

RECTANGULAR DIMONOIDS AS PAIRS OF PARTITION REFINEMENTS, AN ABELIAN COLLAPSE FOR g -DIMONOIDS, AND NEW ENTRIES IN THE ENUMERATION OF DIMONOIDS, g -DIMONOIDS AND DOPPELSEMGROUPS OF SMALL ORDER

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ABSTRACT. Three recent papers of V. M. Gavrylkiv count, up to isomorphism, the dimonoids, the g -dimonoids and the doppelsemigroups of small order, together with their commutative, abelian, rectangular and strong subclasses, and each closes with one problem asking for the next entries of its tables. We prove two structure theorems, settle a published disagreement about the three-element case, and fill twenty of the entries those problems ask for.

The first structure theorem describes the rectangular case completely. For a pair of rectangular semigroups $\dashv, \vdash$ on one set — rectangular meaning $x * y * z = x * z$ — Loday's axiom (D_1) holds if and only if the partition of the carrier into columns of $\vdash$ refines the partition into columns of $\dashv$, and (D_3) holds if and only if the partition into rows of $\dashv$ refines the partition into rows of $\vdash$. A rectangular g -dimonoid is therefore exactly a pair of rectangular semigroups with these two opposite refinements, and the sections that make up the rest of the data of a rectangular semigroup play no part: replacing $\vdash$ by any rectangular semigroup with the same rows and columns preserves the axioms. The second is that an *abelian g -dimonoid is automatically a dimonoid*, so that $\text{agdm}(n) = \text{adm}(n)$ for every n and two of the problems' clauses are one clause; this statement is also proved, independently and earlier, in a companion paper of ours, and the corresponding statements for dimonoids and for doppelsemigroups are due to A. V. Zhuchok and to Zhuchok and Knauer. We also record an arithmetic consequence of the known count of 3-nilpotent semigroups: the abelian layer, which looks thin for $n \leq 6$, is 79.8% of all semigroups of order 7 and 99.74% of those of order 8.

Two published classifications of the three-element doppelsemigroups disagree, one giving 75 classes in 41 commutative ones and 17 dual pairs and the later one giving 77 and 18. We certify the later count inside a proof assistant, directly from the axioms and over all 3^9 multiplication tables, together with the whole duality structure of that order: the 18 dual pairs listed one by one, the iso-dual classes being exactly the commutative ones, and 31 iso-opposite and 15 iso-cross-dual classes, two counts the sources do not print. We also prove, inside the proof assistant, that there are at least 1364 pairwise non-isomorphic rectangular dimonoids of order 7 — a lower bound matching the value $\text{rdm}(7) = 1364$ that our search returns.

By exhaustive computation we determine $\text{dm}(6) = 212\,193\,607$, $\text{gdm}(6) = 559\,338\,535$, $\text{ds}(6) = 212\,886\,176$, $\text{sds}(6) = 212\,396\,016$, $\text{adm}(7) = \text{agdm}(7) = 1\,299\,527$, $\text{ads}(7) = 1\,258\,518$, $\text{adm}(8) = \text{agdm}(8) = 3\,674\,424\,652$, $\text{ads}(8) = 3\,672\,523\,869$, $\text{rdm}(7) = 1364$, $\text{rgdm}(7) = 14\,710$, $\text{rds}(7) = 571$, $\text{cdm}(7) = 83\,580\,226$, $\text{cgdm}(7) = 242\,117\,995$, $\text{cds}(7) = 84\,998\,302$, and, two and three orders beyond the published tables, $\text{rdm}(8) = 3903$, $\text{rdm}(9) = 10\,599$, $\text{rgdm}(8) = 81\,905$, $\text{rgdm}(9) = 472\,836$. The structural results, the order-3 census and the lower bound for $\text{rdm}(7)$ are machine-checked in Lean 4; the computed values of Section 7 are not, and are labelled as such throughout.

1. INTRODUCTION

The objects. In his search for a class of linear algebras standing to Leibniz algebras as associative algebras stand to Lie algebras, Loday [11] separated the left and right multiplications of an associative algebra into two operations $\dashv$ and $\vdash$, and observed that if they satisfy

$$\begin{aligned} (D_1) \quad & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (D_2) \quad & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (D_3) \quad & (x \dashv y) \vdash z = x \vdash (y \vdash z), \end{aligned}$$

then $[x, y] = x \dashv y - y \vdash x$ satisfies the Leibniz identity (we follow the account of [3], whose axiom labels and notation we keep throughout). A *dimonoid* is accordingly a set D with two associative binary operations $\dashv, \vdash$ satisfying (D_1) , (D_2) , (D_3) . A *g-dimonoid* (Movsisyan, Davidov and Safaryan [12]) is a set with two associative operations satisfying (D_1) and (D_3) , that is, all the axioms of a dimonoid except possibly (D_2) . A *doppelsemigroup* [18] is a set with two associative operations that are interassociative:

$$(I) \quad (x \dashv y) \vdash z = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

(In [5] these two identities are themselves labelled (D_1) and (D_2) ; we write (I) for the pair, to keep them apart from Loday's axioms.) Each of the three notions generalises semigroups: any semigroup $(D, \dashv)$ gives the *trivial* structure $(D, \dashv, \dashv)$.

Four modifiers recur in all three settings. A structure $(D, \dashv, \vdash)$ is *commutative* if both $(D, \dashv)$ and $(D, \vdash)$ are commutative semigroups; *abelian* if $x \dashv y = y \vdash x$ for all $x, y \in D$; *rectangular* if both semigroups satisfy $x * y * z = x * z$, that is, every element is a middle unit; and a doppelsemigroup is *strong* if $x \dashv (y \vdash z) = x \vdash (y \dashv z)$. Throughout, “rectangular semigroup” has this meaning and is *not* synonymous with “rectangular band”: as [3] notes, a nontrivial null semigroup is a commutative rectangular semigroup with no identity element.

The tables, and the problems posed. Write $s(n)$, $cs(n)$, $rs(n)$ for the numbers of pairwise non-isomorphic semigroups, commutative semigroups and rectangular semigroups of order n , and

$$dm, cdm, adm, rdm; \quad gdm, cgdm, agdm, rgdm; \quad ds, cds, ads, sds, rds$$

for the corresponding counts of dimonoids, *g*-dimonoids and doppelsemigroups. Here “isomorphism” means a bijection φ with $\varphi(a \dashv b) = \varphi(a) \dashv \varphi(b)$ and $\varphi(a \vdash b) = \varphi(a) \vdash \varphi(b)$ — not anti-isomorphism, and the two operations are not interchangeable, so a structure and its swap are in general not isomorphic.

Three papers of V. M. Gavrylkiv [3, 4, 5] tabulate these counts for small orders and each closes with one problem. We quote the three problems verbatim.

Problem 1.1 ([3]). Determine the numbers $dm(n)$ for $n \geq 6$ and $cdm(n)$, $adm(n)$, $rdm(n)$ for $n \geq 7$.

Problem 1.2 ([4]). Determine the numbers $gdm(n)$ for $n \geq 6$ and $cgdm(n)$, $agdm(n)$, $rgdm(n)$ for $n \geq 7$.

Problem 1.3 ([5]). Determine the numbers $ds(n)$, $sds(n)$ for $n \geq 6$ and $cds(n)$, $ads(n)$, $rds(n)$ for $n \geq 7$.

Table 1 reproduces the published values. The three papers state them for $n \leq 5$ or $n \leq 6$; the semigroup counts s and cs are OEIS A027851 and A001426 and are known further. The rows come, in the numbering of the arXiv v1 sources (which we resolved by compiling their own $\LaTeX$, and which the published versions may change — for [5], whose published version we also read, they do not), from Tables 6–11 of [3] — s , cs , dm , cdm , adm , rdm , with rs printed alongside rdm in Table 11 — from Tables 4–7 of [4], and from Tables 3–7 of [5]. Note that the n -axis of the tables of [5] starts at $n = 0$ and those of the other two at $n = 1$.

What is settled here. Two rows of Table 1 are identical: adm and $agdm$ both read 1, 4, 17, 103, 791, 10870. The two papers print them independently, neither remarks on the coincidence, and each poses the continuation of its own row as a separate problem. Theorem 3.4 says that the coincidence is not a coincidence.

Theorem 1.4. *An abelian g-dimonoid is a dimonoid. The axiom (D_2) that a g-dimonoid is permitted to fail is a consequence of (D_1) , (D_3) and abelianness. Consequently $agdm(n) = adm(n)$ for every n , both as sets of structures on a fixed carrier and up to isomorphism.*

TABLE 1. The published values, as printed in [3, 4, 5]. A question mark marks the first entry the corresponding problem asks for.

n	1	2	3	4	5	6	7
$s(n)$	1	5	24	188	1915	28634	1627672
$cs(n)$	1	3	12	58	325	2143	17291
$rs(n)$	1	3	5	10	14	27	—
$dm(n)$	1	8	52	734	55883	?	
$cdm(n)$	1	3	14	101	1495	102268	?
$adm(n)$	1	4	17	103	791	10870	?
$rdm(n)$	1	6	18	62	166	509	?
$gdm(n)$	1	9	78	1693	162465	?	
$cgdm(n)$	1	4	22	189	3249	256253	?
$agdm(n)$	1	4	17	103	791	10870	?
$rgdm(n)$	1	7	27	128	555	2846	?
$ds(n)$	1	8	77	1217	68177	?	
$cds(n)$	1	6	41	345	3892	134462	?
$ads(n)$	1	3	12	62	446	7503	?
$sds(n)$	1	8	65	841	56428	?	
$rds(n)$	1	3	7	20	51	171	?

The proof, given in Section 3, is three lines once the substitution $\vdash = \dashv^d$ is made, and we do not present it as difficult. What it buys is that Problem 1.2’s clause on $agdm$ and Problem 1.1’s clause on adm are the same question, so that any answer to one answers the other; and the statement has content, because (D_2) is a genuine restriction on g -dimonoids in general: $gdm(3) = 78 > 52 = dm(3)$.

The companion statements for the other two varieties are known: the equivalence “abelian dimonoid $\iff$ right commutative semigroup” is Lemma 3 of A. V. Zhuchok [16] (stated there for a semigroup and its dual, and quoted in the “abelian dimonoid” form by Gavrylkiv [6]), and “abelian doppelsemigroup $\iff$ left and right commutative semigroup” is Lemma 2.1 and Corollary 2.2 of Zhuchok and Knauer [19]. Section 3 states all three uniformly, as they are what turns each abelian count into a count of semigroups in a one-operation identity variety, and that is what makes the computations of Section 7 feasible.

Theorem 1.4 is also proved, independently and slightly earlier, in a companion paper of ours [1], where it is deduced from the observation that under the abelian law each of (D_1) , (D_2) , (D_3) unwinds to the single identity $(xy)z = (xz)y$. We claim no priority for it here, and retain it because it is what the abelian computations of Section 7 rest on; see Remark 3.10.

A second structure theorem, new here, settles the rectangular case outright. Call u and v *column-equivalent* for an operation $*$ if $x * u = x * v$ for every x , and *row-equivalent* if $u * z = v * z$ for every z ; these two partitions of the carrier are the two coordinates of the classical structure theory of rectangular semigroups (Section 4).

Theorem 1.5. *Let $\dashv$ and $\vdash$ be associative rectangular operations on a set D . Then (D_1) holds if and only if every column class of $\vdash$ is contained in a column class of $\dashv$, and (D_3) holds if and only if every row class of $\dashv$ is contained in a row class of $\vdash$. Consequently $(D, \dashv, \vdash)$ is a rectangular g -dimonoid if and only if those two opposite refinements hold, and a rectangular dimonoid if and only if in addition (D_2) holds.*

The criterion mentions only the two partitions, and never the sections — the choice of one representative in each cell of the underlying band, which is the rest of the data of a rectangular semigroup — so replacing $\vdash$ by *any* rectangular semigroup with the same rows and columns preserves the g -dimonoid property (Corollary 4.5). This is what makes the rectangular column of the tables purely combinatorial, and it is what our rectangular computations use.

Section 5 settles a disagreement between two published classifications of the three-element doppelsemigroups. Gavrylkiv and Rendziak [7] state that there are 75 of them, 41 commutative and the rest in 17 dual pairs; the later [5] states 77, 41 and 18. Working from the axioms over all 3^9 multiplication tables inside the proof assistant, we certify the later count and, with it, the entire duality structure of that order: the eighteen dual pairs listed one by one (Table 3), the fact that the iso-dual classes are exactly the 41 commutative ones, and the counts 31 of iso-opposite and 15 of iso-cross-dual classes, which neither source prints.

Section 8 proves inside the proof assistant that $\text{rdm}(7) \geq 1364$: it exhibits 1364 structures on a seven-element set, each certified to be a rectangular dimonoid and no two of them isomorphic. The matching upper bound is not proved; 1364 is the value our search returns for $\text{rdm}(7)$, so the two halves agree but only one of them is machine-checked.

Section 6 draws an arithmetic consequence that none of the three papers records. Every semigroup all of whose triple products are equal is left and right commutative, hence carries an abelian dimonoid and an abelian doppelsemigroup. With Distler and Mitchell’s exact count of the nilpotent semigroups of degree 3 [2] this gives, unconditionally,

$$105\,931\,872\,028\,455 \leq \text{ads}(9) \leq \text{adm}(9) = \text{agdm}(9) \leq 105\,978\,177\,936\,292,$$

an interval of relative width 0.0437%, and it gives the ratios $\text{adm}(7)/s(7) = 0.7984$ and $\text{adm}(8)/s(8) = 0.99739$. So the abelian layer, which is a small fraction of all semigroups for $n \leq 6$ (10870 of 28634 at $n = 6$) and which none of the three papers compares with $s(n)$, is in fact almost everything. The limiting statement $\text{adm}(n)/s(n) \rightarrow 1$ is *conditional*: it follows from the asymptotic that Distler and Mitchell call presumed and explicitly say is not a consequence of Kleitman, Rothschild and Spencer [9]. Only the arithmetic for $n \leq 9$ above is unconditional.

Section 7 reports the computations. They fill twenty of the entries the three problems ask for — eighteen distinct integers, since $\text{adm} = \text{agdm}$ — and carry the rectangular sequences two orders past the point where the problems begin. Table 2 collects them.

TABLE 2. The entries determined here, all by exhaustive computation and none of them machine-checked. Each is an entry one of the three problems asks for; all but $\text{adm}(8) = \text{agdm}(8)$, $\text{ads}(8)$, $\text{rdm}(8)$, $\text{rdm}(9)$, $\text{rgdm}(8)$ and $\text{rgdm}(9)$ are the first such entry of their sequence.

$\text{dm}(6)$	212 193 607	$\text{cdm}(7)$	83 580 226
$\text{gdm}(6)$	559 338 535	$\text{cgdm}(7)$	242 117 995
$\text{ds}(6)$	212 886 176	$\text{cds}(7)$	84 998 302
$\text{sds}(6)$	212 396 016	$\text{rdm}(7)$	1364
$\text{adm}(7) = \text{agdm}(7)$	1 299 527	$\text{rdm}(8)$	3903
$\text{ads}(7)$	1 258 518	$\text{rdm}(9)$	10 599
$\text{adm}(8) = \text{agdm}(8)$	3 674 424 652	$\text{rgdm}(7)$	14 710
$\text{ads}(8)$	3 672 523 869	$\text{rgdm}(8)$	81 905
		$\text{rgdm}(9)$	472 836
		$\text{rds}(7)$	571

What is not claimed. $\text{dm}(7)$ and every cell beyond it are untouched, as is the exact value of $\text{adm}(9)$; Section 9 says what each would cost. The values of Section 7 rest on exhaustive search by unverified programs, cross-validated as described there but not machine-checked; they are stated as computations, not as theorems, and are excluded from the scope of Section 10. In particular the equality $\text{rdm}(7) = 1364$ is a computation; what Section 8 proves is the inequality $\text{rdm}(7) \geq 1364$. The three commutative cells are the least redundantly checked of the values reported here: Remark 7.7 says exactly which check is missing, and publishes the partial counts that would let it be completed.

Provenance, and two caveats. We read [3, 4, 5] from their arXiv sources, retrieved on 3 September 2026, including the Cayley-table files shipped inside the source package of [5], which we used as an independent check (Section 7). Two caveats.

First, [5] cites a submitted, unpublished paper of the same author, *On certain classes of non-isomorphic rectangular doppelsemigroups*, which we were unable to obtain: it is not on arXiv and we did not find it published. As described by the citing paper, its content is a classification of commutative rectangular doppelsemigroups up to isomorphism together with counts of rectangular ideal extensions of left (right) zero doppelsemigroups by null quotients — a different quantity from $\text{rds}(n)$ — and the citing paper, which is later, still poses “determine $\text{rds}(n)$ for $n \geq 7$ ” as open. The openness of the rectangular cells we compute therefore rests on the sources’ own problems, but our search of the literature could not be completed against that text. The related published paper [8] concerns rectangular ideal extensions of left zero *semigroups* by null semigroups, again a different quantity.

Second, we searched the On-Line Encyclopedia of Integer Sequences for each of the sequences we extend and found no entry for any of them, and no entry counting right commutative semigroups. We did find, after computing it three ways ourselves, that $\text{rs}(n)$ — tabulated only to $n = 6$ in [3, 4, 5] and there without an OEIS reference — is A390162, whose b -file reaches $n = 200$; see Section 4. A negative OEIS search is evidence of a kind, not a proof of novelty, and we report it as such.

Changes from the first version. Nothing stated in the first version of this paper is withdrawn or altered: every numbered statement there keeps its number and its wording, and every value it reports stands. What is added is the following. The rectangular refinement theorem (Theorem 1.5 = Theorem 4.3) and its consequences, with the negative controls of Proposition 4.6, are new; the first version computed the rectangular cells from the classification of Section 4 and observed that the axiom tests then simplify, and Theorem 4.3 is the reason why. The order-3 duality census (Theorem 5.3, Table 3, Proposition 5.4) is new; the first version already settled $\text{ds}(3) = 77$ against the published 75, and what is added is the whole involution structure behind that number, including the eighteen dual pairs themselves. Section 8, the machine-checked lower bound $\text{rdm}(7) \geq 1364$, is new, and with it the first version’s closing suggestion that a rectangular cell was the cheapest candidate for formal verification is carried out for the lower half. Computations 7.6 and 7.8 are new: the three commutative cells $\text{cdm}(7)$, $\text{cgdm}(7)$, $\text{cds}(7)$, which the first version explicitly did not touch, and $\text{rdm}(8)$, $\text{rdm}(9)$, $\text{rgdm}(8)$, $\text{rgdm}(9)$, which go two and three orders beyond the last published entries of their sequences. Sections 9 and 10 are rewritten around what is now left and around the enlarged formal development. Remarks 3.10 and 7.9 record the overlap with a companion paper of ours that the first version did not cite: it contains Theorem 1.4 independently and earlier, and it obtains $\text{rds}(7) = 571$ by an entirely different route. The title has been extended to name the new structure theorem.

2. PRELIMINARIES

Let D be a set and $*$ a binary operation on it. The *dual* operation is $x *^d y = y * x$. We use throughout the two identities

- (A) $x * (y * z) = x * (z * y),$
- (B) $(x * y) * z = (y * x) * z.$

A semigroup satisfying (A) is called *right commutative*, and one satisfying (B) *left commutative*, following [19] (whose identities are written $axy = ayx$ and $xya = yxa$). A commutative semigroup satisfies both.

The counts s , cs are OEIS A027851 and A001426; the number of associative operations on a labelled 3-set is A023814(3) = 113. We shall use two further published enumerations. Distler and Mitchell [2] give exact formulae, and tables, for the number of nilpotent semigroups of degree 3 — semigroups with a zero in which every product of three elements is the zero, and some product of two

elements is not — up to equality, up to isomorphism, and up to isomorphism or anti-isomorphism. Their values up to isomorphism at $n = 7, 8, 9$ are 1 199 989, 3 661 522 792 and 105 931 872 028 455; up to isomorphism or anti-isomorphism at $n = 7, 8$ they are 609 797 and 1 831 687 022; and their self-dual counts at $n = 7, 8$ are 19 605 and 1 851 252. The GAP package SMALLSEMI [15] stores one representative of each isomorphism-or-anti-isomorphism class of semigroup of order at most 8: table by table for $n \leq 7$, and at $n = 8$ table by table for the 11 433 106 classes that are not nilpotent of degree 3, the remaining 1 831 687 022 — the degree-3 nilpotent block, whose size is [2] — being held in a compressed form that we never expanded.

3. THE ABELIAN COLLAPSE

Abelianness is not an extra axiom relating two independent operations; it pins the second operation to the first.

Proposition 3.1. *Let $\dashv$ and $\vdash$ be binary operations on a set D . Then $(D, \dashv, \vdash)$ is abelian if and only if $\vdash = \dashv^d$. Moreover, let $*$ be a binary operation on a set E and $\varphi : D \rightarrow E$ a map with $\varphi(x \dashv y) = \varphi(x) * \varphi(y)$ for all $x, y \in D$; then $\varphi(x \dashv^d y) = \varphi(x) *^d \varphi(y)$ for all $x, y \in D$.*

Both halves are immediate. The first is the equivalence “abelian $\iff$ the two semigroups are dual” recorded in each of [3, 4, 5]; the second says that the assignment $\dashv \mapsto (\dashv, \dashv^d)$ is equivariant under relabelling, which is what allows the counts below to be transported to isomorphism classes.

Proposition 3.2. *Let $\dashv$ be a binary operation on D . Then $(D, \dashv, \dashv^d)$ is a dimonoid if and only if $(D, \dashv)$ is a semigroup satisfying (A).*

Proposition 3.3. *Let $\dashv$ be a binary operation on D . Then $(D, \dashv, \dashv^d)$ is a g -dimonoid if and only if $(D, \dashv)$ is a semigroup satisfying (A) — the same condition as in Proposition 3.2.*

Proof of Propositions 3.2 and 3.3. Write $*$ for $\dashv$, so that $x \vdash y = y * x$; the operation $*^d$ is associative exactly when $*$ is. Substituting $\vdash = *^d$ throughout, the three dimonoid axioms read

$$(D_1) : (x * y) * z = x * (z * y), \quad (D_2) : (y * x) * z = (y * z) * x, \quad (D_3) : z * (x * y) = (z * y) * x.$$

Suppose $*$ is associative. Then each of the three is equivalent to (A): rewriting $(x * y) * z$ as $x * (y * z)$ turns (D_1) into $x * (y * z) = x * (z * y)$; rewriting both sides of (D_2) gives $y * (x * z) = y * (z * x)$; and rewriting the right-hand side of (D_3) gives $z * (x * y) = z * (y * x)$.

Now let $(D, *, *^d)$ be a g -dimonoid. Then $*$ is associative and (D_1) holds, hence (A) holds. Conversely, if $(D, *)$ is a semigroup satisfying (A), then $*^d$ is associative and all three axioms hold, so $(D, *, *^d)$ is a dimonoid, and in particular a g -dimonoid. Both propositions follow, the forward direction of each having used only associativity and (D_1) . $\square$

Theorem 3.4. *For binary operations $\dashv, \vdash$ on a set D ,*

$$((D, \dashv, \vdash) \text{ is a } g\text{-dimonoid and abelian}) \iff ((D, \dashv, \vdash) \text{ is a dimonoid and abelian}).$$

In particular an abelian g -dimonoid is a dimonoid, and $\text{agdm}(n) = \text{adm}(n)$ for every n .

Proof. By Proposition 3.1 an abelian structure has $\vdash = \dashv^d$, and then Propositions 3.2 and 3.3 give the same condition on $\dashv$. The two classes of abelian structures on D therefore coincide as sets, and a bijection of carriers carries one to the other, so the counts of isomorphism classes agree. $\square$

The hypothesis of abelianness is doing work: (D_2) is a genuine restriction, since $\text{dm}(3) = 52 < 78 = \text{gdm}(3)$, and [4] exhibits rectangular g -dimonoids that are not dimonoids.

Proposition 3.5. *Let $\dashv$ be a binary operation on D . Then $(D, \dashv, \dashv^d)$ is a doppelsemigroup if and only if $(D, \dashv)$ is a semigroup satisfying both (A) and (B).*

Propositions 3.2 and 3.5 are published. Lemma 3 of [16] reads, verbatim,

Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ be a dual semigroup to $(D, \dashv)$. Then an algebra $(D, \dashv, \vdash)$ is a dimonoid if and only if $(D, \dashv)$ is a right commutative semigroup,

which is Proposition 3.2; and Lemma 2.1 of [19] reads

Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ the dual semigroup to $(D, \dashv)$. Then $(D, \dashv, \vdash)$ is an abelian doppelsemigroup if and only if $(D, \dashv)$ or $(D, \vdash)$ is a left and right commutative semigroup,

whose “constructed in this way” converse is their Corollary 2.2; together these are Proposition 3.5. We restate them here because we need them in this shape, and because the g -dimonoid case (Proposition 3.3) has, to our knowledge, not been recorded.

Proposition 3.6. *For binary operations $\dashv, \vdash$ on D :*

- (1) $(D, \dashv, \vdash)$ is an abelian dimonoid if and only if $\vdash = \dashv^d$ and $(D, \dashv)$ is a semigroup satisfying (A);
- (2) $(D, \dashv, \vdash)$ is an abelian doppelsemigroup if and only if $\vdash = \dashv^d$ and $(D, \dashv)$ is a semigroup satisfying (A) and (B).

Corollary 3.7. *For every n ,*

$$\begin{aligned} \text{adm}(n) &= \text{agdm}(n) = \#\{\text{semigroups of order } n \text{ satisfying (A)}\} / \cong, \\ \text{ads}(n) &= \#\{\text{semigroups of order } n \text{ satisfying (A) and (B)}\} / \cong. \end{aligned}$$

Proof. Proposition 3.6 is a bijection between the abelian structures on D and the operations $\dashv$ in the stated identity variety, and by the second half of Proposition 3.1 it commutes with relabelling; so it descends to isomorphism classes. Theorem 3.4 supplies the first equality. $\square$

Corollary 3.7 is what makes the enumerations of Section 7 possible: a two-operation search over n^{2n^2} candidates becomes a one-operation filter over a library of semigroups. Its numerical content is checked against the two-operation definitions at $n = 3$ in Section 5.

Proposition 3.8. *Every abelian doppelsemigroup is an abelian dimonoid; hence $\text{ads}(n) \leq \text{adm}(n)$ for every n .*

Proposition 3.9. *If $*$ is associative and satisfies (A), then $(x * y) * z = (x * z) * y$ for all x, y, z . In particular this holds in the underlying semigroup of an abelian dimonoid.*

The first assertion of Proposition 3.8 is Proposition 6.1 of [19], “Every abelian doppelsemigroup is an abelian dimonoid. The converse statement is not true in general”; we record it here only because it is what puts ads and adm on the same list of semigroup identities. The inclusion is strict for $n \geq 2$, by the same witness those authors use: the left-zero operation $x * y = x$ satisfies (A) but not (B) on a two-element set, so the left-zero and right-zero dimonoid is an abelian dimonoid that is not an abelian doppelsemigroup. Numerically, $\text{ads}(3) = 12 < 17 = \text{adm}(3)$. The identity of Proposition 3.9 is the case $n = 2$ of the permutation identity numbered (3) in [19], which is there deduced for right commutative semigroups from (A).

Remark 3.10 (a companion paper). Theorem 3.4 is proved independently, and slightly earlier, in our companion paper [1], in the equivalent form that under the abelian law each of (D_1) , (D_2) , (D_3) unwinds, modulo associativity alone, to the single identity $(xy)z = (xz)y$ — the identity of Proposition 3.9, which for a semigroup is (A) rewritten. The two arguments are the same argument. We claim no priority for this theorem here; it is restated because Corollary 3.7, and with it every abelian computation of Section 7, depends on it.

4. RECTANGULAR SEMIGROUPS, AND RECTANGULAR DIMONOIDS

Proposition 4.1. *Let $*$ be a binary operation on S . Then $*$ is associative and satisfies $x*y*z = x*z$ if and only if it satisfies both one-sided identities*

$$(x * y) * z = x * z \quad \text{and} \quad x * (y * z) = x * z.$$

*In that case $x * y = (x * x) * (y * y)$ for all x, y , and $x * x$ is idempotent for every x .*

Proposition 4.1 is elementary but it is what makes rectangular semigroups cheap to enumerate: the product factors through the diagonal map $x \mapsto x * x$, whose image is a rectangular band $L \times R$. Writing λ and ρ for the two coordinates of that band, one has $x * y = e(\lambda x, \rho y)$ where $e : L \times R \rightarrow S$ is the inverse of the coordinate bijection on that image. Both λ and ρ are surjective, and each of the $|L| |R|$ fibres of $x \mapsto (\lambda x, \rho x)$ is non-empty, since $e(l, r)$ lies in the fibre over (l, r) . Consequently the isomorphism classes of rectangular semigroups of order n are exactly the $p \times q$ matrices of *positive* integers of total sum n , up to permutation of rows and of columns.

This classification is published. OEIS A390162, “Number of nonisomorphic semigroups with n elements satisfying the equation $xyz = xz$ ” (A. Howroyd, 14 November 2025), has terms 1, 1, 3, 5, 10, 14, 27, 36, ... (offset 0) and a *b*-file reaching $n = 200$, and carries the comment “Also, $a(n)$ is the number of nonisomorphic matrices with positive integer entries whose sum is n . Isomorphism is up to reordering of rows and columns”. The labelled version is A389829 (D. Pasino, same date), 1, 1, 4, 17, 84, 507, 3668, 31117, ...; it is defined there by associativity together with $x(yz) = xz$, which by Proposition 4.1 is the same condition as rectangularity. A construction of all semigroups whose elements are middle units is due to A. Nagy and O. Nagy [13], cited for that purpose in [3, 4]. We record here only that the values $rs(n)$ printed to $n = 6$ in [3, 4, 5] continue as A390162, so that in particular

$$rs(7) = 36,$$

which we computed three independent ways — from the matrix formula by hand, from the SMALL-SEMI census, and from a direct enumeration of the structures — before locating it in the literature. This is not a new result; we state it because the source papers tabulate rs only to $n = 6$ and give no OEIS reference, and because the same classification is what makes the rectangular cells of Section 7 inexpensive.

Rectangular dimonoids are pairs of partition refinements. We now describe which *pairs* of rectangular semigroups on one carrier satisfy Loday’s axioms. The answer is that the axioms see only the two partitions of Proposition 4.1, and see them in opposite directions.

Definition 4.2. Let $*$ be a binary operation on D and $u, v \in D$. Say that u and v are *column-equivalent* for $*$, written $u \sim_*^c v$, if $x * u = x * v$ for every $x \in D$, and *row-equivalent*, written $u \sim_*^r v$, if $u * z = v * z$ for every $z \in D$. Both are equivalence relations. For two operations $*$ and $\star$ we say that the column partition of $\star$ *refines* that of $*$ if $u \sim_*^c v$ implies $u \sim_\star^c v$, and similarly for rows.

For a rectangular semigroup written as $x * y = e(\lambda x, \rho y)$ in the notation following Proposition 4.1, with e injective, one has $u \sim_*^c v$ exactly when $\rho u = \rho v$ and $u \sim_*^r v$ exactly when $\lambda u = \lambda v$: the two equivalences are the fibres of the two band coordinates, so the $p \times q$ matrix classifying the semigroup records nothing but the sizes of the blocks of these two partitions.

Theorem 4.3. *Let $\dashv$ and $\vdash$ be associative rectangular operations on a set D . Then*

$$\begin{aligned} (x \dashv y) \dashv z &= x \dashv (y \dashv z) \quad \text{for all } x, y, z \iff \text{the column partition of } \vdash \text{ refines that of } \dashv, \\ (x \dashv y) \vdash z &= x \vdash (y \vdash z) \quad \text{for all } x, y, z \iff \text{the row partition of } \dashv \text{ refines that of } \vdash. \end{aligned}$$

Neither equivalence uses the associativity of $\dashv$: in the formal development that hypothesis is carried by both statements and used in neither proof. Associativity of $\vdash$ is used in both, through the second one-sided identity of Proposition 4.1.

Proof. Associativity and $x * y * z = x * z$ give the second one-sided identity $x * (y * z) = x * z$ of Proposition 4.1; we use it for $\vdash$, and the first one-sided identity $(x * y) * z = x * z$ — which for $\dashv$ is rectangularity read as it stands — for $\dashv$.

(D_1) , *forwards*. Assume (D_1) . Since $(x \dashv y) \dashv z = x \dashv z$, it reads $x \dashv z = x \dashv (y \vdash z)$ for all x, y, z . Let $u \sim_{\vdash}^c v$ and let $x \in D$. Then $x \dashv u = x \dashv (x \vdash u) = x \dashv (x \vdash v) = x \dashv v$, the outer two equalities being the displayed identity and the middle one the hypothesis. So $u \sim_{\dashv}^c v$.

(D_1) , *backwards*. For all y, z one has $x \vdash (y \vdash z) = x \vdash z$ for every x , that is, $z \sim_{\vdash}^c (y \vdash z)$. By the refinement hypothesis $z \sim_{\dashv}^c (y \vdash z)$, so $(x \dashv y) \dashv z = x \dashv z = x \dashv (y \vdash z)$.

(D_3) , *forwards*. Assume (D_3) . Since $x \vdash (y \vdash z) = x \vdash z$, it reads $(x \dashv y) \vdash z = x \vdash z$ for all x, y, z . Let $u \sim_{\dashv}^r v$ and let $z \in D$. Then $u \vdash z = (u \dashv u) \vdash z = (v \dashv u) \vdash z = v \vdash z$, using the displayed identity twice and the hypothesis once. So $u \sim_{\vdash}^r v$.

(D_3) , *backwards*. Rectangularity of $\dashv$ gives $(x \dashv y) \dashv w = x \dashv w$ for every w , that is, $(x \dashv y) \sim_{\dashv}^r x$. By the refinement hypothesis $(x \dashv y) \sim_{\vdash}^r x$, so $(x \dashv y) \vdash z = x \vdash z = x \vdash (y \vdash z)$. $\square$

Corollary 4.4. *A pair of associative rectangular operations $(\dashv, \vdash)$ on D is a g -dimonoid if and only if the column partition of $\vdash$ refines that of $\dashv$ and the row partition of $\dashv$ refines that of $\vdash$; and it is a dimonoid if and only if, in addition, (D_2) holds. In particular, for any associative rectangular $\dashv$ the pair $(\dashv, \dashv)$ is a rectangular dimonoid, so $\text{rs}(n) \leq \text{rdm}(n)$ for every n .*

Corollary 4.5 (the sections do not matter). *Let $(\dashv, \vdash)$ be a rectangular g -dimonoid and let $\vdash'$ be any associative rectangular operation on D with the same column classes and the same row classes as $\vdash$. Then $(\dashv, \vdash')$ is a rectangular g -dimonoid.*

A rectangular semigroup consists of two partitions of the carrier together with a section choosing one element in each cell of the resulting grid (Section 4). Corollary 4.5 says that the g -dimonoid axioms constrain the partitions of $\vdash$ and never the section: the rectangular column of the tables of [3, 4, 5] is a count of pairs of partitions with sections attached, in which only the partitions interact. It is also what makes the search of Computation 7.8 cheap: the $O(n^3)$ axiom test of the definition becomes an $O(n^2)$ comparison of two partitions.

The hypotheses of Theorem 4.3 are all used, and its conclusions are not vacuous.

Proposition 4.6. *On the two-element carrier $\{0, 1\}$:*

- (1) *the pair (constant 0, constant 1) is a rectangular g -dimonoid that is not a dimonoid, so (D_2) is a genuine extra condition on rectangular pairs;*
- (2) *the pair (constant 0, left zero) satisfies (D_1) but not (D_3) , and the pair (right zero, constant 0) satisfies (D_3) but not (D_1) , so neither refinement implies the other;*
- (3) *with $\dashv$ the right-zero operation and $\vdash$ the (associative, non-rectangular) meet $x \wedge y$, the column partition of $\vdash$ refines that of $\dashv$ while (D_1) fails; and with $\dashv = \wedge$ and $\vdash$ the left-zero operation, the row partition of $\dashv$ refines that of $\vdash$ while (D_3) fails. So rectangularity of the partner operation cannot be dropped from either equivalence.*

Remark 4.7 (how this was found, and what it is not). The equivalences of Theorem 4.3 were first tested over *every* pair of labelled rectangular semigroups of orders 3, 4 and 5 — 17^2 , 84^2 and 507^2 pairs — with zero mismatches on each of the three statements “ (D_1) iff the column refinement”, “ (D_3) iff the row refinement” and “ g -dimonoid iff both”, and only then proved. We did not find the statement in the literature. A full-text search of a large local mirror of the arXiv found the phrase “rectangular dimonoid” only in papers of the same author; [3] names three rectangular dimonoids as examples and cites, for structural information about them, A. V. Zhuchok’s construction of the free rectangular diband [17], which is a description of a free object and not a criterion for a pair of given operations. A structure theorem for rectangular *doppelsemigroups* is proved in our companion paper [1]: there both operations must be inflations of one and the same band. That is a different statement, and neither implies the other: for dimonoids the two operations need not have the same image, and at $n = 4$ they do not in 800 of the 933 labelled rectangular dimonoid pairs, so the two criteria really do cut out different classes of pairs.

5. CERTIFICATION OF THE ORDER-3 ROW

Every count in Table 1 at $n = 3$, and both of the collapse counts of Corollary 3.7 at $n = 3$, can be obtained by evaluating the axioms over all $3^9 = 19683$ multiplication tables on a three-element set. We did so inside the proof assistant, from the transcribed axioms and nothing else.

Theorem 5.1. *Among the 19683 binary operations on $\{0, 1, 2\}$ exactly 113 are associative, and, counted up to isomorphism,*

$$\begin{aligned} s(3) &= 24, & cs(3) &= 12, & rs(3) &= 5, \\ dm(3) &= 52, & cdm(3) &= 14, & adm(3) &= 17, & rdm(3) &= 18, \\ gdm(3) &= 78, & cgdm(3) &= 22, & agdm(3) &= 17, & rgdm(3) &= 27, \\ ds(3) &= 77, & cds(3) &= 41, & ads(3) &= 12, & sds(3) &= 65, & rds(3) &= 7. \end{aligned}$$

Theorem 5.2. *Counting instead isomorphism classes of order-3 semigroups in the identity varieties of Corollary 3.7 gives 17 for (A) and 12 for $(A) \wedge (B)$; these agree with the two-operation counts $adm(3) = agdm(3) = 17$ and $ads(3) = 12$ of Theorem 5.1. Moreover $dm(3) < gdm(3)$ and $ads(3) < adm(3)$.*

Every value in Theorem 5.1 is published — the order-3 columns of the tables of [3, 4, 5], together with $A027851(3) = 24$, $A001426(3) = 12$ and $A023814(3) = 113$ — so the content of these two statements is not the numbers but the certification: they are derived inside the kernel from the axioms as transcribed, with no appeal to the sources' computations. Isomorphism classes are counted by canonical form: a table is coded as an integer below 3^9 , and the code of a class is the least code obtained from it by the six relabellings of $\{0, 1, 2\}$.

One of these entries is not merely a reproduction. Theorem 4.1 of Gavrylkiv and Rendziak [7] reads “There exist 75 pairwise non-isomorphic three-element doppelsemigroups among which 41 doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into 17 pairs of dual doppelsemigroups.” The later [5] states the same theorem with 77 and 18 pairs, and its abstract says that it refines that classification; the two published values differ by exactly one dual pair. Theorem 5.1 settles the disagreement in the kernel, in favour of the later count: $ds(3) = 77$ with $cds(3) = 41$, and $77 = 41 + 2 \cdot 18$.

Both statements are accompanied by falsifiability controls, also machine-checked: the census returns neither one more nor one less than the published value in three separate cells, six controls in all ($dm(3) \neq 51, 53$; $adm(3) \neq 16, 18$; $rds(3) \neq 6, 8$); the strict inequality $dm(3) < gdm(3)$ shows that (D_2) is a real restriction, so that Theorem 3.4 is not vacuous; and $ads(3) < adm(3)$ shows that (B) is a real restriction on top of (A) .

The duality structure at order 3. The disagreement just described is about a decomposition, not only about a total, so we certify the decomposition. Following [5, §2], a doppelsemigroup $(D, \neg, \vdash)$ has an *opposite* $(D, \vdash, \neg)$, a *dual* $(D, \neg^d, \vdash^d)$ and a *cross-dual* $(D, \vdash^d, \neg^d)$, each again a doppelsemigroup, and it is called *iso-opposite*, *iso-dual* or *iso-cross-dual* when it is isomorphic to the corresponding one. That paper is explicit that these are strictly weaker than the “self-” notions in general, and its Proposition 2.5 shows that self-cross-duality is precisely abelianness. We cite that paper by the numbering of its arXiv v1 source; the published version rennumbers its environments consecutively by kind, so that Proposition 2.5 is there Proposition 3, Example 5.1 is Example 1 and Theorem 6.1 is Theorem 2, while the section numbers and the numbering of the tables are the same in both.

Theorem 5.3. *Among the isomorphism classes of three-element doppelsemigroups:*

- (1) *there are 77 of them, of which 41 are commutative, 24 are trivial, 65 are strong, 12 are abelian and 7 are rectangular;*
- (2) *the iso-dual classes are exactly the 41 commutative ones, and duality permutes the remaining 36 in 18 two-element orbits, listed in Table 3;*
- (3) *hence $77 = 41 + 2 \cdot 18$, and $41 + 2 \cdot 17 \neq 77$;*

- (4) *the 7 rectangular classes are one commutative trivial class, four noncommutative trivial ones forming two dual pairs, and two nontrivial ones forming one dual pair, both iso-opposite.*

Theorem 6.1 of [5] — Theorem 2 of its published version — states the totals 77, 41 commutative, 18 dual pairs, 65 strong, 24 trivial and 7 rectangular with exactly the split of part (iv); the abelian count 12 is $\text{ads}(3)$ of Theorem 5.1. What is added here is that all of it is derived inside the proof assistant from the axioms alone, over all 3^9 multiplication tables; that the eighteen orbits are produced explicitly rather than counted; and the first clause of part (ii), which neither source states. The list is certified to be a list of orbits: every code in it is checked to be a canonical doppelsemigroup pair, duality is checked to map the class list into itself and to be an involution on it, it is checked to have no fixed point among the 36 noncommutative classes, and the eighteen listed pairs are checked to exhaust them. Four falsifiability controls accompany the statement: the class list has no repetitions, so that 77 counts classes and not representatives; it does not have 75 entries; and the number of dual pairs is neither 17 nor 19.

In Table 3 a 3×3 multiplication table t on $\{0, 1, 2\}$ is written as the nine digits $t(0, 0)$, $t(0, 1)$, ..., $t(2, 2)$ in that order, and is coded by the integer

$$\sum_{x,y} t(x, y) 3^{3x+y} < 3^9,$$

a pair of tables being coded by $a \cdot 19683 + b$, where a and b are the codes of $\dashv$ and of $\vdash$. The code of an isomorphism class is the least code of any of its members under the six relabellings of $\{0, 1, 2\}$, so a class *is* its code and the table below is a complete list of representatives. The dual of the listed representative is the class of the pair of transposed tables; its code is given, and the reader can recover its tables by transposing and re-minimising.

TABLE 3. The 18 dual pairs of noncommutative three-element doppelsemigroups. For each pair one representative is printed, with the code of its dual; t, s, r, o mark the trivial, strong, rectangular and iso-opposite classes (the four properties are invariant under duality).

#	code	$\dashv$	$\vdash$	dual code	t	s	r	o
1	567	000000000	000012000	4455		•		
2	15876	000000000	000012012	17820		•		
3	6909084	000111000	000111000	44702364	•	•	•	•
4	6918561	000111000	000111111	44709177			•	•
5	6922206	000111000	000111002	44715486		•		
6	6924393	000111000	000111012	44715729				
7	11160261	000012000	000000000	87687765		•		
8	11160828	000012000	000012000	87692220	•	•		•
9	11175570	000012000	000000012	87701130				•
10	11176137	000012000	000012012	87705585				
11	265189410	000111002	000111000	269808507		•		
12	265202532	000111002	000111002	269808588	•	•		•
13	307765659	010011012	010010010	308236131				
14	307779024	010011012	010011012	308251440	•	•		•
15	308658964	012111012	111111111	337002484		•		
16	308658972	012111012	012111111	337003212				
17	308664804	012111012	012111012	337009764	•	•		•
18	312916548	012012012	012012012	379999620	•	•	•	•

The table is a check on the arithmetic of Theorem 5.3 that a reader can carry out by hand. Six of the eighteen pairs are trivial, so the trivial classes number $12 + 2 \cdot 6 = 24$, the twelve commutative trivial ones being the twelve commutative semigroups of order 3; three of the pairs are rectangular, so the rectangular classes number $1 + 2 \cdot 3 = 7$; and eight of the pairs are iso-opposite, so with the fifteen commutative iso-opposite classes there are 31 in all.

Proposition 5.4. *At order 3 there are 31 iso-opposite classes against 24 trivial ones, and 15 iso-cross-dual classes against 12 abelian ones. In particular exactly three three-element doppelsemigroups are iso-cross-dual without being abelian.*

Neither count is printed by [5], but both follow in one step from the same order-3 classification — whose complete list of representatives that paper ships with its own source package — so we record them as a computation rather than as a result. The two strict inequalities $24 < 31$ and $12 < 15$, and the fact that every trivial class is iso-opposite, are certified alongside the counts; that the 12 abelian classes are among the 15 iso-cross-dual ones, which is what turns $15 - 12$ into the last clause of Proposition 5.4, is immediate from Proposition 3.1 and is not part of the certification. They locate two of that paper’s examples. Its Example 5.1 constructs, from a noncommutative group of even order, a doppelsemigroup that is iso-cross-dual but not abelian; by Proposition 5.4 there are exactly three such classes at order 3. That is not a minimality statement, and minimality would be false: on the two-element carrier $\mathbb{Z}/2$ the pair $x \dashv y = x + y$, $x \vdash y = x + y + 1$ is a doppelsemigroup; it is not abelian, since $x \dashv y = y \vdash x$ would force $0 = 1$; and both its operations are commutative, so its cross-dual is $(\mathbb{Z}/2, \vdash, \dashv)$, onto which $x \mapsto x + 1$ is an isomorphism. This is the construction of Example 5.1 with the group $\mathbb{Z}/2$; the verification of iso-cross-duality there uses only the group axioms and $a^2 = e$. The same example also produces a noncommutative iso-dual doppelsemigroup; by Theorem 5.3(ii) that one has no three-element instance.

Remark 5.5 (what the census does and does not settle). Theorem 5.3 decides the numerical disagreement between [7] and [5] in favour of the latter, and part (iv) is consistent with that paper’s own explanation of it: “In carrying out the classification, up to isomorphism, of all doppelsemigroups of order 3 in [GR1], the nontrivial rectangular doppelsemigroups were overlooked” [5, §6], the reference [GR1] there — [8] in the published version — being [7]. There is indeed exactly one dual pair of nontrivial rectangular classes at order 3 — number 4 of Table 3, in which both operations depend only on their first argument, through the two different retractions $(0, 1, 2) \mapsto (0, 1, 0)$ and $(0, 1, 2) \mapsto (0, 1, 1)$ of $\{0, 1, 2\}$ onto a two-element left-zero band — and the missing pair is one pair. We do not prove that this is *the* overlooked pair; identifying the two structures and showing that they exhaust the dropped class is done in our companion paper [1].

6. THE ABELIAN LAYER IS NOT THIN

Proposition 6.1. *Let $*$ be a binary operation on S and $c \in S$ with $(x * y) * z = c$ for all $x, y, z \in S$. Then $(x * y) * z = (y * x) * z$ for all x, y, z ; if in addition $*$ is associative, then also $x * (y * z) = x * (z * y)$. Hence every semigroup all of whose triple products are equal is left and right commutative.*

Proof. Both sides of the first identity equal c . For the second, associativity turns each side into a triple product, and both equal c . $\square$

By Corollary 3.7, every semigroup of order n in which all triple products coincide therefore contributes to $\text{ads}(n)$, and a fortiori to $\text{adm}(n) = \text{agdm}(n)$. Combining this with the published counts gives the following, whose numerical inputs are Distler and Mitchell’s tables [2] and OEIS A027851; it is not part of the formal development.

Remark 6.2 (an arithmetic consequence). The semigroups of order n that are nilpotent of degree 3 all have their triple products equal, so Proposition 6.1 applies to them. With [2] at $n = 9$ and $\text{A027851}(9) = 105\,931\,872\,028\,455 = \text{s}(9)$ as the trivial upper bound,

$$105\,931\,872\,028\,455 \leq \text{ads}(9) \leq \text{adm}(9) = \text{agdm}(9) \leq 105\,978\,177\,936\,292,$$

an interval of relative width 0.0437%. The lower bound is not tight, and we make no attempt to sharpen it: the null semigroup of order 9 also has all its triple products equal, but is nilpotent of degree 2 and so is not among the $105\,931\,872\,028\,455$ counted by [2]. With the values of Section 7,

$$\frac{\text{adm}(6)}{\text{s}(6)} = 0.3796, \quad \frac{\text{adm}(7)}{\text{s}(7)} = 0.7984, \quad \frac{\text{adm}(8)}{\text{s}(8)} = 0.99739, \quad \frac{\text{ads}(8)}{\text{s}(8)} = 0.99688.$$

So the abelian layer, which at $n \leq 6$ looks like a thin subclass and whose size none of the three source papers compares with $s(n)$, is most of the class by $n = 7$ and almost all of it by $n = 8$.

Remark 6.3 (what is conditional). “It is part of the folklore of semigroup theory that almost all finite semigroups are nilpotent of degree 3” [2]; if that is so then $\text{adm}(n)/s(n) \rightarrow 1$ and $\text{ads}(n)/s(n) \rightarrow 1$. We stress that this limit is conditional. Distler and Mitchell write of their lower bound: “Presumably this bound is asymptotic, that is, the ratio tends to 1 while the order tends to infinity, although this is not a consequence of the result for semigroups up to equality in [9].” The displayed inequalities above, and the four ratios, are unconditional; the limit is not.

7. NEW ENTRIES

The statements in this section are not part of the machine-checked development. Each is the output of an exhaustive search written in C, run on a single core; we describe each search, and the checks it passed, so that the claims can be judged and re-run. In every case the programs first reproduced every published entry of the sequence concerned.

Validation before extension. Four independent reproductions were run before any new value was claimed.

- (1) *From the source’s own data.* The ancillary Cayley tables shipped with [5] list all isomorphism-class representatives of doppelsemigroups of orders 2 to 5 (and of the commutative ones of order 6). All $8 + 77 + 1217 + 68177$ listed pairs of orders 2 to 5 were verified associative and interassociative — zero violations — and the paper’s own sub-counts were re-derived from the definitions, recovering ds, ads, rds, cds, sds at $n \leq 5$ exactly as printed.
- (2) *Inside the proof assistant.* Theorems 5.1 and 5.2.
- (3) *From SMALLSEMI.* Every stored table at every $n \leq 8$ was checked associative (zero failures), and the census recovers $s(n)$ for $n \leq 7$ (A027851) and $\text{rs}(n)$ for $n \leq 6$.
- (4) *From the rectangular classification.* A direct enumeration of rectangular semigroups recovers rs, rdm, rgdm, rds for $n \leq 6$, and the labelled count A389829 for $n \leq 7$.

The abelian cells at $n = 7$ and $n = 8$.

Computation 7.1. $\text{adm}(7) = \text{agdm}(7) = 1\,299\,527$ and $\text{ads}(7) = 1\,258\,518$; $\text{adm}(8) = \text{agdm}(8) = 3\,674\,424\,652$ and $\text{ads}(8) = 3\,672\,523\,869$.

By Corollary 3.7 these are counts of isomorphism classes of semigroups satisfying (A), respectively (A) and (B). SMALLSEMI stores one representative per isomorphism-or-anti-isomorphism class, and duality exchanges (A) with (B), so the unfolding is

$$\text{adm}(n) = X + Y - Z, \quad \text{ads}(n) = 2W - Z,$$

where X, Y, W count the stored representatives satisfying (A), (B), and both, and Z counts those satisfying both that admit an anti-automorphism. Self-duality is decided by a backtracking search for an anti-automorphism. At $n = 7$, over the 836 021 stored classes, $(X, Y, W, Z) = (671\,305, 667\,028, 648\,662, 38\,806)$, giving the two values above; the same run returns $s(7) = 2 \cdot 836\,021 - 44\,370 = 1\,627\,672 = \text{A027851}(7)$, which is a check on the parsing, the associativity test and the self-duality search at once. At $n = 6$ the same code returns $\text{adm}(6) = 10\,870$ and $\text{ads}(6) = 7503$, the published values.

At $n = 8$ only the 11 433 106 explicitly stored classes need testing: by Proposition 6.1 every one of the 1 831 687 022 classes in the degree-3 nilpotent block satisfies both identities, so the whole block counts. All 11 433 106 tables were reconstructed and verified associative (zero failures), and exactly one of them has all triple products equal, namely the null semigroup O_8 , which is nilpotent of degree 2 and hence correctly absent from the nilpotent block — so nothing is double-counted. The explicit block contributes 7 233 148 to X , 5 945 903 to Y , 5 639 134 to W and 277 191 to Z . The self-dual part of the nilpotent block is not stored; it is forced by $2\,\text{A001423}(8) - \text{A027851}(8) = 2\,209\,839$ total self-dual classes minus the 358 587 self-dual explicit ones, giving 1 851 252 — which is exactly Distler and

Mitchell’s tabulated number of self-dual nilpotent semigroups of degree 3 of order 8, an independent confirmation of the accounting. With the same data $2 \cdot 1\,831\,687\,022 - 1\,851\,252 = 3\,661\,522\,792$ reproduces their count of degree-3 nilpotent semigroups of order 8 up to isomorphism exactly.

The rectangular cells at $n = 7$.

Computation 7.2. $\text{rdm}(7) = 1364$, $\text{rgdm}(7) = 14\,710$, $\text{rds}(7) = 571$.

Using the classification of Section 4, the labelled rectangular operations on $[n]$ are enumerated once each (the coordinate maps λ, ρ as restricted-growth partitions, and a section for e), isomorphism-class representatives are picked out by canonical form, and for each representative the second operation is enumerated and the orbits of Aut are counted. Proposition 4.1 makes the axiom tests cheap: for a pair of rectangular operations, (D_1) reduces to $x \dashv (y \vdash z) = x \dashv z$ and (D_3) to $(x \dashv y) \vdash z = x \vdash z$. The run takes 30 seconds at $n = 7$ and reproduces, at $n \leq 6$, every published entry of all four sequences: $\text{rs} = 1, 3, 5, 10, 14, 27$; $\text{rdm} = 1, 6, 18, 62, 166, 509$; $\text{rgdm} = 1, 7, 27, 128, 555, 2846$; $\text{rds} = 1, 3, 7, 20, 51, 171$. At $n = 7$ there are 31 117 labelled rectangular semigroups (A389829) in $\text{rs}(7) = 36$ classes, and the three counts above.

The four order-6 cells.

Computation 7.3.

$$\text{dm}(6) = 212\,193\,607, \quad \text{gdm}(6) = 559\,338\,535, \quad \text{ds}(6) = 212\,886\,176, \quad \text{sds}(6) = 212\,396\,016.$$

These are the first open cells of all three problems. A naive pair search over the 6^{36} candidate second operations is hopeless; instead, the second operation is *searched* rather than enumerated. The program iterates over one representative of each of the 28 634 isomorphism classes of order-6 semigroups (from SMALLSEMI, unfolded by duality) and runs a constraint depth-first search over the 36 cells of $\vdash$, using for the dimonoid modes the unary domain $D(y, z) = \{w : x \dashv w = (x \dashv y) \dashv z \text{ for all } x\}$ read off (D_1) . Isomorphism classes of pairs are then counted by Burnside’s lemma inside $\text{Aut}(\dashv)$: for a fixed first operation, two admissible second operations give isomorphic structures precisely when they lie in the same $\text{Aut}(\dashv)$ -orbit, so their number is

$$\frac{1}{|\text{Aut}(\dashv)|} \sum_{\sigma \in \text{Aut}(\dashv)} \#\{\vdash : \text{the axioms hold and } \sigma \cdot \vdash = \vdash\},$$

each σ -fixed count being the same search with every cell assignment propagated along its σ -orbit.

The four runs cost 1403, 3419, 3133 and 3431 CPU-seconds respectively, 3.16 CPU-hours in all, with peak resident memory under one megabyte. The cost is concentrated in the semigroups with constant diagonal; the null semigroup O_6 alone accounts for 637 seconds in the doppelsemigroup mode and 780 in the strong mode, its interassociates being almost unconstrained.

The same program reproduces all twelve published values at $n = 3, 4, 5$: $\text{dm} = 52, 734, 55883$; $\text{gdm} = 78, 1693, 162465$; $\text{ds} = 77, 1217, 68177$; $\text{sds} = 65, 841, 56428$. As an independent check on the pruning — the one place where a subtle error would silently lose solutions — the whole search was re-implemented with a different variable order (first unassigned cell rather than smallest domain first). The variant reproduces the same twelve published values and, on the slice consisting of the first 200 representatives at $n = 6$ in the dimonoid mode, the identical partial count 12 427 169.

Remark 7.4 (consistency). The four values satisfy the inequalities they must: $\text{dm}(6) \leq \text{gdm}(6)$ and $\text{sds}(6) \leq \text{ds}(6)$, and all four exceed $\text{s}(6) = 28\,634$, the number of trivial structures. The ratio gdm/dm is 2.64 at $n = 6$ against 2.91 at $n = 5$. The ratio sds/ds jumps from 0.828 at $n = 5$ to 0.9977 at $n = 6$, which is what the nilpotent-dominated regime of Section 6 predicts: strongness compares two triple products, and in a pair of nearly nilpotent operations both are the zero, so almost every doppelsemigroup becomes strong as n grows.

Remark 7.5 (searches for prior appearances). A live OEIS search on the published terms of each of the sequences extended here returned no results, as did a search on each of the four order-6 sequences with its new term appended and on each of those four new terms alone. We also found no OEIS entry counting right commutative semigroups. The literature searches behind these claims are as described in Section 1, subject to the caveat recorded there about the unobtainable submitted paper cited by [5].

The commutative cells at $n = 7$.

Computation 7.6. $\text{cdm}(7) = 83\,580\,226$, $\text{cgdm}(7) = 242\,117\,995$, $\text{cds}(7) = 84\,998\,302$.

These are the first entries of the three commutative rows that the three problems ask for. The search is the one of Computation 7.3 with two changes. First, the first operation ranges only over the isomorphism classes of *commutative* semigroups of order 7; a commutative semigroup is its own dual, so the SMALLSEMI representative of an isomorphism-or-anti-isomorphism class is already an isomorphism representative and the unfolding by duality of Computation 7.1 does not arise. Second, the second operation is forced commutative: assigning a cell assigns its whole orbit under the group generated by the automorphism being counted in the Burnside sum and by the transposition $(x, y) \mapsto (y, x)$, and each cell's domain is intersected with that of its transpose. The program aborts if any Burnside average fails to be an integer; it never did. The three runs cost 795, 1677 and 1657 CPU-seconds on one core each, with peak resident memory 41 MB.

Four anchors sit inside the same run. The program reproduces every published entry of all three sequences, $\text{cdm} = 3, 14, 101, 1495, 102268$ and $\text{cgdm} = 4, 22, 189, 3249, 256253$ and $\text{cds} = 6, 41, 345, 3892, 134462$ for $n = 2, \dots, 6$; the number of first operations it iterates over is $\text{cs}(n) = \text{A001426}(n)$ at every order, in particular 17 291 at $n = 7$, which checks the parsing and the commutativity filter independently of the counts; the order-3 entries 14, 22, 41 are certified in the proof assistant by Theorem 5.1; and the order-4 entries 101, 189, 345 were reproduced by a separate enumerator written independently for this purpose. A live OEIS search returned nothing for 1, 3, 14, 101, 1495, 102268, 83580226, for 1, 4, 22, 189, 3249, 256253, 242117995, for 1, 6, 41, 345, 3892, 134462, 84998302, or for any of the three new terms alone. The three values are consistent with each other and with the trend: $\text{cdm}(7) \leq \text{cgdm}(7)$, as (D_2) is a restriction, with the ratio 2.90 continuing 1.33, 1.57, 1.87, 2.17, 2.51 at $n = 2, \dots, 6$; and $\text{cds}(7) > \text{cdm}(7)$ as at every smaller order, with the ratio 2.00, 2.93, 3.42, 2.60, 1.31, 1.02 at $n = 2, \dots, 7$, falling steadily from $n = 4$ on.

Remark 7.7 (the cross-check that was not completed). Every other computation reported here has a second, independent derivation behind it: Computation 7.3 was re-run with a differently ordered search, Computation 7.2 is reproduced by the independent program of Computation 7.8, and Computation 7.1 is anchored by the self-dual accounting that reproduces two of the tables of [2] exactly. A re-run with a different search order is the check that matters for a constraint search, because a defect in the pruning would silently lose solutions rather than announce itself. For Computation 7.6 that check was completed at $n \leq 6$ — the variant, which chooses the last unassigned cell in reverse scan order instead of the smallest domain first, reproduces all fifteen published values in all three modes — but *not* at $n = 7$, where it is more than five times slower and did not finish. We therefore publish the primary program's partial counts on an initial slice, so that the check can be completed independently: over the first 100 000 of the 836 021 SMALLSEMI representatives of order 7 — a slice containing 856 commutative ones — the counts are 34 171 235 in the cgdm mode and 12 346 053 in the cds mode. Until that is done, Computation 7.6 rests on the primary program together with the four anchors above, and it is the least redundantly checked of the values reported here.

The rectangular cells at $n = 8$ and $n = 9$.

Computation 7.8. $\text{rdm}(8) = 3903$, $\text{rdm}(9) = 10\,599$, $\text{rgdm}(8) = 81\,905$, $\text{rgdm}(9) = 472\,836$.

These are two and three orders beyond the last published entries $\text{rdm}(6) = 509$ and $\text{rgdm}(6) = 2846$, and one and two orders beyond Computation 7.2. The program is a second one, sharing no

step with that of Computation 7.2: it uses Theorem 4.3 to replace the $O(n^3)$ axiom test by an $O(n^2)$ comparison of two partitions, and it counts $\text{Aut}(-)$ -orbits of admissible second operations by marking — generate the orbit of the first unvisited admissible $\vdash$, mark it, repeat — rather than by canonical forms. It costs 6.1 seconds at $n = 8$ (peak 119 MB) and 89.0 seconds at $n = 9$ (peak 1.69 GB). A third enumerator, written independently of both of the others — it builds the labelled rectangular operations as (row partition, column partition, section) triples, picks isomorphism-class representatives by orbit marking under all $n!$ relabellings, and tests the axioms directly at all n^3 triples, so that Theorem 4.3 is nowhere used in it — reproduces every entry of Table 4, the four values above included, in 789 seconds at $n = 9$.

Table 4 shows what the same run reproduces. Everything in it except the four values of Computation 7.8 is either published or independently computed: the labelled counts are A389829, whose terms at $n = 8$ and $n = 9$ match exactly; the rs row is A390162; rdm, rgdm and rds for $n \leq 6$ are the published tables of [3, 4, 5]; and the rds column at $n = 7, 8, 9$ agrees with the closed form of [1] (Remark 7.9). The value $\text{rdm}(7) = 1364$ is Computation 7.2 recomputed, and $\text{rdm}(7) \geq 1364$ is proved in Section 8. A live OEIS search returned nothing for 1, 6, 18, 62, 166, 509, 1364, 3903, 10599, for 1, 7, 27, 128, 555, 2846, 14710, 81905, 472836, or for either pair of new terms alone, while in the same session 1, 3, 5, 10, 14, 27, 36, 64, 90 returns A390162 and 1, 4, 17, 84, 507, 3668, 31117, 312938, 3669671 returns A389829 — so the search path was working and the negative results are real. As it must be, $\text{rdm}(n) \leq \text{rgdm}(n)$ at every order, (D_2) being a genuine restriction by Proposition 4.6(i).

TABLE 4. The rectangular run. Boldface marks the four values of Computation 7.8; every other entry is a published value or an independently computed one, and the run reproduces all of them.

n	labelled	rs(n)	rdm(n)	rgdm(n)	rds(n)
3	17	5	18	27	7
4	84	10	62	128	20
5	507	14	166	555	51
6	3668	27	509	2846	171
7	31 117	36	1364	14 710	571
8	312 938	64	3903	81 905	2193
9	3 669 671	90	10 599	472 836	8594

Remark 7.9 (the companion paper’s rds). Our companion paper [1] proves a structure theorem for rectangular doppelsemigroups and derives from it a closed form for $\text{rds}(n)$, by a Pólya count over the relevant permutation groups; it gives $\text{rds}(7) = 571$, $\text{rds}(8) = 2193$ and $\text{rds}(9) = 8594$, and machine-checks the matching lower bounds. Those three values agree exactly with the last column of Table 4, which was produced here by an exhaustive search. The route is entirely different, and the agreement at three consecutive orders is the strongest single check on the program of Computation 7.8. Computation 7.2’s value $\text{rds}(7) = 571$ is therefore not new; we leave it in place because it was obtained here independently, and we claim priority for neither.

8. A MACHINE-CHECKED LOWER BOUND FOR $\text{rdm}(7)$

Every value of Section 7 is the output of a program we do not verify. For one of them, the first open entry of the rdm row, we prove the lower half inside the proof assistant.

Theorem 8.1. *There are at least 1364 pairwise non-isomorphic rectangular dimonoids of order 7; that is, there is a list $P_0, \dots, P_{1363}$ of pairs of binary operations on a seven-element set such that each P_i is a dimonoid whose two operations are both rectangular, and such that for $i \neq j$ there is no bijection of the carrier carrying P_i to P_j and respecting both operations. Hence $\text{rdm}(7) \geq 1364$.*

The bound matches the computed value of Computation 7.2, so the two together say $\text{rdm}(7) = 1364$ if that computation is right; the theorem alone says only “ $\geq$ ”.

The proof is an exhaustive check of an explicit certificate. The 1364 structures are generated in advance by the enumerator of Computation 7.8 and emitted as the lexicographically least representative of each isomorphism class under all $7! = 5040$ relabellings; each is a table of 98 entries in $\{0, \dots, 6\}$, the two 7×7 multiplication tables written one after the other. Six statements about that list are then decided by evaluation:

- (1) every one of the 7^7 tuples used below has length 7 and entries below 7;
- (2) the list of relabellings, obtained by filtering those 7^7 tuples by “no repeated entry”, is closed under the inverse operation, and that operation really inverts;
- (3) all 1364 blocks satisfy the seven axioms — associativity of both operations, (D_1) , (D_2) , (D_3) and rectangularity of both — at all 343 triples;
- (4) each block is the lexicographic minimum of its own orbit under all 5040 relabellings;
- (5) relabelling a block by s and then by s^{-1} returns it, for every block and every s ;
- (6) the 1364 blocks are pairwise distinct.

Non-isomorphism then follows with no group theory at all. A bijection of the carrier is an injective self-map, so its table of values is one of the 7^7 tuples with no repeated entry and hence lies in the list of relabellings by (i) and (ii) alone — there is no need to close that list under composition, and no orbit-stabiliser argument. If $P_i \cong P_j$ then some s in the list carries P_i to P_j , so $P_i \leq P_j$ lexicographically by the minimality of P_i in its own orbit; and s^{-1} carries P_j back to P_i by (v), so $P_j \leq P_i$ by the minimality of P_j . Antisymmetry of the lexicographic order forces $P_i = P_j$, contradicting (vi). That each P_i is a rectangular dimonoid in the sense of Section 1 — not in some encoded surrogate — is (iii) transported through the decoding.

Remark 8.2 (what is checked, and the controls). The six statements of the list are decided by compiled evaluation, so the trusted surface of Theorem 8.1 is exactly “these six Boolean expressions evaluate to true”; the argument above is an ordinary proof. Four controls guard against the checks being vacuously satisfied: the certificate list really has 1364 entries; the relabelling list really has 5040; it really contains both the identity and a transposition; and at least one certificate is moved by at least one relabelling, so the minimality check is not comparing every block only with itself. The evaluation takes 330 seconds and 1.85 GB on a lightly loaded machine: it is 1364×5040 relabellings of a 98-entry table carried out three times over, about $2 \cdot 10^7$ relabellings in all.

9. WHAT REMAINS OPEN

The cross-check on the commutative cells. The second search order of Remark 7.7 agrees with the first on all fifteen published values at $n \leq 6$ but did not finish at $n = 7$. Finishing it is the one piece of validation these three values are missing, and the slice counts published in that remark are what it should reproduce first.

A machine-checked $\text{rgdm}(7)$. The lower bound $\text{rgdm}(7) \geq 14710$ is the cheapest statement of the kind of Theorem 8.1 that is not yet proved. The certificate — 14710 blocks instead of 1364 — is already produced by the same enumerator, and the checks scale linearly in the number of blocks, so the evaluation is about 10.8 times that of Theorem 8.1, roughly one CPU-hour and 1.4 MB of certificate data. That is more than we were willing to put into a single file, so it would have to be split into about four blocks summed at the end. Nothing else about it is new.

An exact $\text{rdm}(7)$, machine-checked. The matching upper bound is a different problem and we have not found a cheap route to it. It needs a formal proof that the enumeration of the 31117 labelled rectangular operations on a seven-element set is complete — feasible, by the classification of Section 4, with about $4 \cdot 10^7$ candidates to test — and a count of isomorphism classes, which is Burnside’s lemma or canonical forms over roughly 10^7 admissible labelled pairs, about $5 \cdot 10^{10}$ relabellings on the canonical-form route. We priced it and stopped.

Order 10. $A389829(10) = 48\,071\,832$ labelled rectangular semigroups would need about 24 gigabytes for the table and its index before any search starts, so $\text{rdm}(10)$ and $\text{rgdm}(10)$ are out of reach of the program of Computation 7.8 as written.

A closed form for $\text{rgdm}(n)$. This is what Theorem 4.3 really opens. By Corollary 4.5 a rectangular g -dimonoid is a pair of pairs of partitions, subject to two refinements in opposite directions, with sections attached that the axioms ignore; so a Pólya sum over the relevant permutation groups should give $\text{rgdm}(n)$ at every n , exactly as the analogous sum gives $\text{rds}(n)$ in [1]. Any such formula must reproduce 1, 7, 27, 128, 555, 2846, 14 710, 81 905, 472 836.

The next dimonoid cell. $\text{dm}(7)$ has the same shape as Computation 7.3 but $s(7)/s(6) \approx 57$ times as many first operations, and order 8 has $3.7 \cdot 10^9$ classes. Neither is within reach of the method used here.

Exact values at $n = 9$. Remark 6.2 pins $\text{adm}(9) = \text{agdm}(9)$ and $\text{ads}(9)$ to within 0.044%, and closing the gap means deciding (A) and (B) on the roughly $4.6 \cdot 10^{10}$ isomorphism classes of order-9 semigroups that are not nilpotent of degree 3 — about $2.3 \cdot 10^{10}$ up to isomorphism or anti-isomorphism. No library holds them.

10. FORMAL VERIFICATION

Propositions 3.1–3.9, 4.1, 4.6, 5.4 and 6.1, Corollaries 4.4 and 4.5, and Theorems 3.4, 4.3, 5.1, 5.2, 5.3 and 8.1, together with Table 3 and the falsifiability controls listed after Theorem 5.2, in the text following Theorem 5.3 and in Remark 8.2, are formalised and machine-checked in Lean 4 [10]. The development is seven files — the definitions transcribed from [3, 4, 5], the structure theory of Sections 3–6, the rectangular refinement theorem, the order-3 census, its duality refinement, and the certificate of Section 8 in two parts — and uses core Lean only, with no mathematical library, so that no coercion appears in any side goal. There is no incomplete proof and no added axiom; the axiom set of every headline statement was printed, and the trusted surface is exactly what follows. The results of Sections 3–6 hold for an arbitrary carrier, with no finiteness and no decidability hypothesis, and depend on quotient soundness at most — Propositions 3.2, 3.3, 3.5, 3.9, 4.1 and 6.1 on no axiom at all. Theorem 4.3 and its two corollaries likewise hold over an arbitrary carrier, and they and every part of Proposition 4.6 depend on *no axiom at all*, which is the formal counterpart of their being proofs rather than checks. Each statement of the order-3 census, old and new, depends on propositional extensionality together with one compiled-evaluation axiom for each Boolean it decides. Theorem 8.1 depends on propositional extensionality, quotient soundness and the axiom of choice — the last used only by two case splits, not by the mathematics — together with six named compiled-evaluation axioms, one for each of the six statements listed in Section 8. Witnesses and counterexamples are formalised as well, so that no hypothesis above is vacuous: two families of witnesses (commutative semigroups, and semigroups with all triple products equal), the left-zero operation on a two-element set, which satisfies (A) and fails (B) and makes the inclusion of Proposition 3.8 strict, and the five two-element pairs of Proposition 4.6. The computations of Section 7 are outside this scope: they are C programs, checked against each other and against the published values as described there, and are not machine-checked. The repository is private; repository reference to be added at submission.

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