambda_i < F_i + 1$ they obtain only for $i = 2$ and $i = 3$ and, when $n \ge 10$, for $i \ge \lfloor n/2 \rfloor + 2$, their Gersgorin bound on the middle band being $\lambda_i < F_i + 2$. We prove that for every n with $3 \le n \le 64$, every complex eigenvalue of $F_R(n)$ is real and lies in $(-1,0) \cup \bigcup_{i=2}^{n} (F_i, F_i + 1)$. Combined with their theorem this gives $-1 < \lambda_1 < 0$ and $F_i < \lambda_i < F_i + 1$ for every $i = 2, \ldots, n$ and every such n, settling the sharp bound at 916 pairs (n,i) that their results leave open. The proof is one certificate per n: an explicit integer polynomial of degree n that annihilates $F_R(n)$, together with $2n$ exact integer sign evaluations. The certificate check is machine-checked, so each n is a theorem rather than the report of a computation. We also record, as a note on quantifiers, that three statements of Doumas and Psarrakos need the hypothesis $n \ge 3$ they do not carry: $F_R(2)$ is the all-ones matrix, singular, with $(1,-1)$ an eigenvector for the eigenvalue 0. Their paper records the case $n = 2$ elsewhere, and its substance is unaffected. Controls show that the enclosing intervals can be neither halved nor shifted.

1. Introduction

1.1. The matrix. Let $F_0 = 0$, $F_1 = F_2 = 1$, $F_{i+1} = F_i + F_{i-1}$ be the Fibonacci numbers. For $n \ge 1$ the *Fibonacci–Redheffer matrix* $F_R(n)$ is the $n \times n$ integer matrix with rows and columns indexed by $1, \ldots, n$ and entries

$$F_R(n)_{ij} = \begin{cases} 1 & \text{if } j = 1, \\ F_i & \text{if } i \mid j, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where the first line takes precedence at $(i,j) = (i,1)$. It was introduced by Doumas and Psarrakos [1]. Its zero pattern is that of Redheffer's matrix R_n — ones in the first column and at the divisibility positions $i \mid j$, zeros elsewhere, the matrix whose determinant is the Mertens function — with the ones of row i replaced by F_i. For instance

$$F_R(8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 13 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \end{pmatrix}.$$

The diagonal of $F_R(n)$ is $(F_1, \ldots, F_n)$, and this is what separates it from the weighted Redheffer matrices of Cardon's framework (§1.4): those carry their weights in the first column only and have a *unit* diagonal, so that 1 is an eigenvalue of multiplicity $n - \lfloor \log_2 n \rfloor - 1$, whereas $F_R(n)$ has n distinct real eigenvalues, one near each Fibonacci number.

The determinant of $F_R(n)$ is computed in [1]:

$$\det F_R(n) = \Big(\prod_{i\le n} F_i\Big) \sum_{k\le n} \frac{\mu(k)}{F_k}, \tag{2}$$

μ being the Möbius function. This is not new mathematics: factoring F_i out of row i turns $F_R(n)$ into a weighted Redheffer matrix with unit diagonal and first column $(1/F_1,\dots,1/F_n)^{\mathsf{T}}$, and (2) becomes Redheffer's weighted determinant identity [2], in the form $\det B_n = \sum_{k\le n} w_k\mu(k)$ recorded by Cardon [3], with $w_k = 1/F_k$. The interesting part of $F_R(n)$ is its spectrum.

1.2. **What is proved in** [1]**, and what is left open.** Write $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ for the eigenvalues of $F_R(n)$ and put $\omega = \lfloor n/2 \rfloor$. We use throughout the following four results of [1], citing each by the number it carries in v2 and, for readability, by a mnemonic letter. All numbering in this paper is that of v2, resolved by compiling the e-print's own LaTeX source; [1] numbers theorems, lemmas, corollaries and remarks in a single sequence, so its Remark 15 is a numbered item of that sequence even though its own text never refers to that number.

(B) (*Bolzano*; [1, Theorem 14]) All eigenvalues of $F_R(n)$ are real and simple, and
$$\lambda_1 < 0, \qquad F_i < \lambda_i < F_{i+1} \quad (i = 2,\dots,n-1), \qquad \lambda_n > F_n.$$
(G) ([1, Remark 15]) For $n \ge 10$, Gersgorin discs give $F_i < \lambda_i < F_i + 1$ for $i = \omega+2,\dots,n$; rows of index $i > n/2$ have off-diagonal row sum 1, because such an i has no multiple j with $i < j \le n$.
(L) ([1, Theorem 16]) $-1 < \lambda_1 < 0$, proved by showing $\det(F_R(n)+I) > 0$.
(F) ([1, Corollary 18]) For $n \ge 10$: $-1 < \lambda_1 < 0$, $1 < \lambda_2 < 2$, $F_i < \lambda_i < F_i + 2$ for $3 \le i \le \omega+1$, and $F_i < \lambda_i < F_i + 1$ for $\omega + 2 \le i \le n$.

The natural sharp statement — that each λ_i sits in the unit interval to the right of F_i — was raised as an open question in v1 of [1], at the close of a remark, where it reads

> "However, we believe (and this is an open question) that $-1 < \lambda_1 < 0$ and $F_i < \lambda_i < F_i + 1$, $i = 2, 3, \dots, n$."

In v1 the two inequalities are set as displayed, numbered equations rather than in line, and are referred to a page later as "our conjectures". That whole passage is *deleted* in v2, which instead proves the first of the two outright, as (L). What (B), (G) and (F) leave of the second is the following. The bound $\lambda_i < F_i + 1$ follows from (B) exactly for $i = 2$ and $i = 3$, and only when $i \le n-1$, because $F_3 = F_2 + 1$ and $F_4 = F_3 + 1$ while $F_{i+1} = F_i + F_{i-1} \ge F_i + 2$ for $i \ge 4$; and it follows from (G) for $i \ge \omega + 2$, but only when $n \ge 10$. So the sharp upper bound is open for

$$4 \le i \le \omega+1 \quad (n \ge 10), \qquad \text{and} \qquad 4 \le i \le n, \text{ together with } i = n = 3, \quad (3 \le n \le 9), \tag{3}$$

where the best previously available upper bound is $F_i + 2$ on the middle band when $n \ge 10$, is F_{i+1} from (B) when $i \le n-1$, and is nothing at all at the top index $i = n$ when $n \le 9$. This is the gap that the present paper closes for $n \le 64$.

1.3. **Results.**

(1) For every n with $3 \le n \le 64$, every complex eigenvalue of $F_R(n)$ is real and lies in $(-1,0) \cup \bigcup_{i=2}^{n} (F_i, F_i + 1)$ (Theorem 4.1).
(2) Together with (B) this gives, for every such n, the sharp enclosure $-1 < \lambda_1 < 0$ and $F_i < \lambda_i < F_i + 1$ for all $i = 2,\dots,n$ (Corollary 4.3). In particular the whole band (3) is settled for $3 \le n \le 64$: 916 pairs (n,i) at which [1] does not give the sharp bound.
(3) The proof is a certificate per n, and its soundness is a single elementary proposition (Proposition 3.1) in which nothing is assumed about the procedure that produces the certificate.
(4) The hypothesis $n \ge 3$ in (1) is necessary, and the same hypothesis is missing from three statements of [1]: $F_R(2)$ is the all-ones matrix, singular, with $(1,-1)$ an eigenvector for the eigenvalue 0 (Proposition 6.1). The substance of [1] is unaffected — what is missing is a quantifier. That paper states its own determinant-sign theorem correctly for $n \ge 3$ and

records the value of $\det F_R(2)$ explicitly elsewhere, so this is a note on three statements, not on the mathematics.

(5) Controls: the enclosing intervals cannot be halved, nor shifted by one, and the certificate is matrix-specific (Proposition 7.1).

Every numbered statement of §§3–7 has been machine-checked in Lean 4 [12] against Mathlib [13], with two exceptions that are flagged where they occur: Corollaries 4.3 and 4.4, which import theorem (B) of [1] from the literature. §8 says precisely which claims rest on exhaustive computation and how large each computation was, and §9 says what the machine checks and what it does not. Prose between the numbered statements is not covered by that claim, and where such a passage carries mathematical weight — three remarks, the deduction that the spectrum of $F_R(2)$ is exactly $\{0, 2\}$, and one paragraph of §10 — it is labelled as being outside the formal development.

1.4. **Related work.** The spectrum of Redheffer's own matrix R_n is well studied and looks nothing like the picture above. Barrett, Forcade and Pollington [4] obtained its characteristic polynomial, in which the eigenvalue 1 has multiplicity $n - \lfloor \log_2 n \rfloor - 1$, and showed that the spectral radius satisfies $\rho(R_n) \sim n^{1/2}$; Jarvis [5] studied a dominant negative eigenvalue of R_n; Barrett and Jarvis [6] refined the asymptotics of the two dominant eigenvalues $\lambda_\pm \sim \pm n^{1/2}$, showed that the remaining $\lfloor \log_2 n \rfloor - 1$ small eigenvalues satisfy $|\lambda| < \log_{2-\varepsilon} n$ for every $\varepsilon > 0$ and n large, and conjectured for those small eigenvalues that $|\lambda| < 1$ and that $\operatorname{Re} \lambda < 1$ — both parts of which Cardon [3] disproved. Vaughan [7] proved $|\lambda| \ll (\log n)^{2/5}$ unconditionally for the small eigenvalues and $\ll \log \log(2 + n)$ under the Riemann hypothesis, and later [8] showed that R_n has nontrivial eigenvalues arbitrarily close to 1 for large n. Cheon and Kim [9] study Riordan–Redheffer matrices and state two eigenvalue conjectures; Clément and Steinerberger [11] study the largest singular vector of R_n. The general framework for weighted Redheffer matrices is Cardon's [3]: the entry at (i, j) is $a_{j/i}$ when $i \mid j$, the weights $w_2, \ldots, w_n$ are confined to the first column, and $a_1 = 1$ is a standing assumption there, so the diagonal is $(1, \ldots, 1)$, the determinant is $\sum_{k \le n} w_k b_k$ with b the Dirichlet inverse of a, and the characteristic polynomial is divisible by $(x - 1)^{n - \lfloor \log_2 n \rfloor - 1}$ whatever the weights are. $F_R(n)$, whose diagonal is $(F_1, \ldots, F_n)$, is not a matrix of that shape; it becomes one only after the row scaling used in §1 for the determinant, and that scaling does not preserve the spectrum. A second generalisation along the same Dirichlet-convolution lines is an undergraduate honours thesis of Gillespie [10]; we could not obtain its text and therefore say nothing about the shape of the matrices it treats. None of the works above gives a unit-width enclosure of an individual indexed eigenvalue, and for R_n and for Cardon's family nothing of that shape could hold: there the eigenvalue 1 carries all but $\lfloor \log_2 n \rfloor + 1$ of the multiplicity, whereas $F_R(n)$ has n distinct real eigenvalues, one near each Fibonacci number.

2. The intervals

Fix $n \ge 3$ and define n open intervals

$$W_1 = (-1, 0), \qquad W_i = (F_i, F_i + 1) \quad (2 \le i \le n). \tag{4}$$

Thus $W_2 = (1, 2)$, $W_3 = (2, 3)$, $W_4 = (3, 4)$, $W_5 = (5, 6)$, $W_6 = (8, 9)$, and so on: the intervals W_2, W_3, W_4 abut, and from there the gaps grow like the Fibonacci numbers. All endpoints are integers, which is what makes the sign conditions below exact integer arithmetic.

Lemma 2.1. *The intervals $W_1, \ldots, W_n$ are pairwise disjoint, and $\sup W_i \le \inf W_{i+1}$ for $1 \le i \le n-1$.*

Proof. $\sup W_1 = 0 < 1 = F_2 = \inf W_2$, and for $2 \le i \le n - 1$, $\sup W_i = F_i + 1 \le F_i + F_{i-1} = F_{i+1} = \inf W_{i+1}$ since $F_{i-1} \ge 1$. Open intervals arranged in this order are pairwise disjoint. □

The union $W_1 \cup \cdots \cup W_n$ has exactly n components and total length n, so it has room for the n eigenvalues and nothing more; this is why a containment statement is already sharp.

3. Certificates for localisation

Everything we prove about the spectrum of $F_R(n)$ goes through one elementary observation: an annihilating polynomial with the right sign pattern at the interval endpoints localises the whole spectrum, and it does so with no assumption about where the polynomial came from. We write P_c for the polynomial with integer coefficient list $c = (c_0, \dots, c_d)$, that is $P_c(x) = c_0 + c_1 x + \cdots + c_d x^d$, and $P_c(A)$ for its value at a square matrix A, computed by Horner's rule.

Proposition 3.1 (Soundness of the certificate)**.** *Let $n \geq 1$, let A be an $n \times n$ integer matrix, let c be a list of at most $n+1$ integers, and let $l_1, h_1, \dots, l_n, h_n$ be integers such that*

(i) $P_c(A) = 0$*;*
(ii) $l_i < h_i$ *for* $1 \leq i \leq n$*, and* $h_i \leq l_{i+1}$ *for* $1 \leq i \leq n-1$*;*
(iii) $P_c(l_i)\, P_c(h_i) < 0$ *for* $1 \leq i \leq n$*.*

Then every $r \in \mathbb{C}$ for which there is a $v \in \mathbb{C}^n$, $v \neq 0$, with $Av = rv$ is real and satisfies $l_i < r < h_i$ for some i.

Proof. If $Av = rv$ with $v \neq 0$ then $P_c(A)v = P_c(r)v$, so (i) gives $P_c(r)v = 0$ and hence $P_c(r) = 0$. By (iii) with $i = 1$, P_c is not the zero polynomial. By (iii) and the intermediate value theorem, applied to P_c as a continuous real function on $[l_i, h_i]$, there is a real $y_i \in (l_i, h_i)$ with $P_c(y_i) = 0$, for each $i = 1, \dots, n$; by (ii) these n intervals are pairwise disjoint, so $y_1 < y_2 < \cdots < y_n$ are n distinct roots of P_c in $\mathbb{C}$. As c has at most $n+1$ entries, $\deg P_c \leq n$, so a nonzero P_c has at most n roots in $\mathbb{C}$; therefore $\{y_1, \dots, y_n\}$ is the entire root set and $r = y_i$ for some i. □

Two features of Proposition 3.1 matter for what follows. First, the coefficient list c is a *hypothesis*, not a construction: a list produced by any procedure whatsoever can be offered, and a wrong list simply fails condition (i) or (iii). Nothing in the proof asserts that P_c is the characteristic polynomial of A — see Remark 4.2. Second, conditions (ii) and (iii) are finitely many comparisons between integers: for a given A and c they are decided by exact integer arithmetic, with no floating point anywhere.

For $A = F_R(n)$ we always take l_i, h_i to be the endpoints of the intervals (4), that is $(l_1, h_1) = (-1, 0)$ and $(l_i, h_i) = (F_i, F_i + 1)$ for $2 \leq i \leq n$; condition (ii) is then Lemma 2.1.

4. The window $3 \leq n \leq 64$

Theorem 4.1 (Localisation)**.** *Let n be an integer with $3 \leq n \leq 64$, let $r \in \mathbb{C}$ and let $v \in \mathbb{C}^n$, $v \neq 0$, satisfy $F_R(n)v = rv$. Then r is real and*

$$r \in (-1, 0) \cup \bigcup_{i=2}^{n} \bigl(F_i,\ F_i + 1\bigr).$$

Proof. For each of the 62 values of n a coefficient list $c^{(n)} = (c_0^{(n)}, \dots, c_n^{(n)})$ of $n+1$ integers is exhibited, and conditions (i)–(iii) of Proposition 3.1 are verified for $A = F_R(n)$, $c = c^{(n)}$ and the endpoints of (4). Condition (i) is one exact evaluation of a Horner chain of n integer matrix products; condition (iii) is $2n$ exact evaluations of an integer polynomial at an integer point; condition (ii) is Lemma 2.1, and is also checked directly for each n. The lists are produced outside the verified development, by exact Lagrange interpolation of $\chi_n(z) = \det(zI - F_R(n))$ through $n+1$ integer nodes, each node value being an exact fraction-free integer determinant; but the proof uses nothing about their provenance, only (i)–(iii). The sizes involved are recorded in §8. □

The certificate is an annihilating polynomial rather than a determinant on purpose: the verification never has to evaluate the characteristic polynomial of an $n \times n$ symbolic matrix, only to multiply integer matrices.

Remark 4.2. Theorem 4.1 is a *containment* statement. It does not assert that each interval W_i contains an eigenvalue, and its proof does not establish that $P_{c^{(n)}}$ is the characteristic polynomial of $F_R(n)$: the identity $P_{c^{(n)}}(F_R(n)) = 0$ alone does not give that, since it is also satisfied by any multiple of the minimal polynomial. See §10 for what would be needed. The counting in Proposition 3.1 runs the other way: the n roots forced into the intervals exhaust the root set because $\deg P_c \leq n$.

Corollary 4.3 (The sharp enclosure)**.** *Let $3 \leq n \leq 64$ and let $\lambda_1 < \cdots < \lambda_n$ be the eigenvalues of $F_R(n)$. Then, granting theorem (B) of* [1]*,*

$$-1 < \lambda_1 < 0 \qquad \textit{and} \qquad F_i < \lambda_i < F_i + 1 \quad (i = 2, \ldots, n).$$

Proof. By (B) the eigenvalues are real and simple, so there are n of them and the indexing makes sense. By Theorem 4.1 each λ_i lies in some interval $W_{\sigma(i)}$ of (4); $\sigma(i)$ is unique by Lemma 2.1. Since $\lambda_1 < 0$ by (B) and $W_j \subseteq (1, \infty)$ for $j \geq 2$, we get $\sigma(1) = 1$, i.e. $-1 < \lambda_1 < 0$. Let $2 \leq i \leq n$. Then $\lambda_i > F_i > 0$ by (B), so $\sigma(i) \geq 2$ and $F_{\sigma(i)} + 1 > \lambda_i > F_i$, whence $F_{\sigma(i)} \geq F_i$ and therefore $\sigma(i) \geq i$, the sequence $F_2 < F_3 < \cdots$ being strictly increasing. If $i \leq n-1$ then $\lambda_i < F_{i+1}$ by (B), and $\lambda_i > F_{\sigma(i)}$ gives $F_{\sigma(i)} < F_{i+1}$, so $\sigma(i) \leq i$. If $i = n$ then $\sigma(n) \leq n$ trivially. Hence $\sigma(i) = i$ for all i, which is the assertion. ☐

Corollary 4.3, and Corollary 4.4 below, are *not* part of the machine-checked development: they combine Theorem 4.1, which is checked, with theorem (B) of [1], which we take from the literature and have not formalised.

Corollary 4.4. *For every n with $3 \leq n \leq 64$ the bound $\lambda_i < F_i + 1$ holds on the whole band* (3) *left open by* [1]*. Counted over that window, this is* 916 *pairs* (n, i) *at which the best previously available upper bound was $F_i + 2$ (for $4 \leq i \leq \omega + 1$, $n \geq 10$) or nothing beyond $\lambda_n > F_n$ and $\lambda_i < F_{i+1}$ (for $3 \leq n \leq 9$).*

The count is the cardinality of $\{(n, i) : 3 \leq n \leq 64,\ 2 \leq i \leq n\}$ minus the pairs covered by (B), (G) and (F), that is minus $i \in \{2, 3\}$ with $i \leq n-1$ and minus $i \geq \omega + 2$ with $n \geq 10$; no other pair (n, i) receives the bound $\lambda_i < F_i + 1$ from those three results. It is 1 pair at each of $n = 3, 4$, rising to 6 at $n = 9$, dropping to 3 at $n = 10$ where (G) becomes available, and reaching 30 at $n = 64$.

Remark 4.5. The enclosure is not an artefact of the small cases: at $n = 64$ the largest eigenvalue lies in $(F_{64}, F_{64} + 1)$, an interval of width 1 around a number of size 1.06×10^{13}, and the certificate that places it there is an integer polynomial of degree 64 whose constant term has 413 decimal digits.

5. The matrix, and its printed data

Before proving anything we checked that the object being studied is the one in [1].

Proposition 5.1. *The matrix defined by* (1) *agrees at $n = 8$, entry for entry, with the matrix $F_R(8)$ printed in* [1]*, and its determinants for $n = 1, \ldots, 6$ are*

$$1, \quad 0, \quad -1, \quad -3, \quad -21, \quad -138,$$

the values that formula (2) *of* [1] *predicts.*

Proof. Six exact integer determinants of matrices of side at most 6, and one entrywise comparison of 64 integers. ☐

Neither (2) nor the values in Proposition 5.1 are new: as noted in §1, (2) is Redheffer's weighted determinant identity [2, 3] after factoring F_i out of row i. We record them because the whole of §4 depends on (1) being the right matrix, and an unverified transcription is exactly the kind of error that a machine check does not catch by itself.

Remark 5.2 (outside the formal development). Two further reproductions were carried out with independent exact-arithmetic code and are *not* machine-checked. First, (2) was confirmed for $n = 1, \ldots, 25$, the determinants being 1, 0, -1, -3, -21, -138, -2034, -42714, -1452276, -77647500, -7033149900, and so on. Second, the eigenvalue table printed in the appendix of [1] for $n = 3, \ldots, 11$ was reproduced by bisection of the exact integer characteristic polynomial. Of its 63 cells, 59 agree with the printed three decimals under round-to-nearest, which is the convention the table follows;

four do not. Three of these are the entry for λ_3 at $n = 9, 10, 11$, printed as 2.189 against exact values 2.18836, 2.18830 and 2.18773, so that the printed digit is wrong under either rounding or truncation. The fourth is λ_2 at $n = 10$, printed as 1.163 against 1.163985; that one is a borderline cell, agreeing with the printed value under truncation and missing it by 1.5×10^{-5} under rounding. All four look like reporting slips rather than mathematical errors, and we record them only because the table was used as a check on our own implementation.

6. The hypothesis $n \geq 3$

Theorem 4.1 carries the hypothesis $n \geq 3$, and the hypothesis is doing real work.

Proposition 6.1. *$F_R(2) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $F_R(1) = (1)$. In particular:*

(1) $\det F_R(2) = 0$*, so* $F_R(2)$ *is singular, and* $\det F_R(1) = 1 > 0$*;*
(2) $(1, -1)^{\mathsf{T}}$ *is an eigenvector of* $F_R(2)$ *for the eigenvalue* 0*, and* $(1, 1)^{\mathsf{T}}$ *is an eigenvector for the eigenvalue* 2*;*
(3) *the conclusion of Theorem 4.1 is false at* $n = 2$*: the eigenvalue* 2 *lies in neither* $(-1, 0)$ *nor* $(F_2, F_2 + 1) = (1, 2)$*, being exactly the excluded right endpoint.*

Proof. The two matrices are read off (1); note that the clause $j = 1$ takes precedence, so every entry of $F_R(2)$ equals 1. Items (1) and (2) are then two determinants of side at most 2 and two matrix–vector products. Item (3) is the observation that $2 \notin (-1, 0) \cup (1, 2)$. $\square$

Part (2) exhibits two eigenvalues of $F_R(2)$ and nothing more; it does not by itself say that they are all of them. That last step is immediate, but it is not part of the formal development, so we mark it: a 2×2 matrix has exactly two eigenvalues counted with multiplicity, and $0 \neq 2$, so the spectrum of $F_R(2)$ is $\{0, 2\}$ and $\lambda_1(F_R(2)) = 0$.

Proposition 6.1 is a correction to three statements of [1], and we state it as small as it is. All three are made there without any restriction on n. The first is (B), that is [1, Theorem 14], which opens "All eigenvalues of the Fibonacci–Redheffer matrix $F_R(n)$ are real and simple" and, for "$\lambda_1 < \lambda_2 < \cdots < \lambda_n$ … the n simple eigenvalues of $F_R(n)$", concludes among other things that $\lambda_1 < 0$. The second is (L), [1, Theorem 16], stated for "the smallest eigenvalue λ_1 of the Fibonacci–Redheffer matrix $F_R(n)$" and concluding $-1 < \lambda_1 < 0$. The third is the first item in the list of "main pillars" on p. 14 of v2: "The determinant of $F_R(n)$ is negative, and thus, $F_R(n)$ is nonsingular and its eigenvalues are nonzero (Theorems 4 and 6)." All three are false as stated at $n = 2$, by Proposition 6.1, and at $n = 1$, where $\det F_R(1) = 1 > 0$ and the only eigenvalue is 1. Each needs the hypothesis $n \geq 3$.

What is missing is a quantifier and not an argument, and [1] is plainly aware of the exception. Its determinant-sign result, [1, Theorem 6], asserts $\sum_{k \leq n} \mu(k)/F_k < 0$ for $n \geq 3$ only, with the footnote "For $k_0 = 2$, the sum is zero"; and five pages before the "main pillars" list, on p. 9, it says outright of the matrix itself that it "is always nonsingular for $n \geq 3$ (also true for the trivial case $n = 1$, while for $n = 2$, the matrix has zero determinant)". So the case $n = 2$ is recorded in [1], and what fails to carry the restriction is the three statements above, all of which come after that sentence. The substance of [1] is unaffected, and Proposition 6.1 is offered as a machine-checked note on quantifiers rather than as a correction to the mathematics.

7. Controls

A certificate scheme that succeeded for structural reasons would prove nothing about $F_R(n)$, so we record that it does not.

Proposition 7.1. (1) The intervals cannot be halved. *At* $n = 3$ *the certificate polynomial* $P_{c^{(3)}}(x) = x^3 - 4x^2 + 3x + 1$ *takes the values* -1 *at* $2 = F_3$ *and* $-7/8$ *at* $5/2$*, of the same sign; so it has no sign change on* $\left(F_3, F_3 + \frac{1}{2}\right)$*.*

(2) The intervals cannot be shifted. *At* $n = 12$*, replacing every interval* (l_i, h_i) *by* $(l_i - 1, h_i - 1)$ *destroys condition (iii) of Proposition 3.1.*

(3) The certificate is matrix-specific. *The coefficient list* $c^{(12)}$ *annihilates* $F_R(12)$ *but does not annihilate the matrix obtained from* $F_R(12)$ *by increasing its* $(1,1)$ *entry by one.*

Proof. Three exact evaluations: two values of a cubic at rational points, 24 integer sign evaluations, and one Horner chain of 12 integer matrix products. □

Part (1) says that the width of the intervals in Theorem 4.1 cannot be reduced by tightening them on the left: the certificate of §3 simply fails on the halved intervals. More is true, though it is a paper argument rather than a machine check: the three roots of $P_{c^{(3)}}$ are simple, since a nonzero polynomial of degree at most 3 with three distinct roots has no others, and there is one in each interval, so the absence of a sign change on $\left(F_3, F_3+\frac{1}{2}\right)$ places the root of W_3 in the right half $\left(F_3+\frac{1}{2}, F_3+1\right)$. Part (2) says that the placement of the intervals is not an artefact of a coarse scheme in which any translate would do; part (3), that condition (i) is a real constraint on the matrix and not a property of the list alone. A fourth control is the non-vacuity of the hypothesis shape of Theorem 4.1: Proposition 6.1(2) exhibits an explicit eigenvector–eigenvalue pair, so the antecedent "$v \neq 0$ and $Av = rv$" is satisfiable.

8. The extent of the computation

We state exactly which claims rest on exhaustive computation, and how large each computation was.

- Proposition 3.1 and Lemma 2.1 rest on no computation; they are uniform in n.
- Theorem 4.1 is the exhaustive claim, and the search is over n, not over eigenvalues: for each of the 62 integers n with $3 \leq n \leq 64$, conditions (i)–(iii) of Proposition 3.1 are checked for that n. Condition (i) costs n integer matrix products, i.e. $\Theta(n^4)$ big-integer multiply–adds, on integers that grow with the coefficients of the certificate: at $n = 64$ the largest coefficient has 413 decimal digits. Condition (iii) is $2n$ evaluations of an integer polynomial of degree n at an integer point — 4154 integer sign evaluations over the whole window. The entire verified development, including the general machinery and all controls, checks in about 27 minutes of wall time on one core.
- The certificate lists themselves were produced outside the verified development, in about 15 minutes of exact rational arithmetic, by Lagrange interpolation of $\chi_n(z) = \det(zI - F_R(n))$ through $n+1$ consecutive integer nodes, the node values being fraction-free (Bareiss) integer determinants; integrality of every interpolated coefficient and monicity were asserted during that computation. None of this is part of any proof: a list that failed condition (i) or (iii) would simply be rejected.
- As an independent check on the interpolation, for every n with $2 \leq n \leq 20$ and each of the $2n$ interval endpoints m, the interpolated value $\chi_n(m)$ was compared against a directly computed exact determinant of $mI - F_R(n)$: 418 agreements between two independent routes.
- Proposition 5.1 is six determinants of side at most 6 and one comparison of 64 integers; Proposition 6.1 is two determinants of side at most 2 and two matrix–vector products; Proposition 7.1 is as described in its proof.
- The two reproductions reported in the remark of §5, and the Gersgorin observation of §10, are outside the formal development.

9. Verification

Proposition 3.1, Theorem 4.1 (as 62 separate statements, one per n), Proposition 5.1, Proposition 6.1 and Proposition 7.1 have been machine-checked in Lean 4 [12] against Mathlib [13]; in Proposition 6.1 the identity $F_R(1) = (1)$ is the only part not separately certified, being one unfolding of (1) at a 1×1 matrix. Lemma 2.1 appears in the development not as a statement about all n but as its 62 instances, the ordering conditions (ii) of Proposition 3.1 being decided outright for each n in the window. The matrix is formalised as what it is, an integer matrix defined by (1) with Mathlib's Fibonacci function, and the certified form of Theorem 4.1 quantifies over complex eigenvectors: for a fixed n, if $v \neq 0$ and $F_R(n)v = rv$ over $\mathbb{C}$, then r is the image of a real number lying strictly inside one of the intervals (4). The development separates the reasoning from the arithmetic. The whole of Proposition 3.1 — the passage from an eigenvector to a root, the intermediate value argument, the disjointness bookkeeping and the root count — and *all* 4154 integer sign evaluations of condition (iii), the entry comparison of Proposition 5.1, the two eigenvector statements and the failure of localisation in Proposition 6.1, and Proposition 7.1(1)–(2), are checked by the kernel alone, their axiom closure being the standard propositional extensionality, quotient soundness and choice. Compiled evaluation enters in exactly one place per n, namely condition (i), the identity $P_{c^{(n)}}(F_R(n)) = 0$; it is recorded by one axiom for each n, and there is nothing else of that kind in the localisation theorems. (The same device is used for the determinants of Propositions 5.1 and 6.1(1) and for Proposition 7.1(3).) There is no `sorry` anywhere in the development. Repository reference to be added at submission.

10. What remains

One eigenvalue per interval. By Remark 4.2, Theorem 4.1 does not by itself say that each W_i contains an eigenvalue; Corollary 4.3 obtains the indexed statement by importing theorem (B) of [1]. Making the development self-contained needs $P_{c^{(n)}} = \chi_n$, which follows from $P_{c^{(n)}}(F_R(n)) = 0$ together with the assertion that no polynomial of degree $< n$ annihilates $F_R(n)$; the latter is certified by a Krylov matrix K with $\operatorname{adj}(K)K = (\det K)I$ and $\det K \neq 0$. Over $\mathbb{Z}$ that certificate is impractical — the adjugate entries already have about 13 000 decimal digits at $n = 40$ — but modulo a prime it is small, and $\det K \neq 0$ modulo one prime suffices. We estimate this at 150–250 lines of work with the minimal and characteristic polynomial machinery of [13]; it is not attempted here.

Widening the window. Nothing structural stops the same certificates at larger n. The cost of condition (i) is $\Theta(n^4)$ big-integer operations with coefficients of $\Theta(n^2)$ digits, so $n = 65, \ldots, 100$ is priced at roughly one core-hour of checking plus a few minutes of interpolation.

A uniform proof. What one wants is Corollary 4.3 for all n, and the natural attempt — Gersgorin discs after a positive diagonal similarity — cannot give it. Indeed, let $S = \operatorname{diag}(s_1, \ldots, s_n)$ with $s_i > 0$. In $S^{-1}F_R(n)S$ the off-diagonal part of row $i \geq 2$ contains the entry s_1/s_i coming from the first column, so a disc of radius < 1 at row i forces $s_i > s_1$; but row 1 of $F_R(n)$ is all ones, so its disc has radius $\sum_{j\geq 2} s_j/s_1 > n-1$, and that disc contains the centres of all the others. No diagonal scaling can therefore separate the spectrum into unit intervals, and a uniform argument must use more than row sums — the natural candidate being the function $Q_{F_R(n)}$ that [1] introduces in the discussion establishing (B), whose real roots are exactly the eigenvalues of $F_R(n)$, together with a quantitative bound on $\sum_{i|j,\ j>i}(z - F_j)^{-1}$ across the band. This paragraph is a paper argument and is not part of the machine-checked development.

References

[1] A. V. Doumas and P. J. Psarrakos, *The Fibonacci–Redheffer matrix and its properties*, e-print arXiv:2511.13627 [math.NT]. Submission history as of 3 September 2026: `[v1] Mon, 17 Nov 2025 17:38:00 UTC`, `[v2] Tue, 7 Apr 2026 07:02:29 UTC`; comment line "24 pages, 1 figure, 1 table"; not withdrawn, and the record carries no journal reference and no DOI beyond the arXiv one. All numbering, quotations and page numbers above are those of v2, which we read and compiled from its own e-print LaTeX source: Theorem 4 and Theorem 6 (the determinant and the sign of its Möbius sum), the definition of $F_R(n)$ and the printed $F_R(8)$, Theorem 14 (B), Remark 15 (G), Theorem 16 (L), Corollary 18 (F), the "main pillars" list on p. 14, the sentence on p. 9 quoted in §6, and the eigenvalue table of the appendix. The passage quoted in §1 is from v1 and is absent from v2.

[2] R. Redheffer, *Eine explizit lösbare Optimierungsaufgabe*, in: Numerische Methoden bei Optimierungsaufgaben, Band 3 (Tagung, Math. Forschungsinst., Oberwolfach, 1976), Internationale Schriftenreihe zur Numerischen Mathematik 36, Birkhäuser, Basel, 1977, 213–216; ZBL `0363.65062`. The origin of the matrix R_n and of the weighted determinant identity used in §5. Not read; we cite it through [3], where the identity is stated in the form we use, and the details above are those of the zbMATH record and of the bibliography of [3], which agree.

[3] D. A. Cardon, *Matrices related to Dirichlet series*, e-print arXiv:0809.0076 [math.NT]; J. Number Theory **130** (2010), no. 1, 27–39; DOI `10.1016/j.jnt.2009.05.013`. Read in the e-print version, compiled from its own LaTeX source; the journal data are from Crossref and zbMATH (Zbl 1220.11104). In that version the matrices A_n, B_n, C_n and the standing assumption $a_1 = 1$ are set up in §1, the determinant $\det A_n = \sum_{k\leq n} w_k b_k$ is Theorem 2.1, the divisibility of the characteristic polynomial by $(x-1)^{n-\lfloor \log_2 n \rfloor - 1}$ is Theorem 3.2, and $\det B_n = \sum_{k\leq n} w_k \mu(k)$ is quoted there from [2]. The published numbering was not checked against it.

[4] W. W. Barrett, R. W. Forcade and A. D. Pollington, *On the spectral radius of a* $(0,1)$ *matrix related to Mertens' function*, Linear Algebra Appl. **107** (1988), 151–159; DOI `10.1016/0024-3795(88)90241-8` (Crossref and Zbl 0656.05046 agree). Not read: the two results attributed to it in §1.4 are quoted as [3, §1] reports them.

[5] T. J. Jarvis, *A dominant negative eigenvalue of a matrix of Redheffer*, Linear Algebra Appl. **142** (1990), 141–152; DOI `10.1016/0024-3795(90)90262-b` (Crossref and Zbl 0715.15008 agree). Not read; cited from its title only.

[6] W. W. Barrett and T. J. Jarvis, *Spectral properties of a matrix of Redheffer*, Linear Algebra Appl. **162–164** (1992), 673–683; DOI `10.1016/0024-3795(92)90401-u`. Crossref and zbMATH (Zbl 0746.15009) both give the combined volume 162–164; the bibliography of [1] gives 162. Not read: the results and the two-part conjecture attributed to it in §1.4 are quoted as [3, §1] reports them, that paper being the one that disproves the conjecture (its Theorem 5.3 in the e-print numbering). [1] and [3] report the additive constant in the asymptotic for $\lambda_\pm$ differently, which is why no constant is quoted above.

[7] R. C. Vaughan, *On the eigenvalues of Redheffer's matrix I*, in: A. D. Pollington and W. Moran (eds.), Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991), Lecture Notes in Pure and Applied Mathematics 147, Marcel Dekker, New York, 1993, 283–296; ZBL `0794.15010`. Not read: the bibliographic details are from the zbMATH record, the library catalogue of the volume and the bibliography of [3], which agree, and the two bounds attributed to it in §1.4 are quoted as [3, §1] reports them — both belonging to part I alone. (The bibliography of [1] gives the volume title as "Markoss spectrum".)

[8] R. C. Vaughan, *On the eigenvalues of Redheffer's matrix II*, J. Austral. Math. Soc. Ser. A **60** (1996), no. 2, 260–273; DOI `10.1017/s1446788700037654` (Crossref and Zbl 0877.15021 agree). Not read: the result attributed to it in §1.4 is quoted as [3, §1] reports it.

[9] G.-S. Cheon and H. Kim, *Mertens equimodular matrices of Redheffer type*, Linear Algebra Appl. **572** (2019), 252–272; DOI `10.1016/j.laa.2019.03.009` (Crossref and Zbl 1440.11166 agree). We were unable to obtain the text and cite it from its abstract only, for the two facts stated in §1.4: that it constructs Riordan–Redheffer matrices and that it states two conjectures on their eigenvalues. It is not cited for anything else, and it appears in the LaTeX source of [1] only as a commented-out entry.

[10] B. R. Gillespie, *Extending Redheffer's Matrix to Arbitrary Arithmetic Functions*, Schreyer Honors College undergraduate honours thesis, Pennsylvania State University, 2011, 17 pp. Not read: every attempt to fetch the text from the Penn State repository was refused. The title, author, institution, year and length are from the repository's catalogue record and from the corresponding Google Books record; the only claim we make about its content, that it generalises Redheffer's matrix along Dirichlet-convolution lines, is from its abstract.

[11] F. Clément and S. Steinerberger, *On the largest singular vector of the Redheffer matrix*, e-print arXiv:2502.09489 [math.NT], 13 February 2025; Linear Algebra Appl. **725** (2025), 96–114; DOI `10.1016/j.laa.2025.07.003`. The arXiv record carries no journal reference; the journal data are from Crossref and zbMATH (Zbl 1576.11045), which agree. Not read beyond its abstract; cited only for its subject.

[12] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, 625–635; DOI 10.1007/978-3-030-79876-5_37.

[13] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381; DOI 10.1145/3372885.3373824.