POSITIVE ROOTS OF THE FLAG-MANIFOLD EINSTEIN POLYNOMIALS OF ARVANITOYEORGOS, SAKANE AND STATHA: AN UNPROVED REMARK FOR $\mathrm{SU}(N)$, AND THE SHARP CONSTANT FOR $\mathrm{SO}(n)$

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Abstract. Two recent papers of Arvanitoyeorgos, Sakane and Statha reduce the existence of non-naturally-reductive invariant Einstein metrics on the compact Lie groups $\mathrm{SU}(N)$ and $\mathrm{SO}(n)$, $N = n = k_1 + (p-1)k$, to the positive real roots of a single univariate polynomial whose coefficients are integer polynomials in (k, p, k_1): a polynomial F_3 of degree 16 for $\mathrm{SU}(N)$, and a polynomial H_1 of degree 8 for $\mathrm{SO}(n)$. The $\mathrm{SO}(n)$ paper proves that H_1 has four positive roots when $k_1 \geq 10kp$; the $\mathrm{SU}(N)$ paper asserts the analogous four-root statement for F_3 when $k_1 \geq 8kp$ inside a remark that explicitly declines to give a proof. We prove the $\mathrm{SO}(n)$ four-root theorem for all real $k \geq 3$, $p \geq 3$ and $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$, and exhibit parameters with $k_1/(kp) = 5.560044$ at which H_1 has only two positive roots; the optimal constant is therefore bracketed to within 0.044%, and its exact asymptotic value is $5.5602472\ldots$. The improvement comes from moving the separating point of the argument, not from a better certificate: we show that the paper's own upper separating point $2/3$ admits no constant below 7.169, replace it by $15/32$, and prove that the paper's lower separating point γ, which it uses without explanation, is the vertex of a perfect square and cannot be simplified at all. On the $\mathrm{SU}(N)$ side we prove the first step of the unproved remark for all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, and the whole of it for six explicit pairs (k, p) and every real k_1 above a threshold well below $8kp$. Finally we show that the two constructions, which use different groups and different flag manifolds and whose polynomials neither paper relates, have literally the same asymptotic four-root threshold: their $k \to \infty$ limit polynomials differ by $x \mapsto 1/x$ and a strictly positive factor. Two errata in the sources' displayed formulas are recorded. Every theorem below is machine-checked in Lean 4; the root counts that bracket the constants from below use Sturm sequences and are flagged where they occur as lying outside the formal development.

1. Introduction

1.1. **The two constructions.** Let $p \geq 3$ and $k_1 \geq 1$ be integers, let $k \geq 2$ for $\mathrm{SU}(N)$ and $k \geq 3$ for $\mathrm{SO}(n)$, and put $N = n = k_1 + (p-1)k$. Arvanitoyeorgos, Sakane and Statha study left-invariant metrics on the compact Lie groups $\mathrm{SU}(N)$ and $\mathrm{SO}(n)$ that are invariant under the adjoint action of a block subgroup with one distinguished block of size k_1 and $p-1$ equal blocks of size k: metrics invariant under $\mathrm{Ad}\big(s(\mathrm{U}(k_1) \times \mathrm{U}(k)^{p-1})\big)$ on $\mathrm{SU}(N)$ in [1], and under $\mathrm{Ad}\big(\mathrm{SO}(k_1) \times \mathrm{SO}(k)^{p-1}\big)$ on $\mathrm{SO}(n)$ in [2]. In both cases the Einstein condition becomes a polynomial system in the finitely many metric parameters, and in both cases the authors eliminate all but one of them. What is left is a single univariate polynomial equation whose

coefficients are integer polynomials in (k, p, k_1):

$$F_3(x_{12}) = \sum_{\ell=0}^{16} a_\ell(k,p,k_1)\, x_{12}^{\ell} = 0 \qquad \text{for } \mathrm{SU}(N),$$

of degree 16, and

$$H_1(x_{23}) = \sum_{e=0}^{8} b_e(k,p,k_1)\, x_{23}^{e} = 0 \qquad \text{for } \mathrm{SO}(n),$$

of degree 8. Under the hypotheses of [1, 2], a positive real root other than the normalising value 1 yields an invariant Einstein metric that is not naturally reductive. So the geometric question "how many such Einstein metrics are there?" becomes the algebraic question

> *how many positive real roots do F_3 and H_1 have, as functions of (k, p, k_1), and how large must k_1 be, relative to kp, for four of them?*

This paper answers the second half of that question about as sharply as it can be answered, for $\mathrm{SO}(n)$ completely and for $\mathrm{SU}(N)$ in part. Nothing here is a statement about Einstein metrics as such (for which see [5], and [4] for the nearest neighbouring results on $\mathrm{SO}(n)$): the passage from a positive root to a metric, and the failure of natural reductivity, are theorems of [1, 2] and are cited rather than reproved (see §7).

1.2. **What the sources leave open.** For $\mathrm{SO}(n)$, [2] proves the four-root statement in full. Its four-metric theorem reads, verbatim:

> "Let $p \geq 3$ and $k \geq 3$. If $k_1 \geq 10kp$, then the Lie group $\mathrm{SO}(n)$ $(n = k_1 + (p-1)k)$ admits at least four $\mathrm{Ad}(\mathrm{SO}(k_1) \times (\mathrm{SO}(k))^{p-1})$-invariant Einstein metrics of the form *[(1) below]*, which are not naturally reductive."

This is Theorem 6.4 of [2]: the sentence, its constant 10 and its proof are identical in the published text and in arXiv:2409.18990v1, and the display it refers to is equation (9) of both. We give the number here because the LaTeX label of that theorem in the arXiv source is `thm6.3` — section 6 runs Proposition 6.1, Theorem 6.2, Remark 6.3, Theorem 6.4 — so a reader working from the source rather than from the paper will mis-cite it. Below we mostly call it *the four-metric theorem* of [2].

The constant 10 is what the paper's argument delivers; nothing in [2], and nothing we could find anywhere else, asks whether it is optimal. That is the question this paper settles.

For $\mathrm{SU}(N)$, the corresponding statement is not proved at all. [1] states it as a remark, verbatim:

> "**Remark 1.** It is possible to show that, for $k_1 \geq 8kp$, $k \geq 2$ and $p \geq 3$, the equation $F_3(x_{12}) = 0$ has at least four positive solutions. We just give a sketch of proof."

The introduction of [1] repeats the assertion and says "We only give a sketch of its proof in Remark 1 of Section 4." The sketch is three unproved assertions — $F_3(2) > 0$ for $k_1 \geq 8kp$; $F_3(\beta) < 0$ for $k_1 \geq 5kp$, where β is the explicit rational function of (k, p, k_1) recalled in (5); and $\beta > 2$ for $k_1 \geq 5kp$ — from which the chain $0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < \beta < \alpha_4 < k_1 kp$ is read off. The paper's own constants

are inconsistent: steps (2) and (3) are stated for $k_1 \geq 5kp$ and the conclusion for $k_1 \geq 8kp$, which is one more sign that the remark was never written out.

1.3. **Results.** Throughout, k, p and k_1 are treated as *real* parameters subject to the stated inequalities; all our theorems are therefore stronger than their integer specialisations, and F_3, H_1 are polynomials in four real variables.

The $\mathrm{SO}(n)$ *constant.* Theorem 3.7 proves the four-root statement for all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{89}{16}kp = 5.5625\,kp$, in the source's own chain with one separating point moved. Proposition 3.9 exhibits $(k, p, k_1) = (10000, 3, 166816)$, ratio $k_1/(kp) = 5.560533$, at which the binding sign of the argument has not yet turned; and exact root counts (Remark 3.10, outside the formal development) give parameters with ratio 5.560044 at which H_1 has only two positive roots. So the optimal constant lies in [5.56004, 5.5625], a bracket of 0.044%, against the published 10.

Where the slack was. Keeping the point $2/3$ that the source runs its chain through, the same style of certificate reaches $\frac{29}{4} = 7.25$ (Theorem 3.2) and no further, since Proposition 3.4 exhibits parameters with ratio 7.1683 at which $H_1(2/3) < 0$. All the slack was in the point, and Theorem 3.6 obtains $\frac{89}{16}$ by replacing $2/3$ by $15/32$.

Why the other separating point cannot be moved. The source's lower separating point is $\gamma = A/(k_1(kp-2))$ with $A = k^2p^2 + k^2p - 2k^2 - 6kp + 4k + 4$, given without explanation. Theorem 4.1 explains it: written as a polynomial in (k_1, u), the quantity $k_1^8 H_1(u/k_1)$ has top k_1-degree part $k_1^9\big((kp-2)u - A\big)^2$, a perfect square. Its discriminant vanishes identically, so the two small roots of H_1 merge as $k_1 \to \infty$ and the window between them closes; γ is the vertex, and every separating point $N(k,p)/k_1$ with $N \neq A/(kp-2)$ eventually leaves that window. Proposition 3.9 makes this concrete for the obvious candidate $N = kp$.

The $\mathrm{SU}(N)$ *remark.* Theorem 5.1 proves step (1) of Remark 1 for all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, sharpening the asserted $8kp$; Theorem 5.2 deduces two positive roots localised in $(0,1)$ and $(1,2)$, against the source's own $(0,1)$ and $(1, k_1kp)$; and Theorem 5.4 proves the whole of Remark 1 — the full chain — for the six pairs $(k,p) \in \{(2,3),(3,3),(4,3),(2,4),(3,4),(2,5)\}$ and every real k_1 above a threshold $m(k,p)$ with $m/(kp)$ between 3.90 and 5.33, against the asserted $8kp$.

A bridge. The two constructions are different — different groups, different flag manifolds, degree 16 against degree 8. [1] does put them side by side, and observes that the two Einstein systems reduce to polynomial equations of degree 8 for $\mathrm{SO}(n)$ and of degree 16 for $\mathrm{SU}(N)$; but neither paper relates the two polynomials. Theorem 6.1 shows that after the substitution $k_1 = wkp$ their top k-degree parts are related by $x \mapsto 1/x$ and multiplication by a factor that is strictly positive for $p > 1$, $w > 0$ and every real x. In the limit the positive roots of F_3 are exactly the reciprocals of those of H_1, with multiplicity, so the two constructions have the *same* asymptotic four-root threshold in $w = k_1/(kp)$.

Finally, Propositions 2.4 and 2.2 record two incorrect displays in the sources, both harmless to their results.

1.4. **Method, and what rests on computation.** Every positivity statement below is a *Handelman certificate* [6] in shifted variables: to prove $P(k,p,k_1) > 0$ on the polyhedron $\{k \geq k_0,\ p \geq 3,\ k_1 \geq c\,kp\}$ we substitute $k = k_0 + X$, $p = 3 + Y$, $k_1 = c\,kp + Z$, clear the denominator, and observe that every coefficient of the

resulting polynomial in (X, Y, Z) is non-negative and that the constant term is positive. Such a certificate is a finite, exactly verifiable object. This is not our invention: the proof in [2] of $H_1(2/3) > 0$ writes

$$H_1(2/3) = \tfrac{1}{3^8}\Big(\textstyle\sum_{j=0}^{4} p_j(k,p)(k_1 - 10kp)^j + 64\,(k_1 - 10kp)^5\Big), \qquad p_j > 0,$$

which is exactly a Handelman certificate in the shifted variable $k_1 - 10kp$. We generalise it in three variables and search over the constant c on a dyadic grid.

Two kinds of claim here rest on exhaustive computation, and we distinguish them throughout. The theorems are *identities*, each checked by the Lean 4 kernel (§7); the *optimality* statements — the lower ends of the brackets, the asymptotic thresholds — use Sturm sequences and so lie outside the formal development, and are stated as labelled remarks.

2. The two polynomials

2.1. $\mathrm{SO}(n)$**.** Let $n = k_1 + (p-1)k$ and let $-B$ be the negative of the Killing form of $\mathfrak{so}(n)$. With the decomposition of $\mathfrak{so}(n)$ associated with the partition $n = k_1 + k + \cdots + k$, [2] considers the left-invariant metrics

(1)
$$\langle\ ,\ \rangle = x_1(-B)|_{\mathfrak{so}(k_1)} + x_2\sum_{i=2}^{p}(-B)|_{\mathfrak{so}(k_i)} + x_{12}\sum_{2\le j\le p}(-B)|_{\mathfrak{m}_{1j}} + x_{23}\sum_{2\le i<j\le p}(-B)|_{\mathfrak{m}_{ij}},$$

which are $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k)^{p-1})$-invariant. The Einstein condition is the system $r_1 = r_2$, $r_2 = r_{12}$, $r_2 = r_{23}$ on the Ricci components; normalising $x_{12} = 1$ turns it into three polynomial equations $g_1 = g_2 = g_3 = 0$ in (x_1, x_2, x_{23}). The factor $x_2 - x_{23}$ of g_3 corresponds to the metrics that are naturally reductive, by the criterion of D'Atri and Ziller [3], and is discarded; what is left is

$$x_2 = \frac{(k-2)x_{23}}{kp - 2 + k_1 x_{23}^2}, \tag{2}$$

and substituting (2) leaves two equations $f_1 = f_2 = 0$ in (x_1, x_{23}), with f_1 quadratic and f_2 linear in x_1.

Proposition 2.1. *Let H_1 be the resultant of f_1 and f_2 in x_1, divided by x_{23} and by $kp-2+k_1x_{23}^2$. Then H_1 is a polynomial of degree 8 in x_{23} with 321 monomials in (k, p, k_1, x_{23}), and its leading coefficient, its constant coefficient and its value at 1 agree, as polynomials in (k, p, k_1), with the three closed forms printed in [2]:*

$$b_8 = k_1^2(pk - 2k + k_1)\big(p^2k^2 - 3pk^2 + 2k^2 - k_1k + k_1pk + k_1^2 - k_1\big),$$
$$H_1(0) = (pk - k + k_1 - 1)\big(p^2k^2 + pk^2 - 2k^2 - 6pk + 4k + 4\big)^2,$$
$$H_1(1) = (k-1)(k-k_1)(kp - k + k_1 - 1)(kp - k + k_1)^2.$$

We stress that H_1 is *recomputed*, in exact integer arithmetic, from the displayed equations of [2]; it is not transcribed from its printed coefficient list. The three agreements of Proposition 2.1 are therefore a check on the derivation rather than an assumption, and it is a check that caught an erratum.

Proposition 2.2. *The value of H_1 at 0 displayed separately in [2], namely $2\big(k^2(p-1)(p+2)+k(3p-2)+1\big)^2\big(2k(p-1)+2k_1+1\big)$, is not $H_1(0)$: at $(k, p, k_1) = (3, 3, 7)$ it equals 677376, while $H_1(0) = 32448$. The correct constant coefficient is the one printed inside the polynomial and reproduced in Proposition 2.1.*

Both expressions are positive under the paper's hypotheses, and [2] uses only the positivity of $H_1(0)$, so no result of [2] is affected.

We write

$$A = A(k,p) = k^2p^2+k^2p-2k^2-6kp+4k+4, \qquad \gamma = \gamma(k,p,k_1) = \frac{A}{k_1(kp-2)}, \tag{3}$$

for the source's lower separating point. [2] notes $A = (k(p-1)-2)(k(p+2)-4)+4(k-1)$ and gives no other explanation of the choice; §4 supplies one.

2.2. SU(N). For SU(N), $N = k_1 + (p-1)k$, the metrics of [1] carry two extra parameters y_1, y_2 coming from the centre of the isotropy subalgebra, and the Einstein condition is a system of five equations $f_1 = \cdots = f_5 = 0$ in $(x_1, x_2, x_{12}, y_1, y_2)$ after the normalisation $x_{23} = 1$. Two of them are linear in (y_1, y_2) and are solved for them; substituting leaves two equations $F_1 = F_2 = 0$, of which F_2 factors, and the branch $x_2 \neq 1$ gives

$$x_2 = \frac{x_{12}^2\big(x_{12}^2(k^2(p-1)+2)+kk_1\big)}{x_{12}^4(k^2(p-1)p-2)+kk_1(2p-1)x_{12}^2+k_1^2}. \tag{4}$$

The equation $F_1 = 0$ is linear in x_1 and the remaining equation $f_3 = 0$ is quadratic in x_1, so eliminating x_1 is a resultant: writing $f_3 = ax_1^2+bx_1+c$ and $F_1 = \alpha x_1+\beta_0$ one has $\alpha^2 f_3 = F_1\cdot(aF_1 - 2a\beta_0 + b\alpha) + \mathrm{Res}$ with $\mathrm{Res} = a\beta_0^2 - b\alpha\beta_0 + c\alpha^2$.

Proposition 2.3. *Write $\mathcal{A}$ and $\mathcal{B}$ for the numerator and the denominator of* (4)*, and $L = k(p-1)x_{12}^2 + k_1$. Substituting* (4) *into* Res *and multiplying by $\mathcal{B}^5$ gives a polynomial of degree* 32 *in x_{12}, and the divisions in*

$$\mathcal{B}^5\,\mathrm{Res} = k_1\,\mathcal{A}\,L^2\,x_{12}^4\,F_3(x_{12})$$

are exact. Here F_3 has degree exactly 16 *in x_{12} and* 1252 *monomials in (k,p,k_1,x_{12}). Its coefficients a_0, a_1, a_{15}, a_{16} agree, as polynomials in (k,p,k_1), with the four closed forms printed in* [1]*; for instance*

$$a_0 = kk_1^7\big(k^2(p-2)+k_1k+1\big)\big(k^2(p-2)(p-1)+kk_1(p-1)+k_1^2+p-2\big),$$

$$a_{16} = k(p-1)\big((p-1)k+k_1\big)\big((p-1)kk_1+1\big)\big(k^4(p-1)^2(p+2)-8k^2+4\big)^2.$$

Proposition 2.4. *The value of F_3 at* 1 *displayed in* [1] *carries the factor $k\big(k^2(p-1)^2(p+2)-4\big)^3$ where the exponent should be* 1*; with exponent* 1 *the display is an identity, and with exponent* 3 *it already fails at $(k,p,k_1) = (2,3,5)$. With the exponent corrected,*

$$\begin{aligned} F_3(1) = {}& (k-1)(k+1)(k-k_1)\big(k(p-1)+k_1-2\big)^2\big(k(p-1)+k_1+2\big)^2 \\ & \times \Big(k_1^3\big(3k^2(p-1)+4\big)+3kk_1^2\big(k^2(p-1)^2+3p-2\big) \\ & \quad + k\big(k^2(p-1)^2(p+2)-4\big)+k_1\big(k^4(p-1)^3+6k^2(p-1)p-4\big) \\ & \quad + kk_1^4\Big). \end{aligned}$$

Again [1] uses only $F_3(1) < 0$, which the corrected display still gives, so no result of [1] is affected. The source's upper separating point is

$$\beta = \frac{\big(k^2(p-1)+2\big)\big(k^2(p-1)p-2\big)}{k(p-1)\big(k^4(p-1)^2(p+2)-8k^2+4\big)}\,k_1. \tag{5}$$

Finally we record the two signs that [1] itself establishes, which we use as controls and as the left half of every chain below.

Lemma 2.5. *For all real $k \geq 2$, $p \geq 3$: $F_3(0) > 0$ whenever $k_1 > 0$, and $F_3(1) < 0$ whenever $k_1 > k$.*

3. SO(n): the four-root theorem at $k_1 \geq \frac{89}{16}kp$

3.1. **The source's own separating point.** We first ask how far the constant can be pushed while keeping the point $2/3$ that [2] uses. The four auxiliary signs certify at $\frac{29}{4}kp$:

Lemma 3.1. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{29}{4}kp$ one has $H_1(0) > 0$, $H_1(1) < 0$, $H_1(2) > 0$ and $0 < \gamma < 2/3$.*

Proof sketch. Each is a Handelman certificate in $X = k - 3$, $Y = p - 3$, $Z = k_1 - \frac{29}{4}kp$: of 61, 70 and 91 monomials for the three values of H_1, and of 9 and 13 monomials for $\gamma > 0$ and $\gamma < 2/3$. In each the coefficients all have the correct sign and the constant term is strictly signed, and the identity between the shifted certificate and 4^d times the quantity in question, for the appropriate d, is a polynomial identity in three variables. These are the sign facts of the proof in [2], there established at $k_1 \geq 10kp$. □

The binding sign is $H_1(2/3) > 0$, and it is where the constant is decided.

Theorem 3.2. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{29}{4}kp = 7.25\,kp$ one has $H_1(2/3) > 0$.*

Proof sketch. A Handelman certificate of 91 monomials in $(k-3,\, p-3,\, k_1 - \frac{29}{4}kp)$, all coefficients non-negative and the constant term positive, satisfying $4^5 \cdot 3^8 \cdot H_1(2/3) =$ cert identically. The finite search behind the constant is a scan of the dyadic grid: the certificate has no negative coefficient at $c = 10$, 8, $\frac{15}{2}$, $\frac{29}{4}$ and acquires one at $c = \frac{57}{8} = 7.125$, so $\frac{29}{4}$ is the smallest constant this certificate reaches. □

Theorem 3.3. *For all real $k \geq 3$, $p \geq 3$, $k_1 \geq \frac{29}{4}kp$, the polynomial H_1 has two positive roots $\alpha_3 \in (2/3, 1)$ and $\alpha_4 \in (1, 2)$.*

Proof. Two applications of the intermediate value theorem to the continuous function H_1, using $H_1(2/3) > 0$ (Theorem 3.2) and $H_1(1) < 0$, $H_1(2) > 0$ (Lemma 3.1). □

The corresponding statement in [2] — its two-metric theorem, which holds under the much weaker hypothesis $k_1 > k \geq 3$ — localises its two roots only in $(0, 1)$ and $(1, 2)$; the left endpoint $2/3$ here is exactly the binding step of the four-metric theorem, at a constant 27.5% below the published one.

Along this route, though, the constant cannot go much further, and the reason is not the certificate.

Proposition 3.4. *$H_1(2/3) < 0$ at $(k, p, k_1) = (400, 3, 8602)$, where $k_1/(kp) = 7.1683\ldots$; and $H_1(2/3) < 0$ at $(100, 3, 2131)$ while $H_1(2/3) > 0$ at $(100, 3, 2132)$. Moreover $H_1(1) = 0$ identically when $k_1 = k$.*

So no constant below 7.168 can serve in the argument of [2] as it is written, and $\frac{29}{4} = 7.25$ is within 1.2% of the best that argument allows. The last statement shows that some hypothesis separating k_1 from k is needed for $H_1(1) < 0$; $k_1 = k$ is

a separate case in [2]. Since exact root counts (Remark 3.10) show that H_1 already has four positive roots at ratios near 5.503, the whole remaining gap is the choice of the point 2/3.

3.2. **Moving the point.** The window in which a constant upper separating point may lie can be computed: for a given ratio $c = k_1/(kp)$ it is $(\max \alpha_2,\ \min \alpha_3)$ over the parameter domain, and both extremes occur at $p = 3$, $k \to \infty$. A search over $3 \le k \le 10^5$, $3 \le p \le 10^3$ gives a window of width 0.018 at $c = \frac{89}{16}$, and no window at all at $c = \frac{177}{32} = 5.53125$, where H_1 already has only two positive roots for $k \ge 300$. The dyadic $15/32 = 0.46875$ lies inside the window all the way down to $c = \frac{89}{16}$. With that point the four auxiliary signs still certify.

Lemma 3.5. *For all real $k \ge 3$, $p \ge 3$, $k_1 \ge \frac{89}{16}kp$ one has $H_1(0) > 0$, $H_1(1) < 0$, $H_1(2) > 0$ and $0 < \gamma < 15/32$.*

Proof sketch. As Lemma 3.1: certificates of 61, 70, 91, 9 and 13 monomials in $X = k - 3$, $Y = p - 3$, $Z = k_1 - \frac{89}{16}kp$, with denominators cleared by powers of 16. The bound $\gamma < 15/32$ replaces the source's $\gamma < 2/3$ because the upper separating point has moved. □

Theorem 3.6. *For all real $k \ge 3$, $p \ge 3$, $k_1 \ge \frac{89}{16}kp = 5.5625\,kp$ one has $H_1(15/32) > 0$; consequently H_1 has two positive roots $\alpha_3 \in (15/32, 1)$ and $\alpha_4 \in (1, 2)$.*

Proof sketch. A Handelman certificate of 91 monomials in $(k-3,\ p-3,\ k_1 - \frac{89}{16}kp)$ with no negative coefficient, satisfying $16^5 \cdot 32^8 \cdot H_1(15/32) = \text{cert}$ identically; the two roots then follow from Lemma 3.5 by the intermediate value theorem. The finite search behind the constant is again a dyadic scan: the certificate acquires a negative coefficient at the next dyadic below $\frac{89}{16}$ that is still above the true optimum, namely $\frac{2847}{512} = 5.560547$. So $\frac{89}{16}$ is this certificate's own limit for this point. □

The last sign of the chain, $H_1(\gamma) < 0$, is the expensive one: clearing the denominator of γ costs an eighth power, and the certificate has 2976 monomials against the 1028 monomials of $\big(k_1(kp-2)\big)^8 H_1(\gamma)$.

Theorem 3.7. *For all real $k \ge 3$, $p \ge 3$, $k_1 \ge \frac{89}{16}kp = 5.5625\,kp$ one has $H_1(\gamma) < 0$, and consequently H_1 has four positive roots*

$$0 < \alpha_1 < \gamma < \alpha_2 < \tfrac{15}{32} < \alpha_3 < 1 < \alpha_4 < 2.$$

Proof sketch. Write $\mathrm{Hg} = \big(k_1(kp-2)\big)^8 H_1(\gamma)$, a polynomial in (k, p, k_1) with 1028 monomials, obtained from H_1 by clearing the denominator of γ with the coefficients $b_0, \dots, b_8$ kept as atoms. The certificate is the identity $16^8 \cdot \mathrm{Hg} = -\text{cert}(k-3,\ p-3,\ k_1 - \frac{89}{16}kp)$ with cert a polynomial of 2976 monomials, no coefficient negative and constant term positive; hence $\mathrm{Hg} < 0$, and $H_1(\gamma) < 0$ because $\big(k_1(kp-2)\big)^8 > 0$ under the hypotheses. Four applications of the intermediate value theorem along $0 < \gamma < 15/32 < 1 < 2$, using Lemma 3.5 and Theorem 3.6, then give the four roots. □

This is the source's four-metric theorem with $10kp$ replaced by $\frac{89}{16}kp$, a reduction of the constant by 44%, and it is the first proof of the statement — at any constant — that has been checked by a proof kernel. The geometric conclusion follows from the source's own construction:

Corollary 3.8. *Let $k \geq 3$, $p \geq 3$ and $k_1 \geq \frac{89}{16}kp$ be integers, and $n = k_1 + (p-1)k$. Then, by the argument of* [2] *that turns a positive root x_{23} of H_1 into a metric* (1) *with $x_{12} = 1$, x_2 given by* (2) *and x_1 determined rationally, the group* $\mathrm{SO}(n)$ *admits at least four* $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k)^{p-1})$*-invariant Einstein metrics of the form* (1) *which are not naturally reductive.*

The four roots of Theorem 3.7 lie in disjoint intervals and none equals 1, which is what the non-natural-reductivity argument of [2] requires. We emphasise that Corollary 3.8 lies outside the formal development: only the four-root statement is machine-checked, and the passage from roots to metrics is quoted from [2]. We also read "at least four metrics" exactly as [2] does, namely as four distinct metrics of the form (1), one for each of the four roots. Whether those four are pairwise non-isometric is not a question a root count can decide; we do not verify it and claim nothing about it. What is proved here, and all that is proved here, is the four-root statement of Theorem 3.7.

Proposition 3.9. *One has $H_1(15/32) < 0$ at $(k, p, k_1) = (10000, 3, 166816)$, where the ratio $k_1/(kp)$ is $5.560533\ldots$, and $H_1(15/32) > 0$ at $(10000, 3, 166817)$. Also $H_1(15/32) > 0$ at $(3, 3, 51)$, where $\frac{89}{16} \cdot 9 = 50.0625 \leq 51$. Finally, at $(k, p, k_1) = (3, 3, 1000)$, where $kp/k_1 = 9/1000$ and $\gamma = 52/7000$, one has $H_1(9/1000) > 0$ while $H_1(52/7000) < 0$.*

The first two statements say that the hypothesis $k_1 \geq \frac{89}{16}kp$ is load-bearing and that the sign at $15/32$ turns exactly there, so the constant is neither vacuous nor slack — for the point $15/32$ it is within 0.03% of the least admissible one, the point's own asymptotic threshold being $5.5611279\ldots$ at $p = 3$, $k \to \infty$. The third says the hypotheses are satisfiable with content. The last pair refutes the natural guess that the lower separating point γ can be replaced by the much simpler kp/k_1: that guess already fails at $(3, 3, 1000)$, while the true $\gamma = A/(k_1(kp-2)) = 52/7000$ still works there. §4 explains why.

Remark 3.10 (outside the formal development). Exact integer Sturm root counts give, at $p = 3$, the least integer k_1 for which H_1 has four positive roots:

k	100	300	1000	3000	10000	30000
k_1	1651	4987	16664	50025	166790	500405
$k_1/(kp)$	5.50333	5.54111	5.55467	5.55833	5.55967	5.56006

The ratio increases in k. At $(k, p, k_1) = (30000, 3, 500404)$, ratio 5.560044, H_1 has only two positive roots, so no constant ≤ 5.56004 can appear in any four-root theorem of this shape. Taking the top k-degree part of H_1 at $p = 3$, $k_1 = 3ck$ and bisecting on the four-root property gives the exact supremum $c^* = 5.5602472\ldots$, at which the two middle roots merge at $0.46327\ldots$; $p = 3$ is the worst case, $p = 4$ and $p = 5$ giving thresholds below 5. Combined with Theorem 3.7, the optimal constant lies in [5.56004, 5.5625], a bracket of 0.044%. These are computations in exact integer arithmetic; they are not part of the formal development, because *Mathlib* has no Sturm sequences. The individual sign evaluations that witness the lower end are machine-checked (Proposition 3.9).

4. Why γ is forced

The lower separating point of the chain cannot be a constant — α_1 and α_2 both shrink like $1/k_1$ — so it must be of the form $N(k, p)/k_1$. The source takes

$N = A/(kp-2)$ and says nothing about the choice. The following identity says that no other choice is possible.

Theorem 4.1. *There is a polynomial $G(k,p,q,u)$, with 321 monomials and of degree 9 in q, such that*

$$G(k,p,q,qx) = q^8 H_1(k,p,q,x) \qquad \text{for all real } k,p,q,x,$$

and

$$G(k,p,q,u) = q^9\big((kp-2)u - A(k,p)\big)^2 + R(k,p,q,u),$$

where R has degree at most 8 in q.

In other words, the top q-degree part of $q^8 H_1(u/q)$ is a quadratic in u whose discriminant vanishes identically: its leading, middle and constant coefficients are $(kp-2)^2$, $-2(kp-2)A$ and A^2. Its vertex is $A/(kp-2)$, which is exactly $k_1\gamma$. Consequently the two small roots of H_1 merge at γ as $k_1 \to \infty$, the interval on which $H_1 < 0$ between them closes, and every point $N(k,p)/k_1$ with $N \neq A/(kp-2)$ eventually leaves it. The obvious simplification $N = kp$, which would shrink the 2976-monomial certificate of Theorem 3.7 to a few hundred monomials, already fails at $(k,p,k_1) = (3,3,1000)$, as Proposition 3.9 records. This is what makes the large certificate unavoidable.

Remark 4.2 (outside the formal development). The same phenomenon holds on the SU(N) side, and there it explains the source's β. Writing

$$F_3(tk_1) = \sum_{\ell,j} a_{\ell,j}(k,p)\, t^\ell k_1^{\ell+j},$$

the top total k_1-degree is 18, and the degree-18 part is $a_{16}t^{14}(t-\beta/k_1)^2$, a perfect square whose vertex is β/k_1; so β is forced there too. Raising the denominator of β to the sixteenth power is then unavoidable, and that alone is why the general SU(N) certificate for the sign of $F_3(\beta)$ runs to 97,178 monomials. The identity is verified in exact integer arithmetic and is not part of the formal development.

5. SU(N): Remark 1

5.1. **Step (1), for all parameters.**

Theorem 5.1. *For all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$ one has $F_3(2) > 0$.*

Proof sketch. A Handelman certificate: substituting $k = 2+X$, $p = 3+Y$, $k_1 = 6kp+Z$ writes $F_3(2)$ as a polynomial in (X,Y,Z) with 638 monomials, no negative coefficient and positive constant term. The finite search behind the constant is a dyadic scan: the certificate has no negative coefficient at $c = 8$, $\frac{23}{4}$, $\frac{91}{16}$, $\frac{181}{32}$, $\frac{45}{8}$ and acquires two at $c = \frac{179}{32} = 5.594$, so the sharp dyadic constant for this certificate is $\frac{45}{8} = 5.625$; the theorem is stated at the rounder 6. □

This is step (1) of Remark 1 of [1], proved, with the constant sharpened from the asserted $8kp$.

Theorem 5.2. *For all real $k \geq 2$, $p \geq 3$, $k_1 \geq 6kp$, the polynomial F_3 has two positive roots $\alpha_1 \in (0,1)$ and $\alpha_2 \in (1,2)$.*

Proof. Two applications of the intermediate value theorem, using Lemma 2.5 and Theorem 5.1; note $k_1 \geq 6kp \geq 18k > k$. □

The two-metric theorem of [1] localises its second root only in $(1, k_1kp)$; the interval $(1, 2)$ is the first half of the chain of Remark 1.

Proposition 5.3. *One has $F_3(2) < 0$ at $(k, p, k_1) = (120, 3, 2001)$, where the ratio $k_1/(kp)$ is $5.5583\ldots$; also $F_3(2) < 0$ at $(2, 3, 25)$ while $F_3(2) > 0$ at $(2, 3, 26)$; and $F_3(1) = 0$ identically when $k_1 = k$.*

So step (1) of Remark 1 fails at ratio 5.5583, no constant at or below that value can serve in it, and the threshold used in Theorem 5.4 below at $(k, p) = (2, 3)$ is exactly where the sign turns. The last statement is the source's own separate case $k_1 = k$ and shows the hypothesis $k < k_1$ of Lemma 2.5 is not removable.

5.2. **The whole of Remark 1, at six pairs.** For fixed (k, p) the denominator of β in (5) is a numeral, the sixteenth power collapses, and each of the three remaining sign conditions of Remark 1 becomes a univariate certificate.

Theorem 5.4. *Let (k, p) and m be one of*

(k,p)	$(2,3)$	$(3,3)$	$(4,3)$	$(2,4)$	$(3,4)$	$(2,5)$
m	26	45	64	33	55	39
$m/(kp)$	4.33	5.00	5.33	4.13	4.58	3.90
$8kp$	48	72	96	64	96	80

Then for every real $k_1 \geq m$ the polynomial F_3 has four positive roots

$$0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < \beta < \alpha_4 < k_1kp.$$

Proof sketch. For each pair, three univariate Handelman certificates in $T = k_1 - m \geq 0$, of degrees 10, 17 and 18 and with integer coefficients of 40 to 100 digits, give $F_3(2) > 0$, $F_3(\beta) < 0$ and $F_3(k_1kp) > 0$; together with Lemma 2.5 and four applications of the intermediate value theorem this is the chain. For $(k, p) = (2, 3)$, for instance, $\beta = \frac{55}{292}k_1$ and $k_1kp = 6k_1$, and $\beta > 2$ then follows from $k_1 \geq 26$ by arithmetic. The thresholds m are not arbitrary: each is the exact integer at which the sign of $F_3(2)$ turns, with $(4, 3)$ the one exception where it is one above. (That last statement rests on exact root counting outside the formal development, except at $(k, p) = (2, 3)$, where Proposition 5.3 records both signs.) □

Each of the six statements is an infinite family — it holds for every real $k_1 \geq m$ — and every threshold is below $5.5\,kp$, against the $8kp$ asserted in [1].

Remark 5.5 (outside the formal development). The general statement of Remark 1 also certifies. Handelman certificates in $(k-2,\ p-3,\ k_1 - c\,kp)$, all verified in exact integer arithmetic, give the four remaining sign conditions: $F_3(2) > 0$ at $c = \frac{45}{8}$ (638 monomials), $\beta > 2$ at $c = 6$, $F_3(\beta) < 0$ at $c = 1$ (97,178 monomials), and $F_3(k_1kp) > 0$ at $c = 1$ (23,826 monomials). Together with Lemma 2.5 they give the whole chain of Remark 1 for all real $k \geq 2$, $p \geq 3$ and $k_1 \geq 6kp$. The constant is 6 and not $\frac{45}{8}$ for a bookkeeping reason and not a mathematical one: 6 is the largest of the four, and it comes from the certificate for $\beta > 2$, which was run only at $c = 6$. Numerically the other thresholds are far smaller — $\beta > 2$ holds from about $2.2\,kp$, and the last two conditions from below kp — so the binding condition is $F_3(2) > 0$, whose certificate reaches $\frac{45}{8}$; but $6kp$ is what the four certificates, as run, establish, and that is the constant we state. We do not include this among the theorems, because the certificate for $F_3(\beta) < 0$ has not been put through a proof kernel: it is a single polynomial identity of 97,178 monomials, and the decomposition that made

the 2976-monomial identity of Theorem 3.7 affordable has not yet been carried out at that scale.

Remark 5.6 (outside the formal development). Exact root counts bracket the $\mathrm{SU}(N)$ constant too. The certificate for $F_3(2) > 0$ stops at $\frac{45}{8} = 5.625$; moving the upper separating point from 2 to $\frac{17}{8} = 2.125$ — which lies inside the window between α_2 and α_3, that window closing at $2.15841\ldots$ at $p = 3$, $k \to \infty$ — the same 638-monomial certificate closes at $\frac{89}{16} = 5.5625$ and fails at $\frac{2847}{512}$ with two negative coefficients. The point $\frac{17}{8}$ has its own least ratio $5.5618\ldots$, and exact root counting gives only two positive roots at $(k, p, k_1) = (1000, 3, 16680)$, ratio 5.560000. So the $\mathrm{SU}(N)$ optimal constant lies in [5.5600, 5.5625]. Section 6 explains why the same constant $\frac{89}{16}$ appears on both sides. These computations are exact but are not part of the formal development.

6. A bridge between the two constructions

Substituting $k_1 = wkp$ — the scaling both sources use — and letting $k \to \infty$ leaves a limit polynomial on each side.

Theorem 6.1. *There are polynomials F_{top} and H_{top} in (p, w, x), with* 394 *and* 121 *monomials, such that, identically in (k, p, w, x),*

$$F_3(k, p, wkp, x) = k^{12} F_{\mathrm{top}}(p, w, x) + F_r,$$

$$H_1(k, p, wkp, x) = k^5 H_{\mathrm{top}}(p, w, x) + H_r,$$

where F_r and H_r are polynomials in (k, p, w, x) of degree at most 10*, resp.* 4*, in k; and, writing $H_{\mathrm{rev}}(p, w, x) = x^8 H_{\mathrm{top}}(p, w, 1/x)$,*

$$F_{\mathrm{top}}(p, w, x) = H_{\mathrm{rev}}(p, w, x) \cdot p\, w\big((p-1)x^2 + pw\big)^4.$$

The factor is strictly positive for $p > 1$, $w > 0$ and every real x; consequently, for $p > 1$, $w > 0$, $x > 0$ and $xy = 1$,

$$F_{\mathrm{top}}(p, w, x) = 0 \iff H_{\mathrm{top}}(p, w, y) = 0.$$

Proof sketch. The two splittings are polynomial identities, obtained by writing the remainders in Horner form in k so that their degrees are visible from the expressions themselves. The factorisation was found by exact multivariate division of F_{top} by H_{rev}: the remainder is zero and the quotient has 15 monomials, which is the displayed product on the nose. Positivity of the factor and the equivalence are then elementary. □

So in the limit the positive roots of the degree-16 $\mathrm{SU}(N)$ polynomial are exactly the reciprocals of the positive roots of the degree-8 $\mathrm{SO}(n)$ polynomial, with multiplicity, and *the two constructions have the same asymptotic four-root threshold in* $w = k_1/(kp)$. That is why two independent bisections both return $5.5602472\ldots$ at $p = 3$ and $4.98878\ldots$ at $p = 4$, and why the merge points are reciprocal: $2.15841\ldots$ for F_3 against $0.46327\ldots$ for H_1 (Remarks 3.10 and 5.6). The positive factor is not arbitrary either: $(p-1)x^2 + pw = L/k$ for the factor $L = k(p-1)x^2 + k_1$ that the $\mathrm{SU}(N)$ elimination divides out twice (Proposition 2.3).

Question 6.2. Is there a geometric reason? The identity of Theorem 6.1 is at present an algebraic coincidence with a suggestive shape.

7. Verification

Every theorem, proposition and lemma stated above has been formally verified in Lean 4 against *Mathlib* [7], with the marked exceptions: Corollary 3.8 and Remarks 3.10, 4.2, 5.5 and 5.6, which lie outside the formal development and are labelled as such where they appear. The development contains no `sorry` and declares no axiom of its own; every headline result depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and nothing here uses `native_decide` or any compiled evaluation, so the trusted base is the Lean kernel alone.

What rests on exhaustive computation. The certificates are the computation. Each positivity theorem is one polynomial identity — between a suitable power of the cleared denominator times the quantity in question, and an explicitly listed polynomial in the shifted variables — together with the observation that the listed polynomial has no coefficient of the wrong sign, which is a finite check on the list. The sizes are recorded with each proof sketch; the largest is Theorem 3.7, a certificate of 2976 monomials against a left-hand side of 1028. The identity of Theorem 3.7 was checked by splitting the certificate by the degree of the shifted variable and expanding one variable at a time, which turns one normalisation of about 1.08 million monomial products into twenty of at most 5566; that is a reformulation of the same identity, not a weakening of it. The negative controls (Propositions 5.3, 3.4, 3.9) are single exact evaluations at explicit integer parameters. The searches over the constant are dyadic scans of the certificate, recorded in the proof sketches; the search for the separating point $15/32$ was a computation of the true window $(\max \alpha_2, \min \alpha_3)$ over $3 \le k \le 10^5$, $3 \le p \le 10^3$; and the optimality brackets of Remarks 3.10 and 5.6 rest on exact integer Sturm root counts up to $k = 3 \cdot 10^4$ for H_1 and $k = 10^3$ for F_3, which are outside the formal development because *Mathlib* has no Sturm sequences. Both polynomials were recomputed from the sources' displayed equations rather than transcribed, and the agreements of Propositions 2.1 and 2.3 are the check on that recomputation; they are what turned up the two errata.

What is not claimed. Nothing above is a statement about Einstein metrics: F_3 and H_1 are *defined*, in the formal development, as the elimination resultants, and the implication "a positive solution of the sources' Einstein system yields a positive root of F_3, resp. H_1" is the sources' own reduction, written out in §2 but not formalised. Likewise the passage from a root to an invariant Einstein metric and the failure of natural reductivity are theorems of [1, 2] and are cited, not reproved; *Mathlib* has no Ricci tensor.

Literature searches. Both sources exist only as version 1 on arXiv (checked live through the arXiv API), so neither has answered the constant question in a revision. For [2] we also read the published text against the arXiv one: the four-metric theorem, its constant 10, its proof, the numbering of section 6 and the erroneous display of Proposition 2.2 all stand unchanged in print. A full-text pass over an arXiv corpus of 881,145 papers finds the strings `2409.18990` and `2502.18491` only in the two papers' own identifier markers, and the symbol $H_1(x_{23})$ only inside [2]; an arXiv API sweep for `all:"naturally reductive" AND all:Einstein`, sorted by date, returns [1] as the newest item on the topic, with nothing since. Semantic Scholar records no citations of [1] and six of [2]; all six were screened and none touches the constant or the root count, with the one reservation recorded below.

No separating point other than $2/3$ for H_1, and none other than 2 and β for F_3, appears anywhere we looked.

What we could not read. Two limits on the paragraph above should be stated plainly. First, one of the six items citing [2] — M. Zhang, J. Tan and N. Xu, *Non-geodesic orbit Einstein–Randers metrics on* $\mathrm{SO}(n)$, Front. Math. (2026), doi: 10.1007/s11464-025-0142-9 — has no arXiv version, and neither the publisher nor Crossref, OpenAlex or Semantic Scholar served us an abstract or any full text. We screened it by title, authors, venue and subject only. It is a Finsler paper about the geodesic-orbit property, so it is unlikely to bear on the constant, but we did not read it. Second, the published text of [1] could not be read: ScienceDirect returns 403 to us and the only repository copy is the arXiv submitted version. So, unlike [2], we cannot exclude a copy-edit that changed Remark 1 in the journal version; everything we quote from [1], including Remark 1 and the display of Proposition 2.4, is quoted from arXiv:2502.18491v1.

The repository is private; repository reference to be added at submission.

References

[1] A. Arvanitoyeorgos, Y. Sakane and M. Statha, *Non naturally reductive Einstein metrics on* $\mathrm{SU}(N)$ *via generalized flag manifolds*, J. Geom. Phys. **216** (2025), Article 105575, 22 pp.; doi:10.1016/j.geomphys.2025.105575. All quotations, displays and numbering are from arXiv:2502.18491v1 (17 February 2025), the only arXiv version.

[2] A. Arvanitoyeorgos, Y. Sakane and M. Statha, *Non naturally reductive Einstein metrics on the orthogonal group via real flag manifolds*, Results Math. **79** (2024), no. 1, Paper No. 42, 18 pp.; doi:10.1007/s00025-023-02068-1 (published online 8 December 2023). All quotations, displays and numbering are from arXiv:2409.18990v1 (17 September 2024), the only arXiv version.

[3] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc. **18** (1979), no. 215, 72 pp.; doi:10.1090/memo/0215.

[4] H. Chen, Z. Chen and S. Deng, *Non-naturally reductive Einstein metrics on* $\mathrm{SO}(n)$, Manuscripta Math. **156** (2018), no. 1–2, 127–136; doi:10.1007/s00229-017-0954-3; arXiv:1701.03806.

[5] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10**, Springer-Verlag, Berlin, 1987.

[6] D. E. Handelman, *Representing polynomials by positive linear functions on compact convex polyhedra*, Pacific J. Math. **132** (1988), no. 1, 35–62; doi:10.2140/pjm.1988.132.35.

[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; doi:10.1145/3372885.3373824.