RESULTANTS OF THE DYNATOMIC POLYNOMIALS OF A CHEBYSHEV MAP

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Abstract. Let φ be a monic polynomial over $\mathbb{Z}$ and $\Phi_{\varphi,n}$ its n-th dynatomic polynomial. Morton and Vivaldi expressed $\operatorname{res}(\Phi_{\varphi,n},\Phi_{\varphi,m})$, for $n \mid m$, as a product of cyclotomic values at the multipliers of the formal period-n points, times an undetermined unit; for the power map $\varphi(x)=x^d$ every multiplier is d^n, the product collapses, and one reads off a closed form. For the monic Chebyshev map E_d, characterised by $E_d(z+z^{-1})=z^d+z^{-d}$, the multipliers are not constant: they take the values $+d^n$, $-d^n$ and, at the two critical fixed points $x=\pm 2$, d^{2n}, and the Morton–Vivaldi product is left unevaluated. We evaluate it. We factor $\Phi_{E_d,n}$ into minimal polynomials of $2\cos(2\pi/N)$ indexed by the order of d in $(\mathbb{Z}/N)^\times/\{\pm 1\}$; we count, in closed form, how the roots split between the three multiplier values; and we obtain

$$\operatorname{res}\big(\Phi_{E_d,n},\Phi_{E_d,m}\big)=\Phi_q(d^n)^{a(d,n)}\,\Phi_q(-d^n)^{b(d,n)}\,\Phi_q(d^{2n})^{c(d,n)},\qquad q=m/n,$$

for $n \mid m$, and 1 for $n \nmid m$, with no unit left over — proved outright at $n=1$, and for $n \geq 2$ verified rather than derived. The resultant is negative exactly when $m=2n$ and $b(d,n)$ is odd. Along the way we record the corresponding statement for the power map, which is a one-line specialisation of the Morton–Vivaldi identity and is not new, together with the criterion $\operatorname{rad}(n) \mid (m/n)$ under which the only bound in print for that case is sharp. Every numerical claim is machine-checked in Lean 4: the Chebyshev law is pinned to 56 exact Sylvester determinants of sizes up to 1020, and on 35 of those cells the value is reached by three routes that share no step beyond the definition of a resultant.

1. Introduction

Let K be a field of characteristic zero and $\varphi \in K[x]$ of degree at least 2. The *n-th dynatomic polynomial* of φ is

$$\Phi_{\varphi,n}(x) \;=\; \prod_{k\mid n}\big(\varphi^{\circ k}(x)-x\big)^{\mu(n/k)}, \tag{1}$$

μ the Möbius function; that this rational function is a polynomial was first proved by Morton and Patel [14], and a proof is in [22, Thm. 4.5] — we follow [2, p. 543] for both attributions, having read neither source. Its roots are the points of *formal* period n. The quantity studied here is

$$\operatorname{res}\big(\Phi_{\varphi,n},\Phi_{\varphi,m}\big),\qquad 1\leq n<m,$$

the determinant of the Sylvester matrix of the two. It carries dynamical information directly: for $n \mid m$, φ has a point of formal period m whose exact period is the proper divisor n exactly when $\operatorname{res}(\Phi_{\varphi,n},\Phi_{\varphi,m})=0$, so the non-vanishing of these resultants is what makes formal period equal to exact period.

The organising identity is due to Morton and Vivaldi [15]. For φ monic over an integral domain and $k \mid m$ they express

$$\operatorname{res}\big(\Phi_{\varphi,k},\Phi_{\varphi,m}\big) \;=\; (\text{unit})\cdot\prod_{\alpha}\Phi_{m/k}\big((\varphi^{\circ k})'(\alpha)\big), \tag{2}$$

where Φ_j is the j-th cyclotomic polynomial and α runs over the roots of $\Phi_{\varphi,k}$ with multiplicity. We could not read [15] itself, and Remark 3.2 says exactly what (2) rests on. Whether (2) is useful depends entirely on whether the multipliers $(\varphi^{\circ k})'(\alpha)$ can be listed. Two families where they can are the ones treated below.

The power map, and what is already known. For $\varphi(x) = x^d$ write $\Phi_{d,n}$ for $\Phi_{\varphi,n}$. Here $(\varphi^{\circ n})'(x) = d^n x^{d^n-1}$, and every root α of $\Phi_{d,n}$ with $n \geq 2$ satisfies $\alpha^{d^n} = \alpha$ and $\alpha \neq 0$, so $\alpha^{d^n-1} = 1$ and the multiplier is d^n at *every* root. The product in (2) collapses to a single power and one reads off the closed form of Theorem 3.1 below. A recent preprint of Kao [7] determines $\operatorname{res}(\Phi_{d,n}, \Phi_{d,m})$ in the two easy regimes — $n = 1$, and $n \nmid m$ — and for $1 < n \mid m$ proves only the lower bound $v_p(\mathrm{res}) \geq v_p(\Phi_m(d)) \cdot \deg \Phi_{d,n}$ for the primes p dividing $\Phi_m(d)$, writing that "it is difficult to determine δ_m explicitly". For x^d it is not difficult, and we say so plainly: Theorem 3.1 is a one-line specialisation of (2) and is *not* new mathematics. We include it, with proof and with exhaustive verification, because it is the base case of the Chebyshev computation and because two of its consequences seem worth recording: the exact value is always a perfect $\deg \Phi_{d,n}$-th power, and the bound of [7] is an equality precisely when $\operatorname{rad}(n) \mid (m/n)$ (Theorem 3.4). Section 3 also records prior art that [7] does not cite (Remark 3.9).

The Chebyshev map, where the product does not collapse. Let $E_d \in \mathbb{Z}[x]$ be the monic Chebyshev polynomial, characterised by

$$E_d(z + z^{-1}) = z^d + z^{-d}, \qquad E_0 = 2, \quad E_1 = x, \quad E_{k+1} = x\,E_k - E_{k-1}, \tag{3}$$

so $E_2 = x^2 - 2$, $E_3 = x^3 - 3x$, $E_4 = x^4 - 4x^2 + 2$, and $E_d \circ E_e = E_{de}$. Differentiating (3) gives $E_a'(z+z^{-1}) = a\,(z^a - z^{-a})/(z - z^{-1})$, so at a periodic point $x = \zeta + \zeta^{-1}$ the multiplier of E_a is $+a$ or $-a$ according as $\zeta^a = \zeta$ or $\zeta^a = \zeta^{-1}$, and it degenerates to a^2 at the two critical fixed points $x = \pm 2$. The product in (2) therefore has three distinct factors and Morton–Vivaldi determine neither their exponents nor the unit. Manes [11] remarks that the dynatomic polynomial of a Chebyshev map "is known to be reducible for infinitely many N", without reference and without a factorisation, and proves only the power-map case.

This paper determines the exponents and the resulting value; the unit is $+1$, proved at $n = 1$ and verified beyond it (Remark 4.8). The three ingredients are:

- a factorisation of $\Phi_{E_d,n}$ into the minimal polynomials Ψ_N of $2\cos(2\pi/N)$, indexed by the order of d in $(\mathbb{Z}/N)^\times/\{\pm 1\}$ (Theorem 4.1) — known at $d = 2$, see Remark 4.2;
- the multiplier assignment, read off the residue of d^n modulo N (Proposition 4.4);
- a closed form for the three multiplicities (Theorem 4.5), which is exactly the step (2) leaves open.

Together they give the law of Theorem 4.7 and the sign rule of Theorem 4.9. The Ψ_N side of the picture is not new — Loper and Werner [10] determine $\operatorname{res}(\Psi_N, \Psi_M)$ completely, in the same normalisation of the Chebyshev polynomials, with no dynamics anywhere in their paper — and we use their theorem both as an external control and as one of three independent routes to the answer.

What is computation and what is proof. Every numerical claim below is a finite, exhaustive computation carried out inside the Lean 4 kernel, and every such claim states its grid. The general statements (Theorems 3.1, 3.3, 3.4, 4.1, 4.5, 4.7, 4.9) carry proofs, given here in sketch; the finite verifications are logically independent of those proofs, since nothing in the verification pipeline uses a factorisation, a multiplier or a closed form — the dynatomic polynomials are built from (1) by composing the map with itself, and the resultant is the determinant of the Sylvester matrix by fraction-free Bareiss elimination. Section 6 says exactly what was checked and how.

2. Preliminaries

Throughout $d \geq 2$ is an integer and Φ_N denotes the N-th cyclotomic polynomial.

Resultants. For $f, g \in \mathbb{Z}[x]$, $\operatorname{res}(f, g)$ is the determinant of their Sylvester matrix. If f is monic of degree D with roots $\alpha_1, \ldots, \alpha_D \in \mathbb{C}$ then $\operatorname{res}(f, g) = \prod_i g(\alpha_i)$, and $\operatorname{res}(f, gh) = \operatorname{res}(f, g)\operatorname{res}(f, h)$.

We use throughout the classical evaluation of the cyclotomic resultants: for $0 < a < b$,

$$\operatorname{res}(\Phi_a, \Phi_b) = \begin{cases} p^{\phi(a)}, & a \mid b \text{ and } b/a \text{ is a power of a prime } p, \\ 1, & \text{otherwise,} \end{cases} \tag{4}$$

ϕ the Euler totient. This has been proved at least five times: by E. T. Lehmer [8], by Diederichsen [3] — where it appears as a class number in a paper on integral group representations, not as a resultant — by Apostol [1], by Louboutin [9] and by Dresden [4], whose Theorem 1 is stated in exactly the form (4) and whose introduction is our source for the list.

Two maps and their dynatomic polynomials. For $\varphi(x) = x^d$ we write $\Phi_{d,n} := \Phi_{\varphi,n}$, and for the monic Chebyshev map E_d of (3) we write $\Phi_{E_d,n}$. Both have degree d, so by (1) both dynatomic polynomials have the same degree,

$$\deg \Phi_{d,n} = \deg \Phi_{E_d,n} = D(d,n) := \sum_{k \mid n} \mu(n/k)\, d^k. \tag{5}$$

The two families therefore sit on the same grid of Sylvester sizes and are directly comparable. They are not the same polynomials: already $\Phi_{E_2,1} = E_2(x) - x = x^2 - x - 2$ against $\Phi_{2,1} = x^2 - x$, and on the grid of Proposition 5.1 they differ at every cell.

Real cyclotomic polynomials. Let $\Psi_N \in \mathbb{Z}[x]$, for $N \geq 3$, denote the minimal polynomial over $\mathbb{Q}$ of the real number $2\cos(2\pi/N)$, so that

$$\Phi_N(z) = z^{\phi(N)/2}\, \Psi_N(z + z^{-1}), \qquad \deg \Psi_N = \phi(N)/2, \tag{6}$$

and set $\Psi_1 = x - 2$, $\Psi_2 = x + 2$, the minimal polynomials of the two critical fixed points ± 2 of every E_d. Thus $\Psi_3 = x + 1$, $\Psi_4 = x$, $\Psi_5 = x^2 + x - 1$, $\Psi_8 = x^2 - 2$, $\Psi_9 = x^3 - 3x + 1$. Loper and Werner [10] call these the minimal polynomials of the maximal real cyclotomic extensions and determine all their pairwise resultants; their $V_n(x) = 2T_n(x/2)$ is our E_n.

The order up to sign. For $\gcd(d, N) = 1$ put

$$\operatorname{ord}_N^{\pm}(d) = \min\{k \geq 1 :\ d^k \equiv \pm 1 \pmod N\}, \tag{7}$$

the order of d in $(\mathbb{Z}/N)^\times/\{\pm 1\}$, with the convention $\operatorname{ord}_1^{\pm}(d) = 1$. This invariant also occurs, in a different problem, in work of Panraksa and Tangboonduangjit [18] on Chebyshev periodic points modulo prime powers. Finally, $\operatorname{rad}(n) = \prod_{p \mid n} p$.

3. The power map: the resultant, and where the published bound is sharp

Theorem 3.1. *Let $d \geq 2$ and $1 \leq n < m$. If $n \nmid m$ then $\operatorname{res}(\Phi_{d,n}, \Phi_{d,m}) = 1$. If $n \mid m$ then*

$$\operatorname{res}\big(\Phi_{d,n}, \Phi_{d,m}\big) = \Phi_{m/n}\big(d^n\big)^{E(d,n)}, \qquad E(d,n) = \begin{cases} \deg \Phi_{d,n} = D(d,n), & n \geq 2, \\ d - 1, & n = 1. \end{cases}$$

In particular the resultant is a perfect $E(d,n)$-th power.

Proof sketch. Apply (2) with $\varphi = x^d$ and $k = n$. For $n \geq 2$ every root α of $\Phi_{d,n}$ satisfies $\alpha^{d^n} = \alpha$ and $\Phi_{d,n}(0) = 1$, hence $\alpha \neq 0$ and $\alpha^{d^n - 1} = 1$, so $(\varphi^{\circ n})'(\alpha) = d^n \alpha^{d^n - 1} = d^n$: the product has $\deg \Phi_{d,n}$ equal factors $\Phi_{m/n}(d^n)$. For $n = 1$, $\Phi_{d,1} = x^d - x$ carries the extra root 0, whose multiplier is 0 and which contributes $\Phi_m(0) = 1$; the remaining $d - 1$ roots contribute $\Phi_m(d)$ each. The unit is $+1$ because both sides are positive. The case $n \nmid m$ follows from (4) together with the factorisation $\Phi_{d,n} = \prod_{\operatorname{ord}_k(d) = n} \Phi_k$ of [7, Lem. 3.1] for $n \geq 2$ (reproduced in Proposition 3.8) and multiplicativity of the resultant, since a non-unit factor $\operatorname{res}(\Phi_a, \Phi_b)$ forces $a \mid b$ or $b \mid a$, hence $n \mid m$ or $m \mid n$. □

Remark 3.2 (Provenance). Theorem 3.1 is not new. It is the specialisation of (2) to $\varphi = x^d$, and (2) is a step inside the proof of [15, Thm. 2.2]; [7] quotes [15] in its own introduction.

On (2) *itself.* We were unable to obtain [15]: it is closed access, with no repository copy and no arXiv version located as of 2026-08-29. Two independent sources put (2) inside that proof. Murakami, Sano and Takehira [16] reproduce it, citing "[15, p. 575, line 14]"; the page-and-line locator is theirs alone. And Doyle, Fili and Hyde [2, Rem. 3.5] write of their own Proposition 3.4 — which gives (2) at $k = 1$ with unit 1, see Remark 4.8 — that it "is implicit in the proof of Theorem 2.2 of Morton and Vivaldi …; see the paragraph starting with display line (2.3)". That such identities carry an unevaluated unit is visible in Morton's own restatement of the Morton–Vivaldi theorem, [13, Thm. A, pp. 321–322]: the delta factors $\Delta_{n,d}$ and $\Delta_{n,n}$ there are "also given by" two product expressions over the periodic points, each carrying a leading factor η_1 or η_2 of which all that is asserted is, verbatim, "here $\eta_1 = \eta_1(d)$ and η_2 lie in the quotient field of $R[\alpha_1, \dots, \alpha_r]$ and $(\eta_1)^d$ and $(\eta_2)^n$ are units".

An elementary derivation avoiding (2) is also available: by (4) and multiplicativity, $v_p(\mathrm{res}) = \sum_{\mathrm{ord}_a(d)=n} \phi(a)\,|T(a,p)|$ with $T(a,p) = \{t \geq 1 : \mathrm{ord}_{ap^t}(d) = m\}$, and writing $k = v_p(a)$, $r = v_p(d^m - 1)$, $r_q = v_p(d^{m/q} - 1)$ one gets $|T(a,p)| = r - \max_{q \mid m/n} r_q$, independent of a. This is exactly the intermediate bound in the proof of [7, Thm. 4.4(2)], before that proof weakens $\max_{q \mid m/n}$ to $\max_{q \mid m}$; the first inequality is in fact an equality, and that single weakening is the whole of the gap measured in Proposition 3.7. In the notation of [7] the restricted index set is $Q(n,m) = \{q \text{ prime},\ q \mid m :\ n \mid m/q\}$, which is exactly the set of primes dividing m/n.

The base of the power has a second, index-restricted form, which is what makes the comparison with [7] transparent.

Theorem 3.3. *For $n \mid m$, $n < m$,*

$$\Phi_{m/n}\big(d^n\big) = \frac{d^m - 1}{\mathrm{lcm}\big\{\, d^{m/q} - 1 \ :\ q \text{ prime},\ q \mid m/n \,\big\}}.$$

At $n = 1$ this is the classical $\Phi_m(d) = (d^m - 1)/\mathrm{lcm}\{d^{m/q} - 1 : q \mid m\}$. Its prime-by-prime form, $v_p(\Phi_m(d)) = \#\{a \geq 1 : \mathrm{ord}_{p^a}(d) = m\}$, is [7, Lem. 3.9].

So the exact value and the published bound are the same quotient with two different index sets: all primes of m for the bound, only the primes of m/n for the truth. That comparison has a clean answer.

Theorem 3.4 (Sharpness)**.** *Let $d \geq 2$ and $1 \leq n < m$ with $n \mid m$. Then $\Phi_m(d)$ divides $\Phi_{m/n}(d^n)$, and*

$$\Phi_m(d) = \Phi_{m/n}\big(d^n\big) \iff \mathrm{rad}(n) \mid \frac{m}{n}.$$

Consequently, in the regime $1 < n \mid m$ of [7, Thm. 4.4(2)]*, that bound read in product form, $\Phi_m(d)^{\deg \Phi_{d,n}}$, always divides $\mathrm{res}(\Phi_{d,n}, \Phi_{d,m})$, and equals it exactly when $\mathrm{rad}(n) \mid (m/n)$. (At $n = 1$ the exponent in Theorem 3.1 is $d - 1$, not $\deg \Phi_{d,1} = d$, so the product form has no meaning there; the divisibility of the bases still holds, and is an equality at every $n = 1$ cell.)*

Proof sketch. By Theorem 3.3 the two sides differ only through the index set of the lcm. Write $k = m/n$ and use $\Phi_k(x^n) = \prod_{\{N\,:\,N/\gcd(N,n)=k\}} \Phi_N(x)$, which is [6, Lem. 1], stated there as $\Phi_a(x^b) = \prod_{\mathrm{lcm}(N,b)=ab} \Phi_N(x)$, and is [5, Lem. 7] in an r/s parameterisation. That index set is $\{(m/n_k)e : e \mid n_k\}$ where n_k is the largest divisor of n coprime to k; it is a singleton if and only if $n_k = 1$, i.e. if and only if $\mathrm{rad}(n) \mid k$. The omitted factors are values $\Phi_N(d)$ with $N \geq 2$, all > 1, which gives both the divisibility and the strictness in the remaining case. □

Proposition 3.5. *On the 826 triples (d, n, m) with $2 \leq d \leq 8$, $n \mid m$, $1 \leq n < m \leq 40$, the divisibility $\Phi_m(d) \mid \Phi_{m/n}(d^n)$ holds at every cell, the criterion of Theorem 3.4 has no exception, and the two are equal on 448 cells and unequal on the other 378.*

Proposition 3.6. *In the regime $1 < n \mid m$, where* [7] *proves only an inequality:*

$$\operatorname{res}(\Phi_{2,2},\Phi_{2,4}) = 5^2, \quad \operatorname{res}(\Phi_{2,2},\Phi_{2,6}) = 21^2 = 441, \quad \operatorname{res}(\Phi_{2,3},\Phi_{2,6}) = 9^6,$$
$$\operatorname{res}(\Phi_{2,2},\Phi_{2,8}) = 17^2, \quad \operatorname{res}(\Phi_{2,4},\Phi_{2,8}) = 17^{12}, \quad \operatorname{res}(\Phi_{3,2},\Phi_{3,4}) = 10^6,$$

and, at Sylvester sizes 510 *to* 720, $\operatorname{res}(\Phi_{2,3},\Phi_{2,9}) = 73^6$, $\operatorname{res}(\Phi_{5,2},\Phi_{5,4}) = 26^{20}$, $\operatorname{res}(\Phi_{3,2},\Phi_{3,6}) = 91^6$, $\operatorname{res}(\Phi_{3,3},\Phi_{3,6}) = 28^{24}$. *Each is an exact determinant of a Sylvester matrix, and each is a perfect* $\deg \Phi_{d,n}$*-th power.*

Proposition 3.7 (How the published bound falls short)**.** *Two distinct failure modes occur. At* $(d,n,m) = (2,3,6)$ *one has* $\Phi_6(2) = 3$ *and* $\deg \Phi_{2,3} = 6$, *so* [7, Thm. 4.4(2)] *gives* $v_3(\mathrm{res}) \geq 6$, *while the exact value is* 3^{12}*: the bound is short by a factor* 729. *At* $(2,2,6)$ *the resultant is* $441 = 3^2 \cdot 7^2$ *while* $\Phi_6(2) = 3$, *so the prime* 7 *lies outside the set of primes the theorem quantifies over and the theorem says nothing about* v_7*; the same happens at* $(3,2,6)$ *with the prime* 13 *and at* $(3,3,6)$ *with the prime* 2.

We record the reproductions of the published material this section rests on, all recomputed from the definitions before anything else was claimed.

Proposition 3.8. *On the* 31 *pairs* (d,n) *with* $2 \leq d \leq 7$, $1 \leq n \leq 9$, $d^n \leq 2000$*:* $\Phi_{d,n} = \prod_{\mathrm{ord}_k(d)=n} \Phi_k$ *for* $n \geq 2$, *with an extra factor* x *at* $n = 1$*;* $\deg \Phi_{d,n} = D(d,n)$ *and* $\Phi_{d,n}$ *is palindromic of even degree for* $n \geq 2$*;* $\Phi_{d,n}(0) = 1$ *and* $\Phi_{d,n}(1) = \Phi_n(d)$ *for* $d, n \geq 2$*; and, for* $n \geq 2$, $\Phi_{d,n}$ *is irreducible over* $\mathbb{Q}$ *if and only if* $d^n - 1$ *is prime, the only such cells in range being* $(2,2),(2,3),(2,5),(2,7)$. *(The restriction to* $n \geq 2$ *is needed:* $\Phi_{d,1} = x^d - x$ *is reducible for every* $d \geq 2$, *while* $d-1$ *is prime for* $d = 3$ *and* $d = 6$*.) Also* $\Phi_{2,6} = \Phi_{63}\Phi_{21}\Phi_9$. *Finally,* $\operatorname{res}(\Phi_{d,1},\Phi_{d,m}) = \Phi_m(d)^{d-1}$ *on the* 19 *cells with* $n = 1$ *of the grid of Section 6, and on that grid* $\operatorname{res}(\Phi_{d,n},\Phi_{d,m}) = 1$ *exactly on the* 24 *cells with* $n \nmid m$.

Remark 3.9 (Prior art). The factorisation and the irreducibility criterion in Proposition 3.8 are [7, Lem. 3.1] — which is stated there for $n \geq 2$ only, the extra factor x at $n = 1$ appearing inside that paper's proof of its Theorem 4.3 — and [7, Prop. 3.5], but they are older. Manes [11] proves, for every $d \geq 2$ and every $N > 1$, that $\Phi_{d,N} = \prod\{\Phi_k : k \mid d^N - 1,\ k \nmid d^j - 1 \text{ for } j \mid N,\ j \neq N\}$ and that $\Phi_{d,N}$ is reducible for every N when $d > 2$ and, for $d = 2$, irreducible exactly when $2^N - 1$ is prime ([11, Lem. 8.1 and Cor. 8.3(a)]); she attributes the factorisation to [13, p. 329], adding "As we could not find a proof, we provide one here for completeness". [7] cites for its Lemma 3.1 only Vivaldi and Hatjispyros [21, §5.1], whose treatment is of $d = 2$, and does not cite [11] at all. This is a missing attribution, not an error.

Proposition 3.10 (Controls)**.** *The Sylvester/Bareiss engine reproduces* (4) *on all* 435 *pairs* $1 \leq a < b \leq 30$, *together with* $\operatorname{res}(x^2-1, x^2+1) = 4$, $\operatorname{res}(x-1, x-2) = -1$, $\operatorname{res}(\Phi_3,\Phi_3) = 0$ *and* $\operatorname{res}(x,\Phi_{2,6}) = \Phi_{2,6}(0) = 1$. *Four false readings of Theorem 3.1 are refuted by single determinants: reading* [7, Thm. 4.4(2)] *as an equality fails at* $(2,2,6)$ *and* $(2,3,6)$*; using* $\deg \Phi_{d,1} = d$ *in place of* $d-1$ *fails at* $(3,1,2)$ *(*64 *against* 16*); replacing* $\Phi_{m/n}(d^n)$ *by* $\Phi_{m/n}(d)$ *fails at* $(2,2,6)$ *(*49 *against* 441*); and using* n *in place of* $\deg \Phi_{d,n}$ *as the exponent fails at* $(2,4,8)$ *(*17^4 *against* 17^{12}*). Neither regime is vacuous on the grid. Finally, the number of irreducible factors of* $\Phi_{2,n}$ *reproduces OEIS A059499 and* $\deg \Phi_{2,n}$ *reproduces OEIS A027375* [23], *for* $n = 1, \ldots, 15$*; at* $n = 1$ *the sequence counts one factor where* $\Phi_{2,1} = x^2 - x$ *has two, the extra* x *again.*

4. The Chebyshev map

We now turn to $\varphi = E_d$, where the multiplier is not constant and (2) does not collapse.

4.1. The factorisation.

Theorem 4.1. *For every* $d \geq 2$ *and* $k \geq 1$,

$$E_{d^k}(x) - x = \prod\{\Psi_N \ : \ N \mid d^k - 1 \text{ or } N \mid d^k + 1\},$$

every root being simple, and consequently

$$\Phi_{E_d,n} = \prod \{ \Psi_N \ : \ \mathrm{ord}_N^{\pm}(d) = n \}, \qquad \deg \Phi_{E_d,n} = D(d,n).$$

Proof sketch. Every $x \in \mathbb{C}$ is $z + z^{-1}$ for some $z \neq 0$, and $u + u^{-1} = v + v^{-1}$ iff $u = v$ or $u = v^{-1}$. So, writing $x = z + z^{-1}$, one has $E_a(x) = x$ iff $z^a = z$ or $z^a = z^{-1}$, i.e. iff $z^{a-1} = 1$ or $z^{a+1} = 1$: the fixed points of E_a are the $\zeta + \zeta^{-1}$ with $\mathrm{ord}(\zeta)$ dividing $a-1$ or $a+1$, and grouping by $N = \mathrm{ord}(\zeta)$ gives the displayed product. The degree count is exact: $\sum_{N|A} \deg \Psi_N = (A + 1 + [2 \mid A])/2$, so the two families $A = a-1$ and $A = a+1$ contribute $a + 1 + [2 \mid a-1]$ in total, while the N counted twice are the divisors of $\gcd(a-1, a+1) \in \{1, 2\}$, whose total degree is $1 + [2 \mid a-1]$; the union has degree exactly $a = \deg(E_a - x)$, and both sides are monic. Taking $a = d^k$, the condition "$N \mid d^k - 1$ or $N \mid d^k + 1$" says $d^k \equiv \pm 1 \pmod N$, and the set of such k is the set of multiples of $\mathrm{ord}_N^{\pm}(d)$. Möbius inversion over $k \mid n$ in (1) gives the second display, and (5) the degree. □

Remark 4.2 (Prior art at $d = 2$). At $d = 2$ Theorem 4.1 is *not* new. Panraksa, Samart and Sriwongsa state, as [17, Thm. 7],

$$\Phi_{E_2,n}(x) = \prod_{k|n} \Big(\prod_{N|2^k-1} \Psi_N(x) \prod_{N|2^k+1} \Psi_N(x) \Big)^{\mu(n/k)},$$

in this normalisation of Ψ_N, and attribute it to [21, §5.2]. Since $\gcd(2^k - 1, 2^k + 1) = 1$ the two inner products share no factor, so the inner product is our $E_{2^k}(x) - x$ and the display is Theorem 4.1 at $d = 2$, both halves of it. We could not obtain [21] — it sits behind the same publisher block as [15], and is advertised as open access without being reachable — so we are relaying an attribution, not reading the original; [7] independently places a decomposition of $\Phi_{2,n}$, for the *power* map, in [21, §5.1]. What Theorem 4.1 adds is every $d \geq 3$ and the indexing by $\mathrm{ord}^{\pm}$, which is what makes the count of Theorem 4.5 possible; and [17] takes no resultants at all — the word does not occur in it.

Remark 4.3 (Novelty, conditionally). The analogue of Theorem 4.1 for x^d is in print at least three times ([13, p. 329], [11, Lem. 8.1], [7, Lem. 3.1]), and the Chebyshev statement itself is in print at $d = 2$ (Remark 4.2); for $d \geq 3$ we have not found it. Two further near misses are worth naming. Silverman's book gives the fixed points of E_d as a *set* — $\{2\cos(2\pi j/(d+1))\}$ together with $\{2\cos(2\pi j/(d-1))\}$, with their multipliers, [22, Prop. 6.8] — and never as a factorisation, never in dynatomic form, and with no Ψ_N anywhere; the words "dynatomic" and "Chebyshev" do not occur together in its body. And Wolfram [20] factors $T_n(x) \pm 1$ and $U_n(x) \pm 1$ into minimal polynomials of $\cos(2\pi/d)$: the same kind of factorisation, applied to a level set of a Chebyshev polynomial rather than to $E_a(x) - x$.

One paper we could not obtain is adjacent enough that we flag it rather than claim past it, namely [19], whose published abstract promises identities for the minimal polynomials of $2\cos(2\pi/n)$ and for Chebyshev polynomials of the first kind, sets of cyclic points of modified versions of these polynomials, and a counting sequence connected with the Carmichael function. That $\mathrm{ord}^{\pm}$ is in it, at least at $d = 2$, is certain: OEIS A216066 [23], contributed by its first author, is $\min\{r \geq 1 : 2^r \equiv \pm 1 \bmod 2n-1\} = \mathrm{ord}_{2n-1}^{\pm}(2)$, and cites "Corollary 5.8 a)" of [19] for a divisibility property of that quantity. Against that, where the same authors print decompositions of the E_d elsewhere they attribute them to a different paper of their own and write them over the roots rather than as products of Ψ_N. As of 2026-08-29 [19] was behind a publisher paywall, is not on arXiv, is not in the local corpus, and we have not read it. *The novelty of Theorem 4.1 should be regarded as conditional on that paper.*

4.2. **The multipliers and their multiplicities.**

Proposition 4.4. *Let* $\mathrm{ord}_N^{\pm}(d) = n$ *and let* λ_N *be the multiplier of* $E_d^{\circ n} = E_{d^n}$ *at the roots of* Ψ_N. *Then*

$$\lambda_N = \begin{cases} +d^n, & d^n \equiv +1 \pmod N,\ N \geq 3, \\ -d^n, & d^n \equiv -1 \pmod N,\ N \geq 3, \\ d^{2n}, & N \in \{1, 2\}, \end{cases}$$

the last case being the critical fixed points $x = \pm 2$. *Equivalently,* Ψ_N *divides* $E'_{d^n} - \lambda_N$. *The case* $N = 2$ *occurs only for odd* d, *and both exceptional* N *occur only at* $n = 1$.

Proof. Differentiating (3) in z gives $E'_a(z + z^{-1})(1 - z^{-2}) = a(z^{a-1} - z^{-a-1})$, i.e. $E'_a(z + z^{-1}) = a\,(z^a - z^{-a})/(z - z^{-1})$ for $z \neq \pm 1$. At $\zeta^a = \zeta$ the quotient is $+1$ and at $\zeta^a = \zeta^{-1}$ it is -1. As $z \to 1$ the quotient tends to a, so $E'_a(2) = a^2$; as $z \to -1$ it tends to $-(-1)^a a$, so $E'_a(-2) = -(-1)^a a^2$, and $N = 2$ occurs only for odd d and only at $n = 1$, where $a = d^n$ is odd and the value is $+a^2$. □

Proposition 4.4 is not new: it is [22, Prop. 6.8] at $a = d^n$, regrouped by N. There — in the same normalisation, T_d being our E_d — the multipliers of E_a at its fixed points are given as $-a$ at the points $2\cos(2\pi j/(a+1))$, $+a$ at the $2\cos(2\pi j/(a-1))$ and a^2 at ± 2; and $E_d^{\circ n} = E_{d^n}$. We record it because it is the analytic input to Theorem 4.5 and is here checked rather than assumed.

Write, for the roots of $\Phi_{E_d,n}$ counted with multiplicity,

$$a(d,n) = \#\{\text{roots with multiplier } +d^n\}, \quad b(d,n) = \#\{-d^n\}, \quad c(d,n) = \#\{d^{2n}\},$$

so that $a + b + c = D(d, n)$. Evaluating the Morton–Vivaldi product requires these three numbers, and this is the step (2) leaves as an unevaluated product.

Theorem 4.5 (The counting rule)**.** *Write* $n = 2^v u$ *with* u *odd and set* $\delta = 1$ *if* $u = 1$ *and* d *is odd,* $\delta = 0$ *otherwise. Then*

$$b(d,n) = \frac{D\big(d^{2^v}, u\big) - \delta}{2}, \qquad c(d,n) = \begin{cases} 1 + [d \text{ odd}], & n = 1, \\ 0, & n \geq 2, \end{cases} \qquad a(d,n) = D(d,n) - b(d,n) - c(d,n).$$

In particular $a(d,n) = b(d,n) = D(d,n)/2$ *for odd* $n \geq 3$, *and* $b(d,n) = D(d,n)/2 + b(d,n/2)$ *for even* n.

Proof sketch. By Theorem 4.1 and Proposition 4.4, $a(d,n)$ and $b(d,n)$ are the sums of $\phi(N)/2$ over the $N \geq 3$ with $\mathrm{ord}_N^{\pm}(d) = n$ and $d^n \equiv +1$, resp. $-1 \pmod N$. If $\mathrm{ord}_N^{\pm}(d) = n$ and $d^n \equiv 1$ then $\mathrm{ord}_N(d) = n$; if $d^n \equiv -1$ then $\mathrm{ord}_N(d) = 2n$. Hence $\{N : \mathrm{ord}_N(d) = n\}$ splits as the $+$-part at level n together with the $-$-part at level $n/2$ (empty for odd n). Since $\sum_{\mathrm{ord}_N(d)=n} \phi(N) = D(d,n)$ for $n \geq 2$ — the degree count behind the power-map factorisation [7, Lem. 3.1] — one gets $2a(d,n) = D(d,n) - 2b(d,n/2)$ for even n and $a = b = D(d,n)/2$ for odd $n \geq 3$, with base $b(d,1) = \frac{1}{2}\sum_{N \mid d+1,\, N \geq 3} \phi(N) = (d - [d \text{ odd}])/2$. The recursion telescopes because $\sum_{j=0}^{v} D(d, 2^j u) = \sum_{k \mid u} \mu(u/k) d^{2^v k} = D(d^{2^v}, u)$. The c-count is Proposition 4.4. □

Remark 4.6. At $d = 2$ the rule reads $b(2,n) = \sum_{e \mid n,\ e \text{ odd}} \mu(e) 2^{n/e-1}$, which is verbatim one of the formulas recorded for OEIS A011946 — so the identification is by formula and not merely numerical. That sequence is defined in OEIS by an unrelated count (of Barlow packings), and is recorded there as $\mathrm{A011946}(n) = n \cdot \mathrm{A000048}(n)$ with $\mathrm{A000048}(n)$ the number of n-bead binary necklaces of primitive period n when the two colours may be interchanged. The companion sequence $a(2,n) = 0, 0, 3, 4, 15, 24, 63, 112, 252, 480, \ldots$ returned no hit in OEIS on 2026-08-29.

4.3. **The resultant.**

Theorem 4.7 (The Chebyshev resultant law)**.** *Let* $d \geq 2$ *and* $1 \leq n < m$, *and write* $q = m/n$ *and* $y = d^n$ *when* $n \mid m$. *Then*

$$\mathrm{res}\big(\Phi_{E_d,n}, \Phi_{E_d,m}\big) = \begin{cases} \Phi_q(y)^{a(d,n)} \cdot \Phi_q(-y)^{b(d,n)} \cdot \Phi_q(y^2)^{c(d,n)}, & n \mid m, \\ 1, & n \nmid m, \end{cases}$$

with (a, b, c) as in Theorem 4.5 and with no unit left over. For $n \geq 2$ the absence of the unit is verified rather than derived; see Remark 4.8.

Proof sketch. For $n \mid m$: $\Phi_{E_d,n}$ is monic, so $\operatorname{res}(\Phi_{E_d,n}, \Phi_{E_d,m}) = \prod_\alpha \Phi_{E_d,m}(\alpha)$ over its roots, which (2) evaluates as a unit times $\prod_\alpha \Phi_q(\lambda_\alpha)$. Proposition 4.4 supplies the λ_α and Theorem 4.5 their multiplicities, which is the displayed product up to the unit; the unit is treated in Remark 4.8. The order of the two arguments is immaterial, the swap costing $(-1)^{D(d,n)D(d,m)}$ with $D(d, m)$ even for every $m \geq 2$.

For $n \nmid m$ the argument does not use (2) at all. By Theorem 4.1 and multiplicativity,

$$\operatorname{res}\big(\Phi_{E_d,n}, \Phi_{E_d,m}\big) = \prod_{\operatorname{ord}_N^{\pm}(d)=n} \; \prod_{\operatorname{ord}_M^{\pm}(d)=m} \operatorname{res}(\Psi_N, \Psi_M).$$

By [10, Thm. 3.4 and Thm. 4.2] each factor is ± 1 unless one of N, M divides the other with prime-power quotient; and $N \mid M$ implies $\operatorname{ord}_N^{\pm}(d) \mid \operatorname{ord}_M^{\pm}(d)$, because $d^k \equiv \pm 1 \pmod{M}$ implies $d^k \equiv \pm 1 \pmod{N}$. A non-unit factor would therefore force $n \mid m$ or $m \mid n$, and $m \mid n$ is impossible for $n < m$. Hence the resultant is ± 1; that it is $+1$ is the content of the verification on the 24 non-divisor cells of the grid. □

Remark 4.8 (The unit). At $k = 1$ the unit in (2) is $+1$, and this is proved. The roots of $\Phi_{\varphi,1} = \varphi(x) - x$ are the fixed points of φ, and Doyle, Fili and Hyde [2, Prop. 3.4] prove that for a polynomial f over a field, a fixed point α of f in the algebraic closure, and any integer $d \geq 2$,

$$\Phi_{f,d}(\alpha) = \Phi_d\big(f'(\alpha)\big)$$

— an identity with no unit in it, their $\Phi_{f,d}$ being (1). Multiplying over the roots of the monic $\varphi(x) - x$ gives $\operatorname{res}(\Phi_{\varphi,1}, \Phi_{\varphi,m}) = \prod_\alpha \Phi_m(\varphi'(\alpha))$ exactly, which, with Proposition 4.4 and Theorem 4.5, is the $n = 1$ line of Theorem 4.7 — 19 of the 32 cells with $n \mid m$ on the grid of Section 6, negative ones included.

For $1 < n \mid m$ we do not derive the unit. (2) leaves it undetermined (Remark 3.2), and our evidence is the verification of Section 6, where the two sides agree exactly — sign included — at all 13 such cells. A reader who wants Theorem 4.7 unconditional should read its $n \geq 2$ half as conditional on that computation.

Theorem 4.9 (The sign)**.** *For $y \geq 2$ and $q \geq 2$ one has the classical rewrites*

$$\Phi_q(-y) = \begin{cases} \Phi_q(y), & 4 \mid q, \\ \Phi_{2q}(y), & q \text{ odd}, \\ \Phi_{q/2}(y), & q \equiv 2 \ (4), \ q > 2, \\ -\Phi_1(y) = -(y-1), & q = 2, \end{cases}$$

so that the right-hand side of Theorem 4.7 is a product of cyclotomic values at $+d^n$ times an explicit sign. Since $\Phi_j(t) > 0$ for every integer $t \geq 2$ and every $j \geq 1$, it follows that $\operatorname{res}(\Phi_{E_d,n}, \Phi_{E_d,m})$ is negative exactly when $m = 2n$ and $b(d, n)$ is odd.

The sign is not a decoration: on the 50-cell grid of Section 6 it is negative at 5 of the 26 cells with $n \mid m$, and the biconditional of Theorem 4.9 was verified there in both directions.

Proposition 4.10 (Named values)**.** *Each of the following is an exact determinant of a Sylvester matrix built from* (1) *by iterating E_d; the column on the right is the value of the same resultant for*

x^d at the same cell, from Proposition 3.6.

(d,n,m)	(a,b,c)	$\operatorname{res}(\Phi_{E_d,n},\Phi_{E_d,m})$	x^d
$(2,1,2)$	$(0,1,1)$	$3^0\cdot(-1)^1\cdot 5^1=-5$	3
$(3,1,2)$	$(0,1,2)$	$4^0\cdot(-2)^1\cdot 10^2=-200$	4^2
$(2,2,4)$	$(0,2,0)$	$5^0\cdot(-3)^2=9$	5^2
$(2,2,6)$	$(0,2,0)$	$21^0\cdot 13^2=169$	21^2
$(2,3,6)$	$(3,3,0)$	$9^3\cdot(-7)^3=-250047$	9^6
$(2,2,8)$	$(0,2,0)$	$17^0\cdot 17^2=289$	17^2
$(3,2,4)$	$(2,4,0)$	$10^2\cdot(-8)^4=409600$	10^6
$(2,4,8)$	$(4,8,0)$	$17^4\cdot(-15)^8=214055187890625$	17^{12}
$(4,2,4)$	$(4,8,0)$	$17^4\cdot(-15)^8$	17^{12}
$(2,3,9)$	$(3,3,0)$	$73^3\cdot 57^3=72043225281$	73^6
$(5,2,4)$	$(8,12,0)$	$26^8\cdot(-24)^{12}$	26^{20}
$(3,2,6)$	$(2,4,0)$	$91^2\cdot 73^4=235165833721$	91^6
$(3,3,6)$	$(12,12,0)$	$28^{12}\cdot(-26)^{12}$	28^{24}
$(2,2,10)$	$(0,2,0)$	$341^0\cdot 205^2=42025$	341^2
$(2,5,10)$	$(15,15,0)$	$33^{15}\cdot(-31)^{15}$	33^{30}

The last two columns show the phenomenon: for x^d the single multiplier makes the answer one perfect power; for E_d it splits. The exception is $(2,2,8)$, where $4\mid q$ and $\Phi_q(-y)=\Phi_q(y)$, so the split is invisible. The last row is the largest determinant evaluated here, of size 1020; its value is negative and has 46 digits.

Remark 4.11. The x^d entries at $(2,2,10)$ and $(2,5,10)$ are the closed form of Theorem 3.1; those two determinants were not evaluated for x^d (see Section 7). All other x^d entries are determinants.

5. Controls

The engine, the objects and the rule were each checked against something outside themselves.

Proposition 5.1 (Engine)**.** *The generic implementation of* (1)*, pointed at $\varphi(x)=x^d$ instead of E_d, reproduces the power-map dynatomic polynomials and the closed form of Theorem 3.1 on 35 cells; on all 43 cells with $2\le d\le 16$, $d^n\le 260$ the division in* (1) *is exact for both maps, and $\Phi_{E_d,n}\neq\Phi_{d,n}$ while their degrees agree. Moreover $E_0,\dots,E_4$ are as displayed in Section 1 and E_5,E_6,E_7 are x^5-5x^3+5x, $x^6-6x^4+9x^2-2$, $x^7-7x^5+14x^3-7x$; $E_d\circ E_e=E_{de}$ for $1\le d,e\le 8$ with $de\le 40$; and hence $E_d^{\circ k}=E_{d^k}$ for $2\le d\le 16$, $d^k\le 130$. The last identity is* [10, Prop. 2.3(3)] *in their normalisation $V_n(x)=2T_n(x/2)$.*

Proposition 5.2 (Objects and published data)**.** *The two printed tables of* [10] *— $V_0,\dots,V_4$ and $\Psi_1,\dots,\Psi_{10}$ — are reproduced exactly from the constructions of Section 2; all 17 published rows of OEIS A187360, the coefficient array of the minimal polynomials of $2\cos(\pi/n)$, agree with Ψ_{2n}; and $\deg\Psi_N=\phi(N)/2$ for $3\le N\le 60$. The count $b(2,n)$ agrees with OEIS A011946 for $n=1,\dots,14$, computed both from the closed form of Theorem 4.5 and by direct enumeration of roots.*

Proposition 5.3 (The published resultant theorem it stands next to)**.** *On all 780 pairs $1\le N<M\le 40$, $\operatorname{res}(\Psi_N,\Psi_M)=\pm 1$ unless $N\mid M$ and M/N is a prime power, which happens at exactly 84 pairs; there $|\operatorname{res}|=p^{\deg\Psi_N}$ with p the prime of M/N, the single exception being $\operatorname{res}(\Psi_1,\Psi_2)=4$. This is* [10, Thm. 3.4 and Thm. 4.2]*. Also $N\mid M\Rightarrow \operatorname{ord}_N^{\pm}(d)\mid\operatorname{ord}_M^{\pm}(d)$, checked for $2\le d\le 8$, $N\le 40$, $M\le 120$.*

Proposition 5.4 (Refutations and non-vacuity)**.** *Four wrong readings of Theorem 4.7, each killed by one determinant: exchanging a and b predicts 640000 at $(3,2,4)$ against the true 409600; transplanting the x^d closed form predicts 25 at $(2,2,4)$ against 9; the symmetric split $a=b$ predicts -15 at $(2,2,4)$ against 9, wrong in sign and in size; and forgetting the critical fixed points predicts -1 at $(2,1,2)$ against -5. On the 26 cells with $n\mid m$ of the grid, these four readings differ from the*

truth on 20, 25, 24 *and* 19 *cells respectively, the last being exactly the cells with* $n = 1$*; so each is refuted across the grid rather than at one lucky place. On the* 54 *counting cells,* $a \neq b$ *at* 35*, and both multiplier signs occur at* 31 *of the* 34 *multiplier cells.*

6. Verification

Every numerical claim of this paper has been checked in Lean 4 in a development that imports no library at all — a deliberate choice, so that each determinant below rests on the definition of the Sylvester matrix and on nothing else, and so that the verification is independent of any library's resultant interface (Section 7 says what such an interface would now supply). Integer polynomials are coefficient arrays, dynatomic polynomials are built from (1) by composing the map with itself and dividing out the $\mu = -1$ factors, and the resultant is the determinant of the Sylvester matrix by fraction-free Bareiss elimination. The development is free of `sorry`; every theorem is a finite integer computation delegated to compiled evaluation, so no theorem uses an axiom beyond `propext`, `Quot.sound` and the per-declaration native-evaluation axiom; in particular classical choice is used nowhere. The finite searches are as follows. Theorem 3.1: 54 cells with $2 \le d \le 7$, Sylvester size up to 720, of which 11 lie in the regime $1 < n \mid m$. Theorems 3.3 and 3.4 and Proposition 3.5: all 826 triples with $2 \le d \le 8$, $n \mid m$, $n < m \le 40$. Theorem 4.7: 56 Sylvester determinants of sizes 4 to 1020, namely 50 cells with $2 \le d \le 7$ and size at most 320 (26 with $n \mid m$, 24 without) plus six larger cells at sizes $510, 620, 702, 720, 992, 1020$. Theorem 4.5: all 54 pairs with $2 \le d \le 12$, $d^n \le 5000$, against the direct enumeration of roots; an independent implementation in Python, outside the formal development, agreed on 267 pairs with $2 \le d \le 30$, $d^n \le 3 \cdot 10^9$. Theorem 4.1: 43 pairs with $2 \le d \le 16$, $d^n \le 260$ for the factorisation, 34 for its un-inverted form. Proposition 4.4: 34 pairs with $2 \le d \le 16$, $d^n \le 130$. Theorem 4.9: the rewrites on 2440 evaluations with $1 \le q \le 40$ and $-30 \le y \le 30$, and the sign biconditional on the 26 divisor cells of the grid. Proposition 5.3: all 780 pairs $1 \le N < M \le 40$.

Three points deserve emphasis. First, the verification of Theorem 4.7 is independent of its proof: the left-hand side is a determinant, and nothing in its computation uses Theorem 4.1, Proposition 4.4 or Theorem 4.5. Second, the value was obtained by a *third* route as well — as $\prod_N \prod_M \operatorname{res}(\Psi_N, \Psi_M)$, using only objects that [10] determines in print — and this route agrees with the determinant and with the multiplier prediction on all 35 cells where it is affordable; the routes share no step beyond the definition of a resultant, which is what makes the counting rule of Theorem 4.5 an identification rather than a fit. Third, the same generic engine pointed at x^d recovers Theorem 3.1, so a defect in the composition or division code would have to be a defect that happens to be correct for the power map too. The repository reference is to be added at submission.

7. Open questions

Larger cells. A natural next cell for the power map is $(d, n, m) = (2, 4, 12)$, where the closed form of Theorem 3.1 predicts the value 273^{12} and both of the primes 3 and 7 fall outside the scope of [7, Thm. 4.4(2)]; that prediction was evaluated outside the formal development and is not backed by a determinant here. The Sylvester matrix there is 4032 square; against the measured cost of the size-1020 determinant and the cubic growth of Bareiss elimination, that is several CPU-hours before entry growth is accounted for, and we did not run it. The cell $(7, 2, 4)$, of size 2394, is affordable but adds no phenomenon not already visible at $(5, 2, 4)$. A modular resultant with a Hadamard bound would reach both, at the cost of a different and longer trust argument.

A uniform formal proof. The verifications above are exhaustive over explicit finite grids, not uniform in (d, n, m). A uniform formal proof of Theorem 4.7 needs the Sylvester resultant together with its multiplicativity and its evaluation over the roots of a monic argument, the relation (6) defining Ψ_N, and either (2) or the pairwise resultants of [10]. Of these, the first is available and the rest are not. Mathlib, the standard mathematical library for Lean, has carried `Polynomial.resultant` — defined there exactly as the determinant of the Sylvester matrix — since a commit of 2 July 2025, first

shipped in the release `v4.22.0` of 14 August 2025 [12], along with $\operatorname{res}(f_1 f_2, g) = \operatorname{res}(f_1, g)\operatorname{res}(f_2, g)$ in either argument and the evaluation $\operatorname{res}(f, g) = \operatorname{lc}(f)^n \prod_\alpha g(\alpha)$ over the roots of a split f, which are the two facts the argument sketched above actually uses. What is absent is the cyclotomic layer. Everything said here about that library was read off one revision, commit `81a5d257` of 13 July 2026, which is its release `v4.32.0`; there, exactly three source files mention a resultant at all, none of those three mentions a cyclotomic polynomial, and no file of the library contains both words except the root import list, which names every module in it. So (4), the polynomials Ψ_N of (6) and the pairwise resultants of [10] would each still have to be developed. Even the purely arithmetic half of Theorem 4.5 needs one input that is not available there, the degree count $\sum_{\operatorname{ord}_N(d)=n} \phi(N) = D(d, n)$; the telescoping identity $\sum_{j=0}^{v} D(d, 2^j u) = D(d^{2^v}, u)$ is elementary by comparison. So the estimate improves rather than vanishes: the Sylvester layer is now an import, the cyclotomic specialisation is still a build. All of this is a statement about a moving target, which is why the revision is named.

Beyond the two families treated here. In both families above the multipliers of the formal period-n points form a small, explicitly indexed set — one value in the power case, three in the Chebyshev case — which is what makes (2) evaluable. A natural next case is a Lattès map, where the periodic points are torsion points of an elliptic curve and the multipliers are again non-constant, but where there is no analogue of [10] to lean on. We have not priced that computation.

The literature question. Remark 4.2 records what the search did turn up — the $d = 2$ factorisation — and Remark 4.3 the two references we could not obtain. Even so, we found no paper that computes a resultant of two Chebyshev dynatomic polynomials: a metadata search for papers mentioning both "dynatomic" and "Chebyshev" returned none against a positive control of 16 for "dynatomic" alone, and a full-text search over a large local arXiv corpus returned 14 documents containing both words, none of which treats the object, against a positive control of 30 for "dynatomic" and "cyclotomic". The citation graph says the same. On 2026-08-29 Semantic Scholar reported no citations of [7] at all; OpenAlex's title-and-abstract index returned 38 works for "dynatomic" and none for "dynatomic" together with "Chebyshev"; and of the 27 works OpenAlex records as citing [15] and the 5 it records as citing [10], not one treats a Chebyshev dynatomic resultant. The last count is the more telling: the five papers citing [10] are about lattice-based cryptography, resultants of multiplication polynomials of Jacobi elliptic functions, and reciprocants. Nobody has taken the Ψ_N resultants into dynamics.

Changes in version 2

This version corrects one statement about the verification tooling and changes no mathematics. The first version asserted, in the paragraph on a uniform formal proof, that "the standard Lean mathematical library contains no resultant of polynomials at all". That was already false when it was written: `Polynomial.resultant`, the determinant of the Sylvester matrix, was merged into Mathlib on 2 July 2025 and shipped in its release `v4.22.0` six weeks later [12], together with multiplicativity and the evaluation over the roots of a split argument. What is true, and is what Section 7 now says, is the narrower statement: the Sylvester layer is upstream, the cyclotomic layer is not, so a uniform formal proof is cheaper than it was priced at and not free.

The complete list of changes is that paragraph, one clause in Section 6 saying why the development imports no library and pointing forward to it, the reference [12] added to support the correction, and this section. Nothing else moves: every theorem, proposition, remark, table and numerical value is unchanged from the first version. The development of Section 6 imports no library by design — so that the resultants verified here are Sylvester determinants and not the output of somebody else's interface — and therefore no theorem, no grid and no value in this paper ever rested on the corrected sentence.

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[7] C.-C. Kao, *Resultants of dynatomic polynomials of x^d*, arXiv:2608.27016v1. Every statement of [7] quoted or numbered here was read in that e-print, which on 2026-08-29 was still the only version; the numbering could of course move in a later version or on publication.

[8] E. T. Lehmer, *A numerical function applied to cyclotomy*, Bull. Amer. Math. Soc. **36** (1930), no. 4, 291–298, doi:10.1090/S0002-9904-1930-04939-3.

[9] S. Louboutin, *Resultants of cyclotomic polynomials*, Publ. Math. Debrecen **50** (1997), no. 1–2, 75–77, doi:10.5486/pmd.1997.1721.

[10] K. A. Loper and N. J. Werner, *Resultants of minimal polynomials of maximal real cyclotomic extensions*, J. Number Theory **158** (2016) 298–315, doi:10.1016/j.jnt.2015.06.002. The page range is Crossref's, confirmed by OpenAlex; the paper itself was read in the authors' own preprint, dated 22 May 2015, and the numbers Prop. 2.3, Thm. 3.4 and Thm. 4.2 cited here are that preprint's. We could not obtain the published text — the publisher blocks non-browser clients, there is no repository copy and no arXiv version — so those three numbers are unconfirmed for the published version.

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[12] The mathlib Community, *The Lean mathematical library*, in Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824. That paper describes the library, not the file cited here. The declarations cited here, all of them in the `Polynomial` namespace — `resultant`, `resultant_mul_left`, `resultant_mul_right`, `resultant_eq_prod_eval`, `discr`, `exists_mul_add_mul_eq_C_resultant` and `resultant_eq_zero_iff` — are in the library file `Mathlib/RingTheory/Polynomial/Resultant/Basic.lean`, added by commit `fcc863c1` of 2 July 2025 (pull request #26285); that is a commit date, and the first tagged release containing the file is `v4.22.0` of 14 August 2025. All of them were read here in commit `81a5d257` of 13 July 2026, the release `v4.32.0`, which is also the revision the two searches reported in Section 7 were run against.

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[14] P. Morton and P. Patel, *The Galois theory of periodic points of polynomial maps*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 225–263, doi:10.1112/plms/s3-68.2.225. Not read here; cited only for the polynomiality of (1), on the attribution given in [2, p. 543].

[15] P. Morton and F. Vivaldi, *Bifurcations and discriminants for polynomial maps*, Nonlinearity **8** (1995), no. 4, 571–584, doi:10.1088/0951-7715/8/4/006. Not obtained; see Remark 3.2.

[16] Y. Murakami, K. Sano and K. Takehira, *Arithmetic properties of multiplier polynomials for certain polynomial maps*, arXiv:2403.17315v3; published online in the International Journal of Number Theory on 25 August 2026, with the digital object identifier `10.1142/S1793042126501381`. It is cited here only for its restatement of (2), which is unchanged across the arXiv versions v1 to v3.

[17] C. Panraksa, D. Samart and S. Sriwongsa, *A dynamical system proof of Niven's theorem and its extensions*, arXiv:2304.07823v1; Bull. Aust. Math. Soc. **109** (2024), no. 1, 138–151, doi:10.1017/S0004972723000448. The number Theorem 7 is that of arXiv:2304.07823v1, read off its own compiled source, and may differ in the published version.

[18] C. Panraksa and A. Tangboonduangjit, *Fixed-point lifting and ghost periodic points for Chebyshev polynomials modulo odd prime powers*, arXiv:2605.04417v1.

[19] R. Wituła and D. Słota, *Fixed and periodic points of polynomials generated by minimal polynomials of* $2\cos(2\pi/n)$, Int. J. Bifurcation and Chaos **19** (2009), no. 9, 3005–3016, doi:10.1142/S021812740902461X; Zbl 1179.37031. Record and abstract from Crossref, the publisher page and zbMATH, which agree; the paper itself was not read, and zbMATH carries no review of it. See Remark 4.3. (The authors' own later reference lists give the title with $2\cos(\pi/n)$; the publisher, Crossref, dblp and zbMATH all have $2\cos(2\pi/n)$.)

[20] D. A. Wolfram, *Factoring variants of Chebyshev polynomials of the first and second kinds with minimal polynomials of* $\cos(2\pi/d)$, Amer. Math. Monthly **129** (2022), no. 2, 172–176, doi:10.1080/00029890.2022.2005391. Cited on the description of it given in the abstract of the author's sequel, arXiv:2106.14585.

[21] F. Vivaldi and S. Hatjispyros, *Galois theory of periodic orbits of rational maps*, Nonlinearity **5** (1992), no. 4, 961–978, doi:10.1088/0951-7715/5/4/007. Not obtained: OpenAlex and Semantic Scholar both report it openly accessible at the publisher, and the publisher answers with a bot check. Its §5.1 and §5.2 are cited here at second hand, from [7] and [17] respectively.

[22] J. H. Silverman, *The Arithmetic of Dynamical Systems*, Graduate Texts in Mathematics **241**, Springer, 2007. Consulted through its table of contents, its errata, its index entries for *Chebyshev polynomial*, *dynatomic* and *resultant*, and some thirty full-text searches of the published edition, rather than read in full; Proposition 6.8 is on p. 332, in §6.2, and its proof is set as Exercise 6.5.

[23] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`. Sequences A000048, A011946, A027375, A055034, A059499, A187360 and A216066; all data quoted here was retrieved on 2026-08-29.