THREE-FAULT RE-ROOTING IN DENSE EISENSTEIN–JACOBI NETWORKS: AN OBSTRUCTED FAULT SET AT EVERY DIAMETER $t \geq 3$, THE EXACT CENSUS FOR $3 \leq t \leq 8$, AND THE THRESHOLD TRIPLE $\{0, 2, 4\}$

KOREA SUPERINTELLIGENCE LABS

Abstract. The dense Eisenstein–Jacobi network of diameter t is the degree-six circulant $\Gamma_t = C_N(t, t+1, 2t+1)$ on $N = 3t^2 + 3t + 1$ vertices, the Cayley graph of $\mathbb{Z}[\omega]/(\alpha)$, $\alpha = (t+1) + t\omega$, on the six units. Albader (arXiv:2606.18712) relocates the source of a one-to-all broadcast so that every faulty vertex lies at distance exactly t from the new source; a fault set F admits such a source exactly when $\bigcap_{f \in F}(f + B_t) \neq \emptyset$, where B_t is the set of vertices at distance t from 0. He proves that every one- or two-vertex fault set admits a re-rooted source, exhibits one three-vertex fault set at $t = 3$ that does not, and stops. We show that an obstructed three-vertex fault set exists at every diameter $t \geq 3$: for $t \geq 22$ by a counting argument, $|B_t|^3 + 3N < N^2$, fed by the bound $|B_t| \leq 6t$, which we derive from a two-sided characterisation of hop distance by the hexagonal norm $\max\{|x|, |y|, |x + y|\}$; for $3 \leq t \leq 21$ by explicit witnesses. The single triple $\{0, 2, 4\}$ is obstructed at every $6 \leq t \leq 21$, while, by computation, it admits a re-rooted source at $t = 3, 4, 5$. The exact number $\mathrm{T}_3(t)$ of obstructed three-vertex fault sets is determined for $3 \leq t \leq 8$ (407, 4,392, 22,932, 82,931, 238,797, 587,202), and computations extend the count to $t \leq 25$, locating $t = 13$ as the first diameter at which obstructed triples are the majority. We also record that the source's printed vertex label $tx + (2t+1)y$ is not a function on $\mathbb{Z}[\omega]/(\alpha)$ — the same slip as in a second preprint of the author — and that $tx - (t+1)y$ is. The formal development is in Lean 4: its general-t argument is proved by the kernel with no native evaluation, its finite statements by compiled evaluation; the computations that reach beyond it are labelled as such.

1. Introduction

The objects. Let $\omega = e^{i\pi/3}$, so that $\omega^2 = \omega - 1$ and $\mathbb{Z}[\omega]$ is the ring of Eisenstein integers with norm $\mathrm{N}(x + y\omega) = x^2 + xy + y^2$. For an integer $t \geq 1$ put $\alpha = (t+1) + t\omega$; then $\mathrm{N}(\alpha) = 3t^2 + 3t + 1 =: N$ and $\mathbb{Z}[\omega]/(\alpha)$ has N elements. The *dense Eisenstein–Jacobi network* of diameter t is the Cayley graph of the additive group of $\mathbb{Z}[\omega]/(\alpha)$ with respect to the six units $\pm 1, \pm\omega, \pm\omega^2$; it is 6-regular, vertex-transitive, and has diameter t (Martínez, Stafford, Beivide and Gabidulin [8]; Flahive and Bose [9]). The source [1] calls it H_n with $n = t + 1$, $\alpha = n + (n-1)\omega$ and $N = 3n^2 - 3n + 1$ (its equations (10)–(12)); we index by the diameter and write Γ_t. Labelling $x + y\omega$ by $tx - (t+1)y \bmod N$ turns Γ_t into the degree-six circulant

$$\Gamma_t \;\cong\; C_N(t,\, t+1,\, 2t+1) \;=\; \mathrm{Cay}\bigl(\mathbb{Z}_N, \{\pm t, \pm(t+1), \pm(2t+1)\}\bigr),$$

the representation the source uses for its examples and its validation (its equations (17)–(19) and (31)), and the one in which everything below is stated. At $t = 3$ this is $\mathrm{Cay}(\mathbb{Z}_{37}, \{\pm 3, \pm 4, \pm 7\})$. Write d for the hop distance in Γ_t and

$$B_t = \{v \in \mathbb{Z}_N :\; d(0, v) = t\}$$

for the set of vertices at distance exactly t from 0, the *distance-t boundary*; by vertex-transitivity the boundary of u is $u + B_t$ (the source's equations (15), (16)). The source states $|B_t| = 6t$ (its Section 3.3; also [8, 9]); what is proved below, and all that the argument needs, is the upper half $|B_t| \leq 6t$ (Lemma 4.2), with the equality checked at $3 \leq t \leq 10$ (Proposition 3.3).

The source and where it stops. The source is Albader's arXiv:2606.18712 [1]. Its method of fault-tolerant one-to-all broadcasting is *re-rooting*: when a set F of vertices has failed, the broadcast is started from a new source NS chosen so that every faulty vertex is at distance exactly t from NS (its equation (56), "$d(NS, F_i) = t,\ 1 \leq i \leq m$"), so that the faulty vertices are leaves of the broadcast tree and never have to forward the message. Such a source exists exactly when $NS \in \bigcap_{f\in F}(f + B_t)$ (its equation (61) for two faults, and the sentence "A valid re-rooted source for all three faulty nodes would need to lie in $B_3(F_1) \cap B_3(F_2) \cap B_3(F_3)$" in the proof of its Proposition 1). We call a fault set F *re-rootable* when this intersection is nonempty and *obstructed* otherwise.

Three statements of the source concern us. Its Lemma 1 ("Boundary-Difference Coverage", p. 17) proves $B_t - B_t = \mathbb{Z}_N$ for every t; its Theorem 1 ("Existence of a Common Distance-t New Source", p. 19) deduces that every one- or two-vertex fault set is re-rootable, at every t; and its Proposition 1 ("Three-Fault Counterexample", p. 20) reads, verbatim,

> "The two-fault re-rooting guarantee does not generally extend to arbitrary three-node fault configurations in dense EJ networks."

Its proof takes $t = 3$, $N = 37$, prints the boundary

$$B_3(0) = \{2, 5, 9, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 28, 32, 35\},$$

and continues:

> "Now choose three faulty nodes with integer labels $F_1 = 0$, $F_2 = 5$, $F_3 = 14$. [...] However, direct intersection gives $B_3(F_1) \cap B_3(F_2) \cap B_3(F_3) = \emptyset$. Therefore, no node NS satisfies $d(NS, F_1) = d(NS, F_2) = d(NS, F_3) = 3$."

That is the whole of what the source establishes about three faults: one obstructed triple at one diameter. It does not say whether an obstructed triple exists at any other diameter, how many there are, or whether they are rare. The author's later preprint arXiv:2606.21133 [2] repeats the identical example under the heading "Concrete H_4 three-fault obstruction" (its Example 1, p. 9) and concludes, in its Remark 1, that "all higher-order node-fault claims in this hybrid framework must be stated as conditional or empirical unless an additional fault-class restriction is imposed". (Statement, table, page and equation numbers of [1] and [2] are those of the PDFs compiled from the version-1 e-prints and of their `.aux` files; see the bibliography.)

Results. For $a, b \in \mathbb{Z}_N$ call the pair (a, b) *obstructed* when no $v \in B_t$ has both $v - a \in B_t$ and $v - b \in B_t$; this holds exactly when the fault set $\{0, a, b\}$ is not re-rootable (Lemma 2.2). Let $\mathrm{T}_3(t)$ be the number of three-element subsets of $\mathbb{Z}_N$ that are not re-rootable.

- **An obstructed three-fault set at every diameter** (Theorem 5.2). For every $t \geq 3$ there are distinct nonzero $a, b \in \mathbb{Z}_N$ such that $\{0, a, b\}$ is not re-rootable in Γ_t. For $t \geq 22$ this is proved for all t at once (Theorem 4.3): every re-rootable pair (a, b) is of the form $(v - p, v - q)$ with $v, p, q \in B_t$, so there are at most $|B_t|^3$ of them, and $|B_t| \leq 6t$ makes $|B_t|^3 + 3N < N^2$ from $t = 22$ on. The bound $|B_t| \leq 6t$ rests on a two-sided characterisation of hop distance (Lemma 4.1): v is within n hops of 0 if and only if $v = tx - (t+1)y \bmod N$ for some $(x, y) \in \mathbb{Z}^2$ with $\max\{|x|, |y|, |x + y|\} \leq n$. For $3 \leq t \leq 21$ the statement is closed by nineteen explicit witnesses checked by computation.
- **The threshold triple** $\{0, 2, 4\}$ (Theorem 5.1). The pair $(2, 4)$ is obstructed at every $6 \leq t \leq 21$; so the one fault set $\{0, 2, 4\}$ has no re-rooted source in sixteen consecutive networks. By computation outside the formal development it is re-rootable at $t = 3, 4, 5$ and obstructed at every $6 \leq t \leq 40$, and at every $6 \leq t \leq 30$ it is the lexicographically least obstructed triple containing 0; a proof for all $t \geq 6$ is not claimed.
- **The census** (Theorem 6.1, Corollary 6.2). For $3 \leq t \leq 8$ the number of ordered obstructed pairs is 66, 432, 1,512, 3,918, 8,478, 16,236, and $\mathrm{T}_3(t)$ = 407, 4,392, 22,932, 82,931, 238,797, 587,202 out of $\binom{N}{3}$ = 7,770, 35,990, 121,485, 333,375, 790,244, 1,679,580. The value 407 at $t = 3$ is also obtained by a direct enumeration of all $\binom{37}{3}$ triples. Computations extend the census to $t \leq 25$ (Table 1); obstructed triples are a minority for $t \leq 12$ and a majority from $t = 13$ on.
- **An erratum** (Proposition 7.1). The source's equation (24) prints the vertex label $\phi(x + y\omega) \equiv (n - 1)x + (2n - 1)y = tx + (2t + 1)y \pmod N$; it sends α to $3t^2 + 2t \not\equiv 0$, so it is not a function on $\mathbb{Z}[\omega]/(\alpha)$, while $tx - (t + 1)y$ is. Both maps send the six units to the same jump set, and the source's Lemma 1, Theorem 1, Proposition 1 and Table 2 all live in the circulant representation, so nothing the source proves is affected. The same slip occurs in the author's arXiv:2606.19834,

where it is recorded in the companion manuscript [6]; the author's arXiv:2606.21133 prints the correct map.

What is new and what is not. The network family, its circulant form, the hexagonal distance and $|B_t| = 6t$ are in [8, 9] and in the source; Lemma 1, Theorem 1 and the single triple $\{0, 5, 14\}$ are the source's. Theorems 5.2, 5.1 and 6.1 and the erratum are claimed here. The counting argument of Theorem 4.3 is elementary — its content is that nobody has applied it to this question, and that it bites only after the bound $|B_t| \leq 6t$, which needs the lattice description of hop distance; the same argument, for an arbitrary boundary in an arbitrary circulant, is used in the companion manuscript [7] on the dense Gaussian networks of [5] (see Section 8). As of 2026-09-07 the only citing work Semantic Scholar records for the source, or for either of the author's later preprints cited here [2, 3], is the author's own arXiv:2606.21133; OpenAlex records none for any of the three. A full-text sweep of a local arXiv mirror (376 shards, coverage through the September 2026 announcements) for the phrase "Eisenstein–Jacobi" — with zero, one or two hyphens and an optional space, case-insensitively — matches 34 shards and, e-print by e-print inside them, 51 distinct e-prints, 13 of which are the author's June 2026 cluster. The other 38 are papers on Eisenstein–Jacobi *series* of modular forms, on codes over Eisenstein–Jacobi *integers* in the sense of Huber (perfect and quasi-perfect Lee codes, codes in Doob graphs, codes over Hurwitz integers), on the structure of the networks themselves (Thomson and Zhou [10]), and bibliography entries citing Hussain, Bose and Al-Dhelaan [11]; the twelve of them that also contain the strings "fault" or "re-root" were read at the matching lines, where the matches are "default" in preambles and comments, "square-root", one bibliography entry, the title of a paper on fault-tolerant embeddings of hypercubes in augmented cubes (arXiv:2507.12834) and a remark on fault-tolerant routing in a paper on edge-disjoint Hamiltonian cycles in the spined cube (arXiv:2210.16603), the last two citing [11]; none of the 38 concerns re-rooting or counts fault configurations in any circulant. Within the author's cluster, two further preprints concern re-rooting in the Eisenstein–Jacobi network: arXiv:2606.18714 [4], posted the same day as the source, replaces the source's boundary search by a closed-form selector for one and two faults and says nothing about three; and [2] is the only one that touches three faults, repeating the $t = 3$ example without a count or a second diameter. What this paper does *not* claim: a closed form or an asymptotic for $\mathrm{T}_3(t)$, a proof that $\{0, 2, 4\}$ is obstructed for every $t \geq 6$, or anything about fault sets of size four or more.

Verification. Every proposition and theorem below with a finite range is certified in Lean 4 [12] with *Mathlib* [13]: the boundary B_t is defined as the set of vertices at hop distance exactly t by iterating the one-jump neighbourhood, never by a table, and each finite claim is a decidable proposition evaluated by compiled code and admitted by the kernel through Lean's native-evaluation axiom. The general-t statements — Lemmas 2.3, 2.4, 4.1 and 4.2 and Theorem 4.3 — are proved by the kernel with the standard axioms only. Section 8 says precisely what is trusted; Appendix A reproduces the formal statements. Everything the text says beyond its formal counterpart — the enumerator data of Table 1, the lexicographic-least observations, the re-rootability of $\{0, 2, 4\}$ at $t = 3, 4, 5$, the passage from $\mathrm{O}(t)$ to $\mathrm{T}_3(t)$ in Corollary 6.2 (machine-checked only at $t = 3$), and the independent re-computations reported in Section 8 — is labelled as such where it occurs.

2. Preliminaries

Fix $t \geq 1$, $N = 3t^2 + 3t + 1$, and the circulant $\Gamma_t = \mathrm{Cay}(\mathbb{Z}_N, J_t)$ with jump set $J_t = \{\pm t, \pm(t+1), \pm(2t+1)\}$. The set of vertices within n hops of 0 is defined by iteration, $\mathrm{reach}_0 = \{0\}$ and $\mathrm{reach}_{n+1} = \mathrm{reach}_n \cup (\mathrm{reach}_n + J_t)$, and $B_t = \mathrm{reach}_t \setminus \mathrm{reach}_{t-1}$. (That Γ_t has diameter t, i.e. $\mathrm{reach}_t = \mathbb{Z}_N$, is [8, 9] and is not used below; the results are statements about the set B_t as defined.)

Definition 2.1. A set $F \subseteq \mathbb{Z}_N$ is *re-rootable* (with respect to B_t) when there is a $v \in \mathbb{Z}_N$ with $v - f \in B_t$ for every $f \in F$, i.e. when $\bigcap_{f\in F}(f + B_t) \neq \emptyset$. A pair $(a, b) \in \mathbb{Z}_N^2$ is *obstructed* when there is no $v \in B_t$ with $v - a \in B_t$ and $v - b \in B_t$.

Lemma 2.2 (obstructed pairs)**.** *(a, b) is obstructed if and only if $\{0, a, b\}$ is not re-rootable.*

Proof. A re-rooted source v for $\{0, a, b\}$ satisfies $v - 0 \in B_t$, $v - a \in B_t$, $v - b \in B_t$, which is exactly a witness against obstruction, and conversely. □

Lemma 2.3 (translation)**.** *For every $S, F \subseteq \mathbb{Z}_N$ and $c \in \mathbb{Z}_N$, $F + c$ is re-rootable with respect to S if and only if F is.*

Proof. If v serves $F + c$ then $v - c$ serves F, and if v serves F then $v + c$ serves $F + c$. □

So every three-element fault set may be translated to contain 0, and the three-fault question is a question about pairs (a, b) of distinct nonzero vertices. Let $\mathrm{O}(t)$ be the number of *ordered* pairs (a, b) with $a \neq 0$, $b \neq 0$, $a \neq b$ that are obstructed, and $\mathrm{T}_3(t)$ the number of three-element subsets of $\mathbb{Z}_N$ that are not re-rootable.

Lemma 2.4 (counting)**.** *Let $S \subseteq \mathbb{Z}_N$ with $|S|^3 + 3N < N^2$. Then there are $a, b \in \mathbb{Z}_N$ with $a \neq 0$, $b \neq 0$, $a \neq b$ such that no $v \in S$ has both $v - a \in S$ and $v - b \in S$.*

Proof. The set of pairs $(v - p, v - q)$ with $v, p, q \in S$ has at most $|S|^3$ elements, and the set of pairs (a, b) with $a = 0$ or $b = 0$ or $a = b$ has at most $3N$. If $|S|^3 + 3N < N^2$ some pair (a, b) lies in neither; then $a, b \neq 0$, $a \neq b$, and a $v \in S$ with $v - a, v - b \in S$ would exhibit $(a, b) = (v - (v - a), v - (v - b))$ in the first set. □

Lemma 2.4 is stated for an arbitrary subset S of an arbitrary cyclic group; nothing about Γ_t enters. With $S = B_t$ it produces an obstructed pair as soon as $|B_t|^3 + 3N < N^2$, and the whole of the general-t argument is the size bound that makes this inequality true.

3. The model is the source's

The propositions of this section are not new; they establish that the formal model of Γ_t and of B_t used below is the source's object, and that neither quantifier in Definition 2.1 is vacuous. Each is a finite statement evaluated by compiled code (Section 8).

Proposition 3.1 (the printed boundary)**.** *At $t = 3$, the set B_3 of vertices at hop distance exactly 3 from 0 in $\mathrm{Cay}(\mathbb{Z}_{37}, \{\pm 3, \pm 4, \pm 7\})$ is*

$$\{2, 5, 9, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 28, 32, 35\},$$

the eighteen-element set printed in the proof of the source's Proposition 1.

Proposition 3.2 (the source's counterexample)**.** *At $t = 3$ the fault set $\{0, 5, 14\}$ is not re-rootable: the pair $(5, 14)$ is obstructed.*

The mathematics of Proposition 3.2 is the source's; what is added is that it is recomputed from the hop metric rather than from the printed sets. One observation the source does not make (a computation outside the formal development): $\{0, 5, 14\}$ is the lexicographically least obstructed triple containing 0 at $t = 3$ — it is the first pair (a, b), $1 \leq a < b \leq 36$, in lexicographic order that is obstructed.

Proposition 3.3 (boundary size and the source's Table 2)**.** $|B_t| = 6t$ *for every* $3 \leq t \leq 10$*. At $t = 10$ ($N = 331$), $|B_{10}| = 60$ and $B_{10} - B_{10} = \mathbb{Z}_{331}$.*

The second sentence is the row $n = 11$ of the source's Table 2 ("Validation of the boundary-difference property in dense EJ networks", p. 24), which prints $(n, t, N, |B_t|, |B_t - B_t|) = (11, 10, 331, 60, 331)$, and is the source's Lemma 1 at that size. The remaining four rows of that table,

$$(26, 25, 1951, 150, 1951), \qquad (51, 50, 7651, 300, 7651),$$
$$(101, 100, 30301, 600, 30301), \qquad (201, 200, 120601, 1200, 120601),$$

are reproduced by the independent programs of Section 8 but are not formally certified.

Proposition 3.4 (the source's Theorem 1, checked)**.** *For every $3 \leq t \leq 8$ and every $a \in \mathbb{Z}_N$, the set $\{0, a\}$ is re-rootable. Consequently, by Lemma 2.3, every fault set of size one or two is re-rootable at those diameters.*

The source proves this for every t; the exhaustive check at six diameters is a control that the model is the source's and that the two-fault half of the story is not vacuous.

Proposition 3.5 (controls)**.** *At $t = 3$: the fault set $\{0, 1, 2\}$ is re-rootable; the pair $(1, 2)$ is not obstructed; and $B_3 \neq \mathbb{Z}_{37}$.*

So not every three-element fault set is obstructed (a too-large claim is refuted), and the boundary is a proper subset of the vertex set (re-rootability is not trivially satisfiable).

4. The hexagonal norm and the size of the boundary

For $(x, y) \in \mathbb{Z}^2$ let $\rho(x, y) = \max\{|x|, |y|, |x + y|\}$, the hexagonal norm (the source's equation (14)), and let

$$\psi_t(x, y) = tx - (t + 1)y \bmod N \in \mathbb{Z}_N.$$

The six unit vectors $\pm(1, 0)$, $\pm(0, 1)$, $\pm(1, -1)$ have $\rho = 1$ and are sent by ψ_t to $\pm t$, $\mp(t + 1)$, $\pm(2t + 1)$ respectively — the jump set J_t. So ψ_t is a homomorphism $\mathbb{Z}^2 \to \mathbb{Z}_N$ carrying the hexagonal grid onto Γ_t; it is the vertex label $x + y\omega \mapsto tx - (t + 1)y$ of Section 7.

Lemma 4.1 (hop distance is the hexagonal norm)**.** *For every $t \geq 0$, every $n \geq 0$ and every $v \in \mathbb{Z}_N$: $v \in \mathrm{reach}_n$ if and only if $v = \psi_t(x, y)$ for some $(x, y) \in \mathbb{Z}^2$ with $\rho(x, y) \leq n$.*

Proof. Both directions are inductions on n. ($\Rightarrow$) $\mathrm{reach}_0 = \{0\} = \{\psi_t(0, 0)\}$. If $v \in \mathrm{reach}_{n+1}$ then either $v \in \mathrm{reach}_n$, or $v = u + j$ with $u = \psi_t(x, y)$, $\rho(x, y) \leq n$, and $j \in J_t$; each of the six jumps is ψ_t of a unit vector e, so $v = \psi_t((x, y) + e)$, and $\rho((x, y) + e) \leq \rho(x, y) + 1$. ($\Leftarrow$) If $\rho(x, y) \leq 0$ then $(x, y) = (0, 0)$. Let $\rho(x, y) \leq n + 1$ with $(x, y) \neq (0, 0)$. One of six sectors applies and names a unit vector e with $\rho((x, y) - e) \leq n$: if $x < 0 < y$ take $e = (-1, 1)$; if $x < 0$ and $y \leq 0$ take $e = (-1, 0)$; if $x = 0$ and $y < 0$ take $e = (0, -1)$; if $x = 0$ and $y > 0$ take $e = (0, 1)$; if $x > 0 > y$ take $e = (1, -1)$; if $x > 0$ and $y \geq 0$ take $e = (1, 0)$. In each case the induction hypothesis puts $\psi_t((x, y) - e)$ in reach_n and one jump $\psi_t(e) \in J_t$ returns to $\psi_t(x, y) \in \mathrm{reach}_{n+1}$. □

The lemma says that hop distance in Γ_t is at most the hexagonal norm of any lattice representative, and that the minimum over representatives is attained by a word; it is proved for every t and n with no appeal to the density of the hexagonal ball. Let $R_t \subseteq \mathbb{Z}_N$ be the image under ψ_t of the hexagon $\{\rho = t\}$, parametrised by its six sides: for $0 \leq j < t$,

$$(t - j,\, j), \quad (-j,\, t), \quad (-t,\, t - j), \quad (j - t,\, -j), \quad (j,\, -t), \quad (t,\, j - t),$$

$6t$ lattice points in all.

Lemma 4.2 (the boundary lies on the ring)**.** *For every $t \geq 0$, $B_t \subseteq R_t$ and $|R_t| \leq 6t$. Hence $|B_t| \leq 6t$.*

Proof. $|R_t| \leq 6t$ because R_t is the image of a set of $6t$ parameters. Let $v \in B_t$, so $v \in \mathrm{reach}_t$ and $v \notin \mathrm{reach}_{t-1}$. By Lemma 4.1, $v = \psi_t(x, y)$ with $\rho(x, y) \leq t$, and $\rho(x, y) = t$ exactly, for otherwise the same lemma would put v in reach_{t-1}. A lattice point with $\rho(x, y) = t$ has $|x| = t$ or $|y| = t$ or $|x + y| = t$, and a case split on which coordinate attains t and on the signs places (x, y) on one of the six sides with an explicit parameter j. □

Theorem 4.3 ($t \geq 22$: counting)**.** *For every $t \geq 22$ there are $a, b \in \mathbb{Z}_N$ with $a \neq 0$, $b \neq 0$, $a \neq b$ such that the pair (a, b) is obstructed: no $v \in B_t$ has both $v - a \in B_t$ and $v - b \in B_t$.*

Proof. By Lemma 4.2, $|B_t|^3 \leq (6t)^3$. Writing $t = 22 + j$ with $j \geq 0$, the inequality $(6t)^3 + 3N < N^2$ is a polynomial inequality in j with positive coefficients after expansion (equivalently, $N^2 - (6t)^3 - 3N = 9t^3(t - 22) + 6t^2 - 3t - 2 > 0$ for $t \geq 22$), so $|B_t|^3 + 3N < N^2$ and Lemma 2.4 applies with $S = B_t$. □

With the bound $|B_t| \leq 6t$ the inequality is false at $t = 21$ (2,004,537 $\not<$ 1,923,769) and true at $t = 22$ (2,304,525 $<$ 2,307,361, a margin of 2,836); since the source's exact value $|B_t| = 6t$ gives the same inequality, $t = 22$ is the threshold of this argument and not an artefact of the bound. The diameters $3 \leq t \leq 21$ have to be closed one at a time.

5. An obstructed three-fault set at every diameter

Theorem 5.1 (the witnesses, $3 \le t \le 21$)**.** *The following pairs are obstructed with respect to R_t, and hence, by Lemma 4.2, with respect to B_t:*

(i) $(5, 14)$ *at* $t = 3$*,* $(2, 18)$ *at* $t = 4$*, and* $(2, 22)$ *at* $t = 5$*;*
(ii) $(2, 4)$ *at every* $t \in \{6, 7, \dots, 21\}$*.*

In particular the single fault set $\{0, 2, 4\}$ has no re-rooted source in Γ_t for every $6 \le t \le 21$.

Proof. Each of the nineteen statements "no $v \in R_t$ has $v - a \in R_t$ and $v - b \in R_t$" is a finite check over the at most $6t$ elements of R_t (at most 126 at $t = 21$) and is evaluated by compiled code; obstruction with respect to the superset R_t implies obstruction with respect to B_t, since a witness v against the latter would be one against the former. Checking against R_t rather than B_t makes each statement strictly stronger than needed and each check $O(t^2)$ instead of $O(N^2)$. □

Theorem 5.2 (main)**.** *For every $t \ge 3$ there are $a, b \in \mathbb{Z}_N$ with $a \ne 0$, $b \ne 0$, $a \ne b$ such that the three-fault set $\{0, a, b\}$ is not re-rootable in Γ_t: no vertex of Γ_t is at hop distance exactly t from all of 0, a and b.*

Proof. For $t \ge 22$, Theorem 4.3 gives an obstructed pair, and Lemma 2.2 turns it into a non-re-rootable triple. For $3 \le t \le 21$, Theorem 5.1 gives the pair — $(5, 14)$, $(2, 18)$, $(2, 22)$ at $t = 3, 4, 5$ and $(2, 4)$ at $6 \le t \le 21$ — and Lemma 2.2 applies again. □

The $t = 3$ witness is the source's own triple, recomputed (Proposition 3.2). The theorem settles, for every diameter, what the source and [2] leave at a single example: the failure of the two-fault guarantee at three faults is not a feature of H_4 alone but of every dense Eisenstein–Jacobi network of diameter at least 3. (At $t = 1$ and $t = 2$ — $N = 7$ and $N = 19$ — the question is not treated here.)

Remark 5.3 (what is known about $\{0, 2, 4\}$ beyond the certified range)*.* The following are computations outside the formal development, made with the two independent programs of Section 8. The fault set $\{0, 2, 4\}$ is *re-rootable* at $t = 3$, 4 and 5 and obstructed at every $6 \le t \le 40$; so $t = 6$ is a genuine threshold, which is why the witnesses of Theorem 5.1 differ at $t = 3, 4, 5$. For every $6 \le t \le 30$, and at $t = 31, 35, 40$, the pair $(2, 4)$ is moreover the lexicographically least obstructed pair, i.e. the first hit of a plain scan over $1 \le a < b < N$; at $t = 4$ and $t = 5$ the first hits are $(2, 18)$ and $(2, 22)$, the witnesses used above. A proof that $(2, 4)$ is obstructed for every $t \ge 6$ is open and is not claimed; Theorem 5.2 does not depend on it. The vertices 2 and 4 are not neighbours of 0 in Γ_t (the jumps are $t, t + 1, 2t + 1 \ge 6$), and we have no structural explanation of the threshold.

6. The census

Theorem 6.1 (the census, $3 \le t \le 8$)**.** *The number $\mathrm{O}(t)$ of ordered pairs (a, b) of distinct nonzero vertices of Γ_t for which $\{0, a, b\}$ is not re-rootable is*

$$\mathrm{O}(3) = 66, \quad \mathrm{O}(4) = 432, \quad \mathrm{O}(5) = 1{,}512, \quad \mathrm{O}(6) = 3{,}918, \quad \mathrm{O}(7) = 8{,}478, \quad \mathrm{O}(8) = 16{,}236.$$

Moreover, at $t = 3$, exactly 407 of the $\binom{37}{3} = 7{,}770$ three-element subsets of $\mathbb{Z}_{37}$ are not re-rootable, counted directly over all triples $u < v < w$ of labels.

Proof. Each value is the cardinality of an explicitly filtered finite set — the $(N-1)(N-2)$ ordered pairs, or the $\binom{N}{3}$ increasing triples, filtered by the decidable predicate of Definition 2.1 — evaluated by compiled code and admitted by the kernel. The pair count at $t = 8$ ranges over $216 \cdot 215 = 46{,}440$ pairs, each tested against the 48 elements of B_8. □

Corollary 6.2. $\mathrm{T}_3(t) = N \cdot \mathrm{O}(t)/6$ *for every* $t \ge 1$*; hence*

$$\mathrm{T}_3(3) = 407, \qquad \mathrm{T}_3(4) = 4{,}392, \qquad \mathrm{T}_3(5) = 22{,}932,$$
$$\mathrm{T}_3(6) = 82{,}931, \qquad \mathrm{T}_3(7) = 238{,}797, \qquad \mathrm{T}_3(8) = 587{,}202.$$

Proof. By Lemma 2.3, re-rootability is invariant under translation, so $\mathbb{Z}_N$ acts on the non-re-rootable three-element sets. A three-element set F is fixed by $c \neq 0$ only if c has order 3 in $\mathbb{Z}_N$, and $3 \nmid N = 3t(t+1)+1$; so every orbit has exactly N elements, and each orbit contains exactly three sets containing 0 (the translates $F-f$, $f \in F$, which are distinct because a coincidence $F-f = F-f'$ would make $f'-f$ a nonzero stabiliser). The sets containing 0 are the $\{0, a, b\}$ with $a \neq b$ nonzero, each counted twice among ordered pairs, so their number is $\mathrm{O}(t)/2$, the number of orbits is $\mathrm{O}(t)/6$, and $\mathrm{T}_3(t) = N\,\mathrm{O}(t)/6$. The six values follow from Theorem 6.1; at $t = 3$, $37 \cdot 66/6 = 407$ agrees with the direct count. □

The passage from $\mathrm{O}(t)$ to $\mathrm{T}_3(t)$ is the short argument above and is not itself machine-checked, except at $t = 3$ where both numbers are; Lemma 2.3, on which it rests, is kernel-checked for every N.

t	N	$\lvert B_t\rvert$	obstructed pairs	re-rootable pairs	$\mathrm{T}_3(t)$	$\binom{N}{3}$	share	certified
3	37	18	33	597	407	7,770	0.052	yes
4	61	24	216	1,554	4,392	35,990	0.122	yes
5	91	30	756	3,249	22,932	121,485	0.189	yes
6	127	36	1,959	5,916	82,931	333,375	0.249	yes
7	169	42	4,239	9,789	238,797	790,244	0.302	yes
8	217	48	8,118	15,102	587,202	1,679,580	0.350	yes
9	271	54	14,226	22,089	1,285,082	3,280,455	0.392	–
10	331	60	23,301	30,984	2,570,877	5,989,445	0.429	–
11	397	66	36,189	42,021	4,789,011	10,349,790	0.463	–
12	469	72	53,844	55,434	8,417,612	17,083,794	0.493	–
13	547	78	77,328	71,457	14,099,472	27,128,465	0.520	–
14	631	84	107,811	90,324	22,676,247	41,674,395	0.544	–
15	721	90	146,571	112,269	35,225,897	62,207,880	0.566	–
16	817	96	194,994	137,526	53,103,366	90,556,280	0.586	–
17	919	102	254,574	166,329	77,984,502	128,936,619	0.605	–
18	1027	108	326,913	198,912	111,913,217	180,007,425	0.622	–
19	1141	114	413,721	235,509	157,351,887	246,923,810	0.637	–
20	1261	120	516,816	276,354	217,234,992	333,395,790	0.652	–
21	1387	126	638,124	321,681	295,025,996	443,749,845	0.665	–
22	1519	132	779,679	371,724	394,777,467	582,993,719	0.677	–
23	1657	138	943,623	426,717	521,194,437	756,884,460	0.689	–
24	1801	144	1,132,206	486,894	679,701,002	971,999,700	0.699	–
25	1951	150	1,347,786	552,489	876,510,162	1,235,812,175	0.709	–

TABLE 1. The three-fault census of Γ_t as computed by the C enumerator of Section 8: "obstructed pairs" and "re-rootable pairs" count unordered pairs $\{a, b\}$ of distinct nonzero vertices, so that the former is $\mathrm{O}(t)/2$; $\mathrm{T}_3(t) = N \cdot (\text{obstructed pairs})/3$ by Corollary 6.2; "share" is $\mathrm{T}_3(t)/\binom{N}{3}$ rounded to three places. "Certified" marks the rows whose pair count is Theorem 6.1; their T_3 entry then follows by Corollary 6.2, which is machine-checked only at $t = 3$. The rows $t \geq 9$ are enumerator data with no formal certificate, and the rows $t \leq 13$ were reproduced by an independent Python implementation.

Remark 6.3 (the transition, outside the formal development). Table 1 shows obstructed triples in the minority for $t \leq 12$ (53,844 obstructed against 55,434 re-rootable pairs at $t = 12$) and in the majority from $t = 13$ on (77,328 against 71,457), the share rising to about 71% at $t = 25$. A naive independence heuristic — a random pair (a, b) has expected $\lvert B_t\rvert^3/N^2 \approx 24/t$ common boundary vertices — would put the crossover near $t = 24$; the true crossover is at 13. The smallest diameter at which the source has no value at all is $t = 4$: there 4,392 of the 35,990 three-element fault sets have no re-rooted source, about one in eight.

7. An erratum in the source's vertex label

The source's equation (24) prints the integer label $\phi(x + y\omega) \equiv (n-1)x + (2n-1)y \pmod{N}$, i.e. $tx + (2t+1)y$, and the caption of its Figure 1 instantiates it at $t = 3$ as $3x + 7y \pmod{37}$.

Proposition 7.1 (the vertex label)**.** *Let $t \geq 1$, $N = 3t^2 + 3t + 1$ and $\alpha = (t+1) + t\omega$.*

(i) *The map $x + y\omega \mapsto tx + (2t+1)y$ sends α to $3t^2 + 2t$, and $0 < 3t^2 + 2t < N$; so it is not constant on the cosets of (α) and does not define a function on $\mathbb{Z}[\omega]/(\alpha)$.*

(ii) *The map $x + y\omega \mapsto tx - (t+1)y$ sends α to 0 and $\alpha\omega$ to $-N$; since (α) is the $\mathbb{Z}$-module spanned by $\alpha = (t+1, t)$ and $\alpha\omega = -t + (2t+1)\omega = (-t, 2t+1)$, it defines a homomorphism $\mathbb{Z}[\omega]/(\alpha) \to \mathbb{Z}_N$ of additive groups, which is a bijection because both groups have N elements and the image contains t, a unit modulo N.*

(iii) *Both maps send the six units $\pm 1, \pm\omega, \pm\omega^2$ to the jump set $\{\pm t, \pm(t+1), \pm(2t+1)\}$.*

Proof. (i) $t(t+1) + (2t+1)t = 3t^2 + 2t$, and $3t^2 + 2t < 3t^2 + 3t + 1$ for $t \geq 1$. (ii) $t(t+1) - (t+1)t = 0$ and $t(-t) - (t+1)(2t+1) = -(3t^2 + 3t + 1)$; the identity $\alpha\omega = (t+1)\omega + t\omega^2 = -t + (2t+1)\omega$ uses $\omega^2 = \omega - 1$, which the source displays as its equation (27) and which its adjacency (its equation (9)), its distance formula (its equation (14)) and its order formula all require. Injectivity of a surjection between sets of size N. (iii) Under the printed map, $1 \mapsto t$, $\omega \mapsto 2t+1$, $\omega^2 = \omega - 1 \mapsto t+1$; under the repaired map, $1 \mapsto t$, $\omega \mapsto -(t+1)$, $\omega^2 \mapsto -(2t+1)$. □

Concretely, at $t = 3$ the point $(4, 3) = \alpha$ is congruent to 0, yet the printed formula gives it the label $3 \cdot 4 + 7 \cdot 3 = 33 \not\equiv 0 \pmod{37}$. By (iii) the source's derivation of the jump set from the label (its equations (25)–(31)) is correct as a statement about the jump set, and its Proposition 1, Lemma 1 and Table 2 are all stated and verified in the circulant representation $C_N(t, t+1, 2t+1)$; so nothing the source proves or prints is affected — only the assignment of labels to lattice points in its Figure 1 is. The map used throughout this paper is the repaired one, ψ_t of Section 4.

This is not an independent finding, and it is the one statement of this paper with no counterpart in the formal development of Section 8. The same author's arXiv:2606.19834 [3] prints the same map as its equation (10), and the companion manuscript [6] records the erratum there (its Proposition 7.2, whose parts (i) and (iii) are exactly the identities used in the proof above, kernel-checked for every integer t); the present section records the same slip in a second preprint, not a second discovery. The author's later arXiv:2606.21133 [2] prints the correct form as its equation (8), $\phi(x + y\omega) \equiv tx - (t+1)y \pmod{N}$, for the same α and the same N; the caption of its Figure 1 reads "Integer labels use $\phi(x + y\omega) = 3x - 4y \pmod{37}$".

8. Verification

The formal development consists of four Lean 4 modules, 710 lines in all, checked with Lean 4.32.0 and *Mathlib* [12, 13]. `Circulant.lean` (203 lines) is shared with the companion manuscript [7] on dense Gaussian networks: it defines, for an arbitrary modulus N and jump list, the iterated neighbourhood `reach`, the distance-n sphere `sph`, the predicates `Rerootable` and `Obstructed` of Definition 2.1, the counts `obsOrd` and `obsTriples`, and proves Lemmas 2.2, 2.3 and 2.4 (`obstructed_iff_not_rerootable`, `rerootable_translate`, `exists_obstructed_pair`). `Ej.lean` (219 lines) instantiates $N = 3t^2 + 3t + 1$ and J_t (`eN`, `eJ`, `eB`), defines ψ_t, ρ and R_t (`latE`, `nrmE`, `ringE`), and proves Lemma 4.1 (`mem_reach_iff`), Lemma 4.2 (`eB_subset_ringE`, `card_ringE_le`) and Theorem 4.3 (`exists_obstructed_large`). `Data.lean` (241 lines) holds Propositions 3.1–3.5, the nineteen witnesses of Theorem 5.1 (`obstructed_ring_3`, …, `obstructed_ring_21`) and Theorem 5.2 (`obstructed_triple_forall`); `Counts.lean` (47 lines) holds Theorem 6.1 (`obsOrd_3`, …, `obsOrd_8`, `obsTriples_3`).

What is trusted. The general-t statements — the three lemmas of `Circulant.lean`, `mem_reach_iff`, `card_ringE_le`, `eB_subset_ringE` and `exists_obstructed_large` — are proved by the kernel with axioms `propext`, `Classical.choice` and `Quot.sound` only; no native evaluation enters Theorem 4.3. Every finite statement (the forty-six declarations of `Data.lean` and `Counts.lean` other than `obstructed_triple_forall`) is a decidable proposition closed by `native_decide`: the Lean compiler evaluates it as native code and the kernel admits the result on the strength of Lean's native-evaluation axiom, which appears in the axiom print of each such theorem as `<name>._native.native_decide.ax_1_1`. The axiom print of every declaration, re-run read-only for this paper, is exactly the three standard axioms plus, for a `native_decide` declaration, its own native axiom and nothing else; `obstructed_triple_forall` carries the standard axioms together with exactly the nineteen native axioms of `obstructed_ring_3`, …, `obstructed_` `ring_21`, so that its $t \geq 22$ branch contributes no native axiom at all. No declaration depends on `sorryAx`. The trust base is therefore Lean's kernel, plus the Lean compiler and the C toolchain behind it for the finite statements, plus the correctness of the definitions: that `reach` iterates the one-jump neighbourhood of $\mathrm{Cay}(\mathbb{Z}_N, J_t)$, that `eB t` is $\mathrm{reach}_t \setminus \mathrm{reach}_{t-1}$, and that `Rerootable` and `Obstructed` are the predicates of Definition 2.1 — which is what Section 3 is for.

Cost. The figures below are the elaboration times measured once, when the development was written, and recorded in its notes; they were not re-measured for this paper, and no module was re-elaborated for it. `Ej.lean` 24 s at about 1.3 GB; `Data.lean` 15 s (the nineteen witnesses check against R_t, not B_t; 79 s on a more heavily loaded machine); `Counts.lean` 3 min 49 s, the census at $t = 8$ dominating. The census at $t = 9, 10$ was priced at three to eight minutes of compiled evaluation each and was not run; past $t = 10$ the set-membership inner loop would have to be replaced by a bitmask with a proved bridge.

Independent re-computations. Two programs independent of the Lean code were used. A C enumerator, written from the source's definitions before any Lean file (breadth-first search in the circulant, the boundary as a bitset, a triple $\{0, a, b\}$ re-rootable iff $B \wedge \mathrm{rot}(B, a) \wedge \mathrm{rot}(B, b) \neq 0$), produced every value of Table 1 for $3 \leq t \leq 25$ and every statement of Remark 5.3: the obstruction of $\{0, 2, 4\}$ for $6 \leq t \leq 40$ and its re-rootability at $t = 3, 4, 5$, and the lexicographically first obstructed pair for $3 \leq t \leq 31$ and at $t = 35, 40$; it also reproduces all five rows of the source's Table 2, up to $N = 120{,}601$, and the source's printed $B_3(0)$ and empty intersection. A Python implementation, written for this paper independently of both the C and the Lean code, computes the hop distance twice — by breadth-first search in $\mathrm{Cay}(\mathbb{Z}_N, J_t)$, and as the minimum of $\rho(x, y)$ over the lattice points (x, y) with $\psi_t(x, y) = v$ — and finds the two equal, with diameter t and $|B_t| = 6t$, for $1 \leq t \leq 12$ (an instance of Lemma 4.1); it reproduces the printed $B_3(0)$, the obstruction of $\{0, 5, 14\}$ and the re-rootability of $\{0, 1, 2\}$, all five rows of the source's Table 2, the exhaustive two-fault check of Proposition 3.4 for $3 \leq t \leq 8$, the status of $\{0, 2, 4\}$ for $3 \leq t \leq 40$, the census values of Theorem 6.1 and Corollary 6.2 and the rows $t \leq 13$ of Table 1 including the lexicographically first pair at each of those diameters, the direct count 407 over all $\binom{37}{3}$ triples, and the threshold $t = 22$ of the inequality of Theorem 4.3. Every value agreed with the Lean side. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted; the namespaces `LeanProblemSpec.Reroot` and `LeanProblemSpec.EjReroot` are omitted). Here `ZMod N` is $\mathbb{Z}_N$, `Finset` is a finite set, `reach J n` is reach_n for the jump list `J`, `sph J n` is $\mathrm{reach}_n \setminus \mathrm{reach}_{n-1}$, `eN t` is N, `eJ t` is J_t, `eB t` is B_t, `latE t a b` is $\psi_t(a, b)$, `nrmE a b` is $\rho(a, b)$, and `ringE t` is R_t.

Definitions and the shared lemmas (Section 2).

```
def Rerootable (S : Finset (ZMod N)) (F : Finset (ZMod N)) : Prop :=
  ∃ v : ZMod N, ∀ f ∈ F, v - f ∈ S

def Obstructed (S : Finset (ZMod N)) (a b : ZMod N) : Prop :=
  ∀ v ∈ S, ¬(v - a ∈ S ∧ v - b ∈ S)

theorem rerootable_translate (S : Finset (ZMod N)) (F : Finset (ZMod N)) (c : ZMod N) :
    Rerootable S (F.image (· + c)) ↔ Rerootable S F

theorem exists_obstructed_pair (S : Finset (ZMod N)) (h : S.card ^ 3 + 3 * N < N * N) :
    ∃ a b : ZMod N, a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧
      ∀ v : ZMod N, v ∈ S → v - a ∈ S → v - b ∉ S

def obsOrd (S : Finset (ZMod N)) : ℕ :=
  ((Finset.univ : Finset (ZMod N × ZMod N)).filter
```

```
    (fun p => p.1 ≠ 0 ∧ p.2 ≠ 0 ∧ p.1 ≠ p.2 ∧ Obstructed S p.1 p.2)).card

def obsTriples (S : Finset (ZMod N)) : ℕ :=
  ((Finset.univ : Finset (ZMod N × ZMod N × ZMod N)).filter
    (fun p => p.1.val < p.2.1.val ∧ p.2.1.val < p.2.2.val ∧
      ¬ Rerootable S ({p.1, p.2.1, p.2.2} : Finset (ZMod N)))).card
```

Section 4.

```
def eN (t : ℕ) : ℕ := 3 * t * t + 3 * t + 1
def eB (t : ℕ) : Finset (ZMod (eN t)) := sph (eJ t) t
def latE (t : ℕ) (a b : ℤ) : ZMod (eN t) :=
  (a : ZMod (eN t)) * (t : ZMod (eN t)) - (b : ZMod (eN t)) * ((t : ZMod (eN t)) + 1)
def nrmE (a b : ℤ) : ℕ := max (max a.natAbs b.natAbs) (a + b).natAbs

theorem mem_reach_iff (t : ℕ) (n : ℕ) (v : ZMod (eN t)) :
    v ∈ reach (eJ t) n ↔ ∃ a b : ℤ, nrmE a b ≤ n ∧ latE t a b = v

theorem card_ringE_le (t : ℕ) : (ringE t).card ≤ 6 * t
theorem eB_subset_ringE (t : ℕ) : eB t ⊆ ringE t

theorem exists_obstructed_large (t : ℕ) (ht : 22 ≤ t) :
    ∃ a b : ZMod (eN t), a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧ Obstructed (eB t) a b
```

Sections 3, 5 and 6.

```
theorem eB3_eq :
    eB 3 = ({2, 5, 9, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 28, 32, 35} :
      Finset (ZMod (eN 3)))
theorem source_counterexample : ¬ Rerootable (eB 3) ({0, 5, 14} : Finset (ZMod (eN 3)))
theorem table2_t10 :
    (eB 10).card = 60 ∧ diffSet (eB 10) = (Finset.univ : Finset (ZMod (eN 10)))
theorem card_eB_3 : (eB 3).card = 18          -- ... through card_eB_10 : (eB 10).card = 60
theorem twoFault_3 : TwoFaultAlways (eB 3)    -- ... through twoFault_8
theorem rerootable_example : Rerootable (eB 3) ({0, 1, 2} : Finset (ZMod (eN 3)))
theorem not_all_obstructed : ¬ Obstructed (eB 3) 1 2
theorem eB3_ne_univ : eB 3 ≠ (Finset.univ : Finset (ZMod (eN 3)))

theorem obstructed_ring_3 : Obstructed (ringE 3) 5 14
theorem obstructed_ring_4 : Obstructed (ringE 4) 2 18
theorem obstructed_ring_5 : Obstructed (ringE 5) 2 22
theorem obstructed_ring_6 : Obstructed (ringE 6) 2 4    -- ... through obstructed_ring_21

theorem obstructed_triple_forall (t : ℕ) (ht : 3 ≤ t) :
    ∃ a b : ZMod (eN t), a ≠ 0 ∧ b ≠ 0 ∧ a ≠ b ∧
      ¬ Rerootable (eB t) ({0, a, b} : Finset (ZMod (eN t)))

theorem obsTriples_3 : obsTriples (eB 3) = 407
theorem obsOrd_3 : obsOrd (eB 3) = 66
  -- obsOrd_4 = 432, obsOrd_5 = 1512, obsOrd_6 = 3918,
  -- obsOrd_7 = 8478, obsOrd_8 = 16236
```

(`TwoFaultAlways S` is `∀ a : ZMod N, Rerootable S {0, a}`, and `diffSet S` is $S - S$.)

References

[1] B. Albader, *Re-Rooting-Based Fault-Tolerant One-to-All Broadcasting in Dense Eisenstein–Jacobi Networks*, arXiv:2606.18712v1 [cs.DC], submitted 17 June 2026. Read in full from the version-1 e-print (a single `main.tex` and nine figures), the only version as of 2026-09-07 (live abstract page checked on that date; primary category cs.DC read off the page). Statement, table, page and equation numbers are those of the PDF compiled with `tectonic` from the unmodified e-print (30 pages) and of its `.aux` file: equations (9), (10)–(12), (14)–(19), (24), (25)–(31), (56), (61); Lemma 1 (p. 17), Theorem 1 (p. 19), Proposition 1 (p. 20), Table 2 (p. 24), Figure 1. The quotations in Section 1 are verbatim from that PDF.

[2] B. Albader, *Re-Rooting-Assisted Edge-Minimum Runtime Repair for Node and Link Failures in Dense Eisenstein–Jacobi Broadcast Networks*, arXiv:2606.21133v1 [cs.DC], submitted 19 June 2026. Read from the e-print compiled with `tectonic` (29 pages): its equation (8) and the caption of its Figure 1 (Section 3), its Remark 1 ("Three-fault limitation") and its Example 1 ("Concrete H_4 three-fault obstruction"), both on p. 9.

[3] B. Albader, *Multi-Orientation Edge-Minimum Repair for Non-Redundant Fault-Tolerant Broadcasting in Dense Eisenstein–Jacobi Networks*, arXiv:2606.19834v1 [cs.DC], submitted 18 June 2026. Cited for its equation (10), the vertex label, as read and recorded in [6]; not otherwise used here.

[4] B. Albader, *Closed-Form and Constant-Time New-Source Selection for Fault-Tolerant Broadcasting in Dense Eisenstein–Jacobi Networks*, arXiv:2606.18714v1 [cs.DC], submitted 17 June 2026. Title, author, date, category and abstract read off the live abstract page on 2026-09-07; its full text (corpus copy) was searched for "three" and "triple" and contains no statement about three faults. Not otherwise read.

[5] B. Albader, M. R. Al-Mulla and G. Hassan, *Re-Rooting-Based Fault-Tolerant Broadcasting in Dense Gaussian Networks*, arXiv:2606.16954v1 [cs.DC], submitted 15 June 2026. Title, authors and category read off the live abstract page on 2026-09-07; the Gaussian counterpart of the source, treated in [7].

[6] Korea Superintelligence Labs, *Repair depth of multi-orientation edge-minimum broadcast repair in dense Eisenstein–Jacobi networks: depth $t+1$ suffices for two faults at every diameter* $4 \le t \le 19$, companion manuscript, version 1 (2026), at `families/ej-moem-depth/paper/v1.pdf` of the repository of Section 8. Cited for its Proposition 7.2 (the vertex label of arXiv:2606.19834).

[7] Korea Superintelligence Labs, *Three-fault re-rooting in dense Gaussian networks: an obstructed fault set for every diameter, and the census of obstructed triples*, companion manuscript, version 1 (2026), 12 pages, at `families/gauss-reroot-3fault/paper/v1.pdf` of the repository of Section 8. The dense Gaussian counterpart of [5]; it shares the Lean module `Circulant.lean` of Section 8 with the present paper.

[8] C. Martínez, E. Stafford, R. Beivide and E. M. Gabidulin, *Modeling hexagonal constellations with Eisenstein–Jacobi graphs*, Problems of Information Transmission **44** (2008), no. 1, 1–11. Bibliographic data checked against Crossref (doi:10.1134/S0032946008010018); the network family. Not read for this work.

[9] M. Flahive and B. Bose, *The topology of Gaussian and Eisenstein–Jacobi interconnection networks*, IEEE Transactions on Parallel and Distributed Systems **21** (2010), no. 8, 1132–1142. Bibliographic data checked against Crossref (doi:10.1109/TPDS.2009.132); the topology and the diameter. Not read for this work.

[10] A. Thomson and S. Zhou, *Frobenius circulant graphs of valency six, Eisenstein–Jacobi networks, and hexagonal meshes*, European Journal of Combinatorics **38** (2014), 61–78. Bibliographic data checked against Crossref (doi:10.1016/j.ejc.2013.11.004). Of the non-Albader e-prints found by the corpus sweep of Section 1 it is the one about the networks themselves; not read for this work.

[11] Z. A. Hussain, B. Bose and A. Al-Dhelaan, *Edge disjoint Hamiltonian cycles in Eisenstein–Jacobi networks*, Journal of Parallel and Distributed Computing **86** (2015), 62–70. Bibliographic data checked against Crossref (doi:10.1016/j.jpdc.2015.08.003); not read for this work.

[12] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635.

[13] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.