SHORT STRATEGIES IN THE EVEN GAME OF BARMPALIAS, ZHANG AND ZHAN

KOREA SUPERINTELLIGENCE LABS

Abstract. In the k-even game of Barmpalias, Zhang and Zhan, Player 1 repeatedly plays a set of k numbers she has not played before and Player 2 answers with an even number in the span of that set which he has not played before; Player 2 loses when he has no legal answer. Player 1 wins for $k = 2$ and $k = 3$, and the case $k \geq 4$ is open. We prove five things about the game. First, Player 1 can finish on her current move exactly when some integer interval has both endpoints unplayed by her, all of its even numbers already played by Player 2, and k of its numbers still unplayed by her. Second, at $k = 3$ Player 1 wins within *four* of her own moves, by an explicit play opening with $\{8, 20, 26\}$; Barmpalias, Zhang and Zhan prove existence only, and reading their argument branch by branch gives five. Third, the step of that argument which fails for $k \geq 4$ can be named exactly: at $k = 3$ the move $\{2n - 1, 2n, 2n + 1\}$ forces a reply that may be a brand-new even number, whereas for $k \geq 4$ every move leaving Player 2 a single reply forces that reply to sit two away from an even number he has already used; and at every $k \geq 1$ such a move never brings Player 1 closer to finishing. Fourth, an interval all of whose even numbers Player 2 has played is at most $2|S_2| + 1$ numbers long, whence Player 1 has no winning strategy of length d once $2d \leq k$: the shortest win, if there is one, is longer than $k/2$. Fifth, and for the whole open range, Player 1 has no winning strategy of length four when $k \geq 4$ — not even from a position in which she has already played — by an explicit three-round rule for Player 2 whose middle step is *answer as far from your first number as her span allows.* All statements are machine-checked in Lean 4, with no appeal to compiled-code evaluation.

1. Introduction

The game. Fix an integer $k \geq 2$. The *k-even game* is played on $\mathbb{N}$ by two players who move alternately, Player 1 first.

- Player 1 plays a set A of k numbers, none of which she has played before.
- Player 2 then plays an even number in the interval $[\min A, \max A]$ which he has not played before.

Each player's constraint refers to their own history only: Player 2 may play a number Player 1 has used, and vice versa. Player 2 loses as soon as he has no legal move; if the play goes on forever, Player 2 wins.

The game was introduced by Barmpalias, Zhang and Zhan [1] in their study of compression of enumerations. There the central open question is whether every computably enumerable set is *well-compressible*; a positive answer would give density of the reducibility $\leq_{rK}$ of Downey, Hirschfeldt and LaForte [4] on the c.e. sets. The authors reduce well-compressibility to a two-player *balance game*, in which Player 1 enumerates a c.e. set and Player 2 tries to keep an even-numbered compression of it in balance on every initial segment, and they prove that a computable winning strategy for Player 2 in the balance game would make every c.e. set well-compressible (their Proposition 5.1). The k-even game is offered as the simplified version of the balance game which, in their words, "captures the essence" of it; the connection is a guide, not a formal implication, and nothing in the present paper bears on the compression question directly. As the authors note, these games depend on the *order* of the enumerations and so fall outside the framework of the recursion-theoretic games of Lachlan [5] and Kummer [6].

What is known, and what is open. Barmpalias, Zhang and Zhan assert the case $k = 2$ — "It is not hard to see that player-1 has a winning strategy for $k = 2$" — and prove the case $k = 3$ (their Theorem 5.3), closing the section with the sentence

> "This proof does not extend to $k > 3$, in which case we do not know the winner."

Every quotation from [1] in this paper, and every section, theorem and proposition number of it cited here, is that of the arXiv version v2 of 17 June 2025, whose wording differs from v1 in several of the places we quote; the published version was not accessible to us (see the bibliography). The same game, under the name *Even Number Game* G_k, is posed as Problem 4 of the Tianyuan workshop problem list [2]:

> "For $k \geq 4$, does Player 1 have a winning strategy in the game G_k?"

Two features of the known positive results shape everything below. The $k = 3$ argument of [1] is a configuration argument, carried out on strings of symbols standing for numbers chosen by one or both players, under a hypothesis of "sufficient space" around each symbol which is nowhere quantified — the source says only that "the exact space required is easily computed in each case", and does not compute it; consequently it exhibits no concrete play and gives no bound on the number of moves Player 1 needs. And the only negative result in [1] about the game concerns the *static* game, in which Player 1 must reveal all of her moves at once: there Player 2 wins even for $k = 2$ (their Proposition 5.2). The static game is a different game, and its analysis says nothing about the length of a Player-1 win in the game proper.

Results. This paper is about the two axes the source leaves untouched: the number of Player 1's own moves, and the structure of the transition from $k = 3$ to $k \geq 4$.

Section 2 fixes the model. Section 3 proves the criterion that everything else is an instance of: Player 1 can finish *on her current move* exactly when some interval $[a, b]$ has a, b unplayed by her, all of its even numbers already played by Player 2, and at least k of its numbers still unplayed by her (Proposition 3.1). The equivalence is a direct unfolding of the rules, and we claim no more for it than that; it is isolated because every bound below is an instance of it, and because [1] states only the two $k = 3$ instances of it, inside its proof.

Section 4 gives explicit strategies. The one for $k = 2$ takes two moves. The one for $k = 3$ (Theorem 4.2) takes four, opens with $\{8, 20, 26\}$, and is a 76-node tree carrying a child for *every* legal reply of Player 2; the deepest line is

$$\{8, 20, 26\} \;\to\; 8 \;\to\; \{4, 14, 16\} \;\to\; 4 \;\to\; \{2, 6, 10\} \;\to\; 2 \;\to\; \{1, 3, 5\}.$$

Reading the argument of [1] branch by branch gives five of Player 1's moves in its worst case (Remark 4.3), so the bound is not merely a formalisation of the source's strategy; nor is the play a shortening of it, since the source's strategy uses only even numbers except at the last step, while the tree here opens with a spread triple and kills seven of its ten branches immediately.

Section 5 isolates the step that breaks. Call a Player-1 move *forcing* if it leaves Player 2 exactly one legal reply. At $k = 3$ the move $\{2n - 1, 2n, 2n + 1\}$ is forcing and its forced reply $2n$ may be a number Player 2 has never touched, so forcing can open a fresh block of even numbers for him — and it is exactly this that the $k = 3$ argument exploits. At $k \geq 4$ this is impossible: a k-set spans at least $k \geq 4$ consecutive integers and hence at least two even ones, so a forced reply always sits two away from an even number Player 2 has already used (Theorem 5.2). Forcing can therefore only extend or merge blocks Player 2 already has, and in particular a $k \geq 4$ strategy cannot even *begin* by forcing (Corollary 5.3). Theorem 5.4 completes the picture at every $k \geq 1$: a forcing move never helps. Writing $\mathrm{Safe}_k(S_1, S_2)$ for "no interval all of whose even numbers Player 2 has used holds k numbers Player 1 has not used", which by Proposition 3.1 is essentially "Player 1 cannot finish on this move", the invariant Safe_k is preserved by every forced reply. The mechanism is that a forcing move is trapped inside the very interval whose freedom it is trying to raise, and spends k of that freedom entering.

The remaining four sections are about the length of a Player-1 win in the open case. Section 6 gives the first such bound: for every $k \geq 4$, Player 1 has no winning strategy of length at most three (Theorem 6.1). Section 7 gives the general one, by counting: an interval all of whose even numbers Player 2 has played carries one of his numbers every second step, so it is at most $2|S_2| + 1$ numbers long, and since his history grows by one per round, Player 1 has no winning strategy of length d as soon as $2(|S_2| + d) \leq k$ (Theorem 7.1). From the empty position this reads $2d \leq k$: the shortest Player-1 win, if there is one, is longer than $k/2$, and the hypothesis is exactly sharp at $k = 2$.

Section 8 develops the mechanism the counting bound does not see. Call an even number *isolated* for Player 2 when neither of its even neighbours is one he has played. An isolated reply builds a corridor only three numbers wide, so for $k \geq 4$ it is always safe (Theorem 8.2); therefore Player 1, to deny Player 2 an isolated reply, must fill the even numbers of her span into the 2-neighbourhood of his history, which costs $\lfloor k/2 \rfloor \leq 3|S_2|$ (Theorem 8.3). That is an explicit Player-2 rule, valid against every Player-1 move with no case analysis, and iterating it gives a no-go for $6d \leq k$ (Theorem 8.4) — the precursor of Theorem 7.1, weaker as a bound but the machinery the last section runs on.

Section 9 is the main new result: for every $k \geq 4$ Player 1 has no winning strategy of length *four* — not even from a position in which she has already played and Player 2 has not (Theorem 9.3). At $k = 4, \ldots, 7$ no counting bound says anything at $d = 4$, and the proof is an explicit three-round rule for Player 2 whose middle step is: *of the even numbers her span offers, answer the one farthest from the number you already hold.* That rule is forced, in the sense that the obvious alternative provably loses: iterating "answer with an isolated even number" reaches a position from which Player 1 wins in two more moves (Remark 9.5). Section 10 reports the finite searches, which are evidence and not theorems, and the questions we leave open; Section 11 describes the formal verification.

Nothing here decides the game for any $k \geq 4$; the problem of [2] remains open.

Changes in this version. Sections 7, 8 and 9 are new, and with them Theorems 7.1, 8.2, 8.3, 8.4 and 9.3; the summary of what is now excluded is Corollary 9.4. Section 10 gains the audit that produced the round-2 rule, and Section 11 the survey of related games (Remark 11.2); the search table of the previous version is now reported in prose. Nothing stated there has been withdrawn. The one theorem carried over that a new one overtakes is Theorem 6.1 (no win of length at most three, for $k \geq 4$), now the $d \leq 3$ case of Theorem 9.3; it is kept, with its proof, for the reasons given where it stands. Theorem 8.4 is new here but already superseded, by Theorem 7.1, and Remark 8.5 says why we keep it too. The question asked in the previous version — whether the no-go ladder continues from three moves to four — is answered here in the affirmative for every $k \geq 4$; the first length not excluded is now five, and only for $4 \leq k \leq 9$ (Question 10.3).

2. The model

Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[a, b] = \{n \in \mathbb{N} : a \leq n \leq b\}$, which is empty when $b < a$. All sets of played numbers are finite. A *position* is a pair (S_1, S_2) of finite subsets of $\mathbb{N}$ — the numbers Player 1 and Player 2 have played so far — with Player 1 to move.

Definition 2.1. Fix $k \geq 2$. A *move* of Player 1 at (S_1, S_2) is a set $A \subseteq \mathbb{N}$ with $|A| = k$ and $A \cap S_1 = \emptyset$; its *span* is $[a, b]$ with $a = \min A$, $b = \max A$. Player 2's *reply set* to the span $[a, b]$ is

$$\mathcal{R}(a, b; S_2) \;:=\; \{e \in [a, b] \;:\; e \text{ even},\; e \notin S_2\}.$$

Player 2 must play some $e \in \mathcal{R}(a, b; S_2)$, and loses if that set is empty.

Definition 2.2. For $d \in \mathbb{N}$ define the predicate $\mathrm{W}_k(d; S_1, S_2)$, "Player 1, to move, forces Player 2 out within d further moves of her own", by recursion on d: $\mathrm{W}_k(0; S_1, S_2)$ is false, and $\mathrm{W}_k(d+1; S_1, S_2)$ holds iff there is a move A with span $[a, b]$ such that

$$\mathrm{W}_k\big(d;\, S_1 \cup A,\, S_2 \cup \{e\}\big) \qquad \text{for every } e \in \mathcal{R}(a, b; S_2).$$

Since a universally quantified statement over the empty set is true, $\mathrm{W}_k(1; S_1, S_2)$ says precisely that Player 1 has a move whose span leaves Player 2 no reply. An immediate induction gives monotonicity: $\mathrm{W}_k(d; S_1, S_2)$ implies $\mathrm{W}_k(d+1; S_1, S_2)$. We say Player 1 *wins G_k in d moves* if $\mathrm{W}_k(d; \emptyset, \emptyset)$.

Remark 2.3. Player 1 has a winning strategy in G_k if and only if $\mathrm{W}_k(d; \emptyset, \emptyset)$ for some d. Indeed, fix a strategy for Player 1 and consider the tree of plays consistent with it: each Player-1 node has one child, and each Player-2 node has one child per legal reply, of which there are finitely many because $\mathcal{R}(a, b; S_2)$ is finite. The strategy wins exactly when every branch is finite, and by König's lemma a finitely branching tree with no infinite branch is finite; the number of Player-1 nodes on its longest branch is such a d. (Player 1 always has a legal move, there being infinitely many numbers, so a branch can end only at a Player-2 node with an empty reply set.) Conversely $\mathrm{W}_k(d; \emptyset, \emptyset)$ unfolds into a strategy. So bounding d is bounding the whole strategy, and the results below about small d are results about the open problem restricted to short strategies. The argument is standard and is deliberately kept outside the formal development: every theorem below is a statement about the predicate $\mathrm{W}_k(d)$ of Definition 2.2, and none of them uses this remark. It is used only in Remark 9.7, which is likewise informal.

Remark 2.4. We take the universe to be $\mathbb{N}$, so that 0 is even and is a legal reply whenever it lies in the span; this matches the condition $n \in 2\mathbb{N} \cap [\min R, \max R]$ used in [1]. The sources are not uniform on this point. The problem list [2] says in one sentence that the two players "take turns to play natural numbers" and in the next that Player 1 "plays a set A of k integers", and the static-game proposition of [1] is stated for "disjoint sets of positive integers of size 2". Nothing below depends on the choice. The explicit strategies of Section 4 use only the numbers 1 to 37, so they are plays in the positive-integer reading as well; and the arguments of Sections 5–9 use only interval arithmetic, written so as to hold as stated over $\mathbb{N}$ at the places where the truncated subtraction of $\mathbb{N}$ could behave differently from that of $\mathbb{Z}$ — for instance the hypothesis $e \geq 2$ in Theorem 5.2, the bound $|J| \leq 7$ in the proof of Theorem 6.1(iii), and the isolation condition of Definition 8.1, which is phrased without subtraction. We claim nothing about any other universe: the formal development is over $\mathbb{N}$ throughout.

For an interval $[a, b]$ and finite $S_1, S_2 \subseteq \mathbb{N}$ we write

$$\mathcal{F}(a, b; S_1) \ := \ [a, b] \setminus S_1,$$

for the numbers of $[a, b]$ still available to Player 1, and we say $[a, b]$ is *covered* by S_2, written $\mathrm{Cov}(a, b; S_2)$, if every even number of $[a, b]$ lies in S_2. Thus $\mathcal{R}(a, b; S_2) = \emptyset$ if and only if $\mathrm{Cov}(a, b; S_2)$.

3. When can Player 1 finish?

Proposition 3.1. *Let $k \geq 2$ and let (S_1, S_2) be a position. Then $\mathrm{W}_k(1; S_1, S_2)$ holds if and only if there are $a \leq b$ with*

$$a \notin S_1, \qquad b \notin S_1, \qquad \mathrm{Cov}(a, b; S_2), \qquad |\mathcal{F}(a, b; S_1)| \ \geq \ k.$$

Proof. Suppose $\mathrm{W}_k(1; S_1, S_2)$, witnessed by a move A with span $[a, b]$. Since $\mathrm{W}_k(0; \cdot, \cdot)$ is false, the reply set of the span must be empty, so $\mathrm{Cov}(a, b; S_2)$. The endpoints a, b lie in A, which is disjoint from S_1, so they are not in S_1; and $A \subseteq \mathcal{F}(a, b; S_1)$, whence $|\mathcal{F}(a, b; S_1)| \geq |A| = k$.

Conversely, given such an interval, the two-element set $\{a, b\}$ is contained in $\mathcal{F}(a, b; S_1)$ and $|\{a, b\}| \leq 2 \leq k \leq |\mathcal{F}(a, b; S_1)|$, so it extends to a subset $A \subseteq \mathcal{F}(a, b; S_1)$ with $|A| = k$. Then A is a legal move, its span is $[a, b]$ because $a, b \in A \subseteq [a, b]$, and its reply set is empty by $\mathrm{Cov}(a, b; S_2)$; so Player 1 finishes with A. $\square$

Nothing in this proof depends on a value of k or on any computation. We state it because every bound in this paper is an instance of it: each of them says that after the play so far no interval can meet all four conditions at once, either because it is too short to be covered or because Player 1's own last move has consumed too much of it.

Remark 3.2. The source [1] states the two $k = 3$ instances of the right-hand side, and only those, as clauses (i) and (ii) at the head of the proof of its Theorem 5.3: Player 2 loses if he has played $2n$ and Player 1 has played none of $2n-1, 2n, 2n+1$ (take $[a, b] = [2n-1, 2n+1]$), and he loses if he has played $2n$ and $2n+2$ while Player 1 has played neither $2n-1$, $2n+1$ nor $2n+3$ (take $[a, b] = [2n-1, 2n+3]$). In the version we cite, the clauses name only Player 2's numbers and leave the freedom of the odd endpoints to the ambient "sufficient space" convention; the arXiv v1 puts it inside the clauses, in its Lemma 3.2. The general equivalence does not appear in either.

Remark 3.3 (the corridor picture; informal). It is useful to read Proposition 3.1 as a statement about *corridors*. A maximal run $2m, 2m+2, \ldots, 2M$ of consecutive even numbers played by Player 2 determines the interval $[2m-1, 2M+1]$, which is covered; and every covered interval containing an even number lies inside exactly one such interval. Writing F for the number of its elements Player 1 has not played, Proposition 3.1 says that Player 1 can finish now precisely when some corridor has $F \geq k$ and its two (odd) endpoints are free for her. Every bound below is this one number being too small. This paragraph is an aid to reading only; the formal statements are the ones displayed above and below.

4. Explicit strategies for $k = 2$ and $k = 3$

A *strategy tree* is a finite tree in which every node carries an explicit k-element move of Player 1 together with one child for *every* number in the reply set of its span, computed from the rules and from the numbers played on the path to that node. A leaf is a node whose reply set is empty. A strategy tree of height d whose root is the initial position witnesses $\mathrm{W}_k(d; \emptyset, \emptyset)$; the verification of this implication, once and for all, is by induction on d and is described in Section 11.

Proposition 4.1. *Player 1 wins the* 2*-even game within two of her own moves. Explicitly: play* $\{2, 4\}$*; if Player 2 answers* 2*, play* $\{1, 3\}$*; if he answers* 4*, play* $\{3, 5\}$.

Proof. The span of $\{2, 4\}$ is $[2, 4]$, whose even numbers are 2 and 4, so those are Player 2's only replies and both are covered by the tree. After the answer 2, the move $\{1, 3\}$ has span $[1, 3]$, whose only even number is 2, already played by Player 2; after the answer 4, the move $\{3, 5\}$ has span $[3, 5]$, whose only even number is 4. In both cases the reply set is empty. Both second moves are legal: $\{1, 3\}$ and $\{3, 5\}$ are disjoint from $\{2, 4\}$. $\square$

Existence of a Player-1 win at $k = 2$ is asserted without proof in [1]; the play above is small enough to find by hand. We record it because it is the base case of the pattern the $k = 3$ tree develops, and because it is the "too large" control on every no-go theorem below: Player 1 *does* win within two moves for some k, so no such theorem can drop its hypothesis on k.

Theorem 4.2. *Player 1 wins the* 3*-even game within* four *of her own moves. A strategy achieving this opens with* $\{8, 20, 26\}$ *and is a tree with* 76 *nodes.*

Proof sketch. The opening move $\{8, 20, 26\}$ has span $[8, 26]$, so Player 2 has ten legal replies, the even numbers $8, 10, \ldots, 26$. Seven of them — $10, 12, 14, 16, 18, 22, 24$ — are numbers Player 1 has not played, and each is answered at once by the move $\{e-1, e, e+1\}$: its span contains the single even number e, which Player 2 has just played, so his reply set is empty. This is the forcing pattern of Remark 3.2. The three replies $8, 20, 26$ are numbers Player 1 has already used, so the corresponding interval $[e-1, e+1]$ holds only the two numbers $e-1, e+1$ free for her, one short of $k = 3$; each of these three branches therefore needs a second move. For instance the reply 20 is answered by $\{12, 14, 28\}$, whose span offers Player 2 eight replies; four of them — 16, 18, 22, 24 — are then dead on her next move, and the other four need one move more. The longest line of the tree is

$$\{8, 20, 26\} \;\rightarrow\; 8 \;\rightarrow\; \{4, 14, 16\} \;\rightarrow\; 4 \;\rightarrow\; \{2, 6, 10\} \;\rightarrow\; 2 \;\rightarrow\; \{1, 3, 5\},$$

which uses four moves of Player 1.

The claim is that the whole tree is correct, and this rests on an exhaustive finite check: at each of its 76 nodes one verifies that the move has three elements, that they are pairwise distinct and disjoint from the numbers Player 1 has played on the path to that node, that the recorded span endpoints belong to the move and bound it, and — the clause that does the work — that the labels of the node's children are *exactly* the list of even numbers in the span not played by Player 2 on that path. This check is a finite Boolean computation and is performed by the proof assistant's kernel; see Section 11. The tree itself was found by a game-tree search (Section 10); nothing about how it was found enters the proof. □

Remark 4.3. Existence of a Player-1 winning strategy at $k = 3$ is Theorem 5.3 of [1]; the bound of four moves is not stated there, and no concrete play is exhibited there at all, because the configuration argument carries an unquantified "sufficient space" hypothesis. Reading that argument branch by branch gives five of Player 1's moves in the worst case: one move to reach the configuration X (clause (iii) of the proof); one more, which either reaches XX or reaches XOX; from XOX a further move to XX; then the move of part (b) of the proof, which forces Player 2 to choose two adjacent even numbers; and then the auxiliary move of clause (ii) by which that configuration is converted into a win in the game proper. The best branch — the one that reaches XX directly — takes four. The arXiv v1 splits the same argument across its Lemmas 3.2–3.4 and has the same branch structure and the same count. This is our reading of the source's argument, not a quotation: the source states no bound at all.

Remark 4.4. The four-move tree is not the source's strategy shortened. The strategy of [1] plays only even numbers until its last step; the tree here opens with the spread triple $\{8, 20, 26\}$, whose span offers Player 2 ten replies at once, and wins seven of those ten branches immediately. The four-move bound also needs room: the same search, under the even-only restriction of Section 10 and over a window of 16 consecutive even numbers, finds no Player-1 win at all within six moves, while the tree above reaches the number 37 (in the branch under Player 2's reply 26).

5. Forcing

Definition 5.1. A move A of Player 1 with span $[a, b]$ is *forcing* at (S_1, S_2) if $\mathcal{R}(a, b; S_2) = \{e\}$ for a single number e; we call e the *forced reply.*

Forcing is the natural way to try to build the block of even numbers Player 1 needs: she would like Player 2's numbers to accumulate in one place, and a forcing move dictates where his next one goes. The two theorems of this section say that this route separates $k = 3$ from $k \geq 4$ sharply, and that at every k it cannot win on its own.

Theorem 5.2. (i) *Let $k = 3$. For every even $e \geq 2$ and every S_2 with $e \notin S_2$, the move $\{e-1, e, e+1\}$ has $\mathcal{R}(e-1, e+1; S_2) = \{e\}$. So at $k = 3$ Player 1 can force a reply that is a* brand-new *even number.*

(ii) *Let $k \geq 4$, let A be a set of k numbers contained in $[a, b]$, and suppose $\mathcal{R}(a, b; S_2) = \{e\}$. Then $e - 2 \in S_2$ (with $e \geq 2$) or $e + 2 \in S_2$. So at $k \geq 4$ a forced reply always sits two away from an even number Player 2 has* already *played.*

Proof. (i) The only even number of $[e - 1, e + 1]$ is e, and $e \notin S_2$.

(ii) Since $A \subseteq [a, b]$ and $|A| = k$, the interval $[a, b]$ has at least $k \geq 4$ elements, so $b \geq a + 3$. The forced reply e is even and lies in $[a, b]$. If $a + 2 \leq e$, then $e - 2$ is an even number of $[a, b]$; were it not in S_2 it would be a second element of the reply set, so $e - 2 \in S_2$. Otherwise $e \leq a + 1$, so $e + 2 \leq a + 3 \leq b$ and $e + 2$ is an even number of $[a, b]$; by the same argument $e + 2 \in S_2$. □

Corollary 5.3. *Let $k \geq 4$. No opening move of Player 1 is forcing: for every set A of k numbers contained in $[a, b]$ and every e, $\mathcal{R}(a, b; \emptyset) \neq \{e\}$. A winning strategy for Player 1 at $k \geq 4$ therefore cannot begin by forcing.*

Proof. Immediate from Theorem 5.2(ii): with $S_2 = \emptyset$ neither $e - 2$ nor $e + 2$ can lie in S_2. □

Theorem 5.2 makes the sentence "this proof does not extend to $k > 3$" precise, at least for the step of the argument of [1] that carries it. That argument begins from the observation that Player 2 loses once he has played an even number Player 1 has not played, i.e. from a fresh block; and a forcing move is how a fresh block is opened. At $k \geq 4$ a move spans at least four consecutive integers, hence at least two even ones, so forcing can only ever extend a block Player 2 already has or merge two of them.

The next theorem says that this is not merely an obstruction to one route but to the whole of it. Write

$$\mathrm{Safe}_k(S_1, S_2) \quad :\Longleftrightarrow \quad \text{for all } a, b : \ \mathrm{Cov}(a, b; S_2) \ \Rightarrow \ |\mathcal{F}(a, b; S_1)| < k.$$

By Proposition 3.1, $\mathrm{Safe}_k(S_1, S_2)$ implies $\neg\, \mathrm{W}_k(1; S_1, S_2)$ for $k \geq 2$: it is the same condition with the requirement that the two endpoints be free dropped, which is the form the counting arguments naturally produce. It holds at the start of the game, and it can fail — for instance $\mathrm{Safe}_3(\emptyset, \{2\})$ is false, since $[1, 3]$ is then covered and holds three numbers free for Player 1.

Theorem 5.4. *Let $k \geq 1$ and let (S_1, S_2) satisfy $\mathrm{Safe}_k(S_1, S_2)$. Let A be a move of Player 1 — so $|A| = k$ and $A \cap S_1 = \emptyset$ — contained in $[a_0, b_0]$, and suppose $\mathcal{R}(a_0, b_0; S_2) = \{e\}$. Then*

$$\mathrm{Safe}_k\big(S_1 \cup A,\ S_2 \cup \{e\}\big).$$

In words: a forcing move never brings Player 1 closer to finishing, at any k and from any position she could not already finish from.

Proof. Let $[a, b]$ be covered by $S_2 \cup \{e\}$.

If $e \notin [a, b]$, then $[a, b]$ is already covered by S_2, and $\mathcal{F}(a, b; S_1 \cup A) \subseteq \mathcal{F}(a, b; S_1)$, so its size is less than k by hypothesis.

Suppose $e \in [a, b]$, and enlarge the interval to $[a', b']$ with $a' = \min(a, a_0)$ and $b' = \max(b, b_0)$. This interval is still covered by $S_2 \cup \{e\}$: an even $f \in [a', b']$ with $f < a$ has $f \geq a' = \min(a, a_0)$, hence $f \geq a_0$, and also $f < a \leq e \leq b_0$, so f lies in the span $[a_0, b_0]$ and hence, the reply set of that span being $\{e\}$, lies in $S_2 \cup \{e\}$; symmetrically for $f > b$; and for $a \leq f \leq b$ this is the hypothesis. Put $F = \mathcal{F}(a', b'; S_1)$. Then $A \subseteq F$, since $A \subseteq [a_0, b_0] \subseteq [a', b']$ and $A \cap S_1 = \emptyset$.

Now split F at e. The interval $[a', e-1]$ is covered by S_2 — each of its even numbers lies in $S_2 \cup \{e\}$ by the previous paragraph and is not e — so $|F \cap [a', e-1]| \leq |\mathcal{F}(a', e-1; S_1)| < k$ by hypothesis, and likewise $|F \cap [e+1, b']| < k$. Since $F \subseteq [a', e-1] \cup \{e\} \cup [e+1, b']$ we get

$$|F| \ \leq \ (k-1) + 1 + (k-1) \ = \ 2k - 1.$$

Finally $\mathcal{F}(a, b; S_1 \cup A) \subseteq F \setminus A$, and $A \subseteq F$ with $|A| = k$, so

$$|\mathcal{F}(a, b; S_1 \cup A)| \ \leq \ |F| - |A| \ \leq \ (2k-1) - k \ = \ k - 1 \ < \ k. \qquad \square$$

The content is the third paragraph: the enlarged interval is the interval Player 1 would have to finish in, it splits at the forced reply into two intervals that were already covered before that reply, and her move is confined to it. A forcing move raises the freedom of a corridor by at most one number — the forced reply itself — while spending k numbers to do so.

6. No win of length at most three when $k \geq 4$

Theorem 6.1. (i) *For every $k \geq 2$, Player 1 does not win G_k on her first move: $\neg\, \mathrm{W}_k(1; \emptyset, \emptyset)$.*

(ii) *For every $k \geq 4$, $\neg\, \mathrm{W}_k(2; \emptyset, \emptyset)$.*

(iii) *For every $k \geq 4$, $\neg\, \mathrm{W}_k(3; \emptyset, \emptyset)$.*

In particular, for $k \geq 4$ Player 1 has no winning strategy of length at most three.

By monotonicity of W_k in d, clause (iii) implies (ii) implies (i) for $k \geq 4$; the three are proved separately because (i) holds in the larger range $k \geq 2$ and because each is the base of the next.

Proof sketch. All three go through Proposition 3.1, and use two pieces of interval arithmetic: an interval $[a, b]$ with $b \geq a + 3$ contains two even numbers f and $f + 2$; and an interval whose even numbers are all equal to one fixed number, or all lie in a pair of even numbers that are *not* two apart, satisfies $b \leq a + 2$ and therefore holds at most three numbers in total.

(i) At the start $S_2 = \emptyset$, so a covered interval contains no even number at all, hence is a single point and holds at most one number free for Player 1; and $k \geq 2$.

(ii) Let A_1 be Player 1's first move, with span $[a_1, b_1]$; since $|A_1| = k \geq 4$ we have $b_1 \geq a_1 + 3$, so the least even number $e_1 \geq a_1$ lies in the span and is a legal reply. Let Player 2 play it. Every interval covered by $\{e_1\}$ has all its even numbers equal to e_1 and so holds at most three numbers, which is less than k; by Proposition 3.1 Player 1 cannot finish.

(iii) Let A_1, $[a_1, b_1]$ and e_1 be as in (ii), and let A_2 with span $[a_2, b_2]$ be Player 1's second move, so again $b_2 \geq a_2 + 3$. Player 2 has a reply: otherwise $[a_2, b_2]$ would be covered by $\{e_1\}$, forcing $b_2 \leq a_2 + 2$. There are two cases.

Some legal reply e_2 is neither $e_1 + 2$ nor $e_1 - 2$. Let Player 2 play it. The two even numbers Player 2 has played then differ by at least 4, so no interval can contain both — an even number strictly between them would be uncovered — and every interval covered by $\{e_1, e_2\}$ has all its even numbers equal to one of them and hence holds at most three numbers. Since $k \geq 4$, Player 1 cannot finish.

Every legal reply is $e_1 \pm 2$. Then every even number of $[a_2, b_2]$ lies in $\{e_1 - 2, e_1, e_1 + 2\}$, so $[a_2, b_2] \subseteq J := [e_1 - 3, e_1 + 3]$, a window of at most seven numbers. Let Player 2 play any legal reply r. Consider an interval $[a, b]$ covered by $\{e_1, r\}$. If $b \leq a + 2$ it holds at most three numbers and we are done. Otherwise it contains two even numbers $f, f + 2$, which must be e_1 and r in some order; every even number of $[a, b]$ is then f or $f + 2$, which forces $a \geq f - 1$ and $b \leq f + 3$, and since $f \in \{e_1 - 2, e_1\}$ we get $[a, b] \subseteq J$ as well. So both $[a, b]$ and Player 1's second move A_2 lie inside the seven-element window J, and every element of A_2 has been played by her, whence

$$|\mathcal{F}(a, b; A_1 \cup A_2)| \ \leq\ |J| - |A_2| \ \leq\ 7 - k \ \leq\ 3 \ <\ k.$$

By Proposition 3.1 Player 1 cannot finish, and (iii) follows. □

The case that carries the theorem is the last one, and its content is the same mechanism as Theorem 5.4: Player 1's move is trapped in the very window she needs to keep free, and at $k \geq 4$ she must spend at least four numbers there while the window holds at most seven. No computation enters any of the three clauses. Two later results overtake this theorem — Theorem 7.1 for $k \geq 6$ and Theorem 9.3 for every $k \geq 4$ — and we keep it because its proof is independent of both and much shorter, because its seven-number window is the first instance of the counting of Section 8 (Remark 8.6), and because clause (i) holds in the range $k \geq 2$, where the game is not open.

Remark 6.2 (controls). Two directions of over-reach are worth excluding explicitly. The hypothesis $k \geq 4$ in Theorem 6.1(iii) is load-bearing: at $k = 2$ Player 1 *does* win within three moves, by Proposition 4.1 and monotonicity. And the theorem is not the empty statement that nothing happens in three moves: Player 2 really does have choices, since for example a first move with span $[2, 4]$ leaves him exactly the two replies 2 and 4. Both facts are part of the formal development.

7. Covered intervals are short

The bound of this section costs four lines and is the strongest general statement we have. Its point is that a covered interval must *spend* one of Player 2's numbers every second step, so his history alone caps how much room Player 1 can ever find.

Theorem 7.1. *Let $S_1, S_2 \subseteq \mathbb{N}$ be finite.*

(i) *If $a \leq b$ and $\mathrm{Cov}(a, b; S_2)$, then $b - a \leq 2|S_2|$; that is, a covered interval holds at most $2|S_2| + 1$ numbers.*

(ii) *If $2|S_2| + 1 < k$ then $\mathrm{Safe}_k(S_1, S_2)$, whatever Player 1 has played.*

(iii) *Let* $k \geq 2$ *and* $d \in \mathbb{N}$. *If* $2(|S_2| + d) \leq k$ *then* $\neg\,\mathrm{W}_k(d; S_1, S_2)$.

In particular, taking $S_1 = S_2 = \emptyset$*: for* $k \geq 2$ *and* $2d \leq k$ *Player 1 has no winning strategy of length* d *in* G_k*, and every Player-1 win of length* d *has* $k < 2d$.

Proof. (i) The interval $[a, b]$ contains at least $\lfloor (b - a + 1)/2 \rfloor$ even numbers, all of which lie in S_2 because $[a, b]$ is covered; so $\lfloor (b - a + 1)/2 \rfloor \leq |S_2|$, which is $b - a \leq 2|S_2|$.

(ii) By (i) a covered interval has at most $2|S_2| + 1 < k$ elements, so certainly fewer than k of them are free for Player 1.

(iii) Induction on d. For $d = 0$ there is nothing to prove, $\mathrm{W}_k(0; \cdot, \cdot)$ being false. For $d + 1$: the budget gives $2|S_2| + 2 \leq k$, so $\mathrm{Safe}_k(S_1, S_2)$ holds by (ii). Let A be any move of Player 1, with span $[a, b]$. Its reply set is not empty — if it were, $[a, b]$ would be covered and would contain the k numbers of A, all free for her, contradicting $\mathrm{Safe}_k(S_1, S_2)$ — so Player 2 may answer with some e. The new position has $|S_2 \cup \{e\}| \leq |S_2| + 1$, so $2(|S_2 \cup \{e\}| + d) \leq k$ and the induction hypothesis applies: A does not witness $\mathrm{W}_k(d + 1; S_1, S_2)$.

The last sentence is (iii) with $S_2 = \emptyset$, read directly and contrapositively. □

Note that (iii) holds from *any* position, not only the initial one, and that it needs no hypothesis whatever on Player 1's history: the only resource it counts is the number of moves Player 2 has made. Written as a bound on the length of a win it says that Player 1 needs more than $k/2$ moves, so that the difficulty of the open problem grows at least linearly with k.

Remark 7.2 (sharpness). The hypothesis $2d \leq k$ cannot be relaxed by two. At $k = 2$ it gives $\neg\,\mathrm{W}_2(1; \emptyset, \emptyset)$, and Proposition 4.1 gives $\mathrm{W}_2(2; \emptyset, \emptyset)$; so "$2d \leq k + 2$" would be false at $k = 2$, $d = 2$. Any improvement of Theorem 7.1 must therefore be k-dependent — it must use something beyond the length of a covered interval — and whether the truth is $d \geq ck$ for some $c > 1/2$, or superlinear, or whether there is no Player-1 win at all for $k \geq 4$, are all open. Both halves of the control are part of the formal development.

Remark 7.3. Theorem 7.1 subsumes Theorem 6.1(i) (take $d = 1$, any $k \geq 2$), Theorem 6.1(ii) (take $d = 2$, $k \geq 4$), and Theorem 6.1(iii) for $k \geq 6$. What it does not reach is the pair of values $k \in \{4, 5\}$ at $d = 3$, and, more importantly, $d = 4$ at $k = 4, \ldots, 7$: the whole content of Section 9 lies in the strip where counting is silent.

8. Isolated replies and crowding

The counting bound of Section 7 ignores Player 2's choices entirely. This section gives him an explicit rule — and it is the rule the depth-four theorem of Section 9 runs on for two of its three rounds.

Definition 8.1. An even number e is *isolated* for S_2 if no $s \in S_2$ satisfies $s + 2 = e$ or $e + 2 = s$; that is, if neither even neighbour of e has been played by Player 2. (The formulation avoids subtraction, so that it reads correctly over $\mathbb{N}$ at $e < 2$.)

Playing an isolated e opens a fresh block rather than extending or merging old ones, and a fresh block is a corridor three numbers wide.

Theorem 8.2. *Let* $k \geq 4$*, let* $\mathrm{Safe}_k(S_1, S_2)$*, let* $S_1 \subseteq S_1'$*, and let* e *be even and isolated for* S_2*. Then* $\mathrm{Safe}_k(S_1', S_2 \cup \{e\})$.

Proof. Let $[a, b]$ be covered by $S_2 \cup \{e\}$. If $e \notin [a, b]$ then $[a, b]$ is already covered by S_2, and $\mathcal{F}(a, b; S_1') \subseteq \mathcal{F}(a, b; S_1)$, so the hypothesis applies. If $e \in [a, b]$, then $[a, b]$ cannot reach $e - 2$: that number is even, it would lie in $[a, b]$, it is not e, so covering would force it into S_2, which isolation forbids. Likewise $[a, b]$ cannot reach $e + 2$. Hence $[a, b] \subseteq [e - 1, e + 1]$, three numbers, and $3 < k$. □

The hypothesis $k \geq 4$ is exactly what makes the game hard for Player 1 above $k = 3$: at $k = 3$ an isolated reply *does* break safety, which is the qualitative content of the source's own remark that its $k = 3$ proof, whose forcing move is $\{2n-1, 2n, 2n+1\}$ against an unanswered even number, "does not extend to $k > 3$". The formal development records this as a control: 2 is isolated for $\emptyset$, $\mathrm{Safe}_3(\emptyset, \emptyset)$ holds and $\mathrm{Safe}_3(\emptyset, \{2\})$ fails. Note also that the theorem asks nothing at all about Player 1's move: the k numbers she has just played are absorbed into S_1'.

So a Player-1 move can only be dangerous if it leaves Player 2 no isolated reply, and that is expensive. Write

$$S_2^+ \;:=\; S_2 \;\cup\; (S_2 + 2) \;\cup\; (S_2 \mathbin{\dot{-}} 2),$$

the 2-neighbourhood of Player 2's history ($\dot{-}$ is truncated subtraction), so that $|S_2^+| \le 3|S_2|$ and every even number that is *not* isolated for S_2 lies in S_2^+.

Theorem 8.3. *Let A be a set of k numbers contained in $[a, b]$ with $b \in A$.*

(i) *If no $e \in \mathcal{R}(a, b; S_2)$ is isolated for S_2, then $\lfloor k/2 \rfloor \le 3|S_2|$.*
(ii) *Let $k \ge 4$ and $\mathrm{Safe}_k(S_1, S_2)$, and suppose $3|S_2| < \lfloor k/2 \rfloor$. Then for* every *such A there is a reply $e \in \mathcal{R}(a, b; S_2)$ with $\mathrm{Safe}_k(S_1 \cup A, S_2 \cup \{e\})$.*

Proof. (i) Since $A \subseteq [a, b]$ and $|A| = k$, the interval $[a, b]$ has at least k elements and hence at least $\lfloor k/2 \rfloor$ even ones. Each even $f \in [a, b]$ either lies in S_2, or is a legal reply and therefore, by hypothesis, is not isolated for S_2; in both cases $f \in S_2^+$. So $\lfloor k/2 \rfloor \le |S_2^+| \le 3|S_2|$.

(ii) If some legal reply is isolated, Theorem 8.2 gives the conclusion for it. Otherwise (i) gives $\lfloor k/2 \rfloor \le 3|S_2|$, contradicting the hypothesis. □

Part (ii) is an explicit rule for Player 2 — *answer with an isolated even number* — valid against every legal move of Player 1 and with no case analysis on that move, for as long as his history is short. Iterating it gives a bound on the length of a Player-1 win, since each round adds one number to S_2 and so consumes one unit of the budget.

Theorem 8.4. *Let $k \ge 4$.*

(i) *If $\mathrm{Safe}_k(S_1, S_2)$ and $3(|S_2| + d) \le \lfloor k/2 \rfloor$, then $\neg\, \mathrm{W}_k(d; S_1, S_2)$.*
(ii) *If $6d \le k$ then $\neg\, \mathrm{W}_k(d; \emptyset, \emptyset)$; equivalently, every Player-1 win of length d in G_k has $k < 6d$.*

Proof. (i) Induction on d, the case $d = 0$ being trivial. Given a move A of Player 1 with span $[a, b]$: if some legal reply e is isolated, Theorem 8.2 gives $\mathrm{Safe}_k(S_1 \cup A, S_2 \cup \{e\})$ and the budget survives, since $|S_2 \cup \{e\}| \le |S_2| + 1$; if none is, Theorem 8.3(i) gives $\lfloor k/2 \rfloor \le 3|S_2|$, which the budget forbids for $d \ge 1$. So every move of Player 1 has a reply after which she still faces a position of the same kind with d lowered by one, and by induction she cannot finish within d moves.

(ii) Apply (i) with $S_1 = S_2 = \emptyset$, where Safe_k holds and $3d \le \lfloor k/2 \rfloor$ follows from $6d \le k$. □

Remark 8.5 (what supersedes what). Theorem 8.4 is strictly weaker than Theorem 7.1: $6d \le k$ implies $2d \le k$, and $3(|S_2| + d) \le \lfloor k/2 \rfloor$ implies $2(|S_2| + d) \le k$, so every position excluded by the former is excluded by the latter, and the latter excludes three times as many lengths at every k. Nothing in Theorem 8.4 is thereby false, and we keep it for two reasons. Its proof is a different mechanism — a rule for Player 2, not a count of his history — and Theorems 8.2 and 8.3 are the machinery on which Section 9 rests, where no counting bound reaches. Historically it was also the first bound in the open case whose strength grows with k, and it is what suggested looking for the shorter argument.

Remark 8.6. The constant 3 in $|S_2^+| \le 3|S_2|$ is attained: for $S_2 = \{2\}$ one has $|S_2^+| = 3$, and every even number of $[0, 5]$ — an interval long enough to carry a move with $k = 6$ — lies in S_2^+. So the bound $6d \le k$ cannot be improved by sharpening this counting alone. The seven-number window in the proof of Theorem 6.1(iii) is the case $|S_2| = 1$ of Theorem 8.3(i): one used even number fattens to three consecutive evens, and an interval whose even numbers all lie there is at most seven numbers long.

9. No win of length four when $k \geq 4$

At $d = 4$ and $4 \leq k \leq 7$ neither Theorem 7.1 nor Theorem 8.4 says anything, and $k = 4$ is the case the source asks about. The result of this section covers the whole open range at once, and it does so from a stronger starting position than the initial one: Player 1 is allowed an arbitrary head start.

It is convenient to name the Player-2 side of W directly.

Definition 9.1. For $d \in \mathbb{N}$ define $\mathrm{Surv}_k(d; S_1, S_2)$, "Player 2 can answer Player 1's next d moves and still leave her unable to finish", by recursion on d: $\mathrm{Surv}_k(0; S_1, S_2)$ is $\mathrm{Safe}_k(S_1, S_2)$, and $\mathrm{Surv}_k(d+1; S_1, S_2)$ holds iff for every move A of Player 1 with span $[a, b]$ there is $e \in \mathcal{R}(a, b; S_2)$ with $\mathrm{Surv}_k\big(d; S_1 \cup A, S_2 \cup \{e\}\big)$.

Lemma 9.2. *For $k \geq 2$, $\mathrm{Surv}_k(d; S_1, S_2)$ implies $\neg\, \mathrm{W}_k(d + 1; S_1, S_2)$.*

Proof. Induction on d. At $d = 0$ this is $\mathrm{Safe}_k(S_1, S_2) \Rightarrow \neg\, \mathrm{W}_k(1; S_1, S_2)$, which is Proposition 3.1. At $d+1$: a witness A for $\mathrm{W}_k(d+2; S_1, S_2)$ must in particular defeat the reply e supplied by $\mathrm{Surv}_k(d+1; \cdot)$, and that reply leads to a position satisfying $\mathrm{Surv}_k(d; \cdot)$; apply the induction hypothesis. □

Theorem 9.3. *Let $k \geq 4$ and let $S_1 \subseteq \mathbb{N}$ be any finite set.*

(i) $\mathrm{Surv}_k(3; S_1, \emptyset)$*, and hence* $\neg\, \mathrm{W}_k(4; S_1, \emptyset)$*: Player 1 has no winning strategy of length four, even from a position in which she has already played and Player 2 has not.*

(ii) $\mathrm{Surv}_k(2; S_1, \{e_0\})$ *for every even* e_0*, and hence* $\neg\, \mathrm{W}_k(3; S_1, \{e_0\})$*: once Player 2 has answered once, she has no winning strategy of length three.*

Corollary 9.4. *For every $k \geq 4$, Player 1 has no winning strategy of length d in G_k whenever $d \leq 4$ or $2d \leq k$; that is, the shortest Player-1 win, if there is one at all, has length greater than* $\max\big(4, \lfloor k/2 \rfloor\big)$.

Proof. The first clause is Theorem 9.3(i) with $S_1 = \emptyset$ and monotonicity of W_k in d; the second is Theorem 7.1. □

Proof sketch of Theorem 9.3. Statement (ii) is the middle of (i), so we describe the three rounds of (i) in order. By Lemma 9.2 it suffices to produce the survival statements. Throughout, Player 1's move is a k-set A with span $[a, b]$, and $b \geq a + k - 1 \geq a + 3$. No step uses a computation; the case analysis is finite and is carried out in full in the formal development.

Round 1 is free. Player 2 answers the opening move with the least even number of its span. Nothing is claimed about that answer; the position reached is $(S_1 \cup A, \{e_0\})$ with e_0 even, and the work starts at round 2.

Round 2: answer as far from your first number as you can. Let e_L be the least even number of the span $[a, b]$ and e_R the greatest, so $e_L \leq a + 1$, $b \leq e_R + 1$ and $e_L + 2 \leq e_R$. Player 2 answers e_R if $e_L + e_R \geq 2e_0$ and e_L otherwise — the reply farther from the number e_0 he already holds. Say it is e_R, the other case being symmetric. Two consequences: her span *reaches* the reply, so one of her own numbers lies at e_R or one past it (namely b, since $e_R \leq b \leq e_R + 1$); and, because e_R was the farther of the two extremes, the distance from e_0 to any legal reply is at most $e_R - e_0$. The first is the hypothesis the round-3 lemmas need; the second is what makes the case $e_R = e_0 + 2$ harmless, her whole move then lying inside the seven numbers around e_0.

Round 3: five shapes. Player 2 now holds $S_2 = \{\alpha, \beta\}$ with $\alpha < \beta$ even. Round 2 has left him one of two things: when the reply it chose was adjacent to e_0 — the case $\beta = \alpha + 2$ below — her whole previous move was trapped beside e_0 and the position is already Safe_k; in every other case one number of Player 1's is pinned at the outer end of the pair. Let A be her move, with span $[a, b]$. If some legal reply is isolated for S_2, Theorem 8.2 finishes with no case analysis at all. Otherwise every even number of $[a, b]$ lies in $S_2^+ = \{\alpha - 2, \alpha, \alpha + 2, \beta - 2, \beta, \beta + 2\}$, which pins $[a, b] \subseteq [\alpha - 3, \beta + 3]$, and the argument splits by the shape of $\{\alpha, \beta\}$:

shape	Player-2 answer	what pays for the corridor
$\beta = \alpha + 2$	outside the block	two of the five numbers of $[\alpha - 1, \alpha + 3]$, by Safe_k
$\beta = \alpha + 4$	$\alpha + 2$, or outside	the merged corridor $[\alpha - 1, \alpha + 5]$, or two used low down
$\beta \geq \alpha + 6$	next to the reached end	her pinned number and the far end of her move

The two asymmetric shapes are proved twice, once in each orientation, because $\mathbb{N}$ has no reflection. Each of the five lemmas is a case analysis on where her span ends (respectively starts) — three cases in the adjacent and in the two wide shapes, five or six in the two gap-4 ones — and each case closes in one of three ways: a corridor of five numbers of which two are already hers, a corridor of seven containing her whole move, or a *forced* reply, where Theorem 5.4 applies unchanged — which happens three times.

The delicate case is $\beta = \alpha + 4$ at $k = 4$, where her move straddles the hole at $\alpha + 2$ and Player 2 must choose between filling it (a block of three, whose corridor holds seven numbers and so needs four of them used) and answering outside it (a block of two, corridor of five, needing two used). The dichotomy that decides is whether a second used number already lies in the low corridor; if none does, then all of her move except its first number lies in the merged corridor, which together with her pinned outer number is k used numbers among seven. No count in any branch depends on the value of k: every one comes out as "at most three numbers free", and $3 < k$ for all $k \geq 4$. □

Remark 9.5 (the round-2 rule is load-bearing)*.* The obvious rule for Player 2 is the one of Theorem 8.3(ii): answer with an isolated even number. It does not reach depth four. Its guarantee holds only while $3|S_2| < \lfloor k/2 \rfloor$, which at $k = 4$ covers his first answer and nothing after it; and where isolated replies do exist, taking them can already be wrong. Consider $S_2 = \{4, 8\}$: the number 4 is isolated for $\emptyset$ and 8 is isolated for $\{4\}$, so each of Player 2's numbers was isolated at the moment he played it. From that position, with $S_1 = \emptyset$, Player 1 wins in two moves at $k = 4$: she plays $\{1, 6, 7, 8\}$, whose span $[1, 8]$ leaves the replies 2 and 6, neither of them isolated for $\{4, 8\}$; the answer 2 leaves the corridor $[2, 5]$ with the four numbers $2, 3, 4, 5$ free for her, and the answer 6 leaves $[3, 9]$ with the four numbers $3, 4, 5, 9$ free. Both continuations are exhibited explicitly and checked in the formal development, as is $\mathrm{W}_4(2; \emptyset, \{4, 8\})$ itself. What this refutes is the round-3 lemma the induction of this section needs: $\mathrm{W}_k(d; \cdot, S_2)$ is antitone in Player 1's history, so $S_1 = \emptyset$ is the position most favourable to her, and every lemma here — Theorem 8.2 above all — is stated uniformly in that history, which is what lets the rounds be composed. It does not refute every conceivable use of isolated replies: a rule that also tracked what Player 1 has spent would not be touched by it, since the position above is not one of an actual play — after two rounds she has played $2k$ numbers. This is why round 2 answers as far away as possible rather than as freshly as possible.

Remark 9.6*.* Theorem 9.3 is not a corollary of any counting bound in this paper: at $k = 4$ and $d = 4$ neither $2d \leq k$ nor $6d \leq k$ holds. Nor can its hypothesis be dropped: at $k = 3$ Player 1 does win within four moves (Theorem 4.2), so "no Player-1 win of length four" is false as a k-free statement. Both facts are recorded as controls in the formal development.

Remark 9.7*.* Combining Corollary 9.4 with Remark 2.3: for $k \geq 4$, if Player 1 has a winning strategy in G_k at all, then some play consistent with it requires at least $\max\bigl(5, \lfloor k/2 \rfloor + 1\bigr)$ of her own moves. Compare $k = 3$, where four moves suffice (Theorem 4.2), and $k = 2$, where two do. In particular the shortest win at any $k \geq 4$, should one exist, is strictly longer than the shortest win at $k = 3$, and the first length not excluded is $d = 5$, only in the strip $4 \leq k \leq 9$.

10. The case $k = 4$: evidence and questions

This section reports computations that are *not* part of the formal development and are not theorems. They are recorded because they say where the open case stands, and because two of them are how the rule proved in Section 9 was found.

10.1. **An even-only reduction, and a search for a Player-1 win.** Restrict Player 1 to even numbers. This is a restriction on her alone, so any win found under it is a win in the real game (the converse is not claimed, and Remark 10.1 says why it may fail). Under the restriction every odd number stays free for her forever, so with the identification cell $i \leftrightarrow$ even $2i$, and with P the set of cells Player 1 has played and T the set played by Player 2, a maximal run R of consecutive cells in T has corridor $[2\min R - 1, 2\max R + 1]$ containing $|R| + 1$ odd numbers, all free, and $|R \setminus P|$ free even numbers. The criterion of Proposition 3.1 becomes: Player 1 finishes now if and only if some run R of T satisfies $|R| + 1 + |R \setminus P| \geq k$. At $k = 3$ its two minimal solutions, $|R| = 1$ with $R \not\subseteq P$ and $|R| = 2$, are exactly the two clauses quoted in Remark 3.2, which is a strong check that the reduction is a reduction of the same game. A game-tree search over the reduction, on a window of consecutive cells and to a bounded number d of Player-1 moves, found **no** Player-1 win at $k = 4$ within five moves on a window of 20 cells (3,307,522 nodes), and none within four moves on a window of 24 (2,952,511 nodes); at 24 cells and $d \leq 5$ it was stopped after 1,001 CPU-seconds with no answer. The first of these is complete for its window, since at $k = 4$ twenty cells hold at most five of her moves, so the depth bound truncates nothing there. The positive control is that the same program, in the same 20-cell window, finds a $k = 3$ win within four moves — the tree of Theorem 4.2 — and finds none at all within six moves on a window of 16 cells, which shows that the window must be wide enough to hold the win. The $d \leq 4$ row is now superseded by Theorem 9.3, which needs no window; the $d \leq 5$ row is the evidence that remains. The whole search programme cost under half a CPU-hour.

Remark 10.1. The even-only reduction understates Player 1. In the real game she may play a move such as $\{2a, 2a+1, 2a+3, 2b\}$: its span is $[2a, 2b]$, exactly as if she had played four even numbers, but she has spent only two of them, and the even numbers she does *not* spend are precisely what keeps a future corridor rich for her. An unrestricted search is out of reach as a naive enumeration: over the integer window corresponding to 20 cells there are $\binom{40}{4} = 91{,}390$ moves per node against $\binom{20}{4} = 4{,}845$ in the reduction, a branching factor about 18.9 times larger at every level. What is needed is a move generator that quotients out interchangeable filler numbers, not a larger budget.

10.2. **The search behind the round-2 rule.** Before any of Section 9 was proved, the three-round question was settled numerically for small k by a separate program implementing the raw rules of Definition 2.1 — the *full* game, Player 1 free to use odd numbers — and sharing no code with the formal development. Two proved facts are used, and only to bound the enumeration: Theorem 8.2, which makes any move offering an isolated reply harmless and so confines the round-3 moves that must be examined to the window $[\min S_2 - 3, \max S_2 + 3]$; and the round-2 window described in the proof sketch. Player 1's history away from the region is taken to be empty, the position most favourable to her. The program searches over a window of consecutive integers centred at Player 2's first number; "all round-2 moves" below means all k-subsets of that window.

run	round-2 moves enumerated	result
$k = 4$	all 1,365 in a window of 15	no move without a safe reply
$k = 5$	all 11,628 in a window of 19	the same
$k = 6$	5,005 in a window of 15 (spot check)	the same

Each round-2 move was followed by a full round-3 search; the $k = 6$ line does not exhaust its window and is a spot check only.

Two further audits of the same program are what produced the design of the proof. First, for every one of the 1,365 round-2 moves at $k = 4$, the *farthest* reply is a good one — which is the rule of round 2 — while 574 of those moves admit no good *adjacent* reply, that is none at $e_0 \pm 2$, so "answer next to your number" cannot be the rule. Second, with an empty Player-1 history the two dangerous shapes $S_2 = \{0, 4\}$ and $S_2 = \{0, 6\}$ do *not* survive, with explicit Player-1 witnesses, whereas all eight variants carrying a single number of hers at the outer end do survive: that is exactly the hypothesis the round-3 lemmas were then given. The whole audit runs in under three CPU-seconds. None of this enters any proof; it is reported because it is where the statement came from, and because it is an independent check of the theorem at $k = 4, 5$.

10.3. **What is still missing.** By Proposition 3.1 with $k = 4$, Player 2 survives exactly as long as every corridor holds at most three numbers free for Player 1. Call this the invariant $F \leq 3$. A reply that opens a new block is safe (Theorem 8.2) and a forced reply is safe (Theorem 5.4); the gap is the unforced non-isolated reply, and there $F \leq 3$ is not preserved by one step, since an unforced extension of a corridor can raise F from 3 to 5. A hand computation, not part of the formal development, exhibits the shape: three blocks of Player-2 numbers separated by single-even gaps, all three corridors with $F = 3$ and both gap numbers free, offer Player 1 exactly the two gap replies, and she can spend four of the eleven free numbers of the region so that either reply merges two corridors into one with at least four free numbers. (Under the stronger invariant $F \leq 2$ the same count is impossible.) This refutes one-step inductivity of $F \leq 3$; it does not say the position is reachable in play. What Section 9 adds is that up to depth four the choice can always be made correctly. What is missing is a rule — or an invariant — that survives every depth.

10.4. **Questions.**

Question 10.2. For $k \geq 4$, does Player 1 have a winning strategy in G_k? *(*[1]*; Problem 4 of* [2]*; open.)*

Question 10.3. For $4 \leq k \leq 9$, is $\mathrm{W}_k(5; \emptyset, \emptyset)$ false — that is, does the ladder of Corollary 9.4 continue one more step? For $k \geq 10$ this is Theorem 7.1. The obstruction is that round 3 would have to be played from $|S_2| = 2$ under a rule, and the shapes of a three-element S_2 number of the order of twenty rather than five.

Question 10.4. Is there a strengthening of the invariant "every corridor holds fewer than k numbers free for Player 1" that Player 2 can maintain for $k \geq 4$? By Theorems 5.2, 5.4 and 8.2 the forced and the isolated branches of such an induction are settled: the forced one at every k, the isolated one at every $k \geq 4$. What is missing is a rule telling Player 2 which reply to choose when every reply he is offered is unforced and non-isolated.

Question 10.5. Does the even-only restriction above cost Player 1 anything? Equivalently: is there a k and a d such that Player 1 wins G_k within d moves but has no such win using only even numbers except at the last step?

11. Verification

Every theorem, proposition, lemma and corollary of Sections 3–9 has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [8, 9]; repository reference to be added at submission. The development contains no `sorry`, and every theorem stated above depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. In particular *nothing* is delegated to compiled-code evaluation. Exactly two families of statements rest on exhaustive computation, and in both the computation is a finite Boolean evaluation carried out by the proof assistant's own kernel: the two strategy trees of Section 4, and the small numerical controls (for instance $\mathrm{W}_4(2; \emptyset, \{4, 8\})$ of Remark 9.5, whose two continuations are checked from the rules). The tree checker recomputes the reply set at every node from Definition 2.1 — it does not read it off the tree — and its soundness, the implication from a `true` verdict at fuel d to $\mathrm{W}_k(d; \emptyset, \emptyset)$, is proved once by induction on d; so no property of the search that produced a tree is trusted. Everything else — the interval characterisation, the forcing dichotomy, the preservation of Safe_k under forced and under isolated replies, the crowding bound, both counting no-gos, and all three rounds of Theorem 9.3 — is an ordinary proof with no finite check anywhere in it; in particular the searches of Section 10 enter no proof. The formal development also carries the controls quoted above (Remarks 7.2, 9.5, 9.6, 8.6) together with four checks that the certificate machinery bites: a tree with one of Player 2's legal replies deleted is rejected; a tree whose second move re-uses a number Player 1 has already played is rejected, even though its span contains only an even number Player 2 has used; the $k = 3$ tree is rejected at fuel three, so four moves are genuinely needed for it; and it is rejected as a $k = 4$ strategy.

Outside the formal development, the $k = 3$ tree was emitted by the search program as an s-expression in the numbers of the real game, re-verified against the raw rules by an independent program sharing no code with the search, and only then translated into the term that Lean checks; the translation was re-run at the end and reproduces that term. The computations of Section 10 are not formalised and are labelled there as evidence.

Remark 11.1 (provenance of the problem list). The Even Number Game problem is posed in [2], the problem list of the 2025 Tianyuan workshop. A 2026 workshop list [3] has been announced with an overlapping author set, and it is easy to take the problem to be posed there too: its submitted source file repeats the whole 2025 list after the `\end{document}` of the article proper, so that the Even Number Game paragraph sits in text that is never typeset. The compiled 2026 article is sixteen pages and contains no occurrence of the phrase "Even Number Game", or indeed of "even number"; its own Barmpalias section is about Turing ideals generated by random reals. We therefore attribute the problem to [2], where it is Problem 4. (Checked against the e-print sources of both listings and against the posted PDF of arXiv:2608.26628v1, 29 August 2026.)

Remark 11.2 (related games). We know of no other work on this game. That is a not-found statement, so we record what was searched, all of it on 30 August 2026 and each index against a positive control. Three citation indexes report no citation of [1] whatever: OpenAlex (`cited_by_count` 0 on both of its records for the paper), Crossref (`is-referenced-by-count` 0 on the published DOI) and Semantic Scholar (`citationCount` 0, with an empty citation list). In the full-text arXiv corpus we hold locally — 881,145 documents, identifiers up to 2608.27451 — the identifier of [1] occurs in exactly three documents: the source itself and the two problem lists, in the 2026 one inside the appended text described in Remark 11.1. Their statements of the problem are byte-identical. The nearest relative we are aware of is the achievement game in which Maker claims single elements of $\{1, \dots, n\}$, or of a cycle, and tries to build a long block of consecutive elements [7]. That framework is different in every respect that matters here: the board is finite, a move claims one element rather than k, claims are exclusive between the players, the game length is fixed in advance, and the quantity studied is a score rather than the number of moves one player needs.

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