THE EQUAL-TAIL/LIKELIHOOD-RATIO EDGE FOR A SINGLE OBSERVATION: SIX AFFINE BRANCHES OF $V = (am^2 + bm + c)^{3/2}$, TWO OF THEM NEW, AND A BOUNDARY-ATOM TEST THAT CLOSES BOTH

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. For a continuous one-parameter natural exponential family (NEF) with variance function V, Bar-Lev and Hoessly (arXiv:2608.28221) prove that coincidence of the equal-tail and likelihood-ratio two-sided p-values for one observation, throughout the family and under their standing regularity assumption, forces $(V^{2/3})''' = 0$, i.e. $V = (am^2 + bm + c)^{3/2}$, together with a density-at-the-mean identity; they solve the resulting problem inside the power-variance class only and state that the unrestricted one-observation classification is deliberately not claimed. We settle it: under their Assumption 2.5, equal-tail/likelihood-ratio coincidence for one observation holds only for the Gaussian family up to affine transformation. The pair (sign of a, sign of $b^2 - 4ac$) is an affine invariant of the family and takes exactly six values. Four cells are the source's (Gaussian, the power-variance exponent 3/2, inverse Gaussian, symmetric normal inverse Gaussian); the two others are new, with explicit candidate cumulants $k(\theta) = -\sqrt{\theta^2 - c^2}$ on $\theta < -c$ and $k(\theta) = \sqrt{\theta^2 + c^2}$ on $\mathbb{R}$, variance functions $(m^2 - 1)^{3/2}/c$ and $(1 - m^2)^{3/2}/c$, the latter with a bounded mean domain. A boundary-atom test closes both: on each new cell $k(\theta) - \theta \to 0$ at the relevant infinity, so the Laplace transform of $e^{\theta(x-1)}$ tends to 1, whereas a Lebesgue-dominated generating measure forces the limit 0. The same test re-proves, without the power-variance classification, the source's exclusion of the exponent 3/2, and it does not fire on the inverse Gaussian or the Gaussian. An integral-free form of the coincidence condition shows, by an exact computation, that the source's local method is exhausted at its own differential equation: every higher-order local condition through order 16 vanishes identically on the solution class. The result does not touch the mean–median conjecture of Letac, Mattner and Piccioni, which is strictly weaker and remains open. The algebraic reduction, the six-cell split, the two new cumulants, the exponent limits, the exclusions and the controls are verified in Lean 4 against *Mathlib*.

1. Introduction

Let μ be a positive Radon measure on $\mathbb{R}$ that is not a Dirac mass, with Laplace transform $L(\theta) = \int e^{\theta x} \mu(dx)$ finite on a set with nonempty interior Θ. The natural exponential family (NEF) generated by μ is $F(\mu) = \{P_\theta(dx) = e^{\theta x - k(\theta)} \mu(dx) : \theta \in \Theta\}$, $k = \log L$; its mean domain is $M = k'(\Theta)$ and its variance function is $V(m) = k''(\psi(m))$, $\psi = (k')^{-1}$. For the two-sided problem $H_0 : \theta = \theta_0$ with a single observation there are four exact p-values: equal-tail (ET), density-ordered, UMPU, and likelihood-ratio (LR). Bar-Lev and Hoessly [1] ask, for each pair of constructions, for which NEFs the two p-values coincide as functions of the observation at every $\theta_0 \in \Theta$, and they answer every pairing except the two involving LR: the three symmetry pairings characterise the Gaussian family (their Corollary 3.4; the two involving the density ordering under their additional unimodality Assumption 3.1), and the UMPU–LR pairing is the theorem of Bar-Lev, Bshouty and Letac [2] (normal, gamma and inverse Gaussian). For the ET–LR pairing they prove (their Theorem 3.8) that, under their standing Assumption 2.5 — a steep NEF whose generating measure has a strictly positive C^2 Lebesgue density on an open interval $S = M$ — coincidence for every θ forces the mean to be a median of every P_θ, and forces the third-order equation

$$9V^2 V''' - 9VV'V'' + 4(V')^3 = 0 \quad \text{on } M, \qquad \text{equivalently} \qquad (V^{2/3})''' = 0, \tag{1}$$

so that $V(m) = (am^2 + bm + c)^{3/2}$ with $am^2 + bm + c > 0$ on M (their equations (11)–(13)). Their Proposition 3.9 adds the density-at-the-mean identity $f_{\psi(m)}(m) = C\,V(m)^{-1/3}$ (their (19)). Their Corollary 3.11 solves the resulting problem inside the power-variance class $V = A(m - b)^p$: only the Gaussian survives. Their Remark 3.13 shows that (1) alone is not sufficient, by exhibiting the symmetric normal inverse Gaussian NEF with $V = (1 + m^2)^{3/2}$. And they leave the unrestricted question open, in so many words (Remark 3.7 and Remark 3.14; the numbers are those of the compiled e-print, see Section 9):

> "A full one-observation classification for an arbitrary NEF is deliberately not claimed."
>
> "Our ET–LR hypothesis is stronger than mean–median coincidence, as witnessed by the additional constraints (11) and (19), but we do not claim that these conditions alone solve the unrestricted NEF problem."

Their Section 6 lists it first among the open questions: "Most notably, a complete unrestricted $n = 1$ classification of ET–LR and density–LR coincidence outside the power-variance or sample-size-stable settings would be of interest."

This paper settles the ET–LR half of that question.

Theorem 1.1 (Theorem 7.1 below)**.** *Let $F(\mu)$ be a continuous one-parameter NEF satisfying Assumption 2.5 of* [1]*. If p_{ET} and p_{LR} coincide, as functions of the observation, at every $\theta \in \Theta$, then $F(\mu)$ is the Gaussian family up to affine transformation.*

The converse is the source's. No power-variance restriction and no passage to large samples is needed; contrast Theorem 4.6 of [1], which reaches the Gaussian conclusion from coincidence along an unbounded set of sample sizes through an Edgeworth expansion. The proof organises the solution set of (1) and adds one analytic test.

- **The differential condition without fractional powers** (Section 3). Writing $V = u^3$ and $W = u^2 = V^{2/3}$, the operator on the left of (1) factors exactly as $\frac{27}{2}u^7 W'''$; so (1) is equivalent to $W''' = 0$ for $u > 0$, and W is a quadratic. This is the algebraic content of the source's identity (18), with the power 2/3 carried by the substitution.
- **Six affine branches** (Section 4). Under an affine change of the observation the quadratic $W = am^2 + bm + c$ is replaced by $\lambda W((m - \beta)/\alpha)$ with $\lambda > 0$, which multiplies a by λ/α^2 and the discriminant $\mathrm{disc} = b^2 - 4ac$ by $(\lambda/\alpha)^2$. So $(\operatorname{sgn} a, \operatorname{sgn} \mathrm{disc})$ is an invariant of the NEF, and on quadratics that are positive somewhere it takes exactly six values, because $a < 0$ forces $\mathrm{disc} > 0$. The source's Corollary 3.11 covers the cells $a = 0$ (both) and $\mathrm{disc} = 0$, its Remark 3.13 the cell $a > 0, \mathrm{disc} < 0$. The cells $a > 0, \mathrm{disc} > 0$ and $a < 0$ are treated nowhere in the source; the second has a bounded mean domain, which the case split of Corollary 3.11 cannot express (its Remark 3.12).
- **Two new candidate cumulants** (Section 5). For every modulus $c > 0$, $k(\theta) = -\sqrt{\theta^2 - c^2}$ on $\theta < -c$ is strictly convex with mean range in $(1, \infty)$ and variance function $(m^2 - 1)^{3/2}/c$, and $k(\theta) = \sqrt{\theta^2 + c^2}$ on $\mathbb{R}$ is strictly convex with mean range in $(-1, 1)$ and variance function $(1 - m^2)^{3/2}/c$. So neither cell can be dismissed on differential-algebraic grounds: the differential equation does not force the Gaussian.
- **A boundary-atom test** (Section 6). On each new cell $k(\theta) - \theta \to 0$ at the relevant infinity, so $\int e^{\theta(x-1)}\mu(dx) = e^{k(\theta)-\theta} \to 1$; but if μ is Lebesgue-dominated and carried on one side of the point 1, dominated convergence gives the limit 0. Hence neither cell admits a generating measure satisfying Assumption 2.5 (Theorem 6.4 for cell (6) and for cell (5) when $c < 1$; for the remaining moduli the same two ingredients are run along $\theta = -(n + 1)N$ in the proof of Theorem 7.1). The same test gives an elementary proof of the source's exclusion of the power-variance exponent 3/2, which the source obtains by citing the power-variance classification of Bar-Lev–Enis and Jørgensen; and it does *not* fire on the inverse Gaussian (where $k \to -\infty$) or on the Gaussian (where the exponent tends to $+\infty$), which are precisely the cells that must be, respectively cannot be, excluded by other means.
- **The classification** (Section 7) assembles these with the source's Theorem 3.8 and Proposition 3.9. It is proved on paper from formally verified ingredients; it is not itself a single formal statement.
- **Why the source's method stops where it does** (Section 8). Substituting the density identity into the null density gives $f_{\psi(m)}(x) = C\,V(x)^{-1/3}e^{r_m(x)}$ with r_m the source's log generalised

likelihood ratio, and the coincidence condition becomes integral-free: the function $\Xi_m(t) = (\psi(m+t)-\psi(m))(V(m+t)/V(m))^{1/3}$ is an odd function of $\zeta_m(t) = \operatorname{sgn}(t)\sqrt{-2V(m)\, r_m(m+t)}$ (the normalisation only fixes the constants A_j below; oddness is unaffected by rescaling either function by a positive constant). Expanding, the first obstruction is exactly (1) — an independent re-derivation of the source's equation — and, by an exact rational computation, every further obstruction vanishes identically on the solution class $V = (am^2 + bm + c)^{3/2}$ through order 16. Whatever separates the branches is global; that is why the atom test is needed. This section is a computation and is labelled as such.

What is new and what is not. Equation (1), its equivalence with $(V^{2/3})''' = 0$, the quadratic-root form, the density identity, the power-variance corollary and the normal inverse Gaussian counterexample are the source's; the variance function $(1 + m^2)^{3/2}$ is classical (Letac attributes it to his IMPA lecture notes of 1992 in [11, Section 3]); the affine invariance of the coincidence hypothesis is the source's Remark 2.7. What we claim is: the fractional-power-free factorisation (elementary, but what makes the rest mechanical); the six-cell organisation of the solution set, with the observation that $a < 0$ forces disc > 0 and that the $a < 0$ cell has a bounded mean domain; the two new branches with their cumulants; the exclusion of both by the boundary-atom test, which is the kernel-checked heart of the paper; the classification theorem the source declines to state; the elementary re-proof of the exponent-3/2 exclusion; and the integral-free form of the coincidence condition with the exhaustion of the local hierarchy.

Two scope statements, stated because they are easy to overclaim. First, Theorem 1.1 does *not* settle the mean–median conjecture of Letac, Mattner and Piccioni [7]: their theorem characterises $N(0, 1)$ by the natural parameter being a median of every tilt, and the mean–median version is stated by them as an open conjecture ("This conjecture … seems harder to establish"); ET–LR coincidence is strictly stronger than mean–median coincidence, as the source records in Remark 3.14, so nothing here bears on it. Second, the density–LR pairing, the source's other open edge, is untouched.

The literature. The class $V = (am^2 + bm + c)^{3/2}$ lies in the class Kokonendji calls *grand-Babel* (his Definition 1.1): $V = R\Delta + Q\sqrt{\Delta}$ with $\deg R \le 1$, $\deg Q \le 2$, $\deg \Delta \le 2$ (here $R = 0$, $Q = \Delta$, $\deg Q = \deg \Delta = 2$). It contains Letac's Babel class, whence the name, and our V is outside all four of its fully described subclasses (Morris: V quadratic [4]; Mora: V cubic [3]; Babel: $\deg Q \le 1$; Seshadri: $\deg \Delta = 1$). Kokonendji [9] writes, at the end of his introduction, of the examples he is about to give: "Ces exemples sont toutefois insuffisants pour pouvoir décrire tous les types de la classe de grand-Babel"; Vandal's compendium [10] says the same in English. We did not find either new branch in the literature: a full-text sweep of a local corpus of arXiv e-prints (Section 9), OpenAlex (the source has no citing work as of 2026-09-07; [7] has three, all read; [8] none; [2] one, Kulinskaya [14], read in full and containing no ET–LR characterisation), Semantic Scholar and the web return nothing about the cumulants $-\sqrt{\theta^2 - c^2}$ and $\sqrt{\theta^2 + c^2}$ as NEF cumulants. One coverage limit is stated because it matters: Letac's IMPA 1992 lecture notes [12] are not online and were not read; they are the one place a table of the class $\Delta^{3/2}$ could hide. If either branch is tabulated there, Theorems 5.2 and 5.3 lose their novelty, but Theorem 6.4 and Theorem 7.1 do not, since no source asserts the exclusion or the classification. The nearest published relative of the present result is [2], which settles a different pairing by a different differential equation.

Verification. The factorisation, the six-cell split, the two new cumulants and their variance functions, the dominated-convergence lemma, the exponent limits, the two exclusions and every control are formalised in Lean 4 against *Mathlib*; every theorem below marked with a Lean name in its proof is machine-checked, its statement is reproduced verbatim in Appendix A, and Section 9 records modules, counts and axioms. The classification (Theorem 7.1) and the integral-free form (Proposition 8.1) are proved on paper; the p-values themselves are not formalised, and the source's Theorem 3.8 and Proposition 3.9 enter as hypotheses.

2. Preliminaries

We use the notation of [1, Section 2.3]. μ is a non-Dirac positive Radon measure on $\mathbb{R}$, $L(\theta) = \int e^{\theta x}\mu(dx)$, $D = \{L < \infty\}$, $\Theta = \operatorname{int} D \neq \varnothing$, $k = \log L$ on Θ (strictly convex and real analytic), $m(\theta) = k'(\theta)$ is the mean of P_θ, $k''(\theta) > 0$ its variance, $M = k'(\Theta)$, $\psi = (k')^{-1} : M \to \Theta$, and $V(m) = k''(\psi(m)) = 1/\psi'(m)$ on M. With C the convex support of μ the NEF is *steep* when $M = \operatorname{int} C$.

Assumption 2.1 (Assumption 2.5 of [1])**.** $F(\mu)$ is steep, and $\mu(dx) = h(x)\,dx$ with h strictly positive and twice continuously differentiable on an open connected support interval S; thus $S = \operatorname{int} C = M$.

For $\theta_0 \in \Theta$ with $m_0 = k'(\theta_0)$ and null law P_{θ_0} with distribution function F_0, the equal-tail p-value of an observation x is $p_{\mathrm{ET}}(x) = 2\min\{F_0(x), 1 - F_0(x)\}$ (the source's (3)); the log generalised likelihood ratio is $r_{\theta_0}(x) = (\theta_0 - \psi(x))x - k(\theta_0) + k(\psi(x))$ (the source's (7)), strictly concave with $r''_{\theta_0} = -\psi' = -1/V < 0$ and maximum 0 at $x = m_0$, and $p_{\mathrm{LR}}(x) = P_{\theta_0}\{r_{\theta_0}(X) \le r_{\theta_0}(x)\}$. Both are exact under Assumption 2.1 (the source's Lemma 2.6). We write r_m for $r_{\psi(m)}$.

Affine equivalence and the generating-measure ambiguity. If $Y = \alpha X + \beta$ with $\alpha \neq 0$, the NEF $F(\nu)$ of Y has $M_\nu = \alpha M_\mu + \beta$ and

$$V_\nu(y) = \alpha^2\, V_\mu\Big(\frac{y-\beta}{\alpha}\Big) \tag{2}$$

([3, Proposition 2.4(iv),(v)]), Assumption 2.1 is preserved, and each of the four p-values is preserved, so that every coincidence relation holds throughout $F(\mu)$ iff it holds throughout $F(\nu)$ (the source's Remark 2.7). Replacing μ by $Ce^{ax}\mu$ ($C > 0$, $a \in \mathbb{R}$) does not change the family: it translates Θ by $-a$, replaces $k(\theta)$ by $k(\theta + a) - \log C$, leaves V and M unchanged, and preserves Assumption 2.1 (the source's Remark 3.10(i)). We use both normalisations freely.

Fact 2.2 (the variance function determines the family)**.** *If two NEFs satisfy $V_1 = V_2$ on a nonempty open subinterval of $M_1 \cap M_2$, then they coincide.*

This is classical: it is Proposition 2.2(iv) of Letac and Mora [3, p. 5], stated there in exactly this form and with the proof omitted as "very much in the spirit of Morris (1982)". Morris gives the construction as running text in his Section 2 [4, p. 67], in the form that the pair (Ω, V) — the variance function *together with* its domain — determines the cumulant generating function, with the explicit warning that V alone need not (his example is $V(\mu) = \mu^2$ on $(-\infty, 0)$ and on $(0, \infty)$). The argument: on the subinterval $\psi' = 1/V$ determines ψ up to an additive constant, hence $k' = \psi^{-1}$ up to a translation of θ and k up to an additive constant, and $L = e^k$ on an open interval determines μ by the uniqueness of Laplace transforms; the translation and the constant are exactly the ambiguity $\mu \mapsto Ce^{ax}\mu$. Only the first two steps are used in the cells of Section 7 where no second NEF is known to exist: there $\psi' = 1/V$ on M determines k on Θ up to that ambiguity, which is all the proof of Theorem 7.1 needs.

The two inputs from the source. We use two results of [1] as black boxes. Their statement numbers, equation numbers and quotations were read off the e-print compiled from the arXiv source.

Theorem 2.3 ([1], Theorem 3.8)**.** *Suppose Assumption 2.1 holds and $p_{\mathrm{ET}} = p_{\mathrm{LR}}$ at every $\theta \in \Theta$ as functions of the observation. Then (i) $m(\theta)$ is a median of P_θ for every θ; (ii)* (1) *holds on M; (iii) equivalently $(V^{2/3})''' = 0$ on M, so there are real a, b, c with $V(m) = (am^2 + bm + c)^{3/2}$ and $am^2 + bm + c > 0$ on M.*

Proposition 2.4 ([1], Proposition 3.9)**.** *Under the same hypotheses there is a constant $C > 0$ with $f_{\psi(m)}(m) = C\,V(m)^{-1/3}$ for $m \in M$; equivalently $h(x) = C\,V(x)^{-1/3}\exp\{k(\psi(x)) - x\psi(x)\}$ on S.*

Throughout, $\mathcal{D}$ denotes the source's operator, evaluated on a value and three derivative values:

$$\mathcal{D}(V, V_1, V_2, V_3) = 9V^2V_3 - 9VV_1V_2 + 4V_1^3, \qquad \mathcal{D}[V](m) = \mathcal{D}\big(V(m), V'(m), V''(m), V'''(m)\big). \tag{3}$$

3. The differential condition without fractional powers

Every positive V is u^3 for $u = V^{1/3} > 0$, and then $V^{2/3} = u^2$. The chain rule gives $V' = 3u^2u'$, $V'' = 6u(u')^2 + 3u^2u''$, $V''' = 6(u')^3 + 18uu'u'' + 3u^2u'''$, and for $W = u^2$, $W''' = 6u'u'' + 2uu'''$.

Theorem 3.1 (the operator factors)**.** *For all real u, u_1, u_2, u_3,*

$$\mathcal{D}\big(u^3,\ 3u^2u_1,\ 6uu_1^2 + 3u^2u_2,\ 6u_1^3 + 18uu_1u_2 + 3u^2u_3\big) = \tfrac{27}{2}\, u^7\,(6u_1u_2 + 2uu_3).$$

The identity is not vacuous: at $(u, u_1, u_2, u_3) = (1, 0, 0, 1)$ both sides equal 27*.*

Proof. A polynomial identity in four variables (formal: `etlrOde_cube`, discharged by `ring`; the control is `etlrOde_cube_nonvacuous`). In terms of V it reads $\mathcal{D}[V] = \frac{27}{2}V^{7/3}(V^{2/3})'''$, which is the source's identity (18); the substitution $V = u^3$ removes every fractional power. □

Theorem 3.2 (reduction to a vanishing third derivative)**.** (i) *Let $u, u_1, u_2, u_3 : \mathbb{R} \to \mathbb{R}$ with $u > 0$ everywhere. Then $\mathcal{D}(u^3, 3u^2u_1, 6uu_1^2 + 3u^2u_2, 6u_1^3 + 18uu_1u_2 + 3u^2u_3) = 0$ at every point if and only if $6u_1u_2 + 2uu_3 = 0$ at every point.*

(ii) *Let $W, W_1, W_2 : \mathbb{R} \to \mathbb{R}$ with $W' = W_1$, $W_1' = W_2$ and $W_2' = 0$ at every point. Then $W(x) = \frac{1}{2}W_2(0)\,x^2 + W_1(0)\,x + W(0)$ for every x.*

Proof. (i) is Theorem 3.1 together with $\frac{27}{2}u^7 \neq 0$ (formal: `etlrOde_zero_iff_third_deriv_zero`); no differentiability is assumed in (i), the u_i being arbitrary functions substituted into the chain-rule expressions. (ii) is three applications of "a function with zero derivative on $\mathbb{R}$ is constant" (formal: `quadratic_of_third_deriv_zero`). Together they are the source's equivalence (11)⇔(12). □

Corollary 3.3 (the quadratic-root form)**.** *Let $u > 0$ be three times differentiable on $\mathbb{R}$ with $u' = u_1$, $u_1' = u_2$, $u_2' = u_3$, and suppose $\mathcal{D}(u^3, 3u^2u_1, 6uu_1^2 + 3u^2u_2, 6u_1^3 + 18uu_1u_2 + 3u^2u_3) = 0$ everywhere. Then there are real a, b, c with $ax^2 + bx + c > 0$ for every x and*

$$u(x)^3 = (ax^2 + bx + c)\sqrt{ax^2 + bx + c} \qquad \textit{for every } x \in \mathbb{R}.$$

Proof. Theorem 3.2(i) gives $W''' = 0$ for $W = u^2$, (ii) makes W a quadratic with $W(x) = u(x)^2 > 0$, and $u^3 = W\sqrt{W}$ (formal: `cube_eq_quadratic_pow_of_etlrOde_zero`). This is the source's (13). The formal statements are on all of $\mathbb{R}$; on a mean domain $M \neq \mathbb{R}$ the same argument applies verbatim, and in Section 7 the interval form is simply Theorem 2.3(iii). □

4. Six affine branches

By (2), an affine change $x \mapsto \alpha x + \beta$ of the observation sends $W = V^{2/3}$ to $W_\nu(y) = \alpha^{4/3}\, W((y-\beta)/\alpha)$, i.e. to $\lambda\, W((y - \beta)/\alpha)$ with $\lambda = \alpha^{4/3} > 0$. Write $\operatorname{disc}(a, b, c) = b^2 - 4ac$.

Proposition 4.1 (affine invariance of the sign data)**.** *Let $\alpha \neq 0$, $\beta, \lambda \in \mathbb{R}$. The quadratic $n \mapsto \lambda\,(a((n-\beta)/\alpha)^2 + b((n-\beta)/\alpha) + c)$ has coefficients*

$$a' = \frac{\lambda a}{\alpha^2}, \qquad b' = -\frac{2\lambda a\beta}{\alpha^2} + \frac{\lambda b}{\alpha}, \qquad c' = \frac{\lambda a\beta^2}{\alpha^2} - \frac{\lambda b\beta}{\alpha} + \lambda c,$$

its discriminant is $\operatorname{disc}(a', b', c') = (\lambda/\alpha)^2 \operatorname{disc}(a, b, c)$, and its leading coefficient is $(\lambda/\alpha^2)\, a$ with $\lambda/\alpha^2 > 0$ whenever $\lambda > 0$. Consequently the pair $(\operatorname{sgn} a, \operatorname{sgn} \operatorname{disc})$ of the quadratic $W = V^{2/3}$ is an affine invariant of the NEF.

Proof. Three rational identities (formal: `affine_transform_eq`, `disc_affine`, `leading_affine`). The source records the affine invariance of the coincidence hypothesis qualitatively (Remark 2.7); this is its quantitative form on the variance-function side, and it is what makes the split below a split of families rather than of coordinates. □

Theorem 4.2 (exactly six cells)**.** *Let $a, b, c \in \mathbb{R}$ and suppose $am_0^2 + bm_0 + c > 0$ for some m_0.*

(i) *If $a < 0$ then* $\operatorname{disc}(a, b, c) > 0$*.*

(ii) *Exactly one of the following holds:*

$$\begin{array}{lll} (1)\ a=0,\ b=0; & (2)\ a=0,\ b\neq 0; & (3)\ a>0,\ \mathrm{disc}<0; \\ (4)\ a>0,\ \mathrm{disc}=0; & (5)\ a>0,\ \mathrm{disc}>0; & (6)\ a<0,\ \mathrm{disc}>0. \end{array}$$

Proof. (i): $4a\,(am_0^2+bm_0+c)=(2am_0+b)^2-\mathrm{disc}$, and the left side is negative (formal: `disc_pos_of_neg_leading`). (ii): trichotomy on a, then on b or on disc, with (i) killing the two cells $a<0, \mathrm{disc}\le 0$ (formal: `six_branches`); the cells are pairwise disjoint (formal: `six_branches_disjoint`, which records the five incompatibilities not immediate from the trichotomies). ☐

Proposition 4.3 (normal forms)**.** *Let $d=\mathrm{disc}(a,b,c)$ and $m_1=-b/(2a)$ when $a\neq 0$.*

(3) *If $a>0$ and $d<0$: $am^2+bm+c=a\big((m-m_1)^2+\frac{-d}{4a^2}\big)$ with $\frac{-d}{4a^2}>0$.*
(4) *If $a>0$ and $d=0$: $am^2+bm+c=a(m-m_1)^2$.*
(5) *If $a>0$ and $d>0$: $am^2+bm+c=a\big((m-m_1)^2-\frac{d}{4a^2}\big)$ with $\frac{d}{4a^2}>0$.*
(6) *If $a<0$ and $am_0^2+bm_0+c>0$ for some m_0: $am^2+bm+c=(-a)\big(\frac{d}{4a^2}-(m-m_1)^2\big)$ with $\frac{d}{4a^2}>0$, and every m with $am^2+bm+c>0$ satisfies $(m-m_1)^2<\frac{d}{4a^2}$: the set where the quadratic is positive is a bounded interval, of half-length $\sqrt{d}/(2|a|)$.*

Proof. Completing the square (formal: `normal_form_nig`, `normal_form_invGaussian`, `normal_form_hyperbolicOuter`, `normal_form_hyperbolicInner`); the boundedness is `hyperbolicInner_bounded`. ☐

Since the mean domain M is an interval on which $W>0$, Proposition 4.1 and the normal forms give, after an affine change of the observation (and a reflection where needed), the following representatives. Here $c>0$ is a free modulus in cells (3), (5), (6), where it is the one affine invariant left inside the cell; in cells (1), (2), (4) the scale is affinely removable.

cell	sign data	$V(m)$	$M\subseteq$	family	in [1]
(1)	$a=b=0$	σ^2	$\mathbb{R}$	Gaussian	Cor. 3.11(a): the answer
(2)	$a=0,\ b\neq 0$	$m^{3/2}/c$	$(0,\infty)$	power variance p=3/2	Cor. 3.11: out, by citation
(3)	$a>0,\ \mathrm{disc}<0$	$(1+m^2)^{3/2}/c$	$\mathbb{R}$	symmetric NIG	Rem. 3.13 ($c=1$): out, by (19)
(4)	$a>0,\ \mathrm{disc}=0$	m^3/c	$(0,\infty)$	inverse Gaussian	Cor. 3.11: out, by (19)
(5)	$a>0,\ \mathrm{disc}>0$	$(m^2-1)^{3/2}/c$	$(1,\infty)$	**new**	not treated
(6)	$a<0$	$(1-m^2)^{3/2}/c$	$(-1,1)$	**new**	not expressible

The last column is the point. Corollary 3.11 of the source is a statement about the power-variance class, which meets the solution set of (1) in cells (1), (2) and (4) only (W constant, linear, or a perfect square); Remark 3.13 treats cell (3) at $c=1$. Cell (6) is the only cell with a bounded mean domain, and the case split of Corollary 3.11 — $M=\mathbb{R}$ in case (a), $M=(b,\infty)$ in case (b), with Remark 3.12 explaining that "no affine transformation maps $\mathbb{R}$ onto a half-line" — cannot even express it. This is a coverage gap, not an error: nothing in the source is wrong, and every number in it that we recomputed reproduces (Section 8).

5. The three square-root branches: explicit cumulants

In cells (3), (5), (6) the representative V is $(\pm 1\pm m^2)^{3/2}/c$, and solving $\psi'=1/V$ and inverting gives the candidate cumulant in closed form:

cell	$V(m)$	$\psi(m)$	$k(\theta)$	Θ	M
(1)	σ^2	m/σ^2	$\sigma^2\theta^2/2$	$\mathbb{R}$	$\mathbb{R}$
(2)	$m^{3/2}/c$	$-2c\,m^{-1/2}$	$-4c^2/\theta$	$(-\infty,0)$	$(0,\infty)$
(3)	$(1+m^2)^{3/2}/c$	$cm/\sqrt{1+m^2}$	$-\sqrt{c^2-\theta^2}$	$(-c,c)$	$\mathbb{R}$
(4)	m^3/c	$-c/(2m^2)$	$-\sqrt{-2c\theta}$	$(-\infty,0)$	$(0,\infty)$
(5)	$(m^2-1)^{3/2}/c$	$-cm/\sqrt{m^2-1}$	$-\sqrt{\theta^2-c^2}$	$(-\infty,-c)$	$(1,\infty)$
(6)	$(1-m^2)^{3/2}/c$	$cm/\sqrt{1-m^2}$	$\sqrt{\theta^2+c^2}$	$\mathbb{R}$	$(-1,1)$

(Each ψ is determined up to an additive constant and each k up to an additive constant, the ambiguity of Section 2; the table fixes both. The columns Θ and M are those of the displayed k; whether the mean domain of an NEF in the cell is all of the displayed M is part of the argument of Section 7.) Row (3) is the source's Remark 3.13 with the modulus restored: the source states it at $c = 1$ with $k(\theta) = 1 - \sqrt{1-\theta^2}$. Rows (5) and (6) appear nowhere in the source. What is proved in this section is that all three square-root rows are strictly convex cumulant candidates with the displayed variance function and the displayed mean range; what separates them is analytic and is Section 6.

Proposition 5.1 (cell (3), the symmetric normal inverse Gaussian; known)**.** *Let $c > 0$ and put $k(\theta) = -\sqrt{c^2-\theta^2}$, $k_1(\theta) = \theta/\sqrt{c^2-\theta^2}$, $k_2(\theta) = c^2/\big((c^2-\theta^2)\sqrt{c^2-\theta^2}\big)$ and $V(m) = (1+m^2)\sqrt{1+m^2}/c$. For $|\theta| < c$: $k'(\theta) = k_1(\theta)$, $k_1'(\theta) = k_2(\theta)$, $k_2(\theta) > 0$, and $V(k_1(\theta)) = k_2(\theta)$.*

Theorem 5.2 (cell (5), the outer hyperbolic branch)**.** *Let $c > 0$ and put $k(\theta) = -\sqrt{\theta^2-c^2}$, $k_1(\theta) = -\theta/\sqrt{\theta^2-c^2}$, $k_2(\theta) = c^2/\big((\theta^2-c^2)\sqrt{\theta^2-c^2}\big)$ and $V(m) = (m^2-1)\sqrt{m^2-1}/c$. Wherever $\theta^2 > c^2$: $k'(\theta) = k_1(\theta)$, $k_1'(\theta) = k_2(\theta)$, $k_2(\theta) > 0$, and $V(k_1(\theta)) = k_2(\theta)$. For $\theta < -c$: $k_1(\theta) > 1$. Thus on $\theta < -c$, k is a strictly convex function whose mean function takes values in $(1,\infty)$ and whose variance function is $(m^2-1)^{3/2}/c$, i.e. $V^{2/3} = (m^2-1)/c^{2/3}$, a quadratic with $a > 0$ and* $\operatorname{disc} > 0$.

Theorem 5.3 (cell (6), the inner hyperbolic branch)**.** *Let $c > 0$ and put $k(\theta) = \sqrt{\theta^2+c^2}$, $k_1(\theta) = \theta/\sqrt{\theta^2+c^2}$, $k_2(\theta) = c^2/\big((\theta^2+c^2)\sqrt{\theta^2+c^2}\big)$ and $V(m) = (1-m^2)\sqrt{1-m^2}/c$. For every $\theta \in \mathbb{R}$: $k'(\theta) = k_1(\theta)$, $k_1'(\theta) = k_2(\theta)$, $k_2(\theta) > 0$, $|k_1(\theta)| < 1$, and $V(k_1(\theta)) = k_2(\theta)$. Thus k is a strictly convex function on all of $\mathbb{R}$ whose mean function takes values in the bounded interval $(-1,1)$ and whose variance function is $(1-m^2)^{3/2}/c$, i.e. $V^{2/3} = (1-m^2)/c^{2/3}$, a quadratic with $a < 0$.*

Proof of Proposition 5.1 and Theorems 5.2, 5.3. All three are instances of the same two derivative formulas: for $p\theta^2+q > 0$, $\frac{d}{d\theta}\sqrt{p\theta^2+q} = p\theta/\sqrt{p\theta^2+q}$ and $\frac{d}{d\theta}\frac{p\theta}{\sqrt{p\theta^2+q}} = \frac{pq}{(p\theta^2+q)\sqrt{p\theta^2+q}}$, with $(p,q) = (-1,c^2)$, $(1,-c^2)$, $(1,c^2)$ and the sign of k chosen so that $k_2 > 0$. The variance-function identity is the algebraic fact $1+k_1^2 = c^2/(c^2-\theta^2)$, resp. $k_1^2-1 = c^2/(\theta^2-c^2)$, resp. $1-k_1^2 = c^2/(\theta^2+c^2)$, whose square root is $c/\sqrt{\cdots}$, so that $V(k_1) = \big(c^2/(\cdots)\big)\cdot\big(c/\sqrt{\cdots}\big)/c = k_2$. The mean ranges: for $\theta < -c$, $\sqrt{\theta^2-c^2} < -\theta$ gives $k_1 > 1$; for all θ, $\theta^2 < \theta^2+c^2$ gives $|k_1| < 1$. (Formal: `kNig_hasDerivAt`, `kNig₁_hasDerivAt`, `vNig_kNig₁`, `kNig₂_pos`; `kOut_hasDerivAt`, `kOut₁_hasDerivAt`, `vOut_kOut₁`, `one_lt_kOut₁`, `kOut₂_pos`; `kIn_hasDerivAt`, `kIn₁_hasDerivAt`, `vIn_kIn₁`, `abs_kIn₁_lt_one`, `kIn₂_pos`.) The derivation of the table — integrate $\psi' = c(\pm1\pm m^2)^{-3/2}$, invert, integrate again — is the paper step that produced the candidates; the formal statements verify the result. ☐

Proposition 5.4 (each cell is realised; the modulus is irrelevant to the equation)**.**

(i) *For all real $\lambda, V, V_1, V_2, V_3$,*

$$\mathcal{D}(\lambda V, \lambda V_1, \lambda V_2, \lambda V_3) = \lambda^3\,\mathcal{D}(V, V_1, V_2, V_3).$$

(ii) *For $c > 0$: $\sqrt{1+m^2}^{\,3} = c\,V_{(3)}(m)$ with $\sqrt{1+m^2}^{\,2} = 1\cdot m^2 + 0\cdot m + 1$ and* $\operatorname{disc}(1,0,1) < 0$*; for $m > 1$, $\sqrt{m^2-1}^{\,3} = c\,V_{(5)}(m)$ with $\sqrt{m^2-1}^{\,2} = 1\cdot m^2 + 0\cdot m - 1$ and* $\operatorname{disc}(1,0,-1) > 0$*; for $|m| < 1$, $\sqrt{1-m^2}^{\,3} = c\,V_{(6)}(m)$ with $\sqrt{1-m^2}^{\,2} = -1\cdot m^2 + 0\cdot m + 1$ and $-1 < 0$. Here $V_{(j)}$ is the variance function of the corresponding row above.*

Proof. (i) is homogeneity of degree three (formal: `etlrOde_smul`); so (cV) satisfies (1) iff V does, and the quadratic-root condition can be checked at $c = 1$. (ii) is $\sqrt{q}^{\,3} = q\sqrt{q}$ and $\sqrt{q}^{\,2} = q$ for $q \geq 0$ (formal: `vNig_quadratic`, `vOut_quadratic`, `vIn_quadratic`). With Theorem 4.2 this shows that cells (3), (5), (6) are non-empty as cells of variance functions of the stated invariants, with explicit witnesses. ☐

6. The boundary-atom test

Let μ generate an NEF with $\int e^{\theta x}\mu(dx) = e^{k(\theta)}$ on Θ, and suppose μ is carried by $(-\infty,\beta]$. Then

$$\int e^{\theta(x-\beta)}\mu(dx) = e^{k(\theta)-\beta\theta} \longrightarrow \mu(\{\beta\}) \qquad (\theta \to +\infty)$$

by dominated convergence, the integrand decreasing pointwise to the indicator of $\{\beta\}$. If μ is Lebesgue-dominated the limit is 0, because $\{\beta\}$ is Lebesgue-null. So a strictly positive limit of $e^{k(\theta)-\beta\theta}$ contradicts Assumption 2.1. The two halves of the test are the following.

Lemma 6.1 (the Lebesgue-dominated side)**.** *Let μ be a measure on $\mathbb{R}$.*

(i) *If g is μ-a.e. strongly measurable, $g < 0$ μ-a.e., and e^g is μ-integrable, then $\int e^{(n+1)g(x)}\mu(dx) \to 0$ as $n \to \infty$.*
(ii) *If $\mu \ll$ Lebesgue, $x \le \beta$ for μ-a.e. x, and $e^{x-\beta}$ is μ-integrable, then $\int e^{(n+1)(x-\beta)}\mu(dx) \to 0$.*
(iii) *If $\mu \ll$ Lebesgue, $x \ge \beta$ for μ-a.e. x, and $e^{\beta-x}$ is μ-integrable, then $\int e^{(n+1)(\beta-x)}\mu(dx) \to 0$.*

Proof. (i): $e^{(n+1)g} = (e^g)^{n+1} \to 0$ pointwise where $g < 0$, dominated by e^g (formal: `tendsto_integral_exp_zero`, via Mathlib's `tendsto_integral_of_dominated_convergence`). (ii), (iii): $\mu(\{\beta\}) = 0$ by absolute continuity, so $x - \beta < 0$, resp. $\beta - x < 0$, μ-a.e. (formal: `tendsto_integral_zero_of_ac_right`, `tendsto_integral_zero_of_ac_left`). ☐

Proposition 6.2 (the exponent limits)**.** *Let $c > 0$.*

(i) *(inner branch) $\sqrt{\theta^2 + c^2} - \theta \to 0$ as $\theta \to +\infty$.*
(ii) *(outer branch) $-\sqrt{\theta^2 - c^2} - \theta \to 0$ as $\theta \to -\infty$.*
(iii) *(exponent 3/2) $-4c^2/\theta \to 0$ as $\theta \to -\infty$ (for every real c).*

Consequently $e^{k(\theta)-\theta} \to 1$ along $\theta \to +\infty$ on cell (6), along $\theta \to -\infty$ on cell (5), and $e^{k(\theta)} \to 1$ along $\theta \to -\infty$ on cell (2).

Proof. (i) $0 \le \sqrt{\theta^2 + c^2} - \theta = c^2/(\sqrt{\theta^2 + c^2} + \theta) \le c^2/(2\theta)$ for $\theta > 0$; (ii) $0 \le -\sqrt{\theta^2 - c^2} - \theta = c^2/(\sqrt{\theta^2 - c^2} - \theta) \le c^2/(-\theta)$ for $\theta < -c$; (iii) is $1/\theta \to 0$ (formal: `tendsto_kIn_sub_id`, `tendsto_kOut_sub_id`, `tendsto_kTweedie`). (i) and (ii) are new in context only because the source never writes the two cumulants down; (iii) is an elementary route to an exclusion the source obtains by citation, see Corollary 6.5. ☐

Proposition 6.3 (controls: no atom on the inverse Gaussian or the Gaussian)**.** *For $c > 0$, $-\sqrt{-2c\theta} \to -\infty$ as $\theta \to -\infty$; and for $\sigma > 0$ and every $\beta \in \mathbb{R}$, $\sigma^2\theta^2/2 - \beta\theta \to +\infty$ as $\theta \to +\infty$.*

Proof. Formal: `tendsto_kInvGauss`, `tendsto_kGauss`. On cell (4), $e^{k(\theta)} \to 0$: the Laplace transform of the inverse Gaussian generating measure vanishes at $-\infty$, there is no boundary atom, and the test says nothing — as it must, since that cell is excluded by the density identity (Proposition 2.4; the source's Remark 3.10(ii)). On cell (1) the exponent tends to $+\infty$ for every candidate boundary: the Gaussian generating measure has unbounded support and the test does not apply — as it must, since the Gaussian is the answer. So the test fires on exactly three of the six cells, and they are the three it should fire on. ☐

Theorem 6.4 (neither new cell admits a Lebesgue-dominated generating measure)**.** *Let $c > 0$, let μ be a measure on $\mathbb{R}$ with $\mu \ll$ Lebesgue, and write $k_{\mathrm{in}}(\theta) = \sqrt{\theta^2 + c^2}$, $k_{\mathrm{out}}(\theta) = -\sqrt{\theta^2 - c^2}$.*

(i) *(cell (6)) It is impossible that $x \le 1$ for μ-a.e. x, that e^{x-1} is μ-integrable, and that*

$$\int e^{(n+1)(x-1)}\,\mu(dx) = \exp\bigl(k_{\mathrm{in}}(n+1) - (n+1)\bigr) \qquad \text{for every } n \in \mathbb{N}.$$

(ii) *(cell (5)) It is impossible that $x \ge 1$ for μ-a.e. x, that e^{1-x} is μ-integrable, and that*

$$\int e^{(n+1)(1-x)}\,\mu(dx) = \exp\bigl(k_{\mathrm{out}}(-(n+1)) + (n+1)\bigr) \qquad \text{for every } n \in \mathbb{N}.$$

Proof. In (i) the left side tends to 0 by Lemma 6.1(ii) with $\beta = 1$ and the right side to $e^0 = 1$ by Proposition 6.2(i) along $\theta = n + 1$; in (ii) the same with Lemma 6.1(iii) and Proposition 6.2(ii) along $\theta = -(n+1)$ (formal: `inner_branch_not_lebesgue`, `outer_branch_not_lebesgue`). ☐

The four hypotheses are the data of an NEF of the cell satisfying Assumption 2.1: Lebesgue domination; the one-sided support forced by the mean domain of the branch ($M \subseteq (-1, 1)$, resp. $M \subseteq (1, \infty)$, and $S = M$); a point of the canonical domain; and the defining Laplace identity along a sequence tending to the relevant infinity. The theorem says the four are jointly impossible. Two caveats, both discharged in Section 7. First, the derivation of the support hypothesis and of the canonical domain from the branch is a paper step, carried out in the proof of Theorem 7.1. Second, the sequence here is $\theta = \pm(n+1)$, which presumes $\pm 1 \in \Theta$: on cell (6) that is automatic, because $\Theta = \mathbb{R}$ there, but on cell (5) one has $\Theta \subseteq (-\infty, -c)$, so $-1 \in \Theta$ forces $c < 1$, and part (ii) is therefore applicable as it stands only to that part of the cell. The whole of cell (5) is excluded in the proof of Theorem 7.1 by the same two ingredients run along $-(n+1)N$ instead — Lemma 6.1(i), which is stated for an arbitrary negative g, together with Proposition 6.2(ii), which is a limit along the full filter at $-\infty$; see Remark 7.2.

Corollary 6.5 (the exponent-3/2 cell, elementarily)**.** *Let $\mu \ll$ Lebesgue be carried by $[0,\infty)$ with e^{-x} μ-integrable and $\int e^{\theta x}\mu(dx) = e^{-4c^2/\theta}$ for $\theta = -1, -2, \ldots$. This is impossible. Hence cell (2) contains no NEF satisfying Assumption 2.1 whose canonical domain contains -1; the cell is excluded for every canonical domain in the proof of Theorem 7.1, by the same argument run along $-(n+1)N$.*

Proof. Lemma 6.1(iii) with $\beta = 0$ gives $\int e^{-(n+1)x}\mu(dx) \to 0$, while Proposition 6.2(iii) gives $e^{-4c^2/\theta} \to 1$ along $\theta = -(n+1)$. The assembly of the two formal statements is a paper step (there is no separate Lean declaration for this corollary). The source excludes this cell in the proof of its Corollary 3.11 by invoking the power-variance classification of Bar-Lev–Enis [5] in Jørgensen's formulation [6] ("the exponent $p = 3/2 \in (1,2)$ belongs to the compound-Poisson–gamma range and therefore has an atom at the boundary b"); the limit above exhibits the atom directly from $k(\theta) = -4c^2/\theta$, with no classification theorem. □

Proposition 6.6 (controls on the hypotheses)**.**
- (i) *Without absolute continuity the conclusion of Lemma 6.1(ii) is false: for the Dirac mass at 1, $\int e^{(n+1)(x-1)}\,\delta_1(dx) = 1$ for every n.*
- (ii) *None of the three square-root variance functions is constant: $V_{(3)}(0) \neq V_{(3)}(3/4)$, $V_{(5)}(5/4) \neq V_{(5)}(5/3)$, $V_{(6)}(0) \neq V_{(6)}(4/5)$ (with values $1/c$ and $125/(64c)$; $27/(64c)$ and $64/(27c)$; $1/c$ and $27/(125c)$).*

Proof. (i) is the atom the test detects (formal: `dirac_integral_eq_one`); it shows the absolute-continuity hypothesis of Theorem 6.4 is load-bearing. (ii) refutes the too-small claim that (1) alone forces the Gaussian: all three are genuinely non-constant solutions of the quadratic-root condition (formal: `vNig_not_constant`, `vOut_not_constant`, `vIn_not_constant`). For cell (3) this is the source's Remark 3.13 ("not merely an algebraically admissible solution but a genuine variance function"), extended to both new cells. □

7. The classification

Theorem 7.1 (the unrestricted one-observation ET–LR classification; proved on paper, not formalised as a single statement)**.** *Let $F(\mu)$ be a continuous one-parameter NEF satisfying Assumption 2.1. If p_{ET} and p_{LR} coincide as functions of the observation at every $\theta \in \Theta$, then $F(\mu)$ is the Gaussian family up to affine transformation. Conversely the Gaussian family has $p_{\mathrm{ET}} = p_{\mathrm{LR}}$ at every θ* [1, Corollary 3.11].

Proof. By Theorem 2.3(iii), $V = W^{3/2}$ on M with $W = am^2 + bm + c$ a quadratic positive on M; by Theorem 4.2, (a, b, c) lies in exactly one of the six cells, and by Proposition 4.1 the cell is an affine invariant, so by the source's Remark 2.7 we may replace $F(\mu)$ by any affine transform. Using Proposition 4.3 and a reflection where needed, V on M is one of the six normal forms of the table of Section 4, with M contained in the interval of positivity displayed there (an interval, since M is connected). By Fact 2.2, $F(\mu)$ is then the NEF of the cumulant in the table of Section 5, up to the ambiguity $\mu \mapsto Ce^{ax}\mu$, which we use to normalise k to the displayed form; the mean domain M is then a subinterval of the displayed M and $\Theta = \psi(M)$.

Cell (1) is the Gaussian family, where coincidence holds.

Cell (2). $V = m^{3/2}/c$ on $M \subseteq (0,\infty)$ and $k(\theta) = -4c^2/\theta$ on $\Theta \subseteq (-\infty, 0)$. Since $S = M$, $\operatorname{supp}\mu \subseteq [0,\infty)$; for $x \geq 0$ and $\theta < \theta_1 \in \Theta$, $e^{\theta x} \leq e^{\theta_1 x}$, so L is finite on $(-\infty, \theta_1]$ and $(-\infty, \theta_1) \subseteq \Theta$; in particular $\inf\Theta = -\infty$. Fix an integer $N \geq 1$ with $-N \in \Theta$. Then e^{-Nx} is μ-integrable, $x > 0$ μ-a.e. (as $\mu(\{0\}) = 0$), and Lemma 6.1(i) with $g(x) = -Nx$ gives $\int e^{-(n+1)Nx}\mu(dx) \to 0$, whereas this integral equals $e^{k(-(n+1)N)} = e^{4c^2/((n+1)N)} \to 1$ by Proposition 6.2(iii), since $-(n+1)N \in \Theta$ for every n. Contradiction. (When $-1 \in \Theta$ one can take $N = 1$, and this is Corollary 6.5.)

Cell (4). $V = m^3/c$ on $M \subseteq (0,\infty)$; the scale c is affinely removable, so by Fact 2.2 and Remark 2.7 of the source we may take the inverse Gaussian family generated by the Lévy density (32) of [1], for which the source's Remark 3.10(ii) shows that Proposition 2.4 would force a generating density proportional to $x^{-1}e^{-1/(4x)}$ against the true $x^{-3/2}e^{-1/(4x)}$ (we reproduced this: $\psi(m) = -1/(4m^2)$, $k(\psi(x)) - x\psi(x) = -1/(4x)$, $V^{-1/3} \propto x^{-1}$). So coincidence fails.

Cell (3). $V = (1+m^2)^{3/2}/c$ on $M \subseteq \mathbb{R}$. By Fact 2.2 this is the symmetric normal inverse Gaussian family of [13] with $\alpha = c$, $\beta = 0$, $\delta = 1$, $\mu = 0$, whose cumulant is $c - \sqrt{c^2-\theta^2}$ on $(-c, c)$ and whose generating density is $h(x) = (c/\pi)\, e^c\, K_1\big(c\sqrt{1+x^2}\big)/\sqrt{1+x^2}$; the mean domain of that family is all of $\mathbb{R}$. With $z = \sqrt{1+m^2}$ one has $\psi(m) = cm/z$, $m\psi(m) - k(\psi(m)) = cz - c$, hence $f_{\psi(m)}(m) = (c/\pi)\, e^{cz} K_1(cz)/z$ while $V(m)^{-1/3} = c^{1/3}/z$; so Proposition 2.4 would force $w \mapsto e^w K_1(w)$ to be constant on $[c,\infty)$, which is false since $e^w K_1(w) \sim \sqrt{\pi/(2w)} \to 0$. At $c = 1$ this is verbatim the source's Remark 3.13 and its equation (24); the general modulus is the same computation with cz in place of z. So coincidence fails. (The normal inverse Gaussian cumulant and density used here are the standard ones, quoted at $(\alpha,\beta,\delta,\mu) = (c,0,1,0)$; their case $c = 1$ is the source's Remark 3.13 with its (23), which we read there. The two are mutually consistent: $k(\theta) = \delta\{\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+\theta)^2}\}$ gives $V(m) = (\delta^2+m^2)^{3/2}/(\alpha\delta^2)$, which is $(1+m^2)^{3/2}/c$ at $(\alpha,\delta) = (c,1)$; we verified that identity. The modulus c is *not* affinely removable, so the case $c = 1$ alone would not settle the cell. See [13].)

Cell (6). $V = (1-m^2)^{3/2}/c$ on $M \subseteq (-1,1)$ and $k = k_{\mathrm{in}}$ after normalisation. M is bounded and $S = M$, so $\operatorname{supp}\mu$ is compact, L is finite everywhere and $\Theta = \mathbb{R}$; then $M = k'(\mathbb{R}) \subseteq (-1,1)$ by Theorem 5.3, and $\operatorname{supp}\mu \subseteq [-1,1]$, so $x \le 1$ μ-a.e.; e^{x-1} is μ-integrable since $1 \in \Theta$; and $\int e^{(n+1)(x-1)}\mu(dx) = e^{k_{\mathrm{in}}(n+1)-(n+1)}$ for every n since $n+1 \in \Theta$. Theorem 6.4(i) says this is impossible.

Cell (5). $V = (m^2-1)^{3/2}/c$ on $M \subseteq (1,\infty)$ (after a reflection if $M \subseteq (-\infty,-1)$) and $k = k_{\mathrm{out}}$ on $\Theta = \psi(M) \subseteq (-\infty,-c)$ after normalisation. Since $S = M$, $\operatorname{supp}\mu \subseteq [m_1,\infty)$ with $m_1 = \inf M \ge 1 > 0$, so $x \ge 1$ μ-a.e.; for $x \ge m_1$ and $\theta < \theta_1 \in \Theta$, $e^{\theta x} \le e^{\theta_1 x}$, so $(-\infty,\theta_1) \subseteq \Theta$ and $\inf\Theta = -\infty$. Fix an integer $N \ge 1$ with $-N \in \Theta$ (necessarily $N > c$). Then $e^{N(1-x)}$ is μ-integrable, $N(1-x) < 0$ μ-a.e. (as $\mu(\{1\}) = 0$), Lemma 6.1(i) with $g(x) = N(1-x)$ gives $\int e^{(n+1)N(1-x)}\mu(dx) \to 0$, and this integral equals $e^{k_{\mathrm{out}}(-(n+1)N)+(n+1)N} \to 1$ by Proposition 6.2(ii), since $-(n+1)N \in \Theta$ for every n. Contradiction. When $-1 \in \Theta$ (which requires $c < 1$) one can take $N = 1$, and the contradiction is exactly Theorem 6.4(ii).

Every cell but (1) is excluded, and (1) is the Gaussian family. □

Remark 7.2 (what is formal in Theorem 7.1 and what is not)*.* Theorem 2.3 and Proposition 2.4 are the source's and are not formalised; Fact 2.2 is classical and not formalised; the derivation of the support and canonical-domain hypotheses of Theorem 6.4 from the mean domain of each cell (the tail bound $e^{\theta x} \le e^{\theta_1 x}$, the compactness of $\operatorname{supp}\mu$ in cell (6)) is a paper step; and the computations for cells (3) and (4) are the source's, reproduced here. Everything else — the factorisation, the six cells, the normal forms, the cumulants and their mean ranges, the exponent limits, the dominated-convergence lemma, the two exclusions and the controls — is machine-checked. One point deserves notice: the formal exclusions use the sequence $\theta = \pm(n+1)$, $n \ge 0$, and so presume that ± 1 lies in the canonical domain. For cell (6) this is automatic ($\Theta = \mathbb{R}$). For cell (5), $\Theta \subseteq (-\infty,-c)$, so $-1 \in \Theta$ forces $c < 1$, and the proof above therefore runs along $-(n+1)N$ for an integer N with $-N \in \Theta$; this is the same argument applied to the general form of Lemma 6.1(i) together with Proposition 6.2(ii), and the formal statement is its case $N = 1$. The same remark applies to cell (2) and Corollary 6.5.

Remark 7.3 (what Theorem 7.1 does not settle)*.* (a) The mean–median conjecture of [7] — if the mean of P_t is a median of P_t for every t, then P is Gaussian — remains open. Its hypothesis is Theorem 2.3(i) alone; ET–LR coincidence adds (1) and the density identity, and the proof above uses all three. Piccioni, Kołodziejek and Letac [8] characterise the Gaussian and gamma families by a fixed α-quantile of every tilt sitting at $b+t$ (their Theorem 1), resp. at a/t (their Theorem 2, for the tilts Q_{-t}, $t > 0$, of a Radon measure on $[0,\infty)$); that is a different hypothesis again and does not settle the mean–median question either. (b) The density–LR pairing of [1] is untouched: the density ordering has no analogue of the quantile symmetry $G_m(u) = G_m(1-u)$ that drives the ET–LR argument (the source's (14)), and its first obstruction is not $(V^{2/3})'''$. (c) Coincidence at a single θ_0, rather than throughout the family, is not addressed.

8. Outside the formal development

Nothing in this section is formalised, and the statements about the local hierarchy are computations, labelled as such.

8.1. **An integral-free form of the coincidence condition.**

Proposition 8.1 (proved on paper only)**.** *Suppose Assumption 2.1 holds and $p_{\mathrm{ET}} = p_{\mathrm{LR}}$ at every $\theta \in \Theta$. Fix $m \in M$. Then:*

(i) $f_{\psi(m)}(x) = C\,V(x)^{-1/3}\,e^{r_m(x)}$ *for $x \in S$, with C the constant of Proposition 2.4.*

(ii) *For every $x < m < y$ in S with $r_m(x) = r_m(y)$,*

$$V(x)^{-1/3}\big(\psi(y) - \psi(m)\big) = V(y)^{-1/3}\big(\psi(m) - \psi(x)\big).$$

(iii) *Equivalently: with*

$$\Xi_m(t) = \big(\psi(m+t) - \psi(m)\big)\Big(\frac{V(m+t)}{V(m)}\Big)^{1/3}, \qquad \zeta_m(t) = \operatorname{sgn}(t)\sqrt{-2V(m)\,r_m(m+t)},$$

the function Ξ_m is an odd function of ζ_m on $S - m$.

Proof. (i) Substitute the second form of Proposition 2.4 into $f_\theta(x) = e^{\theta x - k(\theta)}h(x)$ at $\theta = \psi(m)$; the exponent $(\psi(m) - \psi(x))x - k(\psi(m)) + k(\psi(x))$ is $r_m(x)$ by the source's (7). (ii) Coincidence of the p-values makes the nested rejection regions coincide: for $x < m$ the equal-tail region $(-\infty, x] \cup [q_{1-F(x)}, \infty)$ is the LR region $\{r_m \le r_m(x)\}$, so the right endpoint $y > m$ of the latter satisfies $r_m(y) = r_m(x)$ and $F(x) + F(y) = 1$, where $F = F_{\psi(m)}$. Parametrise both endpoints by the level $s = r_m(x) = r_m(y)$; r_m is strictly monotone on each side of m with $r_m' = \psi(m) - \psi$, so $dx/ds = 1/(\psi(m) - \psi(x))$ and $dy/ds = 1/(\psi(m) - \psi(y))$, and differentiating $F(x(s)) + F(y(s)) = 1$ gives, by (i) and $e^s \ne 0$, $V(x)^{-1/3}/(\psi(m) - \psi(x)) + V(y)^{-1/3}/(\psi(m) - \psi(y)) = 0$, which rearranges to (ii). (iii) $-2V(m)r_m(m+t) = t^2 + O(t^3)$ with r_m strictly concave and maximal at $t = 0$, so ζ_m is real analytic near 0 with $\zeta_m'(0) = 1$, strictly increasing, and $\zeta_m(y-m) = -\zeta_m(x-m)$ exactly when $r_m(x) = r_m(y)$; (ii) says $\Xi_m(y-m) = -\Xi_m(x-m)$ for such pairs. □

Both Ξ_m and ζ_m are built from V alone ($\psi' = 1/V$, $r_m'' = -1/V$), so writing $\Xi_m = \sum_{j \ge 1} A_j\,\zeta_m^j$ near $t = 0$, the coefficients A_j are universal polynomials in $V(m)^{\pm 1}, V'(m), V''(m), \ldots$, and the oddness condition is the hierarchy $A_2 = A_4 = A_6 = \cdots = 0$ of ordinary differential equations in V. The condition $A_{2j} = 0$ corresponds to the source's $G_m^{(2j+1)}(1/2) = 0$ once Proposition 2.4 has been imposed.

8.2. **The hierarchy is exhausted at the source's equation (a computation).**

We computed the A_j in exact rational arithmetic, twice: once symbolically over $\mathbb{Q}[v_0^{\pm 1}, v_1, \ldots, v_8]$ with $v_j = V^{(j)}(m)$ (Laurent-polynomial power series, order 12), and once numerically at rational jets with an independent series engine (order 18), the second reproducing the closed forms of the first at random jets. The results:

$$A_2 = 0 \quad \text{identically;} \tag{4}$$

$$405\,v_0^4\,A_4 = 9v_0^2v_3 - 9v_0v_1v_2 + 4v_1^3 = \mathcal{D}[V](m); \tag{5}$$

$$51030\,v_0^6\,A_6 = 200v_1^5 - 645v_0v_1^3v_2 + 486v_0^2v_1v_2^2 + 261v_0^2v_1^2v_3 - 297v_0^3v_2v_3 - 54v_0^3v_1v_4 + 81v_0^4v_5. \tag{6}$$

Equation (4) says that the order-three condition is already spent on Proposition 2.4; equation (5) is the source's equation (11), re-derived by a route that never forms G_m or a quantile function — this was run before any formal work, as a check that we and the source compute the same object. Equation (6) is a new seventh-order condition. *And it is identically zero on the solution set of* (1): substituting $V = (am^2 + bm + c)^{3/2}$ at rational data points $(a, W'(m), W(m))$, with $W(m)$ a rational square so that everything stays in $\mathbb{Q}$, every even coefficient $A_2, A_4, \ldots, A_{16}$ is exactly 0. The points, grouped by cell of Section 4 (the cell of a point is read off $\operatorname{sgn} a$ and $\operatorname{sgn}(W'(m)^2 - 4aW(m))$, both invariant under the recentring at m): $(0,0,4)$ in cell (1); $(0,3,4)$ in cell (2); $(3,0,1)$, $(1,6,16)$, $(2,5,25)$, $(\frac{1}{2},1,4)$, $(5,-7,\frac{49}{4})$ in cell (3); $(1,4,4)$ in cell (4); $(1,6,4)$, $(1,5,4)$, $(2,7,4)$, $(\frac{1}{3},3,\frac{9}{4})$ in cell (5); $(-1,2,9)$, $(-3,1,\frac{1}{4})$ in cell (6). At the non-solution jet $v_j \equiv 1$ one gets $A_4 = 4/405 \neq 0$, so the engine is not returning zero for free. The two computations took about a second each.

> *Measured, not proved:* on the class $V = (am^2+bm+c)^{3/2}$, every condition of the hierarchy through A_{16} holds identically. So the local expansion at the median cannot separate any of the six cells from any other once (1) and Proposition 2.4 hold; the source's local method is exhausted at its own equation, and whatever separates the branches must be global. That is why Section 6 changes tack. A proof for all orders presumably follows from Proposition 8.1(i) by a symmetry argument; it was not attempted.

8.3. **The source's numerical example.** Section 5.1 of [1] prints, for the Lévy-generated inverse Gaussian family ($\mu = 1$, $\lambda = 1/2$), $p_{\mathrm{ET}}(0.5) \approx 0.9803$, $p_{\mathrm{LR}}(0.5) = p_{\mathrm{UMPU}}(0.5) \approx 0.6171$, $p_{\mathrm{ET}}(1) \approx 0.5724$ and $p_{\mathrm{LR}}(1) = 1$. From the source's distribution function (33) and its reflection formula (34) ($x^* = \mu^2/x$), with the error function as the only special function, we obtain 0.980277, 0.617075, 0.572416 and 1.000000: every printed digit reproduces, and $F_{1,1/2}(1) = 0.713792$ shows the mean 1 is not the median. This is the only numerical content of the source that the present work touches; the hyperbolic-secant example of its Section 5.2 was not recomputed.

8.4. **Open ends.** (a) Prove the exhaustion of the hierarchy for all orders. (b) Formalise the p-values and the source's Theorem 3.8 and Proposition 3.9, so that Theorem 7.1 becomes a single formal statement; the ingredients are a distribution function as a measure-theoretic integral, the strict monotonicity of the source's H_m, and the inverse function theorem for the quantile. (c) Derive the support hypotheses of Theorem 6.4 formally from the mean domain (the tail bound $\mu([b,\infty)) \leq e^{-\theta b}\int e^{\theta x}\mu(dx)$). (d) The density–LR edge.

9. Verification

Theorems 3.1, 3.2, 4.2, 5.2, 5.3, 6.4, Corollary 3.3, Lemma 6.1 and Propositions 4.1, 4.3, 5.1, 5.4, 6.2, 6.3, 6.6 rest on declarations formally verified in Lean 4 against *Mathlib* [15, 16] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. The development is four modules, 1124 lines in all, in the import order `Ode` (219 lines: the operator $\mathcal{D}$, the factorisation, the reduction and the quadratic-root form), `Branches` (189 lines: the discriminant, the affine action, the six cells and the normal forms), `Cumulants` (358 lines: the three cumulants with their first two derivatives as `HasDerivAt` statements, the variance-function identities, the mean ranges, the homogeneity and the cell witnesses) and `Atoms` (358 lines: dominated convergence, the five exponent limits, the two exclusions and the controls). Forty-eight declarations carry the statements of this paper; every proof is checked by the Lean kernel, there is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand, and the axiom print (`#print axioms`) of each of the 48 declarations is `propext`, `Classical.choice`, `Quot.sound`; we re-ran the print for this paper from a scratch file importing `Atoms`. Where a printed statement says more than its Lean counterpart, the proof names the extra step: the interval form in Corollary 3.3, the derivation of the cumulant table in Section 5, the assembly in Corollary 6.5, and the whole of Theorem 7.1 as listed in Remark 7.2. The elaboration of the four modules took about seven processor-minutes in total as recorded at the time of writing, with a peak resident set within the `import Mathlib` baseline; the exact rational computations of Section 8 take seconds.

Statement numbers of [1] were read off the PDF compiled with *tectonic* from the arXiv source of version 1 (top-level file as named in the e-print's manifest), from its `.aux` label table, not from the raw LaTeX; the quotations were checked against the same source. The full-text sweep of the introduction used a local mirror of arXiv e-print sources (376 shards, about 87 GB of text), with the three phrases "equal-tail", "likelihood ratio" and "natural exponential famil" intersected per paper: 170 papers contain the first phrase, and exactly three contain all three — the source, and two papers on confidence intervals (arXiv:1608.07186 on generalized fiducial inference and arXiv:1712.02469 on bootstrap percentile intervals) that use "equal-tail(ed)" for intervals and contain nothing about p-value coincidence. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted), grouped by module. `ℝ` is Mathlib's real line, `Real.sqrt` the real square root (equal to 0 on negative arguments), `HasDerivAt f f' x` the statement that f has derivative f' at x, `Tendsto f atTop (𝓝 0)` that $f \to 0$ at $+\infty$ (`atBot` for $-\infty$), `μ ≪ volume` absolute continuity with respect to Lebesgue measure, `∀ᵐ x ∂μ, p x` that p holds μ-almost everywhere, and `∫ x, f x ∂μ` the Bochner integral. Subscripts on `k` and `V` are derivative orders.

`Ode` (Theorems 3.1, 3.2, Corollary 3.3).

```
def etlrOde (V V₁ V₂ V₃ : ℝ) : ℝ := 9 * V ^ 2 * V₃ - 9 * V * V₁ * V₂ + 4 * V₁ ^ 3

theorem etlrOde_cube (u u₁ u₂ u₃ : ℝ) :
    etlrOde (u ^ 3) (3 * u ^ 2 * u₁) (6 * u * u₁ ^ 2 + 3 * u ^ 2 * u₂)
        (6 * u₁ ^ 3 + 18 * u * u₁ * u₂ + 3 * u ^ 2 * u₃)
      = 27 / 2 * u ^ 7 * (6 * u₁ * u₂ + 2 * u * u₃)

theorem etlrOde_cube_nonvacuous :
    etlrOde ((1:ℝ) ^ 3) (3 * 1 ^ 2 * 0) (6 * 1 * 0 ^ 2 + 3 * 1 ^ 2 * 0)
        (6 * 0 ^ 3 + 18 * 1 * 0 * 0 + 3 * 1 ^ 2 * 1) = 27

theorem quadratic_of_third_deriv_zero {W W₁ W₂ : ℝ → ℝ}
    (h0 : ∀ x, HasDerivAt W (W₁ x) x) (h1 : ∀ x, HasDerivAt W₁ (W₂ x) x)
    (h2 : ∀ x, HasDerivAt W₂ 0 x) (x : ℝ) :
    W x = W₂ 0 / 2 * x ^ 2 + W₁ 0 * x + W 0

theorem etlrOde_zero_iff_third_deriv_zero {u u₁ u₂ u₃ : ℝ → ℝ}
    (hpos : ∀ x, 0 < u x) :
    (∀ x, etlrOde (u x ^ 3) (3 * u x ^ 2 * u₁ x) (6 * u x * u₁ x ^ 2 + 3 * u x ^ 2 * u₂ x)
        (6 * u₁ x ^ 3 + 18 * u x * u₁ x * u₂ x + 3 * u x ^ 2 * u₃ x) = 0)
      ↔ (∀ x, 6 * u₁ x * u₂ x + 2 * u x * u₃ x = 0)

theorem cube_eq_quadratic_pow_of_etlrOde_zero {u u₁ u₂ u₃ : ℝ → ℝ}
    (hpos : ∀ x, 0 < u x) (hu : ∀ x, HasDerivAt u (u₁ x) x)
    (h1 : ∀ x, HasDerivAt u₁ (u₂ x) x) (h2 : ∀ x, HasDerivAt u₂ (u₃ x) x)
    (h : ∀ x, etlrOde (u x ^ 3) (3 * u x ^ 2 * u₁ x) (6 * u x * u₁ x ^ 2 + 3 * u x ^ 2 * u₂ x)
        (6 * u₁ x ^ 3 + 18 * u x * u₁ x * u₂ x + 3 * u x ^ 2 * u₃ x) = 0) :
    ∃ a b c : ℝ, (∀ x, 0 < a * x ^ 2 + b * x + c) ∧
      ∀ x, u x ^ 3 = (a * x ^ 2 + b * x + c) * Real.sqrt (a * x ^ 2 + b * x + c)
```

`Branches` (Proposition 4.1, Theorem 4.2, Proposition 4.3).

```
def disc (a b c : ℝ) : ℝ := b ^ 2 - 4 * a * c

theorem disc_affine (a b c α β lam : ℝ) (hα : α ≠ 0) :
    disc (lam * a / α ^ 2)
        (-2 * lam * a * β / α ^ 2 + lam * b / α)
        (lam * a * β ^ 2 / α ^ 2 - lam * b * β / α + lam * c)
      = (lam / α) ^ 2 * disc a b c

theorem affine_transform_eq (a b c α β lam n : ℝ) (hα : α ≠ 0) :
```

```
    (lam * a / α ^ 2) * n ^ 2 + (-2 * lam * a * β / α ^ 2 + lam * b / α) * n
        + (lam * a * β ^ 2 / α ^ 2 - lam * b * β / α + lam * c)
      = lam * (a * ((n - β) / α) ^ 2 + b * ((n - β) / α) + c)

theorem leading_affine (a α lam : ℝ) (hα : α ≠ 0) (hlam : 0 < lam) :
    lam * a / α ^ 2 = lam / α ^ 2 * a ∧ 0 < lam / α ^ 2

theorem disc_pos_of_neg_leading {a b c m₀ : ℝ} (ha : a < 0)
    (hpos : 0 < a * m₀ ^ 2 + b * m₀ + c) : 0 < disc a b c

theorem six_branches {a b c m₀ : ℝ} (hpos : 0 < a * m₀ ^ 2 + b * m₀ + c) :
    (a = 0 ∧ b = 0) ∨ (a = 0 ∧ b ≠ 0) ∨
    (0 < a ∧ disc a b c < 0) ∨ (0 < a ∧ disc a b c = 0) ∨
    (0 < a ∧ 0 < disc a b c) ∨ (a < 0 ∧ 0 < disc a b c)

theorem six_branches_disjoint (a b c : ℝ) :
    ¬((a = 0 ∧ b = 0) ∧ (a = 0 ∧ b ≠ 0)) ∧
    ¬((a = 0 ∧ b = 0) ∧ (0 < a)) ∧
    ¬((0 < a) ∧ (a < 0)) ∧
    ¬((disc a b c < 0) ∧ (disc a b c = 0)) ∧
    ¬((disc a b c = 0) ∧ (0 < disc a b c))

theorem normal_form_nig {a b c : ℝ} (ha : 0 < a) (hd : disc a b c < 0) (m : ℝ) :
    a * m ^ 2 + b * m + c
      = a * ((m + b / (2 * a)) ^ 2 + (-disc a b c) / (4 * a ^ 2))
    ∧ 0 < (-disc a b c) / (4 * a ^ 2)

theorem normal_form_invGaussian {a b c : ℝ} (ha : 0 < a) (hd : disc a b c = 0) (m : ℝ) :
    a * m ^ 2 + b * m + c = a * (m + b / (2 * a)) ^ 2

theorem normal_form_hyperbolicOuter {a b c : ℝ} (ha : 0 < a) (hd : 0 < disc a b c) (m : ℝ) :
    a * m ^ 2 + b * m + c
      = a * ((m + b / (2 * a)) ^ 2 - disc a b c / (4 * a ^ 2))
    ∧ 0 < disc a b c / (4 * a ^ 2)

theorem normal_form_hyperbolicInner {a b c m₀ : ℝ} (ha : a < 0)
    (hpos : 0 < a * m₀ ^ 2 + b * m₀ + c) (m : ℝ) :
    a * m ^ 2 + b * m + c
      = (-a) * (disc a b c / (4 * a ^ 2) - (m + b / (2 * a)) ^ 2)
    ∧ 0 < disc a b c / (4 * a ^ 2)

theorem hyperbolicInner_bounded {a b c m₀ : ℝ} (ha : a < 0)
    (hpos : 0 < a * m₀ ^ 2 + b * m₀ + c) {m : ℝ} (hm : 0 < a * m ^ 2 + b * m + c) :
    (m + b / (2 * a)) ^ 2 < disc a b c / (4 * a ^ 2)
```

`Cumulants` **(Proposition 5.1, Theorems 5.2, 5.3, Proposition 5.4).**

```
noncomputable def kNig (c θ : ℝ) : ℝ := -Real.sqrt (c ^ 2 - θ ^ 2)

noncomputable def kNig₁ (c θ : ℝ) : ℝ := θ / Real.sqrt (c ^ 2 - θ ^ 2)

noncomputable def kNig₂ (c θ : ℝ) : ℝ :=
  c ^ 2 / ((c ^ 2 - θ ^ 2) * Real.sqrt (c ^ 2 - θ ^ 2))
```

```
noncomputable def vNig (c m : ℝ) : ℝ := (1 + m ^ 2) * Real.sqrt (1 + m ^ 2) / c

noncomputable def kOut (c θ : ℝ) : ℝ := -Real.sqrt (θ ^ 2 - c ^ 2)

noncomputable def kOut₁ (c θ : ℝ) : ℝ := -(θ / Real.sqrt (θ ^ 2 - c ^ 2))

noncomputable def kOut₂ (c θ : ℝ) : ℝ :=
  c ^ 2 / ((θ ^ 2 - c ^ 2) * Real.sqrt (θ ^ 2 - c ^ 2))

noncomputable def vOut (c m : ℝ) : ℝ := (m ^ 2 - 1) * Real.sqrt (m ^ 2 - 1) / c

noncomputable def kIn (c θ : ℝ) : ℝ := Real.sqrt (θ ^ 2 + c ^ 2)

noncomputable def kIn₁ (c θ : ℝ) : ℝ := θ / Real.sqrt (θ ^ 2 + c ^ 2)

noncomputable def kIn₂ (c θ : ℝ) : ℝ :=
  c ^ 2 / ((θ ^ 2 + c ^ 2) * Real.sqrt (θ ^ 2 + c ^ 2))

noncomputable def vIn (c m : ℝ) : ℝ := (1 - m ^ 2) * Real.sqrt (1 - m ^ 2) / c

theorem kNig_hasDerivAt {c θ : ℝ} (hQ : 0 < c ^ 2 - θ ^ 2) :
    HasDerivAt (kNig c) (kNig₁ c θ) θ

theorem kNig₁_hasDerivAt {c θ : ℝ} (hQ : 0 < c ^ 2 - θ ^ 2) :
    HasDerivAt (kNig₁ c) (kNig₂ c θ) θ

theorem vNig_kNig₁ {c θ : ℝ} (hc : 0 < c) (hQ : 0 < c ^ 2 - θ ^ 2) :
    vNig c (kNig₁ c θ) = kNig₂ c θ

theorem kNig₂_pos {c θ : ℝ} (hc : 0 < c) (hQ : 0 < c ^ 2 - θ ^ 2) : 0 < kNig₂ c θ

theorem kOut_hasDerivAt {c θ : ℝ} (hQ : 0 < θ ^ 2 - c ^ 2) :
    HasDerivAt (kOut c) (kOut₁ c θ) θ

theorem kOut₁_hasDerivAt {c θ : ℝ} (hQ : 0 < θ ^ 2 - c ^ 2) :
    HasDerivAt (kOut₁ c) (kOut₂ c θ) θ

theorem vOut_kOut₁ {c θ : ℝ} (hc : 0 < c) (hQ : 0 < θ ^ 2 - c ^ 2) :
    vOut c (kOut₁ c θ) = kOut₂ c θ

theorem one_lt_kOut₁ {c θ : ℝ} (hc : 0 < c) (hθ : θ < -c) : 1 < kOut₁ c θ

theorem kOut₂_pos {c θ : ℝ} (hc : 0 < c) (hQ : 0 < θ ^ 2 - c ^ 2) : 0 < kOut₂ c θ

theorem kIn_hasDerivAt {c θ : ℝ} (hc : 0 < c) :
    HasDerivAt (kIn c) (kIn₁ c θ) θ

theorem kIn₁_hasDerivAt {c θ : ℝ} (hc : 0 < c) :
    HasDerivAt (kIn₁ c) (kIn₂ c θ) θ

theorem vIn_kIn₁ {c θ : ℝ} (hc : 0 < c) : vIn c (kIn₁ c θ) = kIn₂ c θ
```

```
theorem abs_kIn₁_lt_one {c θ : ℝ} (hc : 0 < c) : |kIn₁ c θ| < 1

theorem kIn₂_pos {c θ : ℝ} (hc : 0 < c) : 0 < kIn₂ c θ

theorem etlrOde_smul (lam V V₁ V₂ V₃ : ℝ) :
    etlrOde (lam * V) (lam * V₁) (lam * V₂) (lam * V₃)
      = lam ^ 3 * etlrOde V V₁ V₂ V₃

theorem vNig_quadratic {c m : ℝ} (hc : 0 < c) :
    Real.sqrt (1 + m ^ 2) ^ 3 = c * vNig c m ∧
      Real.sqrt (1 + m ^ 2) ^ 2 = 1 * m ^ 2 + 0 * m + 1 ∧ disc 1 0 1 < 0

theorem vOut_quadratic {c m : ℝ} (hc : 0 < c) (hm : 1 < m) :
    Real.sqrt (m ^ 2 - 1) ^ 3 = c * vOut c m ∧
      Real.sqrt (m ^ 2 - 1) ^ 2 = 1 * m ^ 2 + 0 * m + (-1) ∧ 0 < disc 1 0 (-1)

theorem vIn_quadratic {c m : ℝ} (hc : 0 < c) (hm : |m| < 1) :
    Real.sqrt (1 - m ^ 2) ^ 3 = c * vIn c m ∧
      Real.sqrt (1 - m ^ 2) ^ 2 = (-1) * m ^ 2 + 0 * m + 1 ∧ (-1 : ℝ) < 0
```

`Atoms` **(Lemma 6.1, Propositions 6.2, 6.3, Theorem 6.4, Proposition 6.6).**

```
theorem tendsto_integral_exp_zero {μ : Measure ℝ} {g : ℝ → ℝ}
    (hmeas : AEStronglyMeasurable g μ) (hneg : ∀ᵐ x ∂μ, g x < 0)
    (hbd : Integrable (fun x => Real.exp (g x)) μ) :
    Tendsto (fun n : ℕ => ∫ x, Real.exp ((n + 1 : ℕ) * g x) ∂μ) atTop (𝓝 0)

theorem tendsto_integral_zero_of_ac_right {μ : Measure ℝ} (hac : μ ≪ volume) {β : ℝ}
    (hsupp : ∀ᵐ x ∂μ, x ≤ β)
    (hbd : Integrable (fun x => Real.exp (x - β)) μ) :
    Tendsto (fun n : ℕ => ∫ x, Real.exp ((n + 1 : ℕ) * (x - β)) ∂μ) atTop (𝓝 0)

theorem tendsto_integral_zero_of_ac_left {μ : Measure ℝ} (hac : μ ≪ volume) {β : ℝ}
    (hsupp : ∀ᵐ x ∂μ, β ≤ x)
    (hbd : Integrable (fun x => Real.exp (β - x)) μ) :
    Tendsto (fun n : ℕ => ∫ x, Real.exp ((n + 1 : ℕ) * (β - x)) ∂μ) atTop (𝓝 0)

theorem tendsto_kIn_sub_id {c : ℝ} (hc : 0 < c) :
    Tendsto (fun θ => kIn c θ - θ) atTop (𝓝 0)

theorem tendsto_kOut_sub_id {c : ℝ} (hc : 0 < c) :
    Tendsto (fun θ => kOut c θ - θ) atBot (𝓝 0)

theorem tendsto_kTweedie (c : ℝ) :
    Tendsto (fun θ : ℝ => -4 * c ^ 2 / θ) atBot (𝓝 0)

theorem tendsto_kInvGauss {c : ℝ} (hc : 0 < c) :
    Tendsto (fun θ : ℝ => -Real.sqrt (-2 * c * θ)) atBot atBot

theorem tendsto_kGauss {σ β : ℝ} (hσ : 0 < σ) :
    Tendsto (fun θ : ℝ => σ ^ 2 * θ ^ 2 / 2 - β * θ) atTop atTop
```

```
theorem inner_branch_not_lebesgue {c : ℝ} (hc : 0 < c) {μ : Measure ℝ}
    (hac : μ ≪ volume) (hsupp : ∀ᵐ x ∂μ, x ≤ 1)
    (hbd : Integrable (fun x => Real.exp (x - 1)) μ)
    (hlap : ∀ n : ℕ, ∫ x, Real.exp ((n + 1 : ℕ) * (x - 1)) ∂μ
      = Real.exp (kIn c ((n + 1 : ℕ) : ℝ) - ((n + 1 : ℕ) : ℝ))) : False

theorem outer_branch_not_lebesgue {c : ℝ} (hc : 0 < c) {μ : Measure ℝ}
    (hac : μ ≪ volume) (hsupp : ∀ᵐ x ∂μ, (1 : ℝ) ≤ x)
    (hbd : Integrable (fun x => Real.exp (1 - x)) μ)
    (hlap : ∀ n : ℕ, ∫ x, Real.exp ((n + 1 : ℕ) * ((1 : ℝ) - x)) ∂μ
      = Real.exp (kOut c (-((n + 1 : ℕ) : ℝ)) - (-((n + 1 : ℕ) : ℝ)))) : False

theorem dirac_integral_eq_one (n : ℕ) :
    ∫ x, Real.exp ((n + 1 : ℕ) * (x - 1)) ∂(Measure.dirac (1 : ℝ)) = 1

theorem vNig_not_constant {c : ℝ} (hc : 0 < c) : vNig c 0 ≠ vNig c (3 / 4)

theorem vOut_not_constant {c : ℝ} (hc : 0 < c) : vOut c (5 / 4) ≠ vOut c (5 / 3)

theorem vIn_not_constant {c : ℝ} (hc : 0 < c) : vIn c 0 ≠ vIn c (4 / 5)
```

References

[1] S. K. Bar-Lev and L. Hoessly, *Exact two-sided p-values in natural exponential families: coincidence, non-uniqueness, and sample-size stability*, arXiv:2608.28221v1 [stat.ME] (cross-listed to math.ST), submitted 28 August 2026; still version 1 on the arXiv listing on 2026-09-07, no journal reference. Read in full. All statement, equation and section numbers cited here (Assumption 2.5, Remark 2.7, Theorem 3.5, Remark 3.7, Theorem 3.8 with (11)–(13) and (18), Proposition 3.9 with (19)–(20), Remarks 3.10, 3.12, 3.13 with (22)–(24), Remark 3.14, Theorem 4.6, Section 5.1 with (32)–(34), Section 6) are those of the PDF compiled from the arXiv source of version 1, and the quotations in Section 1 are verbatim from that source. Full-text sweep for the present paper: see Section 9.

[2] S. K. Bar-Lev, D. Bshouty and G. Letac, *Normal, gamma and inverse-Gaussian are the only NEFs where the bilateral UMPU and GLR tests coincide*, Ann. Statist. **30** (2002), no. 5, 1524–1534, DOI 10.1214/aos/1035844987. The source's Theorem 3.5; it settles the UMPU–LR pairing by a different differential equation. Not read in full for this work (only the abstract page and the source's account of it); no claim here depends on its internal argument. Volume, issue, year and DOI checked against Crossref; the page range is as printed in the bibliography of [1].

[3] G. Letac and M. Mora, *Natural real exponential families with cubic variance functions*, Ann. Statist. **18** (1990), no. 1, 1–37, DOI 10.1214/aos/1176347491. Cited for the NEF background, for the affine action (2) (their Proposition 2.4(iv),(v), p. 5) and for Fact 2.2 (their Proposition 2.2(iv), p. 5, which they state without proof, "very much in the spirit of Morris (1982)"). Volume, issue, year and DOI checked against Crossref; the page range and the two propositions were read on the article itself (Project Euclid, open access).

[4] C. N. Morris, *Natural exponential families with quadratic variance functions*, Ann. Statist. **10** (1982), no. 1, 65–80, DOI 10.1214/aos/1176345690. Cited for the Morris class and, with [3], for Fact 2.2: Morris has no numbered statement for it, giving the construction as running text in his Section 2 (p. 67) for the pair (Ω, V) of variance function and domain. Volume, issue, year and DOI checked against Crossref, which carries no page range for this journal and era; the pages and the Section 2 text were read on the article itself (Project Euclid, open access).

[5] S. K. Bar-Lev and P. Enis, *Reproducibility and natural exponential families with power variance functions*, Ann. Statist. **14** (1986), no. 4, 1507–1522, DOI 10.1214/aos/1176350173. The power-variance classification the source invokes to exclude the exponent 3/2; not read for this work. Volume, issue, year and DOI checked against Crossref, pages as printed in [1].

[6] B. Jørgensen, *Exponential dispersion models*, J. Roy. Statist. Soc. Ser. B **49** (1987), no. 2, 127–145, DOI 10.1111/j.2517-6161.1987.tb01685.x. Cited as above; not read for this work. Bibliographic data checked against Crossref (which gives the article proper, without the discussion, as pp. 127–145).

[7] G. Letac, L. Mattner and M. Piccioni, *The median of an exponential family and the normal law*, Statist. Probab. Lett. **133** (2018), 38–41, DOI 10.1016/j.spl.2017.10.002; arXiv:1708.03789 (under the title "The medians for exponential families and the normal law"). Read in full from the arXiv version: its Theorem 1 characterises $N(0,1)$ by the natural parameter being a median of every tilt; the mean–median statement is its explicitly labelled Conjecture, with the sentence quoted in Section 1. Bibliographic data checked against Crossref.

[8] M. Piccioni, B. Kołodziejek and G. Letac, *Location and scale behaviour of the quantiles of a natural exponential family*, ESAIM Probab. Stat. **24** (2020), 244–251, DOI 10.1051/ps/2019009; arXiv:1810.11917. Read in full; its

Theorems 1 and 2 are the location and scale characterisations recalled in Remark 7.3. Bibliographic data checked against Crossref.

[9] C. C. Kokonendji, *Sur les familles exponentielles naturelles réelles de grand-Babel*, Ann. Fac. Sci. Toulouse Math. (6) **4** (1995), no. 4, 763–800, DOI 10.5802/afst.811 (Numdam AFST_1995_6_4_4_763_0). Bibliographic data checked against Crossref and the Numdam record. Only the prose of the scanned article was read (the scan's text layer drops the displayed mathematics), so its variance-function tables were not consulted; the sentence quoted in Section 1 closes his Section 1, on p. 767, and was checked there on the page image of the Numdam scan, not on its text layer.

[10] A. C. Vandal, *A compendium of variance functions for real natural exponential families*, M.A. thesis, Department of Mathematics and Statistics, McGill University, Montréal, 1994. Its abstract states that the members of the grand-Babel class "have not as yet all been enumerated" while the Morris, Mora, Babel and Seshadri subclasses "have been fully described". Title, author, year, degree (Master of Arts), department and abstract were read from the record in McGill's institutional repository (`mcgill.scholaris.ca`, formerly eScholarship@McGill), DOI 10.82308/5147; the thesis itself was not consulted.

[11] G. Letac, *Duality for real and multivariate exponential families*, J. Multivariate Anal. **188** (2022), 104811, DOI 10.1016/j.jmva.2021.104811; arXiv:2104.05510. Read (arXiv version) for the attribution of the variance function $(1+m^2)^{3/2}$: "a distribution P_3 described in Letac [IMPA 1992] generating the exponential family with variance function $(1+m^2)^{3/2}$". The sentence and its position were read on the LaTeX source of the arXiv e-print: it stands in the last subsection ("Warnings for existence and non existence of the dual measures") of Section 3, "Duality", so the section number is that of the arXiv version; the published version is behind a paywall and was not consulted. Bibliographic data checked against Crossref.

[12] G. Letac, *Lectures on natural exponential families and their variance functions*, Monografias de Matemática **50**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1992, 128 pp., ISBN 85-244-0066-8. Not online and not read for this work; cited as the stated coverage limit of the novelty claim. The reference list of [11] gives the title, the volume number 50, the publisher and the year but no series name; the series, the pagination and the ISBN are from the union catalogue record (K10plus, ISBN 8524400668), which prints the series name without accents.

[13] O. E. Barndorff-Nielsen, *Normal inverse Gaussian distributions and stochastic volatility modelling*, Scand. J. Statist. **24** (1997), no. 1, 1–13, DOI 10.1111/1467-9469.00045. Cited by the source for the normal inverse Gaussian family and used in the proof of Theorem 7.1 for the density and cumulant with general (α, δ). Bibliographic data checked against Crossref. The article itself is behind a paywall and was not read: the formulas are quoted as the standard ones, their specialisation $\alpha = \delta = 1$ is the source's Remark 3.13 with its (23), read there, and we verified that the quoted cumulant reproduces the variance function $(\delta^2+m^2)^{3/2}/(\alpha\delta^2)$ that the argument uses. Nothing in this paper turns on the internal argument of [13].

[14] E. Kulinskaya, *On two-sided p-values for non-symmetric distributions*, arXiv:0810.2124 [math.ST], 2008. Read in full: it compares doubled-tail and conditional two-sided p-values and contains no equal-tail/likelihood-ratio characterisation; it is the one work OpenAlex lists as citing [2].

[15] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635.

[16] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.