$n(5,4,2) = 24$, AND TWENTY-FOUR CELLS OF THE EDGE-GIRTH-REGULAR TABLE SETTLED WITHOUT EXHAUSTIVE SEARCH

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Abstract. An $\mathrm{egr}(v,k,g,\lambda)$ graph is a k-regular graph on v vertices of girth exactly g in which every edge lies on exactly λ cycles of length g; $n(k,g,\lambda)$ denotes the least order of such a graph. Goedgebeur and Jooken tabulate $n(k,g,\lambda)$ by exhaustive generation and leave many cells of the table with a two-sided gap. We exhibit a 5-regular triangle-free graph on 24 vertices in which every edge lies on exactly two 4-cycles. It improves the published upper bound $n(5,4,2) \le 32$ to 24 and, together with a matching lower bound proved here, gives $n(5,4,2) = 24$. The lower bound uses no search: it is a codegree count of the kind used by Araujo-Pardo, Kiss and Porupsánszki, combined with the divisibility $2g \mid vk\lambda$. We derive both from the definition of an edge-girth-regular graph, along with the counting parts of the Jajcay–Kiss–Miklavič proposition at girths 3 and 4 and the Drglin–Filipovski–Jajcay–Raiman lower bound, and we deduce the exact value of $n(k,g,\lambda)$ at twenty-four cells of the table — twenty-three of them published values, whose two halves rest here on no unformalized computation. All statements below are machine-checked in Lean 4.

1. Introduction

Let G be a finite simple graph. Following Jajcay, Kiss and Miklavič [8], G is an $\mathrm{egr}(v,k,g,\lambda)$ *graph* if it has v vertices, is k-regular, has girth exactly g, and every edge of G lies on exactly λ cycles of length g. Write

$$n(k,g,\lambda) \; = \; \min\{\, v \;:\; \text{an } \mathrm{egr}(v,k,g,\lambda) \text{ graph exists} \,\},$$

with $n(k,g,\lambda) = \infty$ when there is none. The triple (k,g,λ) will be called a *cell*; a graph realising it is a *witness*. Edge-girth-regular graphs sit between the cages and the vertex-transitive graphs: the cage problem is the case in which the count λ is not prescribed, and every edge-transitive graph is edge-girth-regular for some λ.

Goedgebeur and Jooken [7] computed $n(k,g,\lambda)$ by exhaustive generation over the range $k = 3$ with $g \le 8$, $k = 4$ with $g \le 6$, and $k = 5, 6$ with $g \le 5$, and published the result as their Tables 1–4: for each cell, a lower bound from the literature, a lower bound from their own generation, an upper bound from their own search, and an upper bound from the literature. Two later papers move entries of that table. Droogendijk [5] proves $n = \infty$ at $(3,7,6)$, $(3,8,14)$, $(4,4,4)$, $(4,5,8)$ and $(6,5,24)$, and Exoo, Goedgebeur, Jooken, Stubbe and Van den Eede [6] improve 21 upper bounds and close $n(4,5,3) = n(4,5,4) = 30$. After all of this, 66 cells of the tabulated range still carry a two-sided gap or no known example at all — counting each λ of a range row such as "$7 \le \lambda \le 8$" separately, and counting a cell as settled when n is known exactly or is proved to be ∞.

Three of those gaps organise the present paper.

Question 1.1. The lowest-lying two-sided gap in the table is $22 \le n(6,4,8) \le 26$; no other cell with a finite bracket and an unknown value has either end so small. Is there a 6-regular triangle-free graph on fewer than 26 vertices in which every edge lies on exactly eight 4-cycles?

Question 1.2. For $(6,4,7)$ the published bracket is $24 \le n(6,4,7) \le 28$. Which end is right?

Question 1.3. For the five cells $(5,4,6)$, $(5,4,8)$, $(5,4,10)$, $(6,4,11)$ and $(6,4,15)$ no witness is known at any order and no nonexistence proof is known. Is n finite at any of them?

None of the three is answered here. What is settled is a fourth cell, one order below the smallest of those brackets, and a general reorganisation of the arithmetic that the published tables are built on.

Results. *A new exact value.* Goedgebeur and Jooken record $24 \le n(5,4,2) \le 32$, the lower bound from their generation and the upper bound from a 32-vertex graph; [6] has no $(5,4,\cdot)$ entry, so 32 was the published record. Theorem 6.1 exhibits an $\mathrm{egr}(24,5,4,2)$ graph, so $n(5,4,2) \le 24$, and proves $n(5,4,2) \ge 24$ outright. Hence $n(5,4,2) = 24$, with an upper bound improved by 8 and with neither half resting on an exhaustive generation.

The counting proposition and the order tests. Section 4 derives the three counting parts of [7, Proposition 1], due to [8], at girths 3 and 4: the number of g-cycles through a vertex is $k\lambda/2$, the number of g-cycles in the graph is $vk\lambda/(2g)$, and $k\lambda$ is even. Together with Theorem 3.2 that is all of Proposition 1 at girth 4; at girth 3 its remaining part, $\lambda \le k-1$, is not proved here. The integrality of the second count is the divisibility $2g \mid vk\lambda$ that [7] uses to skip orders; the proof given here replaces the free dihedral action by a chain of three involutions at $g = 4$ and by a rotation partition at $g = 3$.

Two lower bounds. Section 5 proves the bound of Drglin, Filipovski, Jajcay and Raiman [4] at $g = 3,4$ — the formula generating the literature column of [7] — and the codegree bound $v \ge k^2+1-k\lambda/2$ at $g = 4$, whose count is the one used by Araujo-Pardo, Kiss and Porupsánszki [1]. Neither dominates the other. The second is what closes the cell $(5,4,2)$, and it also gives $n(6,4,2) \ge 32$, two more than the value obtained in [7] by generation — not new mathematics, but a consequence of a bound that appeared after those tables were compiled.

Exact values without a search. Section 7 lists twenty-four cells at which both $n(k,g,\lambda) \le V$ and $n(k,g,\lambda) \ge V$ are proved: the upper half by a witness, the lower half by the bounds of Section 5 raised to the next admissible order by the divisibility of Section 4. Twenty-three of the twenty-four values are published; what is new about them is that neither half now rests on an unformalized computation, the upper half being a finite check on one stored graph and the lower half a proof. At two of the twenty-four the published lower bound was itself an exhaustive generation and is here replaced by a proof; at the other twenty-two it was a bound from the literature, and that bound is proved here too.

Certified witnesses, controls, and the frontier. Section 8 records 82 witness graphs from the two source data sets, verified against the definition, together with three further witnesses produced here by a Cayley-graph search — these establish no new bound, but two of them are provably not isomorphic to the published graphs at the same parameters. Section 9 records the negative controls that make the whole development refutable, and Section 10 states what is certified about Questions 1.1–1.3.

Remark 1.4. A citation hazard, recorded because it is easy to inherit. Conjecture 2 of [7] asserts $n = \infty$ at the four cubic cells $(3,7,6)$, $(3,8,10)$, $(3,8,12)$ and $(3,8,14)$. The introduction of [5] recalls that conjecture and says "This paper proves that conjecture to be true"; its abstract claims two of the four, and its numbered statements settle exactly five cells, one section at a time: $(4,4,4)$, $(3,8,14)$, then $(4,5,8)$ and $(6,5,24)$ together, then $(3,7,6)$. Nothing there covers $(3,8,10)$ or $(3,8,12)$, and those two cells are counted as open in the tally above.

2. Preliminaries

Throughout, G is a finite simple graph on the vertex set $\{0,1,\dots,v-1\}$, given by its adjacency matrix A, and $N(u) = \{x : A_{ux} = 1\}$. For disjoint S,T we write $e(S,T)$ for the number of edges with one end in S and the other in T. A *cycle of length L through the edge* $\{i,j\}$ is a cycle of G of length L containing that edge, counted as an undirected subgraph.

2.1. **The four conditions.** We use the definition in the following unbundled form, which is what all statements below are proved from. For $L \ge 3$, deleting the edge $\{i,j\}$ from a cycle of length L through it leaves a path of length $L-1$ from j to i whose vertices are pairwise distinct, and conversely such a path closes up to a cycle of length L through $\{i,j\}$; the hypothesis $L \ge 3$ is exactly what forbids the edge $\{i,j\}$ from occurring among the steps of the path. So the number of L-cycles through $\{i,j\}$ is the number of such paths, and we may say: A is an $\mathrm{egr}(v,k,g,\lambda)$ graph when

- (E1) $A_{ii} = 0$ for all i, and $A_{ij} = A_{ji}$;
- (E2) $|N(i)| = k$ for every i;
- (E3) for every edge $\{i,j\}$ and every L with $3 \le L < g$, no L-cycle passes through $\{i,j\}$;

(E4) every edge lies on exactly λ cycles of length g;
(E5) $\lambda > 0$ and $g \geq 3$.

Conditions (E3) and (E5) together say that the girth is exactly g, and neither may be dropped: without (E3) a graph of girth 4 can be reported as one of girth 6, and without (E5) a graph of girth 5 can be reported as one of girth 4 with $\lambda = 0$. Both failures are exhibited by concrete graphs in Section 9.

2.2. **The vacuous order.** One consequence of the quantifier structure has to be stated once, because it is invisible in the informal reading and it changes the shape of every lower bound below.

Proposition 2.1. *For every k, every $g \geq 3$ and every $\lambda > 0$, the empty graph on 0 vertices satisfies (E1)–(E5). Consequently the assertion "there exists v with an $\mathrm{egr}(v,k,g,\lambda)$ graph" is true for all such k, g, λ, and both the definition of $n(k,g,\lambda)$ and every nonexistence statement below are to be read with $v > 0$.*

Proof. Each of (E1)–(E4) is universally quantified over vertices, of which there are none, and (E5) is a condition on numbers. □

Proposition 2.1 is the reason the hypothesis $0 < v$ appears in Theorems 3.1, 5.1, 5.3 and in every statement of the form "no $\mathrm{egr}(v,k,g,\lambda)$ graph has $v < V$" below: without it those statements are false as written, not merely unproved.

3. The local picture at girth four

Everything about girth 4 used below comes from one closed form.

Theorem 3.1. *Let A be any adjacency matrix and $\{i,j\}$ any pair. Then the number of triangles through $\{i,j\}$ is $|N(i) \cap N(j)|$, and the number of 4-cycles through $\{i,j\}$ is the number of pairs (x,y) with $x \in N(j) \setminus \{i\}$, $y \in N(i) \setminus \{j\}$ and $x \sim y$.*

Let now G be an $\mathrm{egr}(v,k,4,\lambda)$ graph. Then adjacent vertices of G have no common neighbour, so for every edge $\{i,j\}$ the sets $N(j) \setminus \{i\}$ and $N(i) \setminus \{j\}$ are disjoint and each has $k-1$ elements, and

$$e\big(N(j) \setminus \{i\},\, N(i) \setminus \{j\}\big) \;=\; \lambda \qquad \text{for every edge } \{i,j\}. \tag{1}$$

Summing (1) over the vk ordered edges, the number of 4-cycles of G counted with a distinguished directed edge is exactly $\lambda v k$.

Sketch. The two closed forms are the path description of Section 2 at $L = 3$ and $L = 4$, written out: a 4-cycle through $\{i,j\}$ is $i - j - x - y - i$. If two adjacent vertices had a common neighbour there would be a triangle through the edge joining them, contradicting (E3) at $L = 3$. Given that, $i \notin N(i)$ and $N(i) \cap N(j) = \emptyset$ give the cardinalities and the disjointness, and (1) is (E4) rewritten. The last sentence is (E4) summed over the vk ordered pairs spanning an edge, using (E2). No computation. □

Identity (1) is the local form of the defining condition: at girth 4, being edge-girth-regular means that the bipartite graph between the two punctured neighbourhoods of an edge has exactly λ edges, for every edge. Two immediate consequences:

Theorem 3.2. *If an $\mathrm{egr}(v,k,4,\lambda)$ graph exists with $v > 0$ and $k > 0$, then $\lambda \leq (k-1)^2$ and $v \geq 2k$. Consequently, if $\lambda > (k-1)^2$ then $n(k,4,\lambda) = \infty$; in particular there is no 6-regular graph of girth 4 in which every edge lies on 26 four-cycles, and no 5-regular one with 17.*

Sketch. The λ edges of (1) live in a $(k-1) \times (k-1)$ grid, and the two neighbourhoods $N(i)$, $N(j)$ of an edge are disjoint k-sets. No computation. The first inequality is [7, Proposition 1(iv)], due to [8], at $g = 4$, and the second is the Moore bound $M(k,4) = 2k$. □

Both bounds are attained: $K_{6,6}$ is an $\mathrm{egr}(12,6,4,25)$ graph, so $\lambda = (k-1)^2$ and $v = 2k$ occur simultaneously.

4. Counting the girth cycles

4.1. **The handshake identity.** The first count needs no girth hypothesis at all, and it is the check that (E1) and (E2) really describe a loopless undirected k-regular graph.

Proposition 4.1. *Let G be an* $\mathrm{egr}(v,k,g,\lambda)$ *graph. Then G is k-regular in the usual sense and $vk = 2|E(G)|$; in particular $2 \mid vk$. Hence for every g and every λ there is no* $\mathrm{egr}(v,k,g,\lambda)$ *graph with vk odd — for instance none on 5 vertices with $k=3$, and none on 33 vertices with $k=5$.*

Proof. Immediate from the degree-sum formula. No computation, and no data. □

Remark 4.2. This is the parity test that [7, §4] applies when it increases the order of a search, and that its tables apply to the lower bounds; both places cite [7, Proposition 1] for it. Proposition 1 as printed has five parts — the three counts of Theorem 4.3 below, together with $\lambda \le (k-1)^{g/2}$ for even g and $\lambda \le (k-1)^{(g-1)/2}$ for odd g — and none of them is "vk is even". The parity half of that test is the handshake identity, which is what Proposition 4.1 proves.

4.2. **Proposition 1 at girth four.** The three counting parts of [7, Proposition 1] come next. Let $F(G)$ be the set of *rooted directed* 4-cycles of G: quadruples (i,j,x,y) of pairwise distinct vertices with $i \sim j \sim x \sim y \sim i$.

Theorem 4.3. *Let G be an* $\mathrm{egr}(v,k,4,\lambda)$ *graph with $v>0$. Then*

(i) every vertex of G lies on exactly $k\lambda/2$ cycles of length 4;
(ii) G has exactly $vk\lambda/8$ cycles of length 4; in particular $8 \mid vk\lambda$;
(iii) $k\lambda$ is even.

Consequently, if k and λ are both odd there is no $\mathrm{egr}(v,k,4,\lambda)$ *graph at any positive order: for example none with $(k,\lambda)=(5,3)$ and none with $(3,1)$. Combining (iii) with Theorem 3.2, the admissible λ at $k=3$ are exactly $\{2,4\}$, and at $k=5$ exactly the eight even values $2,4,\dots,16$.*

Sketch. By Theorem 3.1, $|F(G)| = \lambda vk$. Every 4-cycle of G is rooted in exactly eight ways, and the usual proof of this quotients $F(G)$ by the free action of the dihedral group of order 8. We use instead that $8 = 2\cdot 2\cdot 2$: the three reflections

$$(i,j,x,y) \mapsto (i,y,x,j), \qquad (i,j,x,y) \mapsto (x,j,i,y), \qquad (i,j,x,y) \mapsto (j,i,y,x)$$

each map $F(G)$ to itself, using only the symmetry of A, and each is an involution without fixed points on the set left by the previous ones, because $j \ne y$, $i \ne x$ and $i \ne j$ respectively. So each halves that set, cutting it down by the conditions $j<y$, then $i<x$, then $i<j$. The order matters: the second reflection leaves the pair (j,y) untouched, and the third exchanges the two conditions already imposed. The residue has one element per undirected 4-cycle, which is (ii). For (i) and (iii), the same first reflection fixes the root and halves the $k\lambda$ rooted 4-cycles based at a given vertex. No computation, and no data. □

Theorem 4.4. *Let G be an* $\mathrm{egr}(v,k,3,\lambda)$ *graph with $v>0$. Then every vertex of G lies on exactly $k\lambda/2$ triangles, G has exactly $vk\lambda/6$ triangles, and $k\lambda$ is even; in particular $6 \mid vk\lambda$.*

Sketch. Rooted directed triangles are triples (i,j,x) of distinct vertices with $i \sim j \sim x \sim i$, and the count λvk of Theorem 3.1 holds verbatim at length 3, its proof using only (E2) and (E4). Here the symmetry group has order 6, which is not a power of 2, so the reflection chain of Theorem 4.3 does not apply. Instead the rotation $\rho(i,j,x) = (j,x,i)$ partitions the rooted triangles into three parts according to which of the three positions holds the smallest entry, and carries the parts bijectively onto one another; the rotation step needs no hypothesis on A whatsoever. The reversal $(i,j,x) \mapsto (i,x,j)$ then halves the part in which the minimum comes first, by the same involution argument as before. No computation, and no data. □

4.3. **Which orders a cell can have.** Goedgebeur and Jooken describe their protocol as increasing the order of the search "skipping orders for which vk is not even or $2g$ is not a divisor of $vk\lambda$" [7, §4], and say of the tables that the lower bounds "use the fact vk is even and that $2g$ is a divisor of $vk\lambda$ … without repeating this reference everywhere". With Proposition 4.1 and Theorems 4.3–4.4 that arithmetic is available as a deduction rather than as a convention, and it decides which orders a search must visit.

Proposition 4.5. *(i) If an* $\mathrm{egr}(v,6,4,7)$ *graph exists then* $4 \mid v$*. Hence there is no* $\mathrm{egr}(v,6,4,7)$ *graph with* $24 < v < 28$*: the published bracket* $24 \le n(6,4,7) \le 28$ *of Question 1.2 has exactly one undecided order, namely* 24*, and either verdict there closes the cell.*

(ii) If an $\mathrm{egr}(v,5,4,2)$ *graph exists then* $4 \mid v$*.*

(iii) At the cell $(6,4,8)$ *of Question 1.1 the criterion is vacuous:* $8 \mid v \cdot 6 \cdot 8$ *for every* v*, so no order at all is excluded there.*

Proof. For (i), $8 \mid 42v$ forces $4 \mid v$; for (ii), $8 \mid 10v$ forces $4 \mid v$; (iii) is immediate. No computation. □

Part (iii) is worth stating beside (i): a divisibility criterion is only informative where it fires, and at the hardest cell of the table it does not.

5. Two lower bounds

5.1. **The bound behind the literature column.** Every girth-4 row of [7, Tables 1–4] carries a lower bound taken from the literature, and all but six of those entries are attributed to [4, Theorem 2.3]:

$$n(k,g,\lambda) \;\ge\; M(k,g) + \left\lceil \frac{2\big((k-1)^{g/2}-\lambda\big)}{k} \right\rceil \qquad (g \text{ even}),$$

$$n(k,g,\lambda) \;\ge\; M(k,g) + (k-1)^{(g-1)/2} - \lambda \qquad (g \text{ odd}),$$

where $M(k,g)$ is the Moore bound. At $g=4$, where $M(k,4)=2k$, and at $g=3$, where $M(k,3)=1+k$, we prove it in the following division-free forms.

Theorem 5.1. *Let G be an* $\mathrm{egr}(v,k,4,\lambda)$ *graph with $v>0$ and $k>0$. Then*

$$2k^2 + 2(k-1)^2 \;\le\; kv + 2\lambda, \tag{2}$$

which is $v \ge 2k + 2\big((k-1)^2-\lambda\big)/k$; since v is an integer, (2) carries the ceiling. It implies $v \ge 2k$, and it is strictly stronger: $14 \ge 2\cdot 6$ while there is no $\mathrm{egr}(14,6,4,7)$ *graph. For an* $\mathrm{egr}(v,k,3,\lambda)$ *graph with $v>0$ and $k>0$ one has $2k \le v+\lambda$.*

Sketch. Fix an edge $\{u,w\}$. By Theorem 3.1 the sets $N(u)$ and $N(w)$ are disjoint k-sets, so the vertex set splits as $N(u) \sqcup N(w) \sqcup O$ with $|O| = v-2k$; the Moore bound is the statement $|O| \ge 0$, and (2) counts the edges leaving into O. Take $x \in N(w)$: its k neighbours avoid $N(w)$, one of them is w, and the others lie in $N(u) \setminus \{w\}$ or in O. Summing over the $k-1$ vertices of $N(w) \setminus \{u\}$, the middle terms contribute exactly λ by (1), so $(k-1)^2 - \lambda$ edges run from $N(w) \setminus \{u\}$ into O, and as many from $N(u) \setminus \{w\}$. Counted from the other side, each vertex of O absorbs at most k of them, the two punctured neighbourhoods being disjoint; hence $2\big((k-1)^2-\lambda\big) \le k(v-2k)$. At $g=3$ the two neighbourhoods of an edge meet in exactly the λ triangles through it, by the first part of Theorem 3.1, and inclusion–exclusion gives $2k-\lambda \le v$. No computation, and no data. The hypothesis $\lambda \le (k-1)^{g/2}$ carried by [4, Theorem 2.3] is not needed, because Theorem 3.2 supplies it. □

Inequality (2) is attained — by $K_{6,6}$ at $(k,\lambda) = (6,25)$, where both sides equal 122, and by an $\mathrm{egr}(10,5,4,16)$ graph — so no constant in it can be improved; and the strengthening obtained by adding 1 to its left-hand side is false, refuted by those same graphs. The girth-3 form is attained by K_5, an $\mathrm{egr}(5,4,3,3)$ graph, where $2 \cdot 4 = 5 + 3$.

Remark 5.2. Outside the formal development we checked (2) against every girth-4 row of [7, Tables 1–4]: rounding its value up to the first order satisfying $2 \mid vk$ and $8 \mid vk\lambda$ reproduces the printed entry at all 31 rows attributed there to [4, Theorem 2.3], and the 6 rows where it differs are exactly the 6 attributed to a different result, in each of which the printed entry is larger. All seven girth-3 rows attributed to that theorem are likewise reproduced exactly, this time with no rounding needed. Of the four girth-3 rows attributed elsewhere, $(4,3,1)$ and $(6,3,1)$ go to [4, Proposition 3.1], and $(5,3,2)$ and $(6,3,2)$ to [3, Lemma 1], a paper on edge-regular graphs — the class an $\mathrm{egr}(v,k,3,\lambda)$ graph belongs to, since at $g=3$ the condition (E4) says that adjacent vertices have exactly λ common neighbours. Our value reproduces the printed entry at $(4,3,1)$ and $(5,3,2)$ and falls below it at $(6,3,1)$ and $(6,3,2)$. The bound is not always the best known: an eigenvalue argument of Porupsánszki [12] gives 24 against its 20 at the cell $(6,4,2)$.

5.2. **The codegree bound.** At small λ a different count is much stronger. For distinct vertices u and w write $\mathrm{codeg}(u,w)$ for $|N(u)\cap N(w)|$.

Theorem 5.3. *Let G be an $\mathrm{egr}(v,k,4,\lambda)$ graph with $v>0$ and $k>0$. Then*

$$2k^2+2 \ \le\ 2v+k\lambda, \qquad \textit{that is} \qquad v \ \ge\ k^2+1-\tfrac{k\lambda}{2}. \tag{3}$$

Sketch. Fix a vertex u and let D_2 be the set of $w\neq u$ with $\mathrm{codeg}(u,w)>0$. Two double counts, both from the definition. First, $\sum_{w\neq u}\mathrm{codeg}(u,w)=k(k-1)$: the left side counts the paths $u-a-w$ of length two out of u from the far end, the right side from the near end. Second, $\sum_{w\neq u}\big(\mathrm{codeg}(u,w)^2-\mathrm{codeg}(u,w)\big)=k\lambda$: the left side counts triples (w,a,b) with $a\neq b$ in $N(u)\cap N(w)$, and $(u,j,x,y)\mapsto(x,j,y)$ is a bijection from the rooted 4-cycles based at u, of which there are $k\lambda$ by Theorem 4.3(i), onto those triples. Since $2m\le 2+(m^2-m)$ for every natural m, summing over D_2 gives $2|D_2|+k\lambda\ge 2k(k-1)$. Girth 4 keeps D_2 disjoint from $N(u)\cup\{u\}$, so $|D_2|+1+k\le v$, and (3) follows. No computation, and no data. ☐

At $\lambda=0$ the right-hand side of (3) is the Moore bound k^2+1 for girth 5: each 4-cycle at an edge is allowed to cost $k/2$ vertices of it. The bound is attained at an $\mathrm{egr}(16,5,4,4)$ graph, where $2\cdot 25+2=52=2\cdot 16+5\cdot 4$, and the $+1$ strengthening is refuted by that graph. Neither (2) nor (3) dominates the other: at $(6,4,8)$ they give 18 and 13, at $(5,4,2)$ they give 16 and 21.

Remark 5.4. Theorem 5.3 claims no new mathematics. The count is the one used by Araujo-Pardo, Kiss and Porupsánszki in [1, Theorem 4.4], whose specialisation to edge-girth-regular graphs is their Corollary 4.5; they finish with the arithmetic–quadratic mean inequality, which is stronger for large λ, where the pointwise estimate $2m\le 2+m(m-1)$ used above is stronger for small λ, which is the regime that matters below — at $(5,4,2)$ their corollary gives 20 where (3) gives 21. A generalisation of the same count to vertex-girth-regular graphs is given in [9, §4], which cites [1, Theorem 4.4] for it. It is worth recording that the tables of [7] do not cite [1], which appeared eleven days after the first version of [7] and is absent from its bibliography; this is why their literature column sits below (3) at the small-λ girth-4 cells. Beyond $(5,4,2)$, treated in Section 6, (3) gives $n(6,4,3)\ge 28$ and $n(6,4,4)\ge 25$, which are the values [7] obtains by generation, and $n(6,4,2)\ge 32$, which is two more.

6. $n(5,4,2)=24$

At the cell $(5,4,2)$ an $\mathrm{egr}(v,5,4,2)$ graph is a 5-regular triangle-free graph in which every edge lies on exactly two 4-cycles. Goedgebeur and Jooken record $24\le n(5,4,2)\le 32$ [7, Table 3]; the paper [6], which improves 21 upper bounds elsewhere in the table, has no $(5,4,\cdot)$ row.

Theorem 6.1. $n(5,4,2)=24$. *Explicitly, the graph H on $\{0,\dots,23\}$ whose neighbourhoods are*

0: $1,2,3,4,5$	*8:* $1,11,12,20,23$	*16:* $4,5,7,12,21$
1: $0,6,7,8,9$	*9:* $1,11,14,18,21$	*17:* $4,7,11,14,22$
2: $0,6,10,11,12$	*10:* $2,4,13,20,23$	*18:* $4,9,19,22,23$
3: $0,7,13,14,15$	*11:* $2,8,9,13,17$	*19:* $5,6,12,14,18$
4: $0,10,16,17,18$	*12:* $2,8,15,16,19$	*20:* $5,8,10,14,15$
5: $0,13,16,19,20$	*13:* $3,5,10,11,22$	*21:* $6,9,15,16,23$
6: $1,2,19,21,22$	*14:* $3,9,17,19,20$	*22:* $6,13,15,17,18$
7: $1,3,16,17,23$	*15:* $3,12,20,21,22$	*23:* $7,8,10,18,21$

is an $\mathrm{egr}(24,5,4,2)$ *graph, and there is no* $\mathrm{egr}(v,5,4,2)$ *graph with* $0 < v < 24$.

Proof. Upper bound. That H satisfies (E1)–(E5) with $(v,k,g,\lambda) = (24,5,4,2)$ is a finite check on one explicit graph: H is loopless and symmetric, all 24 degrees equal 5, and for each of the 120 ordered pairs spanning an edge one enumerates the paths of length 2 and of length 3 between its ends with pairwise distinct vertices, obtaining 0 and 2 respectively. This is the only exhaustive computation the theorem rests on. It was carried out three times by programs sharing no code: by a from-scratch checker that rebuilds the adjacency list and counts simple paths, by the constraint model that produced H, and by compiled evaluation of (E1)–(E5) inside the proof assistant (Section 11).

Lower bound. Let $0 < v < 24$ and suppose an $\mathrm{egr}(v,5,4,2)$ graph exists. By Theorem 5.3, $2 \cdot 25 + 2 \leq 2v + 5 \cdot 2$, so $v \geq 21$. By Proposition 4.5(ii), $4 \mid v$. Hence $v \geq 24$, a contradiction. No computation. □

Remark 6.2 (how H was found). H is not needed by the proof in any form other than as the table above, but the search that produced it is worth recording. The question "is there an $\mathrm{egr}(v,k,4,\lambda)$ graph?" was written as a propositional formula: one variable per unordered pair of vertices, ternary clauses for triangle-freeness, a sequential counter per vertex for k-regularity, one variable per pair $\big(\{u,w\},\{a,b\}\big)$ of disjoint pairs meaning "$u - w - a - b - u$ is a 4-cycle", and a second sequential counter per edge slot enforcing (1) in both directions, guarded by that slot's own edge variable. Girth exactly 4 needs no further clause: by (E5) some edge lies on a 4-cycle. Two sound symmetry-breaking families were added, one fixing $N(0)$ and $N(1)$ and one requiring the adjacency matrix not to increase under any transposition of two vertices. CaDiCaL 2.1.2 [2] returned a model at $v = 24$ after 666 seconds; the model is H. The controls run before it were $v = 16$ and $v = 20$, the two admissible orders below 24 that the generation of [7] completed, and both came back unsatisfiable, in 0 and 2 seconds. The unsatisfiability verdicts are solver output; nothing below depends on them, and the lower half of Theorem 6.1 is a deduction, not a search.

7. Cells settled without exhaustive search

Of the 82 witnesses of Section 8, 56 sit at cells whose exact value is published; for those the witness proves one half of $n(k,g,\lambda) = v$, and the other half was until now a computation this development did not carry out. Where that half is arithmetic — the bound of Theorem 5.1 raised to the first order satisfying the tests of Section 4 — the arithmetic can be redone as a proof, and where the value so obtained is an order at which a witness is available, the cell is settled outright.

Theorem 7.1. *The following values of* $n(k,g,\lambda)$ *hold, each with both inequalities proved:*

$$\begin{aligned} g = 3: \quad & n(3,3,2) = 4,\ n(4,3,1) = 9,\ n(4,3,2) = 6,\ n(4,3,3) = 5,\ n(5,3,2) = 12, \\ & n(5,3,4) = 6,\ n(6,3,3) = 9,\ n(6,3,4) = 8,\ n(6,3,5) = 7; \\ g = 4: \quad & n(3,4,2) = 8,\ n(3,4,4) = 6,\ n(4,4,5) = 10,\ n(4,4,6) = 10,\ n(4,4,9) = 8, \\ & n(5,4,2) = 24,\ n(5,4,4) = 16,\ n(5,4,12) = 12,\ n(5,4,16) = 10,\ n(6,4,5) = 24, \\ & n(6,4,9) = 20,\ n(6,4,16) = 15,\ n(6,4,17) = 16,\ n(6,4,20) = 14,\ n(6,4,25) = 12. \end{aligned}$$

Sketch. Each value V is proved as a conjunction: a witness at order V, and the assertion that no $\mathrm{egr}(v,k,g,\lambda)$ graph has $0 < v < V$. The upper halves are the finite checks of Section 8, one per graph, except at $(5,4,2)$ where the witness is the graph H of Theorem 6.1. The lower halves use no computation: they are Theorem 5.1 at $g = 4$ and its girth-3 form at $g = 3$, except at $(5,4,2)$ and $(6,4,5)$ where they are Theorem 5.3; in each case the resulting inequality is then raised to the first order admitted by Proposition 4.1 and Theorems 4.3(ii), 4.4. Five of the values need that rounding visibly: the raw inequalities at $(5,4,4)$, $(6,4,9)$, $(6,4,17)$, $(4,3,1)$ and $(5,3,2)$ give 15, 18, 15, 7 and 8, lifted to 16, 20, 16, 9 and 12. □

Every value in Theorem 7.1 except $n(5,4,2) = 24$ is a published value, read off [7, Tables 1–4]. What the theorem adds for those twenty-three is that neither half now rests on an unformalized computation. It is worth being exact about which of those lower halves were searches in the first place. At twenty-two of the twenty-four cells the value V is already the entry in the literature lower-bound column of [7, Tables 1–4], attributed there to [4, Theorem 2.3] at twenty of them, to [4, Proposition 3.1] at $(4,3,1)$ and to [3, Lemma 1] at $(5,3,2)$; at those cells the published lower half was a bound and not a search, and what is added here is a proof of a bound that reaches the same number. At the remaining two, $(5,4,2)$ and $(6,4,5)$, the published lower bound 24 is the exhaustive generation of [7], and those are exactly the two cells where Theorem 5.3 supplies the proof. The method has a definite limit, and it is visible at the cell that Theorem 6.1 settles: order 20 satisfies (2) at $(5,4,2)$, and satisfies both order tests, so the argument of Theorem 7.1 provably cannot reach 24 there — which is why the published lower bound at that cell is a generation and why Theorem 5.3 is needed.

Two further controls are worth recording. The hypothesis $0 < v$ in the lower halves is not decoration: by Proposition 2.1 the statement "no $\mathrm{egr}(v,6,4,25)$ graph has $v < 12$" is false without it. And the lower bounds are met exactly, not approached: at $(6,4,25)$ order 11 is refuted and order 12 is attained.

8. Certified witnesses

Theorem 8.1. *The* 82 *graphs of the data sets accompanying* [7] *and* [6] *that this development transcribes, of orders* 4 *to* 64*, each satisfy (E1)–(E5) at their stated parameters. Each therefore gives* $n(k,g,\lambda) \le v$*;* 56 *of them stand at cells with a published exact value, and the remainder witness current record upper bounds. Among them are the two cells closed in* [6]*,* $n(4,5,3) = n(4,5,4) = 30$*.*

Sketch. One finite check per graph, of exactly the shape described in Theorem 6.1: for each ordered pair spanning an edge, an enumeration of the simple paths of every length below g and of length $g-1$ between its ends. The graphs are stored as row bitmasks of the adjacency matrix. Before being believed, every one was also checked outside the proof assistant by an independent implementation in C, which reproduced all 475 published parameter labels of the data set of [7] and all 21 of that of [6] with no mismatch. The lower bounds matching the 56 exact values are taken from those papers and are not reproduced here; twenty-three of them become theorems in Section 7. □

Theorem 8.2. *The Cayley graphs*

$$\mathrm{Cay}\big(\mathbb{Z}_{24} \rtimes_7 \mathbb{Z}_2, \{1,4,7,44\}\big), \quad \mathrm{Cay}\big(\mathbb{Z}_{18} \rtimes_7 \mathbb{Z}_3, \{3,4,17,51\}\big), \quad \mathrm{Cay}\big(\mathbb{Z}_{32} \rtimes_{17} \mathbb{Z}_2, \{2,5,11,23,61,62\}\big)$$

— elements written as $it + j$ *for* (i,j) *— are respectively an* $\mathrm{egr}(48,4,6,6)$*, an* $\mathrm{egr}(54,4,6,7)$ *and an* $\mathrm{egr}(64,6,5,10)$ *graph. Writing* $c_L(G)$ *for the total, over ordered pairs spanning an edge, of the number of* L*-cycles through that edge, the first has* $c_7 = 3360$ *against* 3136 *for the graph of* [6] *at the same parameters, and the third has* $c_6 = 21888$ *against* 19968*.*

Sketch. Finite checks, as in Theorem 8.1, one per graph and one per invariant value. The three graphs were found by an exhaustive enumeration of Cayley graphs over metacyclic groups $\mathbb{Z}_m \rtimes_r \mathbb{Z}_t$, dicyclic groups and small permutation groups, each times a cyclic factor, over every inverse-closed connection set of the required size, with the cycle counts evaluated at the identity only, which is legitimate by vertex-transitivity. The ranges swept were, by (k,g): orders 39–96 for $(4,6)$, 32–70 for $(5,5)$, 42–94 for $(6,5)$, 22–84 for $(6,4)$, 24–104 for $(5,4)$, 30–60 for $(4,5)$, 42–60 for $(3,7)$ and 56–70 for $(3,8)$; the whole sweep took about 1.5 core-hours. □

Remark 8.3. Theorem 8.2 establishes *no new bound*: all three orders coincide exactly with upper bounds published a month earlier in [6, Table 4], which were obtained by voltage-graph lifts rather than by Cayley constructions. Its content is a re-derivation by an independent method, and the observation that at $(4,6,6)$ and $(6,5,10)$ the published bound is met by at least two non-isomorphic graphs. That last conclusion carries a caveat which we state because it is load-bearing: the invariant values in Theorem 8.2 are machine-checked, but the step from "the invariants differ" to "the graphs are not isomorphic" uses the invariance of cycle counts under isomorphism, which is standard and is not part of the formal development. At $(4,6,7)$ the invariant does not separate the two graphs and nothing is claimed.

9. Controls

A development consisting of upper bounds is satisfied by a misplaced quantifier: an upper bound can be witnessed but never contradicted, so a wrong reading of the definition would leave every certificate true and every certificate meaningless. The following statements are the guard.

Proposition 9.1. *(i) Deleting a single edge from the Petersen graph gives a graph that is not an* $\operatorname{egr}(10, 3, 5, 4)$ *graph, and neither is the empty graph on* 10 *vertices.*

(ii) The Petersen graph is not an $\operatorname{egr}(10, k, g, \lambda)$ *graph at* $k = 4$*, at* $\lambda \in \{3, 5\}$*, at* $g = 4$*, or at* $g = 6$*: both directions of the girth condition bite.*

(iii) Condition (E3) is load-bearing: $K_{3,3}$ *has girth* 4*, yet each of its nine edges lies on exactly four* 6*-cycles, so with (E3) deleted it would pass as an* $\operatorname{egr}(6, 3, 6, 4)$ *graph. The full definition rejects it.*

(iv) Condition (E5) is load-bearing: the Petersen graph has girth 5 *and hence no* 4*-cycles at all, so with* $\lambda > 0$ *deleted it would pass as an* $\operatorname{egr}(10, 3, 4, 0)$ *graph, the count being vacuously constant. The full definition rejects it.*

Sketch. Finite checks on the graphs named, of the same shape as in Theorem 8.1, run against the full definition and, in (iii) and (iv), against the definition with the named clause deleted. □

Proposition 4.1, Theorems 3.2, 4.3 and 4.4 supply the complementary kind of control: nonexistence statements holding at every order simultaneously, proved with no reference to any graph.

10. The frontier

We close by stating exactly what is certified about Questions 1.1–1.3, since the interest of the table is in what remains.

Proposition 10.1. *There is an* $\operatorname{egr}(26, 6, 4, 8)$ *graph, so* $n(6, 4, 8) \le 26$*, and by Proposition 4.5(iii) the order tests exclude no order at all at that cell. For the five cells of Question 1.3, Theorem 5.1 gives* $n(5, 4, 6) \ge 16$*,* $n(5, 4, 8) \ge 14$*,* $n(5, 4, 10) \ge 16$*,* $n(6, 4, 11) \ge 20$ *and* $n(6, 4, 15) \ge 16$ *after rounding, and every* 5*-regular witness of girth* 4 *has even order, whatever* λ*.*

Combined with the published lower bound $n(6, 4, 8) \ge 22$, which is the exhaustive generation of [7] and is not reproduced here, Proposition 10.1 leaves Question 1.1 with exactly four undecided orders, 22 to 25; settling any one of them settles the cell. The published lower bounds at the five cells of Question 1.3, and at $(6, 4, 7)$ and $(6, 4, 8)$, are all larger than the values proved above, and all come from that same generation.

Remark 10.2 (not theorems). Two observations from searches run alongside the present work are recorded because they bear on Question 1.3, and neither is proved. First, over an exhaustive sweep of the Cayley families of Theorem 8.2 — $(6, 4)$ to order 84 and $(5, 4)$ to order 104 — the values $\lambda \in \{11, 15, 18, 19\}$ for $k = 6$ and $\lambda \in \{6, 8, 10\}$ for $k = 5$ never occurred, while neighbouring values of λ occurred freely; this is a statement about one family of graphs and is not evidence of nonexistence. Second, the constraint model of Remark 6.2, pointed at those five cells, returned unsatisfiable at every admissible order it was given. The orders given were, cell by cell: $20, 24, 28, 32$ at $(5, 4, 6)$; the nine even orders $20, 22, \ldots, 36$ at $(5, 4, 8)$; $20, 24, \ldots, 40$ at $(5, 4, 10)$; and 20 alone at $(6, 4, 11)$ and at $(6, 4, 15)$. Each run took between 14 seconds and 26 minutes, and each cell was first controlled at the largest order the generation of [7] reports finishing, where the model must agree with them and does. These verdicts are the output of a single solver on a single encoding, with no proof certificate checked and nothing formalised; they are recorded as a priced frontier and are *not* bounds — in particular nothing below, and no statement in Proposition 10.1, uses them. The local structure at these parameters is close to forced, which suggests that the right target is a nonexistence proof of the kind used in [5] rather than a longer ladder of refutations.

11. Verification

Every statement in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [10, 11], and the development is free of `sorry`; a repository reference to be added at submission. The reasoning layer is kernel-clean: Propositions 2.1, 4.1, 4.5, Theorems 3.1, 3.2, 4.3, 4.4, 5.1, 5.3 and the lower halves of Theorems 6.1 and 7.1 depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound` (Proposition 4.5(iii) on `propext` alone). What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite checks on individual graphs: one per witness in Theorems 8.1 and 8.2 and in Proposition 9.1, one for the graph H of Theorem 6.1, one for each invariant value in Theorem 8.2, and one for the witness half of each value in Theorem 7.1; each such check evaluates (E1)–(E5) on a stored adjacency matrix by enumerating simple paths, so it is a bounded computation and not a proof search. Every graph was checked outside the proof assistant before it was entered: the witnesses of Theorem 8.1 by a from-scratch C implementation that reproduced all $475 + 21$ published parameter labels of the two data sets with no mismatch, the graph H by two further independent programs (Remark 6.2), and the counting identities of Theorems 4.3, 4.4 and 5.3 by a Python enumerator run over the 38 transcribed graphs of girth 3 or 4 and, for the codegree identities, at every vertex of the 27 of those that have girth 4, with no violation.

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