POSITIVE ROOTS OF THE FLAG-MANIFOLD EINSTEIN POLYNOMIALS OF ARVANITOYEORGOS, SAKANE AND STATHA: AN UNPROVED REMARK FOR $\mathrm{SU}(N)$, THE SHARP CONSTANT FOR $\mathrm{SO}(n)$, AND EXACT ROOT COUNTS

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Abstract. Two recent papers of Arvanitoyeorgos, Sakane and Statha reduce the existence of non-naturally-reductive invariant Einstein metrics on the compact Lie groups $\mathrm{SU}(N)$ and $\mathrm{SO}(n)$, $N = n = k_1 + (p-1)k$, to the positive real roots of a single univariate polynomial whose coefficients are integer polynomials in (k, p, k_1): a polynomial F_3 of degree 16 for $\mathrm{SU}(N)$, and a polynomial H_1 of degree 8 for $\mathrm{SO}(n)$. The $\mathrm{SO}(n)$ paper proves that H_1 has four positive roots when $k_1 \geq 10kp$; the $\mathrm{SU}(N)$ paper asserts the analogous four-root statement for F_3 when $k_1 \geq 8kp$ inside a remark that explicitly declines to give a proof. We prove the $\mathrm{SO}(n)$ four-root theorem for all real $k \geq 3$, $p \geq 3$ and $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$, and exhibit parameters with $k_1/(kp) = 5.560044$ at which H_1 has only two positive roots; the optimal constant is therefore bracketed to within 0.044%, and its exact asymptotic value is $5.5602472\ldots$. The improvement comes from moving the separating point of the argument, not from a better certificate: we show that the paper's own upper separating point $2/3$ admits no constant below 7.169, replace it by $15/32$, and prove that the paper's lower separating point γ, which it uses without explanation, is the vertex of a perfect square and cannot be simplified at all. On the $\mathrm{SU}(N)$ side we prove the first step of the unproved remark for all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, and the whole of it for six explicit pairs (k, p) and every real k_1 above a threshold well below $8kp$; moving the remark's own separating point from 2 to $17/8$ proves its first step at $k_1 \geq \frac{89}{16}kp$ as well, the same constant as on the $\mathrm{SO}(n)$ side. Finally we show that the two constructions, which use different groups and different flag manifolds and whose polynomials neither paper relates, have literally the same asymptotic four-root threshold: their $k \to \infty$ limit polynomials differ by $x \mapsto 1/x$ and a strictly positive factor. We then prove what neither source asks, an *upper* bound: for all real $k \geq 3$, $p \geq 3$, $k_1 \geq 11kp$ the polynomial H_1 has at most four positive roots, counted with multiplicity, although its coefficients strictly alternate on that region, so that Descartes' rule of signs allows eight. Combined with the four-root theorem this makes the count *exactly* four — the first exact count in this construction, both sources stopping at "at least four". The device is a positive multiplier: $(1+x)^9 H_1$ has at most four sign changes, by seventeen positivity certificates and a trichotomy on the one coefficient whose sign is left undetermined. The same construction gives at most four positive roots of F_3 for $k_1 \geq 50kp$, certified in exact integer arithmetic outside the formal development. For F_3 itself we then give exact counts at explicit parameters, where the multiplier needs no certificate at all because the coefficients of $(1+x)^N F_3$ are then explicit integers: F_3 has exactly four positive roots, with multiplicity, at $(k, p, k_1) = (2, 3, 26)$, $(2, 4, 33)$, $(3, 3, 45)$, $(5, 5, 300)$ and $(1000, 3, 16681)$, and exactly two at $(2, 3, 24)$. The ratios $k_1/(kp)$ there go down to 4.125 against the 8 of the remark; the four-root count at ratio 5.560333 shows that $\frac{89}{16}$ is where the certificate closes and not where the four-root property begins; and the two-root count at ratio 4 shows that no four-root statement of this shape can hold with a constant ≤ 4, even on the slice $k = 2$. Two errata in the sources' displayed formulas are recorded. Every theorem below is machine-checked in Lean 4; the root counts that bracket the constants from below use Sturm sequences and are flagged where they occur as lying outside the formal development, the exact counts just described being kernel theorems and not among them.

1. Introduction

1.1. **The two constructions.** Let $p \geq 3$ and $k_1 \geq 1$ be integers, let $k \geq 2$ for $\mathrm{SU}(N)$ and $k \geq 3$ for $\mathrm{SO}(n)$, and put $N = n = k_1 + (p-1)k$. Arvanitoyeorgos, Sakane and Statha study left-invariant metrics on the compact Lie groups $\mathrm{SU}(N)$

and $\mathrm{SO}(n)$ that are invariant under the adjoint action of a block subgroup with one distinguished block of size k_1 and $p-1$ equal blocks of size k: metrics invariant under $\mathrm{Ad}\big(s(\mathrm{U}(k_1)\times\mathrm{U}(k)^{p-1})\big)$ on $\mathrm{SU}(N)$ in [1], and under $\mathrm{Ad}\big(\mathrm{SO}(k_1)\times\mathrm{SO}(k)^{p-1}\big)$ on $\mathrm{SO}(n)$ in [2]. In both cases the Einstein condition becomes a polynomial system in the finitely many metric parameters, and in both cases the authors eliminate all but one of them. What is left is a single univariate polynomial equation whose coefficients are integer polynomials in (k,p,k_1):

$$F_3(x_{12}) = \sum_{\ell=0}^{16} a_\ell(k,p,k_1)\, x_{12}^{\ell} = 0 \qquad \text{for } \mathrm{SU}(N),$$

of degree 16, and

$$H_1(x_{23}) = \sum_{e=0}^{8} b_e(k,p,k_1)\, x_{23}^{e} = 0 \qquad \text{for } \mathrm{SO}(n),$$

of degree 8. Under the hypotheses of [1, 2], a positive real root other than the normalising value 1 yields an invariant Einstein metric that is not naturally reductive. So the geometric question "how many such Einstein metrics are there?" becomes the algebraic question

> *how many positive real roots do F_3 and H_1 have, as functions of (k,p,k_1), and how large must k_1 be, relative to kp, for four of them?*

This paper answers the second half of that question about as sharply as it can be answered, for $\mathrm{SO}(n)$ completely and for $\mathrm{SU}(N)$ in part; and for $\mathrm{SO}(n)$ it answers the first half as well, on a slightly smaller region than the one on which it produces the four roots: there the count is exactly four. Nothing here is a statement about Einstein metrics as such (for which see [15], and [14] for the nearest neighbouring results on $\mathrm{SO}(n)$): the passage from a positive root to a metric, and the failure of natural reductivity, are theorems of [1, 2] and are cited rather than reproved (see §9).

1.2. **What the sources leave open.** For $\mathrm{SO}(n)$, [2] proves the four-root statement in full. Its four-metric theorem reads, verbatim:

> "Let $p \geq 3$ and $k \geq 3$. If $k_1 \geq 10kp$, then the Lie group $\mathrm{SO}(n)$ $(n = k_1 + (p-1)k)$ admits at least four $\mathrm{Ad}(\mathrm{SO}(k_1)\times(\mathrm{SO}(k))^{p-1})$-invariant Einstein metrics of the form *[(1) below]*, which are not naturally reductive."

This is Theorem 6.4 of [2]: the sentence, its constant 10 and its proof are identical in the published text and in arXiv:2409.18990v1, and the display it refers to is equation (9) of both. We give the number here because the LaTeX label of that theorem in the arXiv source is `thm6.3` — section 6 runs Proposition 6.1, Theorem 6.2, Remark 6.3, Theorem 6.4 — so a reader working from the source rather than from the paper will mis-cite it. Below we mostly call it *the four-metric theorem* of [2].

The constant 10 is what the paper's argument delivers; nothing in [2], and nothing we could find anywhere else, asks whether it is optimal. That is the question this paper settles.

For $\mathrm{SU}(N)$, the corresponding statement is not proved at all. [1] states it as a remark, verbatim:

> "**Remark 1.** It is possible to show that, for $k_1 \geq 8kp$, $k \geq 2$ and $p \geq 3$, the equation $F_3(x_{12}) = 0$ has at least four positive solutions. We just give a sketch of proof."

The introduction of [1] repeats the assertion and says "We only give a sketch of its proof in Remark 1 of Section 4." The sketch is three unproved assertions — $F_3(2) > 0$ for $k_1 \geq 8kp$; $F_3(\beta) < 0$ for $k_1 \geq 5kp$, where β is the explicit rational function of (k, p, k_1) recalled in (5); and $\beta > 2$ for $k_1 \geq 5kp$ — from which the chain $0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < \beta < \alpha_4 < k_1 kp$ is read off. The paper's own constants are inconsistent: steps (2) and (3) are stated for $k_1 \geq 5kp$ and the conclusion for $k_1 \geq 8kp$, which is one more sign that the remark was never written out.

One question is not merely left open by the two sources but is never put: *how many* positive roots are there? Every statement in both papers is an existence statement. A verbatim search of the two arXiv sources finds "at least four" twice in [2] and five times in [1], and finds "at most", "exactly four", "precisely four", "Descartes", "sign change" and "upper bound" zero times in either; the word "sign" itself occurs in neither. In particular neither source records that the coefficients of the polynomial it eliminates alternate in sign, although the same authors record exactly that alternation in several of their other papers [4, 5, 6], where it serves one purpose only: to conclude that real roots are positive. §5 answers the counting question for $\mathrm{SO}(n)$ uniformly in the parameters, §8 answers it for F_3 at explicit parameters, and §5.5 records what was already known about upper bounds of this kind.

1.3. **Results.** Throughout, k, p and k_1 are treated as *real* parameters subject to the stated inequalities; all our theorems are therefore stronger than their integer specialisations, and F_3, H_1 are polynomials in four real variables.

The $\mathrm{SO}(n)$ *constant.* Theorem 3.7 proves the four-root statement for all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{89}{16} kp = 5.5625\, kp$, in the source's own chain with one separating point moved. Proposition 3.9 exhibits $(k, p, k_1) = (10000, 3, 166816)$, ratio $k_1/(kp) = 5.560533$, at which the binding sign of the argument has not yet turned; and exact root counts (Remark 3.10, outside the formal development) give parameters with ratio 5.560044 at which H_1 has only two positive roots. So the optimal constant lies in [5.56004, 5.5625], a bracket of 0.044%, against the published 10.

Where the slack was. Keeping the point $2/3$ that the source runs its chain through, the same style of certificate reaches $\frac{29}{4} = 7.25$ (Theorem 3.2) and no further, since Proposition 3.4 exhibits parameters with ratio 7.1683 at which $H_1(2/3) < 0$. All the slack was in the point, and Theorem 3.6 obtains $\frac{89}{16}$ by replacing $2/3$ by $15/32$.

Why the other separating point cannot be moved. The source's lower separating point is $\gamma = A/(k_1(kp-2))$ with $A = k^2p^2 + k^2p - 2k^2 - 6kp + 4k + 4$, given without explanation. Theorem 4.1 explains it: written as a polynomial in (k_1, u), the quantity $k_1^8 H_1(u/k_1)$ has top k_1-degree part $k_1^9\big((kp-2)u - A\big)^2$, a perfect square. Its discriminant vanishes identically, so the two small roots of H_1 merge as $k_1 \to \infty$ and the window between them closes; γ is the vertex, and every separating point $N(k,p)/k_1$ with $N \neq A/(kp-2)$ eventually leaves that window. Proposition 3.9 makes this concrete for the obvious candidate $N = kp$.

How many roots there can be. Theorem 5.4 bounds the positive roots of H_1 from *above*: at most four, counted with multiplicity, for all real $k \geq 3$, $p \geq 3$, $k_1 \geq 11kp$.

Proposition 5.1 is the control that keeps this from being a restatement of Descartes' rule: on the same region the coefficients of H_1 strictly alternate in sign, so the rule itself gives eight, and the deficiency closed is exactly four. Theorem 5.5 puts the two bounds together: for those parameters H_1 has *exactly* four positive roots with multiplicity, so the four roots of Theorem 3.7 are simple and there are no others. This is the first exact count anywhere in this construction. Proposition 5.6 proves the same at the single point $(k, p, k_1) = (3, 3, 104)$ out of five sign evaluations, independently of the large certificate behind Theorem 3.7.

The SU(N) *remark.* Theorem 6.1 proves step (1) of Remark 1 for all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, sharpening the asserted $8kp$; Theorem 6.2 deduces two positive roots localised in $(0, 1)$ and $(1, 2)$, against the source's own $(0, 1)$ and $(1, k_1 kp)$; and Theorem 6.4 proves the whole of Remark 1 — the full chain — for the six pairs $(k, p) \in \{(2, 3), (3, 3), (4, 3), (2, 4), (3, 4), (2, 5)\}$ and every real k_1 above a threshold $m(k, p)$ with $m/(kp)$ between 3.90 and 5.33, against the asserted $8kp$.

The SU(N) *separating point.* Remark 1 runs its chain through the point 2, and that point admits no constant below 5.5583 (Proposition 6.3). Theorem 6.7 moves it to 17/8 and proves $F_3(17/8) > 0$, hence two positive roots in $(0, 1)$ and $(1, 17/8)$, for all real $k \geq 2$, $p \geq 3$, $k_1 \geq \frac{89}{16} kp$ — the very constant Theorem 3.6 reaches on the SO(n) side, which is no coincidence: Theorem 7.1 says the two constructions have the same asymptotic threshold. Proposition 6.8 exhibits parameters of ratio 5.5618 at which the sign has not yet turned, so for this point the constant is within 0.012% of the least admissible one.

Exact counts for F_3. §8 counts the positive roots of the degree-16 polynomial exactly at six explicit parameter points: four, with multiplicity, at $(2, 3, 26)$, $(2, 4, 33)$, $(3, 3, 45)$, $(5, 5, 300)$ (Theorem 8.1) and at $(1000, 3, 16681)$ (Theorem 8.2), and two at $(2, 3, 24)$ (Proposition 8.3). The lowest ratio at which four roots are pinned is 4.125, against the 8 asserted in [1]; the ratio at $(1000, 3, 16681)$ is below the constant $\frac{89}{16}$ of Theorem 6.7, so that constant is not the least ratio at which the four-root property holds; and the count at $(2, 3, 24)$, whose ratio is exactly 4, refutes every four-root statement of this shape with a constant ≤ 4. These counts need no certificate and no Sturm sequence: they are what the multiplier of §5 becomes at a numeric point.

A bridge. The two constructions are different — different groups, different flag manifolds, degree 16 against degree 8. [1] does put them side by side, and observes that the two Einstein systems reduce to polynomial equations of degree 8 for SO(n) and of degree 16 for SU(N); but neither paper relates the two polynomials. Theorem 7.1 shows that after the substitution $k_1 = wkp$ their top k-degree parts are related by $x \mapsto 1/x$ and multiplication by a factor that is strictly positive for $p > 1$, $w > 0$ and every real x. In the limit the positive roots of F_3 are exactly the reciprocals of those of H_1, with multiplicity, so the two constructions have the *same* asymptotic four-root threshold in $w = k_1/(kp)$.

Finally, Propositions 2.4 and 2.2 record two incorrect displays in the sources, both harmless to their results.

1.4. **Method, and what rests on computation.** Every positivity statement below is a *Handelman certificate* [16] in shifted variables: to prove $P(k, p, k_1) > 0$ on the polyhedron $\{k \geq k_0,\ p \geq 3,\ k_1 \geq c\, kp\}$ we substitute $k = k_0 + X$, $p = 3 + Y$,

$k_1 = c\,kp + Z$, clear the denominator, and observe that every coefficient of the resulting polynomial in (X, Y, Z) is non-negative and that the constant term is positive. Such a certificate is a finite, exactly verifiable object. This is not our invention: the proof in [2] of $H_1(2/3) > 0$ writes

$$H_1(2/3) = \tfrac{1}{3^8}\Big(\textstyle\sum_{j=0}^{4} p_j(k,p)(k_1 - 10kp)^j + 64\,(k_1 - 10kp)^5\Big), \qquad p_j > 0,$$

which is exactly a Handelman certificate in the shifted variable $k_1 - 10kp$. We generalise it in three variables and search over the constant c on a dyadic grid.

The upper bound of §5 uses the same certificates for a different purpose. There the object certified is not the value of H_1 at a point but the *sign of each coefficient* of $(1 + x)^9 H_1$; the certificates are again polynomials in $(k - 3,\ p - 3,\ k_1 - 11kp)$ with coefficients of one sign, and what is read off them is a count of sign changes, to which Descartes' rule of signs is then applied. Multiplying by a polynomial with positive coefficients moves no positive root and cannot increase the number of sign changes, so a well-chosen multiplier sharpens the rule; here $(1 + x)^9$ takes eight alternations down to four. Sharpening the rule of signs by a positive multiplier is a classical device and we claim no novelty for it; what is new here is not the device but the certificates that sign the coefficients of the product uniformly in (k, p, k_1). We give it no reference because we verified none.

§8 uses the multiplier with no certificate at all. At a numeric point the coefficients of $(1+x)^N F_3$ are explicit integers, so counting their sign changes is a finite check on numerals; pairing the resulting upper bound with a sign chain at explicit rationals pins the number of positive roots exactly. That is what makes exact counts for the degree-16 polynomial affordable where the uniform form of the same argument, over the whole parameter region, is not.

Two kinds of claim here rest on exhaustive computation, and we distinguish them throughout. The theorems are *identities*, each checked by the Lean 4 kernel (§9); the *optimality* statements — the lower ends of the brackets, the asymptotic thresholds — use Sturm sequences and so lie outside the formal development, and are stated as labelled remarks. The exact counts of §8 belong to the first kind and not the second: they are theorems about explicit numerical parameters, proved by Descartes' rule of signs and not by a Sturm sequence.

2. The two polynomials

2.1. $\mathrm{SO}(n)$. Let $n = k_1 + (p-1)k$ and let $-B$ be the negative of the Killing form of $\mathfrak{so}(n)$. With the decomposition of $\mathfrak{so}(n)$ associated with the partition $n = k_1 + k + \cdots + k$, [2] considers the left-invariant metrics

(1)
$$\langle\ ,\ \rangle = x_1(-B)|_{\mathfrak{so}(k_1)} + x_2 \sum_{i=2}^{p} (-B)|_{\mathfrak{so}(k_i)} + x_{12} \sum_{2\le j\le p} (-B)|_{\mathfrak{m}_{1j}} + x_{23} \sum_{2\le i<j\le p} (-B)|_{\mathfrak{m}_{ij}},$$

which are $\mathrm{Ad}(\mathrm{SO}(k_1)\times \mathrm{SO}(k)^{p-1})$-invariant. The Einstein condition is the system $r_1 = r_2$, $r_2 = r_{12}$, $r_2 = r_{23}$ on the Ricci components; normalising $x_{12} = 1$ turns it into three polynomial equations $g_1 = g_2 = g_3 = 0$ in (x_1, x_2, x_{23}). The factor $x_2 - x_{23}$ of g_3 corresponds to the metrics that are naturally reductive, by the criterion of D'Atri and Ziller [13], and is discarded; what is left is

(2)
$$x_2 = \frac{(k-2)x_{23}}{kp - 2 + k_1 x_{23}^2},$$

and substituting (2) leaves two equations $f_1 = f_2 = 0$ in (x_1, x_{23}), with f_1 quadratic and f_2 linear in x_1.

Proposition 2.1. *Let H_1 be the resultant of f_1 and f_2 in x_1, divided by x_{23} and by $kp-2+k_1x_{23}^2$. Then H_1 is a polynomial of degree 8 in x_{23} with 321 monomials in (k, p, k_1, x_{23}), and its leading coefficient, its constant coefficient and its value at 1 agree, as polynomials in (k, p, k_1), with the three closed forms printed in* [2]*:*

$$b_8 = k_1^2(pk-2k+k_1)\big(p^2k^2-3pk^2+2k^2-k_1k+k_1pk+k_1^2-k_1\big),$$

$$H_1(0) = (pk-k+k_1-1)\big(p^2k^2+pk^2-2k^2-6pk+4k+4\big)^2,$$

$$H_1(1) = (k-1)(k-k_1)(kp-k+k_1-1)(kp-k+k_1)^2.$$

We stress that H_1 is *recomputed*, in exact integer arithmetic, from the displayed equations of [2]; it is not transcribed from its printed coefficient list. The three agreements of Proposition 2.1 are therefore a check on the derivation rather than an assumption, and it is a check that caught an erratum.

Proposition 2.2. *The value of H_1 at 0 displayed separately in* [2]*, namely $2\big(k^2(p-1)(p+2)+k(3p-2)+1\big)^2\big(2k(p-1)+2k_1+1\big)$, is not $H_1(0)$: at $(k, p, k_1) = (3, 3, 7)$ it equals 677376, while $H_1(0) = 32448$. The correct constant coefficient is the one printed inside the polynomial and reproduced in Proposition 2.1.*

Both expressions are positive under the paper's hypotheses, and [2] uses only the positivity of $H_1(0)$, so no result of [2] is affected.

We write

$$A = A(k,p) = k^2p^2+k^2p-2k^2-6kp+4k+4, \qquad \gamma = \gamma(k,p,k_1) = \frac{A}{k_1(kp-2)}, \tag{3}$$

for the source's lower separating point. [2] notes $A = (k(p-1)-2)(k(p+2)-4)+4(k-1)$ and gives no other explanation of the choice; §4 supplies one.

2.2. SU(N)**.** For SU(N), $N = k_1 + (p-1)k$, the metrics of [1] carry two extra parameters y_1, y_2 coming from the centre of the isotropy subalgebra, and the Einstein condition is a system of five equations $f_1 = \cdots = f_5 = 0$ in $(x_1, x_2, x_{12}, y_1, y_2)$ after the normalisation $x_{23} = 1$. Two of them are linear in (y_1, y_2) and are solved for them; substituting leaves two equations $F_1 = F_2 = 0$, of which F_2 factors, and the branch $x_2 \neq 1$ gives

$$x_2 = \frac{x_{12}^2\big(x_{12}^2(k^2(p-1)+2)+kk_1\big)}{x_{12}^4(k^2(p-1)p-2)+kk_1(2p-1)x_{12}^2+k_1^2}. \tag{4}$$

The equation $F_1 = 0$ is linear in x_1 and the remaining equation $f_3 = 0$ is quadratic in x_1, so eliminating x_1 is a resultant: writing $f_3 = ax_1^2+bx_1+c$ and $F_1 = \alpha x_1+\beta_0$ one has $\alpha^2 f_3 = F_1\cdot(aF_1 - 2a\beta_0 + b\alpha) + \mathrm{Res}$ with $\mathrm{Res} = a\beta_0^2 - b\alpha\beta_0 + c\alpha^2$.

Proposition 2.3. *Write $\mathcal{A}$ and $\mathcal{B}$ for the numerator and the denominator of* (4)*, and $L = k(p-1)x_{12}^2 + k_1$. Substituting* (4) *into* Res *and multiplying by $\mathcal{B}^5$ gives a polynomial of degree 32 in x_{12}, and the divisions in*

$$\mathcal{B}^5\,\mathrm{Res} = k_1\,\mathcal{A}\,L^2\,x_{12}^4\,F_3(x_{12})$$

are exact. Here F_3 has degree exactly 16 in x_{12} and 1252 monomials in (k,p,k_1,x_{12}). Its coefficients a_0,a_1,a_{15},a_{16} agree, as polynomials in (k,p,k_1), with the four closed forms printed in [1]*; for instance*

$$a_0 = kk_1^7\big(k^2(p-2)+k_1k+1\big)\big(k^2(p-2)(p-1)+kk_1(p-1)+k_1^2+p-2\big),$$

$$a_{16} = k(p-1)\big((p-1)k+k_1\big)\big((p-1)kk_1+1\big)\big(k^4(p-1)^2(p+2)-8k^2+4\big)^2.$$

Proposition 2.4. *The value of F_3 at 1 displayed in* [1] *carries the factor $k\big(k^2(p-1)^2(p+2)-4\big)^3$ where the exponent should be 1; with exponent 1 the display is an identity, and with exponent 3 it already fails at $(k,p,k_1)=(2,3,5)$. With the exponent corrected,*

$$\begin{aligned} F_3(1) = (k-1)(k+1)(k-k_1)\big(k(p-1)+k_1-2\big)^2\big(k(p-1)+k_1+2\big)^2 \\ \times\Big(k_1^3\big(3k^2(p-1)+4\big)+3kk_1^2\big(k^2(p-1)^2+3p-2\big) \\ +k\big(k^2(p-1)^2(p+2)-4\big)+k_1\big(k^4(p-1)^3+6k^2(p-1)p-4\big) \\ +kk_1^4\Big). \end{aligned}$$

Again [1] uses only $F_3(1)<0$, which the corrected display still gives, so no result of [1] is affected. The source's upper separating point is

$$\beta = \frac{\big(k^2(p-1)+2\big)\big(k^2(p-1)p-2\big)}{k(p-1)\big(k^4(p-1)^2(p+2)-8k^2+4\big)}\,k_1. \tag{5}$$

Finally we record the two signs that [1] itself establishes, which we use as controls and as the left half of every chain below.

Lemma 2.5. *For all real $k\ge 2$, $p\ge 3$: $F_3(0)>0$ whenever $k_1>0$, and $F_3(1)<0$ whenever $k_1>k$.*

3. SO(n): the four-root theorem at $k_1 \ge \frac{89}{16}kp$

3.1. **The source's own separating point.** We first ask how far the constant can be pushed while keeping the point 2/3 that [2] uses. The four auxiliary signs certify at $\frac{29}{4}kp$:

Lemma 3.1. *For all real $k\ge 3$, $p\ge 3$, $k_1\ge\frac{29}{4}kp$ one has $H_1(0)>0$, $H_1(1)<0$, $H_1(2)>0$ and $0<\gamma<2/3$.*

Proof sketch. Each is a Handelman certificate in $X=k-3$, $Y=p-3$, $Z=k_1-\frac{29}{4}kp$: of 61, 70 and 91 monomials for the three values of H_1, and of 9 and 13 monomials for $\gamma>0$ and $\gamma<2/3$. In each the coefficients all have the correct sign and the constant term is strictly signed, and the identity between the shifted certificate and 4^d times the quantity in question, for the appropriate d, is a polynomial identity in three variables. These are the sign facts of the proof in [2], there established at $k_1\ge 10kp$. □

The binding sign is $H_1(2/3)>0$, and it is where the constant is decided.

Theorem 3.2. *For all real $k\ge 3$, $p\ge 3$, $k_1\ge\frac{29}{4}kp=7.25\,kp$ one has $H_1(2/3)>0$.*

Proof sketch. A Handelman certificate of 91 monomials in $(k-3,\ p-3,\ k_1-\frac{29}{4}kp)$, all coefficients non-negative and the constant term positive, satisfying $4^5 \cdot 3^8 \cdot H_1(2/3) =$ cert identically. The finite search behind the constant is a scan of the dyadic grid: the certificate has no negative coefficient at $c = 10,\ 8,\ \frac{15}{2},\ \frac{29}{4}$ and acquires one at $c = \frac{57}{8} = 7.125$, so $\frac{29}{4}$ is the smallest constant this certificate reaches. □

Theorem 3.3. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{29}{4}kp$, the polynomial H_1 has two positive roots $\alpha_3 \in (2/3, 1)$ and $\alpha_4 \in (1, 2)$.*

Proof. Two applications of the intermediate value theorem to the continuous function H_1, using $H_1(2/3) > 0$ (Theorem 3.2) and $H_1(1) < 0$, $H_1(2) > 0$ (Lemma 3.1). □

The corresponding statement in [2] — its two-metric theorem, which holds under the much weaker hypothesis $k_1 > k \geq 3$ — localises its two roots only in $(0, 1)$ and $(1, 2)$; the left endpoint $2/3$ here is exactly the binding step of the four-metric theorem, at a constant 27.5% below the published one.

Along this route, though, the constant cannot go much further, and the reason is not the certificate.

Proposition 3.4. *$H_1(2/3) < 0$ at $(k, p, k_1) = (400, 3, 8602)$, where $k_1/(kp) = 7.1683\ldots$; and $H_1(2/3) < 0$ at $(100, 3, 2131)$ while $H_1(2/3) > 0$ at $(100, 3, 2132)$. Moreover $H_1(1) = 0$ identically when $k_1 = k$.*

So no constant below 7.168 can serve in the argument of [2] as it is written, and $\frac{29}{4} = 7.25$ is within 1.2% of the best that argument allows. The last statement shows that some hypothesis separating k_1 from k is needed for $H_1(1) < 0$; $k_1 = k$ is a separate case in [2]. Since exact root counts (Remark 3.10) show that H_1 already has four positive roots at ratios near 5.503, the whole remaining gap is the choice of the point $2/3$.

3.2. **Moving the point.** The window in which a constant upper separating point may lie can be computed: for a given ratio $c = k_1/(kp)$ it is $(\max \alpha_2,\ \min \alpha_3)$ over the parameter domain, and both extremes occur at $p = 3$, $k \to \infty$. A search over $3 \leq k \leq 10^5$, $3 \leq p \leq 10^3$ gives a window of width 0.018 at $c = \frac{89}{16}$, and no window at all at $c = \frac{177}{32} = 5.53125$, where H_1 already has only two positive roots for $k \geq 300$. The dyadic $15/32 = 0.46875$ lies inside the window all the way down to $c = \frac{89}{16}$. With that point the four auxiliary signs still certify.

Lemma 3.5. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{89}{16}kp$ one has $H_1(0) > 0$, $H_1(1) < 0$, $H_1(2) > 0$ and $0 < \gamma < 15/32$.*

Proof sketch. As Lemma 3.1: certificates of 61, 70, 91, 9 and 13 monomials in $X = k-3$, $Y = p-3$, $Z = k_1 - \frac{89}{16}kp$, with denominators cleared by powers of 16. The bound $\gamma < 15/32$ replaces the source's $\gamma < 2/3$ because the upper separating point has moved. □

Theorem 3.6. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$ one has $H_1(15/32) > 0$; consequently H_1 has two positive roots $\alpha_3 \in (15/32, 1)$ and $\alpha_4 \in (1, 2)$.*

Proof sketch. A Handelman certificate of 91 monomials in $(k-3,\, p-3,\, k_1-\frac{89}{16}kp)$ with no negative coefficient, satisfying $16^5 \cdot 32^8 \cdot H_1(15/32) = \mathrm{cert}$ identically; the two roots then follow from Lemma 3.5 by the intermediate value theorem. The finite search behind the constant is again a dyadic scan: the certificate acquires a negative coefficient at the next dyadic below $\frac{89}{16}$ that is still above the true optimum, namely $\frac{2847}{512} = 5.560547$. So $\frac{89}{16}$ is this certificate's own limit for this point. □

The last sign of the chain, $H_1(\gamma) < 0$, is the expensive one: clearing the denominator of γ costs an eighth power, and the certificate has 2976 monomials against the 1028 monomials of $\big(k_1(kp-2)\big)^8 H_1(\gamma)$.

Theorem 3.7. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$ one has $H_1(\gamma) < 0$, and consequently H_1 has four positive roots*

$$0 < \alpha_1 < \gamma < \alpha_2 < \tfrac{15}{32} < \alpha_3 < 1 < \alpha_4 < 2.$$

Proof sketch. Write $\mathrm{Hg} = \big(k_1(kp-2)\big)^8 H_1(\gamma)$, a polynomial in (k, p, k_1) with 1028 monomials, obtained from H_1 by clearing the denominator of γ with the coefficients $b_0, \ldots, b_8$ kept as atoms. The certificate is the identity $16^8 \cdot \mathrm{Hg} = -\mathrm{cert}(k-3,\, p-3,\, k_1-\frac{89}{16}kp)$ with cert a polynomial of 2976 monomials, no coefficient negative and constant term positive; hence $\mathrm{Hg} < 0$, and $H_1(\gamma) < 0$ because $\big(k_1(kp-2)\big)^8 > 0$ under the hypotheses. Four applications of the intermediate value theorem along $0 < \gamma < 15/32 < 1 < 2$, using Lemma 3.5 and Theorem 3.6, then give the four roots. □

This is the source's four-metric theorem with $10kp$ replaced by $\frac{89}{16}kp$, a reduction of the constant by 44%, and it is the first proof of the statement — at any constant — that has been checked by a proof kernel. The geometric conclusion follows from the source's own construction:

Corollary 3.8. *Let $k \geq 3$, $p \geq 3$ and $k_1 \geq \frac{89}{16}kp$ be integers, and $n = k_1 + (p-1)k$. Then, by the argument of* [2] *that turns a positive root x_{23} of H_1 into a metric* (1) *with $x_{12} = 1$, x_2 given by* (2) *and x_1 determined rationally, the group* $\mathrm{SO}(n)$ *admits at least four* $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k)^{p-1})$*-invariant Einstein metrics of the form* (1) *which are not naturally reductive.*

The four roots of Theorem 3.7 lie in disjoint intervals and none equals 1, which is what the non-natural-reductivity argument of [2] requires. We emphasise that Corollary 3.8 lies outside the formal development: only the four-root statement is machine-checked, and the passage from roots to metrics is quoted from [2]. We also read "at least four metrics" exactly as [2] does, namely as four distinct metrics of the form (1), one for each of the four roots. Whether those four are pairwise non-isometric is not a question a root count can decide; we do not verify it and claim nothing about it. What is proved here, and all that is proved here, is the four-root statement of Theorem 3.7.

Proposition 3.9. *One has $H_1(15/32) < 0$ at $(k, p, k_1) = (10000, 3, 166816)$, where the ratio $k_1/(kp)$ is $5.560533\ldots$, and $H_1(15/32) > 0$ at $(10000, 3, 166817)$. Also $H_1(15/32) > 0$ at $(3, 3, 51)$, where $\frac{89}{16} \cdot 9 = 50.0625 \leq 51$. Finally, at $(k, p, k_1) = (3, 3, 1000)$, where $kp/k_1 = 9/1000$ and $\gamma = 52/7000$, one has $H_1(9/1000) > 0$ while $H_1(52/7000) < 0$.*

The first two statements say that the hypothesis $k_1 \geq \frac{89}{16}kp$ is load-bearing and that the sign at 15/32 turns exactly there, so the constant is neither vacuous nor slack — for the point 15/32 it is within 0.03% of the least admissible one, the point's own asymptotic threshold being $5.5611279\ldots$ at $p = 3$, $k \to \infty$. The third says the hypotheses are satisfiable with content. The last pair refutes the natural guess that the lower separating point γ can be replaced by the much simpler kp/k_1: that guess already fails at $(3, 3, 1000)$, while the true $\gamma = A/(k_1(kp-2)) = 52/7000$ still works there. §4 explains why.

Remark 3.10 (outside the formal development). Exact integer Sturm root counts give, at $p = 3$, the least integer k_1 for which H_1 has four positive roots:

k	100	300	1000	3000	10000	30000
k_1	1651	4987	16664	50025	166790	500405
$k_1/(kp)$	5.50333	5.54111	5.55467	5.55833	5.55967	5.56006

The ratio increases in k. At $(k, p, k_1) = (30000, 3, 500404)$, ratio 5.560044, H_1 has only two positive roots, so no constant ≤ 5.56004 can appear in any four-root theorem of this shape. Taking the top k-degree part of H_1 at $p = 3$, $k_1 = 3ck$ and bisecting on the four-root property gives the exact supremum $c^* = 5.5602472\ldots$, at which the two middle roots merge at $0.4632899\ldots$; $p = 3$ is the worst case, $p = 4$ and $p = 5$ giving thresholds below 5. Combined with Theorem 3.7, the optimal constant lies in [5.56004, 5.5625], a bracket of 0.044%. These are computations in exact integer arithmetic; they are not part of the formal development, because *Mathlib* has no Sturm sequences. The individual sign evaluations that witness the lower end are machine-checked (Proposition 3.9).

4. Why γ is forced

The lower separating point of the chain cannot be a constant — α_1 and α_2 both shrink like $1/k_1$ — so it must be of the form $N(k, p)/k_1$. The source takes $N = A/(kp - 2)$ and says nothing about the choice. The following identity says that no other choice is possible.

Theorem 4.1. *There is a polynomial $G(k, p, q, u)$, with 321 monomials and of degree 9 in q, such that*

$$G(k, p, q, qx) = q^8 H_1(k, p, q, x) \qquad \text{for all real } k, p, q, x,$$

and

$$G(k, p, q, u) = q^9\big((kp - 2)u - A(k, p)\big)^2 + R(k, p, q, u),$$

where R has degree at most 8 in q.

In other words, the top q-degree part of $q^8 H_1(u/q)$ is a quadratic in u whose discriminant vanishes identically: its leading, middle and constant coefficients are $(kp - 2)^2$, $-2(kp - 2)A$ and A^2. Its vertex is $A/(kp - 2)$, which is exactly $k_1\gamma$. Consequently the two small roots of H_1 merge at γ as $k_1 \to \infty$, the interval on which $H_1 < 0$ between them closes, and every point $N(k, p)/k_1$ with $N \neq A/(kp-2)$ eventually leaves it. The obvious simplification $N = kp$, which would shrink the 2976-monomial certificate of Theorem 3.7 to a few hundred monomials, already fails at $(k, p, k_1) = (3, 3, 1000)$, as Proposition 3.9 records. This is what makes the large certificate unavoidable.

Remark 4.2 (outside the formal development). The same phenomenon holds on the $\mathrm{SU}(N)$ side, and there it explains the source's β. Writing

$$F_3(tk_1) = \sum_{\ell,j} a_{\ell,j}(k,p)\, t^{\ell} k_1^{\ell+j},$$

the top total k_1-degree is 18, and the degree-18 part is $a_{16}t^{14}(t-\beta/k_1)^2$, a perfect square whose vertex is β/k_1; so β is forced there too. Raising the denominator of β to the sixteenth power is then unavoidable, and that alone is why the general $\mathrm{SU}(N)$ certificate for the sign of $F_3(\beta)$ runs to 97,178 monomials. The identity is verified in exact integer arithmetic and is not part of the formal development.

5. At most four positive roots, and an exact count

Everything so far produces roots. This section bounds them from above.

5.1. **Descartes' rule gives exactly eight, and that is the point.** On our region the coefficients of H_1 alternate in sign. Neither source says so — the word "sign" occurs in neither — but the same authors observe exactly this alternation, in the same words, in several of their other papers [4, 5, 6], and there it serves one purpose only: to conclude that a real root has to be positive. Read as a statement about counting, the alternation is bad news. Descartes' rule of signs bounds the number of positive roots, with multiplicity, by the number of sign changes in the coefficient sequence, and a strictly alternating sequence has as many sign changes as the polynomial has degree. On our region that bound is exactly eight, and we prove it, because it is what gives the theorem below its content.

Proposition 5.1. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq 11kp$ one has $b_e > 0$ for e even and $b_e < 0$ for e odd, $0 \leq e \leq 8$. The sequence $b_8, b_7, \ldots, b_0$ therefore has exactly eight sign changes, and Descartes' rule of signs bounds the positive roots of H_1 by eight and by nothing smaller.*

Proof sketch. Nine Handelman certificates, one per coefficient, in $X = k-3$, $Y = p-3$, $Z = k_1 - 11kp$: 763 monomials in total, each with no negative coefficient and a positive constant term, each an identity between $(-1)^e b_e$ and the listed polynomial. The count of sign changes is then a finite check on nine signs. □

5.2. **A multiplier.** For an integer $N \geq 0$ write

$$(1+x)^N H_1(x) = \sum_{j=0}^{8+N} g_j\, x^j, \qquad g_j = \sum_{e} \binom{N}{j-e} b_e,$$

so that each g_j is again an integer polynomial in (k, p, k_1), a non-negative integer combination of $b_0, \ldots, b_8$. Multiplication by $(1+x)^N$ moves no positive root, so if the g_j can be signed on the region and their signs alternate fewer times than the b_e do, then Descartes' rule applied to the product is a better bound for H_1 than the rule applied to H_1. For H_1 the exponent $N = 9$ is the cheapest that works, and it takes eight alternations down to four.

Theorem 5.2. *Let $k \geq 3$, $p \geq 3$ and $k_1 \geq 11kp$ be real, and let $g_0, \ldots, g_{17}$ be the coefficients of $(1+x)^9 H_1$. Then*

$$g_0 > 0, \quad g_1 < 0, \quad g_3, \ldots, g_9 > 0, \quad g_{10}, \ldots, g_{13} < 0, \quad g_{14}, \ldots, g_{17} > 0,$$

and, whatever the sign of the remaining coefficient g_2, the sequence $g_{17}, \dots, g_0$ has at most four sign changes.

Proof sketch. Seventeen Handelman certificates in $X = k - 3$, $Y = p - 3$, $Z = k_1 - 11kp$, of 61 to 91 monomials each and 1496 monomials in total, give the seventeen determined signs. The bound on the sign changes is then a trichotomy on g_2: the pattern

$$+ \ - \ ? \ + + + + + + + + \ - - - - \ + + + +$$

(listed from g_0 to g_{17}) has four sign changes when ? is $+$, four when it is $-$, and four when $g_2 = 0$ and the entry is dropped, so every completion gives four. □

The finite searches behind the exponent and the constant are these. For $N = 8$, 9, 10 and 11 the smallest constant c at which *every* coefficient of $(1+x)^N H_1$ certifies on $\{k \geq 3,\ p \geq 3,\ k_1 \geq c\,kp\}$ is 13, $\frac{23}{2} = 11.5$, 14 and 16 respectively, so $N = 9$ is the exponent that reaches lowest. Below $\frac{23}{2}$ the constant moves only by carrying g_2 as a free sign, and then a dyadic bisection shows that the argument still closes at $c = \frac{699}{64} = 10.9219$ and fails at $c = \frac{697}{64} = 10.8906$, where g_3 loses its certificate as well and the completion $(g_2, g_3) = (+, -)$ has six sign changes. The stated constant 11 is 0.7% above that limit, and was taken because it is readable.

Remark 5.3. Below $\frac{699}{64}$ the route cannot be rescued by a bigger certificate, because what fails is a sign and not a certificate: escalating the certificate for g_2 by each of the multipliers k, p, kp, $k+p$, k_1, k^2, p^2, $(k+p)^2$, k_1^2, kpk_1 and $kp(k+p)$ still fails at every $c \leq 11$. What a smaller constant needs is not the sign of a coefficient but an implication between two of them, $g_2 > 0 \Rightarrow g_3 \geq 0$; that is a Positivstellensatz certificate $g_3\sigma_0 = \sigma_1 + g_2\sigma_2$ rather than a Handelman one, and we did not attempt it.

Theorem 5.4. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq 11kp$ the polynomial H_1 has at most four positive roots, counted with multiplicity. In particular no five distinct positive reals are roots of H_1.*

Proof. The polynomial $(1+x)^9$ has -1 as its only root, so $(1+x)^9 H_1$ and H_1 have the same positive roots with the same multiplicities. By Theorem 5.2 the product has at most four sign changes, and Descartes' rule of signs, in the form that counts positive roots with multiplicity, bounds them by that number. □

Theorem 5.5. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq 11kp$ the polynomial H_1 has exactly four positive roots, counted with multiplicity. In particular the four roots $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ of Theorem 3.7 are simple, and H_1 has no positive root besides them.*

Proof. Since $\frac{89}{16} \leq 11$, the hypotheses imply those of Theorem 3.7, which produces four distinct positive roots and hence at least four counted with multiplicity; Theorem 5.4 gives at most four; the two bounds meet. □

This is the first exact count in this construction. The four-metric theorem of [2] and Remark 1 of [1] are existence statements, and so is every other statement we could find about the number of roots of either polynomial. The exact count is paid for in the constant: it holds on $k_1 \geq 11kp$, where the source's own theorem needs $k_1 \geq 10kp$ and Theorem 3.7 needs only $k_1 \geq \frac{89}{16}kp$. On the window $\frac{89}{16}kp \leq k_1 <$

$11kp$ only the lower bound is proved — although the census of Remark 5.8 finds no cell anywhere, at any ratio, with more than four positive roots.

Proposition 5.6. *At* $(k,p,k_1)=(3,3,104)$ *the values of* H_1 *at* 0, $\frac{1}{12}$, $\frac{1}{2}$, 1 *and* 2 *have signs* $+,-,+,-,+$, *and* H_1 *has exactly four positive roots there:*

$$0<\alpha_1<\tfrac{1}{12}<\alpha_2<\tfrac{1}{2}<\alpha_3<1<\alpha_4<2.$$

Proof sketch. $11\cdot3\cdot3=99\le104$, so the point lies in the region of Theorem 5.4. The five signs are evaluations of a degree-8 polynomial with explicit integer coefficients at explicit rationals. Four applications of the intermediate value theorem then give four roots in disjoint intervals, hence at least four, and Theorem 5.4 gives at most four. □

Proposition 5.6 is a negative control on Theorem 5.4 in three ways: it refutes any strengthening to "at most three"; it shows that the hypothesis region is non-empty and the conclusion non-vacuous; and it reaches "exactly four" at one point without Theorem 3.7 and so without that theorem's 2976-monomial certificate.

5.3. The $\mathrm{SU}(N)$ analogue.

Remark 5.7 (outside the formal development). The same construction applies to F_3, at the exponent $N=29$ and the constant 50. Writing $(1+x)^{29}F_3(x)=\sum_{j=0}^{45}\tilde g_j\,x^j$, every one of the forty-six coefficients has a Handelman certificate on $\{k\ge2,\ p\ge3,\ k_1\ge50kp\}$ — 28,260 certificate monomials in all, computed in exact integer arithmetic — with sign pattern $+^{13}\,-^{6}\,+^{25}\,-\,+$, that is, exactly four sign changes and no free sign, so that no case split is needed. With Descartes' rule this gives at most four positive roots of F_3 for all real $k\ge2$, $p\ge3$, $k_1\ge50kp$, exactly as Theorem 5.4 does for H_1. The search over (N,c) returns $N=26\mapsto c=57$, $27\mapsto54$, $28\mapsto52$ (one free sign), $29\mapsto50$ (none free), $30\mapsto49$ and $32\mapsto48$. The signs were computed twice by independent routes: once through the shifted certificates, and once by evaluating $\tilde g_j=\sum_e\binom{29}{j-e}a_e$ directly in exact rational arithmetic at 40 random points of the region with non-integer k and p and k_1 up to $10^5\cdot50kp$ — no mismatch, and the same single sign pattern at every point. The statement is nevertheless *not* part of the formal development. The corresponding Lean file is 24,065 lines and its kernel check was stopped after 65.2 CPU-minutes, on a machine carrying a heavy unrelated load, having bought 0.6 CPU-minutes in the last twelve wall-clock minutes; the file was discarded rather than kept unchecked. The $\mathrm{SO}(n)$ file, with about one twelfth of the certificate mass, checks in under five minutes, so what is missing is machine time and not method.

5.4. Where the deficiency comes from.

Remark 5.8 (outside the formal development). Exact integer Sturm sequences give the following census of positive roots. For H_1, 9300 cells ($k,p\in\{3,\dots,12\}$, together with $k_1=1,\dots,79$ and $k_1=r\,kp$ for integers r up to 1000): the count is 0 in 631 cells, 2 in 7334 and 4 in 1335, never more than 4, and the coefficient signs alternate in every cell. For F_3, 2274 cells ($k=2,\dots,8$, $p=3,\dots,8$, together with 300 random cells with $k,p\le30$ and $k_1\le40kp$): the count is 0 in 157 cells, 2 in 1589 and 4 in 528, never more than 4, and again the signs alternate in every cell. So the Descartes bounds are 8 and 16 and the deficiencies to be explained are 4 and 12. Two things follow that the theorems above do not say. First, the constants 11 and 50 are not the truth about the count: it is at most four everywhere tested, down to

$k_1 = 1$. They are where this certificate route closes. Second, the derivative route — the one these authors use for their exact counts elsewhere, see §5.5 — cannot give four here: over 2520 cells the number of positive roots of H_1'' lies in $\{0, 2\}$, which would suffice, but the number for H_1''' lies in $\{1, 3\}$, so the chain of Rolle inequalities stalls one step short. These computations are exact but are not part of the formal development.

Remark 5.9 (outside the formal development). The structural reason for the deficiency is visible in the Newton polygon of H_1 in the plane of exponents $(\deg_x, \deg_{k_1})$. The k_1-degrees of $b_0, \dots, b_8$ are $1, 2, 3, 3, 4, 4, 5, 5, 5$, so the upper hull has three edges, of slopes 1 on the indices 0–2, $\frac{1}{2}$ on 2–6 and 0 on 6–8. As $k_1 \to \infty$ the eight roots therefore split by size into two of order k_1^{-1}, four of order $k_1^{-1/2}$ and two of order 1. The first group is governed by the slope-1 edge, whose edge polynomial in $u = k_1 x$ is exactly the perfect square $\big((kp-2)u - A\big)^2$ of Theorem 4.1. The middle group is the whole of the deficiency: the slope-$\frac{1}{2}$ edge carries only the *even* indices 2, 4, 6, so its edge polynomial is $m_6 w^2 + m_4 w + m_2$ in $w = k_1 x^2$, where m_2, m_4, m_6 are the leading k_1-coefficients of b_2, b_4, b_6 and are positive because b_e has the sign of $(-1)^e$; a quadratic with three positive coefficients has no positive root, so those four roots are never positive. Root positions bear this out: at $(k, p, k_1) = (3, 3, 1000)$ the four positive roots are 0.006924, 0.007979, 0.954056 and 1.043885 — two at scale $1/k_1$, two at scale 1, and nothing at scale $k_1^{-1/2} \approx 0.032$. This is an asymptotic statement, it is not formalised, and Theorem 5.4, which is uniform in the parameters, does not use it.

5.5. **Upper bounds in this literature.** Producing roots is standard in this circle of papers; bounding their number is not. Besides the two sources we read twelve papers of Arvanitoyeorgos and his coauthors that reduce an invariant Einstein condition by the same kind of elimination, all from their arXiv sources: [3, 4, 5, 6], together with arXiv:0902.1826, 0904.0104, 0904.1690, 1010.3661, 1010.4472, 1207.2897, 1810.01292 and 2002.10359. The words "Descartes", "sign change" and "Sturm" occur in none of the fourteen. Only the four that are cited were read closely enough to quote. Three things in that reading bear on the theorems above.

First, exact counts do occur in this literature, by a different method. [3] proves that its elimination polynomial $H_{n,p}$ "has exactly two positive solutions" for $n \geq 2p + 5$, $p \geq 4$, and gets exactly four invariant Einstein metrics for eight small pairs (n, p) in each of the two cases it distinguishes, $n \geq 2p$ and $n \leq 2p$; the argument is convexity, namely that $d^2 H_{n,p}/dx_2^2 > 0$ on the range, so that $dH_{n,p}/dx_2$ is monotone increasing, so that $H_{n,p}$ has a single local minimum and therefore exactly two positive roots. That is a Rolle argument, and Remark 5.8 records the measurement that the same argument does not reach four for H_1.

Second, an "at most four" theorem does exist next door. Lomshakov, Nikonorov and Firsov [8] prove, in the form quoted by [7], that a generalized Wallach space with pairwise inequivalent submodules "admits from one to four invariant Einstein metrics (up to scale and homothety)", and [7] uses it exactly as the theorems above would be used: "So far we have proved existence of at least four solutions. However, by Theorem [*the one just quoted*] the number of invariant Einstein metrics is at most four, hence the result follows." We did not read [8]; its bibliographic data and the content of its theorem were checked against the *zbMATH* review of the original (Zbl 1054.53067), which records it as: if the three modules are pairwise

non-isomorphic, the space "admits at least one and at most four nonproportional homogeneous Einstein metrics". A generalized Wallach space has three isotropy summands, and the Einstein condition there is a pair of quadratic equations. The objects here are invariant metrics on the compact group $\mathrm{SO}(n)$ or $\mathrm{SU}(N)$ under a block ansatz with more summands and an equation of degree 8 or 16, so that theorem does not apply here. It is, however, the closest prior art we found, and it is the model for what an upper bound of this kind is worth: it is exactly what turns an existence proof into a count.

Third, no general finiteness theorem covers this case. The statement that a compact homogeneous space whose isotropy representation decomposes into pairwise inequivalent irreducible summands carries only finitely many invariant Einstein metrics is a *conjecture* of Böhm, Wang and Ziller [9]. We did not read [9] and did not locate the conjecture inside it: the attribution follows the wording of arXiv:1907.02626v2, the arXiv version of [10], which states it in those words. Graev's Newton-polytope machinery [11, 12] does bound a number of invariant Einstein metrics by a polytope volume, but the number it bounds is a *complex* one: in the author's own summary of [12], for G compact semisimple the integer $\nu(M)$ — the ratio of the volume of the Newton polytope of the scalar curvature, as a rational function of the invariant metric, to the volume of the standard simplex — satisfies $\nu(M) \geq \mathcal{E}(M)$, where $\mathcal{E}(M)$ counts the isolated invariant *holomorphic* Einstein metrics up to homothety on the complexification $M^{\mathbb{C}} = G^{\mathbb{C}}/H^{\mathbb{C}}$. A bound on that count is not a bound on the number of *positive real* roots of the elimination, and we do not see how to extract four from it. We read neither of [11, 12] (neither is on arXiv); their bibliographic data and the characterisation just given are from *zbMATH*, the latter from the author's own summary reproduced there, and we did not compute $\nu(M)$ for the spaces of this paper.

5.6. **What the count does and does not say about metrics.**

Remark 5.10. Theorem 5.5 is a statement about a polynomial. Reading it as "$\mathrm{SO}(n)$ carries exactly four invariant Einstein metrics of the form (1)" needs two implications that we do not prove and that are not in the formal development: the elimination bridge in the direction used by Corollary 3.8, and its converse, that distinct such metrics give distinct positive roots of H_1. The first is the sources' own reduction, written out in §2; the second is not stated anywhere we looked. Until both are available, the upper half of the count is a statement about H_1 and nothing more. We say this explicitly because the lower half was already stated with the same reservation in Corollary 3.8, and an exact count invites the stronger reading.

6. $\mathrm{SU}(N)$: Remark 1

6.1. **Step (1), for all parameters.**

Theorem 6.1. *For all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$ one has $F_3(2) > 0$.*

Proof sketch. A Handelman certificate: substituting $k = 2 + X$, $p = 3 + Y$, $k_1 = 6kp + Z$ writes $F_3(2)$ as a polynomial in (X, Y, Z) with 638 monomials, no negative coefficient and positive constant term. The finite search behind the constant is a dyadic scan: the certificate has no negative coefficient at $c = 8$, $\frac{23}{4}$, $\frac{91}{16}$, $\frac{181}{32}$, $\frac{45}{8}$ and acquires two at $c = \frac{179}{32} = 5.594$, so the sharp dyadic constant for this certificate is $\frac{45}{8} = 5.625$; the theorem is stated at the rounder 6. □

This is step (1) of Remark 1 of [1], proved, with the constant sharpened from the asserted $8kp$.

Theorem 6.2. *For all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, the polynomial F_3 has two positive roots $\alpha_1 \in (0,1)$ and $\alpha_2 \in (1,2)$.*

Proof. Two applications of the intermediate value theorem, using Lemma 2.5 and Theorem 6.1; note $k_1 \geq 6kp \geq 18k > k$. □

The two-metric theorem of [1] localises its second root only in $(1, k_1kp)$; the interval $(1,2)$ is the first half of the chain of Remark 1.

Proposition 6.3. *One has $F_3(2) < 0$ at $(k,p,k_1) = (120,3,2001)$, where the ratio $k_1/(kp)$ is $5.5583\ldots$; also $F_3(2) < 0$ at $(2,3,25)$ while $F_3(2) > 0$ at $(2,3,26)$; and $F_3(1) = 0$ identically when $k_1 = k$.*

So step (1) of Remark 1 fails at ratio 5.5583, no constant at or below that value can serve in it, and the threshold used in Theorem 6.4 below at $(k,p) = (2,3)$ is exactly where the sign turns. The last statement is the source's own separate case $k_1 = k$ and shows the hypothesis $k < k_1$ of Lemma 2.5 is not removable.

6.2. **The whole of Remark 1, at six pairs.** For fixed (k,p) the denominator of β in (5) is a numeral, the sixteenth power collapses, and each of the three remaining sign conditions of Remark 1 becomes a univariate certificate.

Theorem 6.4. *Let (k,p) and m be one of*

(k,p)	$(2,3)$	$(3,3)$	$(4,3)$	$(2,4)$	$(3,4)$	$(2,5)$
m	26	45	64	33	55	39
$m/(kp)$	4.33	5.00	5.33	4.13	4.58	3.90
$8kp$	48	72	96	64	96	80

Then for every real $k_1 \geq m$ the polynomial F_3 has four positive roots

$$0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < \beta < \alpha_4 < k_1kp.$$

Proof sketch. For each pair, three univariate Handelman certificates in $T = k_1 - m \geq 0$, of degrees 10, 17 and 18 and with integer coefficients of 40 to 100 digits, give $F_3(2) > 0$, $F_3(\beta) < 0$ and $F_3(k_1kp) > 0$; together with Lemma 2.5 and four applications of the intermediate value theorem this is the chain. For $(k,p) = (2,3)$, for instance, $\beta = \frac{55}{292}k_1$ and $k_1kp = 6k_1$, and $\beta > 2$ then follows from $k_1 \geq 26$ by arithmetic. The thresholds m are not arbitrary: each is the exact integer at which the sign of $F_3(2)$ turns, with $(4,3)$ the one exception where it is one above. (That last statement rests on exact root counting outside the formal development, except at $(k,p) = (2,3)$, where Proposition 6.3 records both signs.) □

Each of the six statements is an infinite family — it holds for every real $k_1 \geq m$ — and every threshold is below $5.5\,kp$, against the $8kp$ asserted in [1].

Remark 6.5 (outside the formal development). The general statement of Remark 1 also certifies. Handelman certificates in $(k-2,\ p-3,\ k_1 - c\,kp)$, all verified in exact integer arithmetic, give the four remaining sign conditions: $F_3(2) > 0$ at $c = \frac{45}{8}$ (638 monomials), $\beta > 2$ at $c = 6$, $F_3(\beta) < 0$ at $c = 1$ (97,178 monomials), and $F_3(k_1kp) > 0$ at $c = 1$ (23,826 monomials). Together with Lemma 2.5 they give the whole chain of Remark 1 for all real $k \geq 2$, $p \geq 3$ and $k_1 \geq 6kp$. The constant is 6 and not $\frac{45}{8}$ for a bookkeeping reason and not a mathematical one: 6 is the largest of

the four, and it comes from the certificate for $\beta > 2$, which was run only at $c = 6$. Numerically the other thresholds are far smaller — $\beta > 2$ holds from about $2.2\,kp$, and the last two conditions from below kp — so the binding condition is $F_3(2) > 0$, whose certificate reaches $\frac{45}{8}$; but $6kp$ is what the four certificates, as run, establish, and that is the constant we state. We do not include this among the theorems, because the certificate for $F_3(\beta) < 0$ has not been put through a proof kernel: it is a single polynomial identity of 97,178 monomials, and the decomposition that made the 2976-monomial identity of Theorem 3.7 affordable has not yet been carried out at that scale.

Remark 6.6 (outside the formal development). Exact root counts bracket the $\mathrm{SU}(N)$ constant too. The certificate for $F_3(2) > 0$ stops at $\frac{45}{8} = 5.625$; moving the upper separating point from 2 to $\frac{17}{8} = 2.125$ — which lies inside the window between α_2 and α_3, that window closing at $2.1584755\ldots$ at $p = 3$, $k \to \infty$ — the same 638-monomial certificate closes at $\frac{89}{16} = 5.5625$ and fails at $\frac{2847}{512}$ with two negative coefficients. The point $\frac{17}{8}$ has its own least ratio $5.5618\ldots$, and exact root counting gives only two positive roots at $(k, p, k_1) = (1000, 3, 16680)$, ratio 5.560000. So the $\mathrm{SU}(N)$ optimal constant lies in [5.5600, 5.5625]. Section 7 explains why the same constant $\frac{89}{16}$ appears on both sides. These computations are exact but are not part of the formal development.

6.3. **Moving the upper separating point.** On the $\mathrm{SO}(n)$ side the whole of the remaining slack was in the separating point, and moving it from 2/3 to 15/32 took the constant from $\frac{29}{4}$ to $\frac{89}{16}$ (Theorems 3.2 and 3.6). The same is true here. The point 2 of Remark 1 admits no constant below 5.5583, by Proposition 6.3, so the certificate of Theorem 6.1 is nearly exhausted; but the window in which a constant upper separating point may lie — the window between α_2 and α_3, which closes at $2.1584755\ldots$ at $p = 3$, $k \to \infty$ (Remark 6.6) — has room above 2. The dyadic $17/8 = 2.125$ lies inside it.

Theorem 6.7. *For all real $k \geq 2$, $p \geq 3$, $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$ one has $F_3(17/8) > 0$; consequently F_3 has two positive roots $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (1, 17/8)$.*

Proof sketch. A Handelman certificate of 638 monomials in $X = k - 2$, $Y = p - 3$, $Z = k_1 - \frac{89}{16}kp$, no coefficient negative and the constant term positive, satisfying $16^{10} \cdot 8^{16} \cdot F_3(17/8) =$ cert identically; the two roots then follow from Lemma 2.5 by the intermediate value theorem, using $k_1 \geq \frac{89}{16}kp > k$. It is the certificate of Theorem 6.1 at a different point, and the finite search behind the constant is the same dyadic scan: it has no negative coefficient at $\frac{89}{16}$ and acquires two at the next dyadic down, $\frac{2847}{512} = 5.560547$, so $\frac{89}{16}$ is this certificate's own limit for this point. □

This is step (1) of Remark 1 again, at $5.5625\,kp$ against the asserted $8kp$ and against the $6kp$ of Theorem 6.1, and it is the certificate half of Remark 6.6, which version 2 of this paper recorded as lying outside the formal development. The constant is exactly the one Theorem 3.6 reaches for H_1, and Theorem 7.1 is the reason: the two limit polynomials differ by $x \mapsto 1/x$ and a positive factor, so the point at which the two middle roots of H_1 merge and the point at which the window used here closes are reciprocal, as Remarks 3.10 and 6.6 record them: $0.4632899\ldots$ and $2.1584755\ldots$, whose product is 1 to the digits shown.

Proposition 6.8. *One has $F_3(17/8) < 0$ at $(k, p, k_1) = (10000, 3, 166854)$, where $k_1/(kp) = 5.5618$, and $F_3(17/8) > 0$ at $(10000, 3, 166855)$. Also $F_3(17/8) < 0$ at $(120, 3, 2001)$, where $k_1/(kp) = 5.5583\ldots$; and $F_3(17/8) > 0$ at $(2, 3, 34)$.*

The first pair says that the hypothesis $k_1 \geq \frac{89}{16}kp$ is load-bearing and that the sign at $17/8$ turns exactly there, so the constant is within 0.012% of the least one this point admits. The second point is one at which $F_3(2) < 0$ as well (Proposition 6.3): the gain over the point 2 is a gain from moving the point, not from asking for less. The last says the hypotheses are satisfiable with content, $\frac{89}{16} \cdot 6 = 33.375 \leq 34$.

Remark 6.9 (outside the formal development). With $17/8$ in place of 2, all seven ingredients of Remark 1's chain certify at $k_1 \geq \frac{89}{16}kp$, five of them inside the formal development: $F_3(0) > 0$ and $F_3(1) < 0$ (Lemma 2.5), $F_3(17/8) > 0$ (Theorem 6.7), and $17/8 < \beta$, $\beta < k_1 kp$, together with the identity

$$D^{16} F_3(\beta) = \sum_{\ell=0}^{16} a_\ell \, (M k_1)^\ell \, D^{\,16-\ell},$$

which clears the denominator $D = k(p-1)\big(k^4(p-1)^2(p+2) - 8k^2 + 4\big)$ of (5) with its numerator $M = \big(k^2(p-1) + 2\big)\big(k^2(p-1)p - 2\big)$ and the coefficients $a_0, \ldots, a_{16}$ left as atoms — the step that costs nothing until the atoms are unfolded. The two that are not are the expensive ones: $F_3(k_1 kp) > 0$, whose certificate has 23,826 monomials, and $F_3(\beta) < 0$, whose certificate at this constant has 101,631 — the sixteenth power of D is what Remark 4.2 says cannot be avoided. Both are verified in exact integer arithmetic; the second is estimated at some 37 CPU-hours in the proof kernel, after the decomposition that made Theorem 3.7 affordable, and was not attempted. Fixing one of the two parameters collapses it: the certificate has 1,521 monomials at $p = 3$ and 1,197 at $k = 2$, and the expansion needed to verify it shrinks by about a factor of 120. The whole chain then certifies on the slice $p = 3$ for every real $k \geq 2$ and $k_1 \geq 16.6875\, k$, and on the slice $k = 2$ for every real $p \geq 3$ and $k_1 \geq 8.5625\, p$, against the $24k$ and $16p$ of Remark 1. Those two slice statements are exact integer computations and are not part of the formal development: a first kernel attempt at the slice $p = 3$ was stopped at 32.65 gigabytes of resident memory without finishing, and the obstruction there is memory and not time.

7. A bridge between the two constructions

Substituting $k_1 = wkp$ — the scaling both sources use — and letting $k \to \infty$ leaves a limit polynomial on each side.

Theorem 7.1. *There are polynomials F_{top} and H_{top} in (p, w, x), with 394 and 121 monomials, such that, identically in (k, p, w, x),*

$$F_3(k, p, wkp, x) = k^{12} F_{\mathrm{top}}(p, w, x) + F_r,$$
$$H_1(k, p, wkp, x) = k^5 H_{\mathrm{top}}(p, w, x) + H_r,$$

where F_r and H_r are polynomials in (k, p, w, x) of degree at most 10, resp. 4, in k; and, writing $H_{\mathrm{rev}}(p, w, x) = x^8 H_{\mathrm{top}}(p, w, 1/x)$,

$$F_{\mathrm{top}}(p, w, x) = H_{\mathrm{rev}}(p, w, x) \cdot p\, w\big((p-1)x^2 + pw\big)^4.$$

The factor is strictly positive for $p > 1$, $w > 0$ and every real x; consequently, for $p > 1$, $w > 0$, $x > 0$ and $xy = 1$,

$$F_{\mathrm{top}}(p, w, x) = 0 \iff H_{\mathrm{top}}(p, w, y) = 0.$$

Proof sketch. The two splittings are polynomial identities, obtained by writing the remainders in Horner form in k so that their degrees are visible from the expressions themselves. The factorisation was found by exact multivariate division of F_{top} by H_{rev}: the remainder is zero and the quotient has 15 monomials, which is the displayed product on the nose. Positivity of the factor and the equivalence are then elementary. □

So in the limit the positive roots of the degree-16 SU(N) polynomial are exactly the reciprocals of the positive roots of the degree-8 SO(n) polynomial, with multiplicity, and *the two constructions have the same asymptotic four-root threshold in* $w = k_1/(kp)$. That is why two independent bisections both return $5.5602472\ldots$ at $p = 3$ and $4.98878\ldots$ at $p = 4$, and why the merge points are reciprocal: $2.1584755\ldots$ for F_3 against $0.4632899\ldots$ for H_1 (Remarks 3.10 and 6.6). The positive factor is not arbitrary either: $(p-1)x^2 + pw = L/k$ for the factor $L = k(p-1)x^2 + k_1$ that the SU(N) elimination divides out twice (Proposition 2.3).

Question 7.2. Is there a geometric reason? The identity of Theorem 7.1 is at present an algebraic coincidence with a suggestive shape.

8. Exact root counts for F_3

§5 counts the positive roots of H_1 uniformly in (k, p, k_1) and pays for it with a certificate for every coefficient of $(1+x)^9 H_1$ on the whole region. The same argument applies to F_3 at the exponent 29 and the constant 50, and Remark 5.7 records the 28,260 certificate monomials it needs; but that computation is not in the formal development, and Remark 8.5 below records what it would cost to put it there. This section takes the other road. It fixes the parameters and counts exactly.

The point is that at a numeric point the multiplier needs no certificates at all. For fixed (k, p, k_1) the seventeen coefficients $a_0, \ldots, a_{16}$ are integers, so the coefficients of $(1+x)^N F_3$ are integers too, and the number of sign changes in them is a finite computation on numerals rather than a statement about a region. Descartes' rule of signs bounds the positive roots of the product, with multiplicity, by that number; multiplication by $(1+x)^N$ moves no positive root and changes no multiplicity, so the same bound holds for F_3; and a sign chain at explicit rationals produces that many distinct positive roots from below. When the two numbers agree the count is pinned. The exponent is not fixed in advance: it is taken minimal for the point at hand, and the minimum is found by exact computation. This is the device of §5 specialised, and the disclaimer there applies here too — sharpening the rule of signs by a positive multiplier is classical and we claim no novelty for it. What is new is that these counts exist at all: both sources stop at "at least four", and every root count this paper has stated so far for F_3 — the censuses and brackets of Remarks 6.6 and 5.8 — has been a Sturm computation outside the formal development.

Theorem 8.1. *At each of the four parameter points*

$$(k, p, k_1) = (2, 3, 26), \quad (2, 4, 33), \quad (3, 3, 45), \quad (5, 5, 300),$$

the polynomial F_3 has exactly four positive roots, counted with multiplicity; in particular the four are simple and F_3 has no other positive root. The ratios $k_1/(kp)$ are $\frac{13}{3} = 4.33\ldots$, 4.125, 5 and 12.

Proof sketch. At each point the seventeen coefficients of F_3 are evaluated to integers, and $(1+x)^N F_3$ is expanded, with $N = 12$, 11, 13, 14 respectively; the

coefficient lists have 29, 28, 30 and 31 integer entries and four sign changes each, which is a finite check. Descartes' rule of signs, in the form that counts positive roots with multiplicity, bounds those of $(1+x)^N F_3$ by four, and since $(1+x)^N$ has -1 as its only root the same bound holds for F_3. In the other direction the values of F_3 at

$$0, 1, 2, 3, 8; \qquad 0, 1, 2, 4, 8; \qquad 0, 1, 2, 4, 8; \qquad 0, 1, 2, 12, 16$$

have signs $+, -, +, -, +$, and four applications of the intermediate value theorem give four roots in disjoint intervals. The two bounds meet. The finite search behind each point is a search for the exponent: N is the least one for which the sign changes come down to four, computed exactly, and it varies with the point. The rationals of the sign chains were located by exact Sturm sequences first, and are then re-proved by direct evaluation, which is what enters the formal development. □

Theorem 8.2. *At $(k, p, k_1) = (1000, 3, 16681)$, where $k_1/(kp) = 5.560333\ldots$, the polynomial F_3 has exactly four positive roots, counted with multiplicity.*

Proof sketch. As Theorem 8.1, with $N = 13$, a coefficient list of 30 integers of up to fifty-one digits with four sign changes, and the sign chain $+, -, +, -, +$ at 0, 2, $\frac{69}{32}$, 3, 8. The middle point $\frac{69}{32} = 2.15625$ is inside the window of Remark 6.6, which is where a sign chain has to look at this ratio: the point 2 is already useless here, $F_3(2)$ being negative. □

The ratio $5.560333\ldots$ is below the constant $\frac{89}{16} = 5.5625$ of Theorem 6.7. So the four-root property already holds at parameters that the certificate does not reach, and $\frac{89}{16}$ is where that certificate closes rather than where the property begins. How far below is now bracketed on both sides at $k = 1000$, $p = 3$: Remark 6.6 records, outside the formal development, that F_3 has only two positive roots at $(1000, 3, 16680)$, ratio 5.5600.

Proposition 8.3. *At $(k, p, k_1) = (2, 3, 24)$, where $k_1/(kp) = 4$ exactly, F_3 has exactly two positive roots, counted with multiplicity. Consequently no statement of the form*

$$\text{if } k_1 \geq c\,kp \text{ then } F_3 \text{ has four positive roots}$$

is true with $c \leq 4$, on the whole parameter region or on the slice $k = 2$ alone; and, with Theorem 8.1, at $(k, p) = (2, 3)$ the count passes from two to four somewhere between $k_1 = 24$ and $k_1 = 26$.

Proof sketch. The same three steps, with $N = 67$: the coefficient list of $(1+x)^{67} F_3$ at this point has 84 integer entries and exactly two sign changes, so Descartes' rule gives at most two positive roots with multiplicity, and the signs $+, -, +$ of F_3 at 0, 4, 8 give two distinct roots. The exponent is minimal, and it is large for a reason: see Remark 8.4. □

Proposition 8.3 is the negative control that the four-root statements of this paper need on the $\mathrm{SU}(N)$ side, and Theorem 8.1 is the control on the other side: it refutes any strengthening to "at most three" at those points, and it shows that the hypotheses of the four-root statements are satisfiable with content. Together the two bracket the transition at $(k, p) = (2, 3)$ inside two integer steps. We repeat the caution of Remark 5.10: these are statements about a polynomial. Reading them as counts of invariant Einstein metrics needs the elimination bridge in both directions, which we do not prove.

Remark 8.4 (outside the formal development). The route has a limit and we measured it rather than guessing it. The exponent needed is small when the positive roots are real and well separated, and grows without visible bound as the parameters approach a four-root threshold from below, where two roots are about to be born: $N = 12$ at $(2, 3, 26)$, ratio 4.33, just above the threshold, against $N = 67$ at $(2, 3, 24)$, ratio 4, and $N = 217$ at $(2, 3, 25)$, ratio 4.1667, where the count is still two. At $(1000, 3, 16680)$, ratio 5.5600, where exact Sturm counting gives two positive roots and the two roots about to be born are a complex pair very close to the positive axis, no exponent N in $0 \le N \le 600$ brings the number of sign changes of $(1+x)^N F_3$ below four; that scan is exhaustive over the range, and the minimum over it is four. So this device is not a substitute for a Sturm sequence. It pins the count away from a threshold, cheaply and in the kernel, and it says nothing at a threshold. This is why the lower end of the $\mathrm{SU}(N)$ bracket in Remark 6.6 is still a Sturm computation while its four-root end is now Theorem 8.2.

Remark 8.5 (outside the formal development). The uniform $\mathrm{SU}(N)$ upper bound of Remark 5.7 was priced again, and the earlier price was wrong in kind. Half of its forty-six certificates, in one module, reached 27.1 gigabytes of resident memory in 39.9 CPU-minutes without finishing, against an estimate of 45 to 75 CPU-minutes for the whole file; the binding resource is memory, which grows by about a gigabyte per certificate held in one module above a baseline of some five gigabytes, so the remedy is to split the certificates across a dozen modules of four — about eight CPU-minutes and nine gigabytes each, of the order of two CPU-hours in all. Two scans bound the alternatives. Continuing the search over the exponent past $N = 32$ gives $N = 36, 44, 56, 72$, all of which need a constant above 50 while the certificate mass grows from 28,260 monomials to 55,694, so the pair $(N, c) = (29, 50)$ of Remark 5.7 is the best this route has; and, unlike the four-root certificates of Remark 6.9, which collapse by two orders of magnitude on a slice, the upper-bound certificates do not improve on a slice at all — $p = 3$ and $k = 2$ still need $c = 50$ at $N = 29$. The reason is structural: what is certified is the sign of each coefficient, and fixing a parameter does not change which coefficients are marginal. Both figures, like those of Remark 5.7, were measured on a shared machine carrying a heavy unrelated load, and both are CPU time and peak resident memory rather than wall-clock time. The memory figures are what the conclusion rests on, and they are the figures least affected by the load; the CPU times should be read as upper bounds.

The counts of this section are cheap. Each costs about a CPU-minute, and the five points of Theorems 8.1 and 8.2 together with all their sign chains were verified in some four CPU-minutes and eight gigabytes, where the uniform certificates for the same polynomial passed twenty-seven gigabytes without finishing (Remark 8.5). Nothing in the argument is special to F_3: the same three steps apply verbatim to H_1, where they would count the positive roots at a numeric point without the region-wide certificate that Proposition 5.6 leans on, and to whatever univariate polynomial the same elimination produces for another compact group — $\mathrm{Sp}(n)$, E_6, G_2 — or for the generalized Wallach spaces of §5.5. We have carried out none of those, and we make no claim that the corresponding elimination polynomials exist in the literature: we did not find one for any of those three groups in the fourteen papers we read, and name them only as the groups a reader would try next.

Changes from version 1 of this paper. Nothing stated in version 1 is withdrawn: every theorem, proposition, lemma, corollary and remark of that version stands here with its statement unchanged, and every attribution and every scope restriction it carries is unchanged too. What is new is the whole of §5, the paragraph of §1 that records the counting question as one the sources never put, the paragraph of the results list that summarises the new theorems, the paragraph of §1 on the multiplier device, the corresponding sentences of the abstract, and the references [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One sentence of version 1 is widened rather than corrected: version 1 said that this paper answers "the second half" of its own question — how large k_1 must be for four roots — and not the first, how many roots there are; §5 now answers the first half as well for $\mathrm{SO}(n)$, on the smaller region $k_1 \geq 11kp$, and the sentence in §1 says so. The counts and the exceptions listed in §9 are larger accordingly.

Changes from version 2 of this paper. Nothing stated in version 2 is withdrawn: every theorem, proposition, lemma, corollary and remark of that version stands here with its number and its statement unchanged — apart from two decimal truncations corrected at the end of this subsection — and so does every attribution and every scope restriction. What is new is §6.3 (Theorem 6.7, Proposition 6.8, Remark 6.9), the whole of §8, the two paragraphs of the results list that summarise them, the sentence of §1 that records where the counting question is answered for F_3, the paragraph of §1 on the multiplier at a numeric point, and the corresponding sentences of the abstract.

Two things version 2 stated as lying outside the formal development are now theorems, and both remarks are left as published except for the decimal correction recorded below. The certificate half of Remark 6.6 — that the 638-monomial certificate closes at $\frac{89}{16}$ with the point $17/8$ — is Theorem 6.7; the label of that remark now applies only to its Sturm half. The four-root end of the bracket it records at $k = 1000$, $p = 3$ is Theorem 8.2; the two-root end is still a Sturm computation, and Remark 8.4 explains why the method of §8 cannot reach it. Remark 8.5 corrects a cost, not a claim: the content of Remark 5.7 is unchanged, and the uniform $\mathrm{SU}(N)$ upper bound is still not part of the formal development.

Nothing here contradicts version 2. In particular Theorem 8.2, whose parameters have ratio below $\frac{89}{16}$, is not in tension with Theorem 6.7: that theorem's constant is a sufficient condition uniform in three parameters, not the threshold at a single point. §5 gave the first exact count in this construction, for H_1; §8 gives the first ones for the degree-16 polynomial F_3. That is deliberately the narrower of the two readings: what §8 claims is the first exact counts for F_3, not the first anywhere in this construction, since the latter is what §5 already claimed for H_1 in version 2.

Two decimal expansions printed in version 2 are corrected here, and they are the only statements of that version that change. Remark 3.10 printed the point at which the two middle roots of H_1 merge as $0.46327\ldots$ and Remark 6.6 printed the point at which the F_3 window closes as $2.15841\ldots$; Theorem 7.1 makes the two reciprocal, and the printed pair was not. Recomputation gives $0.4632899\ldots$ and $2.1584755\ldots$, which are. Both truncations were wrong in the fifth decimal; no constant, no bracket and no theorem depends on either digit.

9. Verification

Every theorem, proposition and lemma stated above has been formally verified in Lean 4 against *Mathlib* [17], with the marked exceptions: Corollary 3.8 and Remarks 3.10, 4.2, 5.3, 5.7, 5.8, 5.9, 5.10, 6.5, 6.6, 6.9, 8.4 and 8.5, which lie outside the formal development and are labelled as such where they appear. The development contains no `sorry` and declares no axiom of its own; every headline result depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and nothing here uses `native_decide` or any compiled evaluation, so the trusted base is the Lean kernel alone. That includes the counts of §8: the number of sign changes in an explicit list of integers is evaluated by the kernel on numerals, in lists of at most eighty-four entries, and the nineteen theorems added to the development since version 2 were each checked individually for their axiom dependencies, with the same three axioms and no others. Descartes' rule of signs is used in the form in which *Mathlib* states it, `Polynomial.roots_countP_pos_le_signVariations`, which counts positive roots with multiplicity; that this was available is what made §5 formalisable without first proving the rule.[1]

What rests on exhaustive computation. The certificates are the computation. Each positivity theorem is one polynomial identity — between a suitable power of the cleared denominator times the quantity in question, and an explicitly listed polynomial in the shifted variables — together with the observation that the listed polynomial has no coefficient of the wrong sign, which is a finite check on the list. The sizes are recorded with each proof sketch; the largest is Theorem 3.7, a certificate of 2976 monomials against a left-hand side of 1028. The identity of Theorem 3.7 was checked by splitting the certificate by the degree of the shifted variable and expanding one variable at a time, which turns one normalisation of about 1.08 million monomial products into twenty of at most 5566; that is a reformulation of the same identity, not a weakening of it. In §5 the certified quantities are the coefficients themselves: seventeen certificates of 61 to 91 monomials, 1496 in all, for the coefficients of $(1+x)^9 H_1$, and nine more, 763 monomials in all, for the coefficients of H_1. What is then done with them is again a finite check — a count of sign changes in a list of eighteen, respectively nine, signs, carried out once for each of the three possible signs of the one undetermined coefficient. The search behind the pair $(N, c) = (9, 11)$ is the scan recorded after Theorem 5.2, together with the dyadic bisection down to $\frac{699}{64}$ and the eleven escalation attempts of Remark 5.3; the census of Remark 5.8 is 9300 exact Sturm counts for H_1 and 2274 for F_3. The negative controls (Propositions 6.3, 3.4, 3.9) are single exact evaluations at explicit integer parameters, and so is the sign chain of Proposition 5.6. The searches over the constant are dyadic scans of the certificate, recorded in the proof sketches; the search for the separating point 15/32 was a computation of the true window $(\max \alpha_2, \min \alpha_3)$ over $3 \le k \le 10^5$, $3 \le p \le 10^3$; and the optimality brackets of Remarks 3.10 and 6.6 rest on exact integer Sturm root counts up to $k = 3 \cdot 10^4$ for H_1 and $k = 10^3$ for F_3, which are outside the formal development because *Mathlib* has no Sturm sequences. Both polynomials were recomputed from the sources' displayed equations rather than transcribed, and the agreements of Propositions 2.1

[1] One practical note for a reader of the development. The modules that prove the lower bound (Theorem 3.7) declare their results inside a nested namespace, while those of §5 do not, so the lower-bound theorem has to be named with its qualifying prefix where the two are composed in the proof of Theorem 5.5. Nothing mathematical turns on it; it cost one failed check.

and 2.3 are the check on that recomputation; they are what turned up the two errata.

In §8 the computation is smaller and of a different kind. At each of the six parameter points what is checked is a list of 28 to 84 integers — the coefficients of $(1+x)^N F_3$ — together with its sign changes, and the signs of F_3 at five, respectively three, explicit rationals; there is no certificate and no polynomial identity in the parameters. The finite searches there are searches for the exponent: it is minimal at each point, and at $(1000, 3, 16680)$ the scan is exhaustive over $0 \le N \le 600$ and finds none that works (Remark 8.4); that scan, like the searches for the exponent, is a computation in exact integer arithmetic and not a kernel check. The costs quoted at the end of §8 and in Remark 8.5 are measurements of kernel checks that were actually run, not estimates, with the one exception stated there: the two CPU-hours for the split certificates, which is an extrapolation from the one module that was run.

What is not claimed. Nothing above is a statement about Einstein metrics: F_3 and H_1 are *defined*, in the formal development, as the elimination resultants, and the implication "a positive solution of the sources' Einstein system yields a positive root of F_3, resp. H_1" is the sources' own reduction, written out in §2 but not formalised. Likewise the passage from a root to an invariant Einstein metric and the failure of natural reductivity are theorems of [1, 2] and are cited, not reproved; *Mathlib* has no Ricci tensor.

Literature searches. Both sources exist only as version 1 on arXiv (checked live through the arXiv API), so neither has answered the constant question in a revision. For [2] we also read the published text against the arXiv one: the four-metric theorem, its constant 10, its proof, the numbering of section 6 and the erroneous display of Proposition 2.2 all stand unchanged in print. A full-text pass over an arXiv corpus of 881,145 papers finds the strings `2409.18990` and `2502.18491` only in the two papers' own identifier markers, and the symbol $H_1(x_{23})$ only inside [2]; an arXiv API sweep for `all:"naturally reductive" AND all:Einstein`, sorted by date, returns [1] as the newest item on the topic, with nothing since. Semantic Scholar records no citations of [1] and six of [2]; all six were screened and none touches the constant or the root count, with the one reservation recorded below. Both checks were repeated when §8 was added: the two `abs` pages still show version 1 only, and a second citation index records no citing work for [1] and four for [2] — the SU(N) paper itself and its preprint, and the two citing papers named at the end of this section — all of them already screened. A full-text search of the shards added to the corpus since the sweep just described finds neither identifier and neither polynomial. No separating point other than $2/3$ for H_1, and none other than 2 and β for F_3, appears anywhere we looked. The search behind the upper bound is the one described in §5.5: the two sources and twelve further papers of the same group, all read from their arXiv sources, contain no occurrence of "Descartes", "sign change" or "Sturm"; "at most", "exactly four", "precisely four" and "upper bound" occur in neither source. A caution for anybody repeating that search: in this literature "positive roots" almost always means positive roots of a root system, and the statements about polynomials are phrased with "positive solutions".

What we could not read. Three limits on the paragraph above should be stated plainly. First, one of the six items citing [2] — M. Zhang, J. Tan and N. Xu,

Non-geodesic orbit Einstein–Randers metrics on $\mathrm{SO}(n)$, Front. Math. (2026), doi: 10.1007/s11464-025-0142-9 — has no arXiv version, and neither the publisher nor Crossref, OpenAlex or Semantic Scholar served us an abstract or any full text. We screened it by title, authors, venue and subject only. It is a Finsler paper about the geodesic-orbit property, so it is unlikely to bear on the constant, but we did not read it. Second, the published text of [1] could not be read: ScienceDirect returns 403 to us and the only repository copy is the arXiv submitted version. So, unlike [2], we cannot exclude a copy-edit that changed Remark 1 in the journal version; everything we quote from [1], including Remark 1 and the display of Proposition 2.4, is quoted from arXiv:2502.18491v1. Third, the citation check was re-run for the counting question, and the same limit applies to it: of the works recorded as citing [2], the one independent research paper is M. Wu, J. Tan and N. Xu, *Non-naturally reductive Einstein metrics on* $\mathrm{SO}(n)$, Differential Geom. Appl. **101** (2025), Paper No. 102294, doi:10.1016/j.difgeo.2025.102294 (authors, title, volume and article number from Crossref), whose abstract announces at least two metrics under a different, four-block ansatz. It is paywalled and not on arXiv, and we read only its title and abstract. Nothing is recorded as citing [1].

The repository is private; repository reference to be added at submission.

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