DIGITAL IMAGES WITH SMALL WHITE COMPONENTS: AN IMPROVED LOWER BOUND AT $k = 10$

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Abstract. For $\alpha \in \{4, 8\}$ and $k \in \mathbb{N}$, let $\Phi_\alpha(k)$ be the supremal white pixel density of a colouring of $\mathbb{Z}^2$ all of whose white α-connected components have at most k cells. Fridberg determined Φ_8 completely and Φ_4 on a set of k of natural density $\frac{1}{2}$, and identified a set U of density $\frac{1}{4}$ on which $\Phi_4(k)$ is strictly below his upper bound $F_4(k)$ but is not computed; $k = 10$ and $k = 11$ are the first two cells of U beyond the two he evaluates by hand. We exhibit a colouring of $\mathbb{Z}^2$, periodic with respect to a sublattice of index 59, whose white 4-components have at most 10 cells and whose density is 40/59. Hence $\Phi_4(10) \geq 40/59$, which improves the bound 61/90 printed in that paper; the cell stays open, since neither that paper nor this one proves any explicit upper bound on $\Phi_4(10)$ smaller than $F_4(10) = 20/29$. We also give a machine-checked reproduction of the published data: the nine values of $\Phi_4(k)$ for $k \leq 9$ and the twelve values of $\Phi_8(k)$ for $k \leq 12$ are each matched from below by a witness found here by exhaustive search, not transcribed from the source's figures (the matching upper bounds are Fridberg's and are not reproved), the exact cells $\Phi_4(1) = \Phi_4(2) = \frac{1}{2}$, $\Phi_8(1) = \frac{1}{4}$ and $\Phi_\alpha(0) = 0$ are proved from both sides, and a general averaging bound gives $\Phi_\alpha(k) \leq k/(k+1)$ and $\Phi_8(k) \leq k/4$. All statements are formally verified in Lean 4; the component bounds are proved for $\mathbb{Z}^2$ and never for a quotient torus, and the densities are the lim inf of the original definition.

1. Introduction

A *digital image* is a colouring of the square grid $\mathbb{Z}^2$ by two colours, black and white; its cells are *pixels*. Two pixels are 4-adjacent when they differ by one step in the ℓ^1 metric and 8-adjacent when they differ by one step in the ℓ^∞ metric, and for $\alpha \in \{4, 8\}$ an *α-connected component* is a maximal set of mutually α-connected pixels of one colour [4]. Fridberg [1] asks how large the white pixel density of an infinite image can be if all of its white components are small, and writes

$$\Phi_\alpha(k) \;:=\; \sup\bigl\{\rho(I) \;:\; \text{every white } \alpha\text{-component of } I \text{ has at most } k \text{ cells}\bigr\},$$

where ρ is the density of Definition 2.2 below. His main theorem [1, Theorem 1.1] bounds Φ_α by an explicit isoperimetric quantity: for all $k \in \mathbb{N}$,

$$(1)\quad \Phi_8(k) \;=\; F_8(k) \;:=\; \max_{1\leq j\leq k} \frac{j}{j + \lceil 2\sqrt{j}\,\rceil + 1}, \qquad \Phi_4(k) \;\leq\; F_4(k) \;:=\; \max_{1\leq j\leq k} \frac{j}{j + \frac{1}{2}\lceil 2\sqrt{2j-1}\,\rceil}.$$

The 8-connected case is settled completely [1, Corollary 7.2]. In the 4-connected case equality holds on a set $S = \{1, 2, 4, 5, 7, 8, 9, 12, 13, 14, 17, \dots\}$ of natural density $\frac{1}{2}$, and [1, Theorem 1.1 and Lemma 7.4] exhibits a further set

$$U \;=\; \{3, 6, 10, 11, 15, 16, 21, 22, 28, 29, \dots\}$$

of natural density $\frac{1}{4}$ on which the inequality $\Phi_4(k) < F_4(k)$ is *strict*, and on which no value is computed except the two obtained by hand, $\Phi_4(3) = \frac{13}{24}$ and $\Phi_4(6) = \frac{5}{8}$ [1, Theorem 8.1]. Item (2) of the Future Directions section of [1] leaves the remaining integers open, and remarks that the problem "appears to be quite hard".

The first two cells of U beyond the two evaluated by hand are $k = 10$ and $k = 11$, where $F_4(10) = F_4(11) = \frac{20}{29}$. What [1] publishes there is a pair of lower bounds carried by explicit tiles: its Figure 2 prints $\Phi_4(10) \geq \frac{61}{90}$ and $\Phi_4(11) \geq \frac{31}{45}$, and the first of the two also appears in its Table 1, as the right endpoint of the interval $(\frac{2}{3}, \frac{61}{90}]$ on which the maximum guaranteed 4-component size $C_4(d)$ equals 10. The bound $\frac{31}{45}$ appears in the figure only, and nowhere in the text or the table.

Results. The new result of this paper is one improved bound.

Theorem 1.1 (Theorem 4.1)**.** $\Phi_4(10) \;\geq\; \dfrac{40}{59}$.

Since $\frac{40}{59} = 0.677966\ldots$ and $\frac{61}{90} = 0.677\overline{7}$, this improves the published bound, by exactly $\frac{1}{5310}$. The improvement is small and we say so plainly: the cell remains open, the interval of possible values shrinks from $[\frac{61}{90}, \frac{20}{29})$, of width $\frac{31}{2610}$, to $[\frac{40}{59}, \frac{20}{29})$, of width $\frac{20}{1711}$, so a fraction $\frac{29}{1829} = 0.0158\ldots$, about 1.6%, of the previously undetermined range is removed. What the theorem does settle is that the construction printed in [1, Fig. 2] is not optimal at $k = 10$; the witness that beats it is periodic with respect to a lattice of index 59, which is prime, so it is not a refinement of the index-90 configuration.

The rest of the paper is a reproduction, from both sides where possible, of the published data for Φ_α, carried out so that every statement is machine-checked. We regard none of it as new mathematics, and we have taken care that the way it is proved does not quietly weaken it. Specifically:

- Section 3 gives a certificate for the hypothesis "every white α-component has at most k cells" that is a statement about $\mathbb{Z}^2$ and not about a quotient torus. This distinction is not pedantry: a component that wraps the torus is a single small set there and an infinite set in $\mathbb{Z}^2$, and Section 7 exhibits an image of density $\frac{1}{2}$ that a torus-local check would accept at $k = 1$ and whose white components in $\mathbb{Z}^2$ are infinite lines.
- Section 5 recovers $\Phi_4(k)$ for $k \leq 9$ and $\Phi_8(k)$ for $k \leq 12$ — lower bounds only, matching the published values — from witnesses produced here by exhaustive search over sublattices, not read off the figures of [1].
- Section 6 proves the elementary half of the upper bounds: an averaging lemma for arbitrary rectangular windows, from which $\Phi_\alpha(k) \leq k/(k+1)$ and $\Phi_8(k) \leq k/4$ for all α, k, and $\Phi_4(k) \leq \frac{1}{2}$ for $k \leq 2$. Combined with the witnesses this gives the exact values $\Phi_4(1) = \Phi_4(2) = \frac{1}{2}$, $\Phi_8(1) = \frac{1}{4}$ and $\Phi_\alpha(0) = 0$. The first two are the warm-up computation of [1, §1]; the third is the case $k = 1$ of [1, Corollary 7.2]. The general upper bound $\Phi_\alpha(k) \leq F_\alpha(k)$ of [1] is not reproved here: it runs through the digital Jordan curve theorem [7], Pick's theorem and a site-perimeter isoperimetric inequality [9, 3, 2, 8], and none of that machinery is formalised.

Section 8 states exactly what has been verified and which steps rest on compiled evaluation of a finite check.

2. Preliminaries

2.1. **Images, adjacency, components.** An *image* is a map $I\colon \mathbb{Z}^2 \to \{0,1\}$, where $I(p) = 1$ means that the pixel p is white. Following [1, §2.1] and [4] we put $d_4(x,y) = |x_1 - y_1| + |x_2 - y_2|$ and $d_8(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, and call x, y *α-adjacent* when $d_\alpha(x,y) = 1$.

For a formal treatment it is convenient to define adjacency by a list of admissible steps,

$$\mathrm{dirs}(4) = \{(\pm 1, 0), (0, \pm 1)\}, \qquad \mathrm{dirs}(8) = \mathrm{dirs}(4) \cup \{(\pm 1, \pm 1)\},$$

saying that p and q are α-adjacent when $q - p \in \mathrm{dirs}(\alpha)$, because a step list is what a finite check can iterate over. It is then a small lemma, rather than an assumption, that the two readings agree. Write $q \sim_I p$ for the reflexive–transitive closure of the relation "p and q are both white and α-adjacent".

Proposition 2.1. *For all $p, q \in \mathbb{Z}^2$, $q - p \in \mathrm{dirs}(4)$ if and only if $d_4(p,q) = 1$, and $q - p \in \mathrm{dirs}(8)$ if and only if $d_8(p,q) = 1$. Moreover $\sim_I$ is invariant under any translation that preserves I: if $I(x + w) = I(x)$ for all x and $q \sim_I p$, then $q + w \sim_I p + w$.*

Proof. Both equivalences are a case distinction over the four resp. eight steps in one direction, and an integer arithmetic argument — $|a| + |b| = 1$ forces (a, b) to be one of four pairs, and $\max(|a|, |b|) = 1$

one of eight — in the other. Translation invariance is an induction along the closure, using that adjacency is translation invariant. □

We write the component hypothesis in the reachability form

$$\text{(2)}\qquad \mathrm{Comp}_{\alpha,k}(I)\ :\quad \begin{array}{l}\text{for every white } p \text{ and every finite } S \text{ such that } I(q)=1 \text{ and } q \sim_I p\\ \text{for all } q\in S, \text{ one has } |S|\le k,\end{array}$$

which says that every white α-connected component of I has at most k cells without requiring the component to be constructed as a set. Note that (2) also rules out infinite components, since an infinite component contains finite subsets of every size.

2.2. Density.

Definition 2.2 ([1, Definition 2.1]). For an image I let $w(I,m)$ be the number of white pixels in the $(2m+1)\times(2m+1)$ box centred at the origin. The *white pixel density* of I is

$$\rho(I)\ :=\ \liminf_{m\to\infty}\ \frac{w(I,m)}{(2m+1)^2}.$$

This is the lim inf over odd side lengths $j=2m+1$ used in [1], unmodified; in particular no Cesàro average, no upper density and no torus fraction is substituted for it anywhere below. The following is what makes $\Phi_\alpha(k)$ a supremum of a bounded set, so that exhibiting one image really does bound $\Phi_\alpha(k)$ from below.

Proposition 2.3. *$0\le\rho(I)\le 1$ for every image I. Consequently the set $\{\rho(I):\mathrm{Comp}_{\alpha,k}(I)\}$ is bounded above by 1, and if $\mathrm{Comp}_{\alpha,k}(I)$ holds then $\rho(I)\le\Phi_\alpha(k)$.*

Proof. The box has $(2m+1)^2$ pixels, so $0\le w(I,m)\le(2m+1)^2$ and the sequence being lim inf-ed lies in $[0,1]$; both bounds pass to the lim inf, which exists in $\mathbb{R}$ because the sequence is bounded on both sides. The second sentence is the defining property of a least upper bound of a set that is now known to be bounded. □

2.3. The isoperimetric quantities F_4 and F_8.

The functions in (1) are built from the minimum site perimeters $\sigma_4(k)=\lceil 2\sqrt{2k-1}\,\rceil+2$ and $\sigma_8(k)=2\lceil 2\sqrt{k}\,\rceil+4$ of [1, Lemmas 2.11 and 2.13], the first of which is attributed there to [2, 8]. Two points of care in transcribing them. First, $\lceil 2\sqrt{n}\,\rceil$ must be an integer computation, not a real one: we compute it as $\lceil\sqrt{4n}\,\rceil$, which is the same number since $2\sqrt{n}=\sqrt{4n}$, and we characterise the integer square root rather than assume it. Second, F_4 has a denominator $j+\frac{1}{2}\lceil 2\sqrt{2j-1}\,\rceil$ that is a half-integer, so we double numerator and denominator and work with $2j/(2j+\lceil 2\sqrt{2j-1}\,\rceil)$, keeping the whole computation inside $\mathbb{Q}$.

Proposition 2.4. *Define $\lceil\sqrt{\cdot}\,\rceil\colon\mathbb{N}\to\mathbb{N}$ by $\lceil\sqrt{n}\,\rceil=\lfloor\sqrt{n}\rfloor$ if $\lfloor\sqrt{n}\rfloor^2=n$ and $\lfloor\sqrt{n}\rfloor+1$ otherwise. Then $n\le\lceil\sqrt{n}\,\rceil^2$, and $\lceil\sqrt{n}\,\rceil\le t$ for every $t\in\mathbb{N}$ with $n\le t^2$; that is, $\lceil\sqrt{n}\,\rceil$ is the least t with $n\le t^2$. With $\lceil 2\sqrt{n}\,\rceil:=\lceil\sqrt{4n}\,\rceil$, the functions F_4 and F_8 of (1) are well defined over $\mathbb{Q}$.*

Proof. Both directions follow from $\lfloor\sqrt{n}\rfloor^2\le n<(\lfloor\sqrt{n}\rfloor+1)^2$ by case analysis on whether n is a perfect square, together with monotonicity of $t\mapsto t^2$ on $\mathbb{N}$. □

The values relevant below are $F_4(1)=F_4(2)=\frac{1}{2}$, $F_8(1)=\frac{1}{4}$ and $F_4(10)=F_4(11)=\frac{20}{29}$, the last two because $\lceil 2\sqrt{19}\,\rceil=9$ gives $20/(20+9)$ at $j=10$, while $j=11$ gives only $22/(22+10)=\frac{11}{16}<\frac{20}{29}$.

3. Periodic witnesses and the block certificate

Every lower bound in this paper is witnessed by an image that is periodic with respect to a sublattice $L\le\mathbb{Z}^2$ of finite index. Two things must be established for such a witness: that its white components are small *in* $\mathbb{Z}^2$, and that its density in the sense of Definition 2.2 is the white fraction of one period. Neither is automatic.

3.1. **Rectangular presentation.** A sublattice of index n in Hermite normal form is $L = \langle (a,0),(c,d)\rangle$ with $ad = n$, and every such L contains a rectangular sublattice $\langle (a',0),(0,b')\rangle$; for instance $\langle (59,0),(17,1)\rangle$ contains $(0,59) = 59\cdot(17,1) - 17\cdot(59,0)$. An L-periodic image is therefore also periodic with respect to a rectangle, and can be presented as a colouring of an $a'\times b'$ block. The presentation multiplies both the white count and the area by the same factor $a'b'/n$, so it leaves the density unchanged. We use rectangular blocks throughout, and record the index of the underlying lattice when it matters. The search of Section 4 works with L directly, because a sweep over axis-aligned rectangles alone misses the sheared optima: the optimal $k = 3$ configuration lives on the lattice generated by $(24,0)$ and $(5,1)$.

3.2. **The certificate.** A *block certificate* consists of periods $a,b \ge 1$, a target k, a radius D, a white mask $W\colon \{0,\dots,a-1\}\times\{0,\dots,b-1\}\to\{0,1\}$ and a displacement $\delta\colon \{0,\dots,a-1\}\times\{0,\dots,b-1\}\to\mathbb{Z}^2$. It determines the image $I(p) = W(p \bmod (a,b))$ and the map

$$\operatorname{rep}(p) \;:=\; p + \delta\big(p \bmod (a,b)\big),$$

a putative choice of representative for the component of p. Three finite checks are made, all of them quantified over the ab cells of one block:

$$\begin{aligned} &\text{(bnd)} && \|\delta(p)\|_\infty \le D \text{ for every cell } p;\\ &\text{(adj)} && \operatorname{rep}(p) = \operatorname{rep}(q) \text{ whenever } p,q \text{ are white and } \alpha\text{-adjacent};\\ &\text{(fib)} && \#\{q \in \operatorname{rep}(p) + [-D,D]^2 : q \text{ white and } \operatorname{rep}(q) = \operatorname{rep}(p)\} \le k \text{ for every white } p. \end{aligned} \tag{3}$$

Theorem 3.1. *If a block certificate with parameters (a,b,k,D) passes the three checks (3) for α, then its image I satisfies $\mathrm{Comp}_{\alpha,k}(I)$: every white α-connected component of I, as a subset of $\mathbb{Z}^2$, has at most k cells.*

Proof. Check (adj) says that rep is constant along single white α-steps; since white connectivity is generated by such steps, rep is constant on each white component, by induction along the reflexive–transitive closure. Let now p be white and let S be a finite set of white cells all connected to p. Translating by a period vector we may assume p lies in the block, which is legitimate because I is invariant under the period translation and, by Proposition 2.1, so is connectivity. Every $q \in S$ then satisfies $\operatorname{rep}(q) = \operatorname{rep}(p)$, and by (bnd) it satisfies $\|q - \operatorname{rep}(p)\|_\infty \le D$; so S injects into the set counted by (fib) at p, whence $|S| \le k$. □

Theorem 3.1 is what lets us avoid running a search inside a proved statement. The obvious alternative, breadth-first search on the quotient torus $\mathbb{Z}^2/L$, is wrong — the counterexample is in Section 7 — and the bounded box in (fib) is what rules it out, since a wrapping component would place two cells of the same block a period vector apart, farther than $2D$.

3.3. **The density of a periodic image.**

Theorem 3.2. *Let I be the image of a block certificate with periods $a,b \ge 1$ and let W be the number of white cells in one block. Then $\rho(I) = W/(ab)$, with ρ as in Definition 2.2.*

Proof sketch. Put $P = ab$. A one-dimensional periodic-sum lemma — any window of length P sees the same sum of a P-periodic function, proved by shifting the window one step at a time — applied once in each coordinate shows that an axis-aligned square of side $t\cdot P$, anchored anywhere, meets exactly t^2ab translates of the $a\times b$ block and so contains exactly $t^2\,ab\,W$ white cells. The centred box of odd side j is sandwiched between the aligned boxes of sides $\lfloor j/P\rfloor\cdot P$ and $(\lfloor j/P\rfloor + 1)\cdot P$, giving $L\,(j-P)^2 \le w \le L\,(j+P)^2$ with $L = W/(ab)$, and dividing by j^2 and letting $j\to\infty$ squeezes the lim inf onto L. □

Combining Theorems 3.1 and 3.2 with Proposition 2.3: a block certificate that passes its checks proves $\Phi_\alpha(k) \ge W/(ab)$. All lower bounds below are of this form.

4. An improved lower bound at $k = 10$

Theorem 4.1. $\Phi_4(10) \geq \dfrac{40}{59}$. *Moreover* $\dfrac{61}{90} < \dfrac{40}{59} < F_4(10)$*, so the bound improves the one printed in* [1, Fig. 2 and Table 1] *and is consistent with the strict inequality* $\Phi_4(10) < F_4(10) = \frac{20}{29}$ *of* [1, Lemma 7.4].

Proof. The witness is the image I_{10} of a block certificate with $a = b = 59$, $k = 10$, $D = 3$, arising from a colouring of the fundamental domain of the sublattice $\langle(59, 0), (17, 1)\rangle$ of index 59 that has 40 white cells; presented on the 59×59 block (Section 3.1) it has $W = 2360 = 40 \cdot 59$ white cells out of 3481. The three checks (3) hold for $\alpha = 4$, and $W = 2360$; both are finite computations, evaluated by compiled code as described in Section 8. Theorem 3.1 then gives $\mathrm{Comp}_{4,10}(I_{10})$, Theorem 3.2 gives $\rho(I_{10}) = 2360/3481 = 40/59$, and Proposition 2.3 concludes. The two comparisons are arithmetic: $40 \cdot 90 - 61 \cdot 59 = 3600 - 3599 = 1$, so $\frac{40}{59} - \frac{61}{90} = \frac{1}{5310} > 0$, and $40 \cdot 29 = 1160 < 1180 = 20 \cdot 59$. □

Remark 4.2 (how the witness was found, and how far the search reached). The certificate is checked, not trusted, so the search that produced it is outside the proof; we record it because it is the honest measure of how much room is left. The search space is a sublattice $L = \langle(a, 0), (c, d)\rangle$ of $\mathbb{Z}^2$ in Hermite normal form together with a white subset of its fundamental domain, with component size measured *in the universal cover*: the flood fill records, for each cell of the quotient, the lattice offset at which it was first reached, and a second arrival with a different offset marks the component infinite and rejects the colouring.

Exhaustive enumeration — every L of index at most 24, and for each of them every colouring of the fundamental domain — settles all indices up to 24; there the best achievable density at $k = 10$ is only $\frac{2}{3}$. Index 59 is far past exhaustive reach: a single lattice of that index already carries $\binom{59}{40} \approx 1.4 \times 10^{15}$ colourings of its fundamental domain with the white count needed here. The witness of Theorem 4.1 was therefore produced by ruin-and-recreate local search, run over every index from 25 to 64 and with dedicated high-iteration runs at indices 45, 60 and 90, always maintaining feasibility. It was then re-verified by two independent implementations before being turned into a certificate.

The measured limitation is worth stating: at index 90, where the published $\frac{61}{90}$ lives, the same local search plateaus at $\frac{59}{90}$ and never reproduces $\frac{61}{90}$. So $\frac{40}{59}$ is a find at a different index rather than a converged answer, and we make no claim that $\frac{40}{59}$ is optimal, nor that indices between 59 and 90 have been explored to exhaustion. Closing the cell $k = 10$ needs enumeration and an upper bound argument, not more annealing.

Finally the state of the literature, since what an improvement is worth depends on nothing better being in print. The source [1] is on arXiv in version v1 only — its submission history lists the single entry of 12 December 2025 — has no journal version, and has no citing or successor work of any kind, in particular none improving the lower bound at $k = 10$: checked 28 August 2026 against the arXiv abstract page, the arXiv listing of that author's papers, OpenAlex and Semantic Scholar, the last two reporting zero citations. "Not found" is not "not there", and we claim only the two checkable facts: $\frac{40}{59}$ is achievable, which is proved below, and it exceeds the number printed in [1, Fig. 2], which is arithmetic.

At $k = 11$ the same method reproduces the published bound but does not improve on it, and we record that as such.

Proposition 4.3. $\Phi_4(11) \geq \dfrac{31}{45}$, *and* $\dfrac{31}{45} < F_4(11) = \dfrac{20}{29}$.

Proof. A block certificate with $a = b = 45$, $k = 11$, $D = 4$, coming from a colouring with 31 white cells of the fundamental domain of $\langle(45, 0), (19, 1)\rangle$ of index 45, passes the three checks for $\alpha = 4$ and has $W = 1395 = 31 \cdot 45$ white cells in the 45×45 block, so $\rho = 1395/2025 = 31/45$; conclude as in Theorem 4.1. Finally $31 \cdot 29 = 899 < 900 = 20 \cdot 45$. □

This is exactly the value printed in [1, Fig. 2], and it was found here by a dedicated high-iteration run at index 45, independently of the figure. We stress that $\frac{31}{45}$ is *not* new; the only reason to record

Proposition 4.3 is that it is the source's own bound, recovered by the same search that produced Theorem 4.1, which is some evidence that the search is not systematically weak at these indices. Both cells remain open:

$$\frac{40}{59} \ \le\ \Phi_4(10)\ <\ \frac{20}{29}, \qquad \frac{31}{45}\ \le\ \Phi_4(11)\ <\ \frac{20}{29},$$

where the strict upper bounds are [1, Lemma 7.4] and are not reproved here.

5. The published tables, reproduced

Theorem 5.1. *For $k = 1, \dots, 9$ respectively,*

$$\Phi_4(k)\ \ge\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{13}{24},\ \tfrac{4}{7},\ \tfrac{5}{8},\ \tfrac{5}{8},\ \tfrac{7}{11},\ \tfrac{2}{3},\ \tfrac{2}{3}.$$

Theorem 5.2. *For $k = 1, \dots, 12$ respectively,*

$$\Phi_8(k)\ \ge\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{4}{9},\ \tfrac{5}{11},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{8}{15},\ \tfrac{9}{16},\ \tfrac{9}{16},\ \tfrac{11}{19},\ \tfrac{3}{5},$$

and each of these twelve numbers equals $F_8(k)$ as defined in (1)*.*

Proof sketch for Theorems 5.1 and 5.2. Each of the twenty-one bounds is one block certificate, checked as in Theorem 4.1: three finite checks and one white count, then Theorems 3.1 and 3.2 and Proposition 2.3. The rectangular blocks used have $4, 4, 576, 49, 64, 64, 121, 48, 48$ cells in the 4-connected case, and $8, 18, 32, 27, 121, 48, 48, 75, 64, 64, 361, 100$ cells in the 8-connected case; the underlying period lattices have index at most 24 and 20 respectively, as in Remark 5.3. The last assertion of Theorem 5.2 is the evaluation of the list $(F_8(1), \dots, F_8(12))$ from the definition (1), a finite rational computation. □

By [1, Theorem 1.1 and Theorem 8.1] the nine numbers of Theorem 5.1 are the exact values of Φ_4 — seven of them because $k \in S$, and $k = 3, 6$ by the hand computation of [1, Theorem 8.1] — and by [1, Corollary 7.2] the twelve numbers of Theorem 5.2 are the exact values of Φ_8. The matching upper bounds are not proved here.

Remark 5.3. The witnesses behind Theorems 5.1 and 5.2 were not read off the figures of [1]. They come from the exhaustive sweep of Remark 4.2: over all sublattices of index at most 24 for $\alpha = 4$, and at most 20 for $\alpha = 8$, the maximum achievable densities are exactly the numbers listed above. That is an independent reproduction of both published tables, including the two cells $\Phi_4(3) = \frac{13}{24}$ and $\Phi_4(6) = \frac{5}{8}$ that [1] computes by hand. The sweep must range over Hermite normal forms and not only over axis-aligned rectangles: the optimum at $k = 3$ sits on the lattice generated by $(24, 0)$ and $(5, 1)$, and a sweep over rectangular periods alone would have missed $\frac{13}{24}$. The two reach figures — index at most 24 for $\alpha = 4$, at most 20 for $\alpha = 8$ — are the reach recorded when that enumeration was run, on 22 August 2026, and are not re-run here; what is asserted is exhaustiveness up to those indices and no further. It is the sweep, not any witness, that sits outside the formal development: each witness is checked in Lean.

6. The elementary upper bounds, and four exact cells

The upper bounds of [1] are structural. There is however one elementary route, the "averaging argument for the density" that [1, §1] uses in its warm-up. We state it in full generality, for an arbitrary window shape and an arbitrary budget; an $r \times s$ *window* is a block of rs cells with a given lower-left corner.

Theorem 6.1. *Let $r, s \ge 1$ and $c \ge 0$. If every $r \times s$ window of I contains at most c white cells, then $\rho(I) \le c/(rs)$. Consequently, if every image satisfying $\mathrm{Comp}_{\alpha,k}$ has at most c white cells in every $r \times s$ window, then $\Phi_\alpha(k) \le c/(rs)$.*

Proof sketch. Because the boxes of Definition 2.2 have odd side, they are never tiled exactly by $r \times s$ windows and a boundary term is unavoidable. Cover the box $[-m, m]^2$ by the windows anchored at the corners $q - ((q_1 + m) \bmod r,\ (q_2 + m) \bmod s)$. The map sending a cell to its window corner has image of size at most $(\lfloor 2m/r \rfloor + 1)(\lfloor 2m/s \rfloor + 1)$, and the fibre over each corner is contained in the window there, hence carries at most c white cells; so $w(I, m) \le c(\lfloor 2m/r \rfloor + 1)(\lfloor 2m/s \rfloor + 1)$. Dividing by $(2m+1)^2$, absorbing the boundary into $(2m+1+t)^2$ with $t = \max(r, s) - 1$ and letting $m \to \infty$ gives the bound. The transfer to $\Phi_\alpha(k)$ needs the defining set to be nonempty, which the all-black image supplies, and bounded, which is Proposition 2.3. □

Three instances follow, each general in its parameter. The tool for all three is that a window fully occupied by white cells produces a *connected* white set: it suffices to order the cells so that each is α-adjacent to p or to a later one, which covers every connected configuration and not only those with a Hamiltonian path.

Theorem 6.2. $\Phi_\alpha(k) \le \dfrac{k}{k+1}$ *for every α and every k. Equivalently, every image of white density greater than $k/(k+1)$ has a white α-connected component of more than k cells.*

Proof. Apply Theorem 6.1 with the $(k+1) \times 1$ window and budget k: if all $k+1$ cells of a horizontal row of length $k+1$ are white, they form a single white α-connected component of size $k+1$, contradicting $\mathrm{Comp}_{\alpha,k}$. □

Theorem 6.3. $\Phi_8(k) \le \dfrac{k}{4}$ *for every k.*

Proof. The four cells of a 2×2 window are pairwise 8-adjacent, so its white cells form a single 8-connected component; hence at most k of them are white, and Theorem 6.1 with $r = s = 2$, $c = k$ applies. □

Theorem 6.4. $\Phi_4(k) \le \dfrac{1}{2}$ *for $k \le 2$. Equivalently, an image of white density greater than $\frac{1}{2}$ has a white 4-connected component of at least three cells.*

Proof. Any three distinct cells of a 2×2 window contain a 4-path of length three: the two diagonal pairs are the only non-adjacent pairs, so among three cells one is 4-adjacent to the other two. Hence at most two cells of a 2×2 window are white when $\mathrm{Comp}_{4,2}$ holds, and Theorem 6.1 gives $\Phi_4(2) \le \frac{2}{4}$; monotonicity of Φ_4 in k covers $k \le 2$. □

Theorem 6.4 is the second half of the warm-up of [1, §1], and Theorem 6.3 at $k = 1$ is the case $k = 1$ of [1, Corollary 7.2]. Combined with the witnesses of Theorem 5.1 and Theorem 5.2 they pin three values exactly, and Theorem 6.2 at $k = 0$ pins a fourth with no search at all.

Theorem 6.5. $\Phi_4(1) = \Phi_4(2) = \frac{1}{2}$, $\Phi_8(1) = \frac{1}{4}$, *and $\Phi_\alpha(0) = 0$ for every α. Moreover $\Phi_4(1) = F_4(1)$, $\Phi_4(2) = F_4(2)$ and $\Phi_8(1) = F_8(1)$, which are the instances of (1) at these k.*

Proof. Antisymmetry: Theorem 6.4 against the checkerboard witness of Theorem 5.1 at $k = 1, 2$; Theorem 6.3 at $k = 1$ against the witness of Theorem 5.2 at $k = 1$. For $k = 0$, Theorem 6.2 gives $\Phi_\alpha(0) \le 0$ and the all-black image, which satisfies every component bound vacuously and has density 0, gives the reverse. The identifications with F_4, F_8 are the evaluations $F_4(1) = F_4(2) = \frac{1}{2}$ and $F_8(1) = \frac{1}{4}$, obtained from Proposition 2.4 with $\lceil 2\sqrt{1}\, \rceil = 2$ and $\lceil 2\sqrt{3}\, \rceil = 4$. □

None of Theorem 6.5 is new: $\Phi_4(1) = \Phi_4(2) = \frac{1}{2}$ is [1, §1], and $\Phi_4(1) = \frac{1}{2}$ is in any case the maximum density of an independent set in the grid graph, which is bipartite with classes of density $\frac{1}{2}$, while $\Phi_8(1) = \frac{1}{4}$ is the classical maximum density of pairwise non-attacking kings. What is contributed is that all four hold for arbitrary colourings of $\mathbb{Z}^2$ under Definition 2.2 and are checked by a proof assistant.

Finally we record the one case of the isoperimetric ingredient that is proved here.

Proposition 6.6. *Let α be arbitrary and suppose every white α-component of I is a single cell. Then every α-neighbour of a white cell is black.*

Proof. If p and an α-neighbour $q \neq p$ were both white, then $\{p, q\}$ is a set of two white cells connected to p, contradicting $\mathrm{Comp}_{\alpha,1}$. □

This is σ_α at $j = 1$, the one case where the site-perimeter bound [1, Lemmas 2.11, 2.13] is tight against a window argument. The general $\sigma_\alpha(k)$ is not formalised here.

Remark 6.7 (the reach of window averaging). It is worth saying how far Theorem 6.1 can go, since the answer bounds how much of [1] an elementary argument could recover. The following is an unverified Python computation, run on 22 August 2026 outside the formal development and not machine-checked; it is stated as a measurement, not as a theorem, and a referee who wants to rely on it must re-run it. For every rectangle with $rs \leq 16$ and every colouring of it, take the largest white count whose in-window α-components all have at most k cells; this is a necessary condition on any window of a valid image, so the resulting $N/(rs)$ is always a legitimate upper bound on $\Phi_\alpha(k)$. Maximising over shapes, the best bound obtainable at $rs \leq 16$ is $\frac{1}{2}$ at $(\alpha, k) = (4, 1)$ and $(4, 2)$ and $\frac{1}{4}$ at $(8, 1)$ — tight in all three cases — and is strictly weaker than the value printed in [1] at every other (α, k) with $k \leq 9$. At $(4, 3)$, for instance, the best area-16 rectangle gives $\frac{7}{12}$ against the true $\frac{13}{24}$. So window averaging settles exactly the three cells of Theorem 6.5 and nothing else at that reach, which is consistent with the remark in [1, §1] that the warm-up "does not appear to generalize". The structural reason is visible in the extremal configurations: already at $k = 3$ the optimum has period lattice of index 24, which no window of area 16 can see.

7. Negative controls

A development in which every theorem is an inequality about a supremum can be satisfied by a misplaced quantifier while proving nothing. We therefore record explicitly that the statements above are not vacuous and that their hypotheses are load-bearing, beginning with the witnesses.

Proposition 7.1. *Let I_3 be the $k = 3$ witness of Theorem 5.1, of density $\frac{13}{24}$. Then:*

(1) *$\rho(I_3) \neq \frac{4}{7}$ and $\rho(I_3) \neq \frac{1}{2}$, and $\frac{1}{2} < \frac{13}{24} < \frac{4}{7}$, so neither refutation is vacuous: the witness reaches neither the next tabulated value above nor the one below.*

(2) *Turning one black cell of its block white makes check* (adj) *fail, and it really does raise the white count of the block, from* 312 *to* 313. *Asking the same colouring to certify $k = 2$ makes check* (fib) *fail.*

(3) *Let J be the horizontal stripe image $J(x, y) = [\,y \text{ even}]$. Then $\rho(J) = \frac{1}{2}$, the same density as the genuine $\Phi_4(1)$ witness, so density alone distinguishes nothing; but* $\mathrm{Comp}_{4,3}(J)$ *is* false, *and check* (adj) *rejects J.*

(4) $(0, 0)$ *and* $(1, 1)$ *are* 8*-adjacent and not* 4*-adjacent, so the two adjacency relations of Proposition 2.1 are genuinely different.*

Proof. (1) and (4) are arithmetic and a two-case check. (2) is a finite evaluation of the checks on the perturbed block. For (3): on the 1×2 block the white part of J is a single cell, so a check performed on the quotient torus would accept J at $k = 1$; but in $\mathbb{Z}^2$ the cells $(0, 0), (1, 0), (2, 0), (3, 0)$ are white and successively 4-adjacent, and they are four distinct cells all connected to $(0, 0)$, refuting $\mathrm{Comp}_{4,3}(J)$. □

Part (3) is the reason for the design of Section 3: it is the exact failure mode a torus-local count would exhibit, it is invisible to the density, and the certificate rejects it.

The second group concerns the upper bounds.

Proposition 7.2. (1) *Not too small: $\Phi_4(1)$ and $\Phi_4(2)$ are not less than $\frac{1}{2}$, and $\Phi_8(1)$ is not less than $\frac{1}{4}$.*

(2) *Not too large: $\Phi_4(2) \neq \frac{13}{24}$, $\Phi_4(1) < \frac{2}{3}$ and $\Phi_8(1) < \frac{1}{3}$, so the upper bounds of Section 6 bite.*

(3) *The component hypothesis is not removable: the all-white image has density* 1 *and fails* $\mathrm{Comp}_{\alpha,k}$ *for every* α *and* k*, so no unconditional density bound below* 1 *holds.*
(4) *The window budgets are sharp: the budget* 2 *for the* 2×2 *window at* $\alpha = 4$*,* $k = 2$ *cannot be lowered to* 1*; the budget* k *for the* 2×2 *window at* $\alpha = 8$ *cannot be lowered at* $k = 1$*; the row budget* k *of Theorem 6.2 cannot be lowered at* $k = 1$*.*
(5) *The ranges of* k *are sharp:* $\Phi_4(k) \le \frac{1}{2}$ *fails at* $k = 3$ *and* $\Phi_8(k) \le \frac{1}{4}$ *fails at* $k = 2$*, so the hypotheses of Theorems 6.4 and 6.3 are doing work; and the row bound of Theorem 6.2 is not tight at* $\alpha = 4$*,* $k = 2$*, where* $\Phi_4(2) = \frac{1}{2} < \frac{2}{3}$*.*

Proof. (1), (2) and (5) follow from Theorem 6.5 and the witnesses of Theorem 5.1 at $k = 3$ and Theorem 5.2 at $k = 2$. For (3), the all-white image has $w(I, m) = (2m + 1)^2$ for every m, so its density is 1, and a row of $k + 1$ white cells refutes $\mathrm{Comp}_{\alpha,k}$. For (4), each lowered budget would give through Theorem 6.1 an upper bound strictly below the corresponding witness density. □

8. Verification

Every statement in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [5, 6]; the development is free of `sorry`, and the repository reference is to be added at submission. The reasoning layer is kernel-checked with no extra axioms: Proposition 2.1, Proposition 2.3, Proposition 2.4, the certificate soundness Theorem 3.1, the density evaluation Theorem 3.2, the averaging Theorem 6.1 and its three instances Theorems 6.2, 6.3, 6.4, the value $\Phi_\alpha(0) = 0$, Proposition 6.6, the refutation of $\mathrm{Comp}_{4,3}$ for the stripe image in Proposition 7.1(3) and Proposition 7.2(3) all depend on nothing beyond `propext`, `Classical.choice` and `Quot.sound`. In particular no upper bound in this paper uses compiled evaluation at any point; the exact values of Theorem 6.5 inherit only the evaluation axioms of the witness supplying their lower half.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite data checks, two per witness: the conjunction of the three checks (3) on the cells of one block, and the white count W of that block. There are twenty-three witnesses in all — nine for Theorem 5.1, twelve for Theorem 5.2, one for Theorem 4.1 and one for Proposition 4.3. A few further Booleans are evaluated the same way: the list of twelve values of F_8 in Theorem 5.2; the comparisons of $\frac{40}{59}$ and $\frac{31}{45}$ against $F_4(10)$ and $F_4(11)$ in Theorem 4.1 and Proposition 4.3; and, in Proposition 7.1, the perturbed-block checks and the white count of part (2), together with the density and the rejection of the stripe image in part (3). Each individual density bound therefore carries exactly the two evaluation axioms of its own witness, and nothing else.

The searches that *produce* the witnesses are outside the verified perimeter, by design: their output is a certificate, and the certificate is checked. The two witnesses of Theorem 4.1 and Proposition 4.3 were each re-verified by two independent implementations, an exhaustive C checker and a separate re-implementation in another language, before being turned into certificates. The two sweeps of Remarks 5.3 and 6.7 are likewise outside the formal development, and each is labelled as such where it appears.

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