THE MOST COMMON DEGREE SEQUENCE OF A LABELLED GRAPH

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Abstract. Draw a graph on the labelled vertex set $\{1, \ldots, n\}$ uniformly at random — equivalently, sample the Erdős–Rényi model $G(n, 1/2)$ — and record the multiset of its vertex degrees. Which multiset is the most likely one? Ekhad and Zeilberger asked and answered the corresponding question for round-robin tournaments up to $n = 15$, continuing a programme of MacMahon; for graphs we could not find the question asked anywhere, and its natural guess is wrong. We determine, for every $n \le 12$, the exact maximum of $N(d)$, the number of labelled graphs whose degree sequence is d, together with the complete set of maximisers. At $n = 9$, for instance, the unique maximiser is $(6, 5, 5, 4, 4, 4, 3, 3, 2)$ with 941 371 200 realisations, while the 4-regular sequence — the naive answer — has only 1 024 380, smaller by a factor of about 919. Over the computed range the maximiser set has one, two or three elements and is always closed under $d \mapsto (n-1)-d$. We determine, for the same range, the exact number of labelled graphs whose maximum degree is attained by a single vertex: the finite counterpart of the result of Erdős and Wilson that almost all graphs have such a vertex. Both are contrasted with the degree-*vector* question, whose answer is a (near-)regular sequence and whose asymptotic form is known. The method is a dynamic program that seats one vertex at a time and carries only the sorted multiset of the degrees so far; its state space is the set of realisable degree sequences (222 117 of them at $n = 12$) rather than the 2^{66} graphs. Every result stated here is machine-checked in Lean 4.

1. Introduction

The objects. Fix $n \ge 1$ and write $[n] = \{1, \ldots, n\}$. A *labelled graph* on $[n]$ is a choice, for each of the $\binom{n}{2}$ unordered pairs of distinct vertices, of whether that pair is an edge. We write $\mathcal{G}_n$ for the set of these, so that

$$|\mathcal{G}_n| \;=\; 2^{\binom{n}{2}},$$

and drawing an element of $\mathcal{G}_n$ uniformly at random is exactly sampling the Erdős–Rényi model $G(n, 1/2)$. The *degree sequence* of $G \in \mathcal{G}_n$ is the multiset $D(G) = \{\{\deg_G(1), \ldots, \deg_G(n)\}\}$; the *degree vector* is the ordered tuple $\vec{D}(G) = (\deg_G(1), \ldots, \deg_G(n))$. For a multiset d of n natural numbers and a vector $v \in \mathbb{N}^n$ put

$$N(d) = \#\{G \in \mathcal{G}_n : D(G) = d\}, \qquad N^{\mathrm{v}}(v) = \#\{G \in \mathcal{G}_n : \vec{D}(G) = v\}.$$

Thus $N(d)/2^{\binom{n}{2}}$ is the probability that $G(n, 1/2)$ has degree sequence d. The two functions differ by the number of orderings of d: summing N^{v} over the distinct orderings of d gives $N(d)$, and since relabelling the vertices acts transitively on those orderings,

$$N(d) \;=\; N^{\mathrm{v}}(v) \cdot \frac{n!}{\prod_x m_x(d)!}, \qquad v \text{ any ordering of } d, \tag{1}$$

where $m_x(d)$ is the multiplicity of x in d. Identity (1) is classical; it is used in this paper only where it is quoted explicitly, and it is *not* part of the formal development (Section 5).

Two statistics of the uniform distribution on $\mathcal{G}_n$ are the subject of this paper.

Question 1.1. Which degree sequence d maximises $N(d)$? What is the maximum, and how many maximisers are there?

Question 1.2. How many labelled graphs on n vertices have a *unique* vertex of maximum degree?

Where the questions come from. Question 1.1 is in print for tournaments. Ekhad and Zeilberger [2] compute, for round-robin tournaments on $n \le 15$ players, the most commonly occurring score vectors, explicitly continuing MacMahon's programme [7]: MacMahon "considered round-robin tournaments for n players for n from 2 to 9, and for each found all the possible score vectors, and determined the most popular one, as well as the number of times it shows up", and "[i]n this article, we continue this work up to $n = 15$". Their answer is indexed as sequence A274098 of [11]. A graph is the same kind of object with a different match rule — each pair of vertices meets once, and the two outcomes are "no edge" and "edge" — so the question transfers verbatim. We were unable to find it asked: searches of the OEIS on the modal values and on the number of maximisers, full-text searches of the arXiv for the phrasings *most common / modal / most likely degree sequence*, a sweep of a local arXiv full-text corpus for co-occurrences of those phrasings with *degree sequence*, and a scan of Zeilberger's personal-journal index all returned nothing for graphs. The one substantive hit, Krivelevich, Sudakov and Wormald [6], is about degree *vectors* and is asymptotic; we discuss it in Section 5, where it is prior art for a question with a different answer.

Question 1.2 has a famous asymptotic answer and, as far as we could find, no exact one. Erdős and Wilson [3] proved — it is the lemma of their note, and they prove it for labelled graphs first — that almost all graphs have a unique vertex of maximum degree, and used it to show that almost all graphs are of class one; the statement is restated in that form by McDiarmid and Yolov [9]. Its $G(n,p)$ form is Theorem 3.9(i) of Bollobás [1]: if $p \le \frac{1}{2}$ and $pn/\log n \to \infty$, then almost every G_p has a unique vertex of maximum degree (and a unique vertex of minimum degree). The tournament analogue — a unique player with the maximum score — was settled by Malinovsky and Moon [8], and its exact counts are sequence A013976 of [11], known to $n = 20$. The corresponding column for graphs is what Theorem 4.2 supplies. In the literature we surveyed, "unique vertex of maximum degree" occurs as a structural *hypothesis* — in edge colouring, path decomposition, reconstruction, antimagic labelling — and never as something to count.

The set of labelled realisations of a fixed degree sequence is itself a current object of study: Hladík and Fink [4] prove that the *realization graph* of every degree sequence has a Hamilton path. Its vertex set has size $N(d)$. That paper neither counts the set nor optimises over d; we mention it because it shows the object is live, not because it is prior art.

What is settled here. For every $n \le 12$ we determine the maximum of N over all degree sequences, the exact set of maximisers (Theorem 3.1), and the number of labelled graphs with a unique maximum-degree vertex (Theorem 4.2). The answers, and the two features that make Question 1.1 more than a table lookup, are:

- *The mode is not the regular sequence, and not the balanced one.* At $n = 9$ the modal sequence is $(6,5,5,4,4,4,3,3,2)$ with 941 371 200 realisations, while the 4-regular sequence has 1 024 380: a factor of about 919. The reason is (1): N counts every ordering of d, so a sequence with many distinct values carries a large multinomial factor, and the mode sits where that factor and the per-vector probability balance. Where that balance falls is not guessable, and it is not at a regular sequence.
- *The mode is often not unique, and the maximisers come in complementary pairs.* Writing d^c for the sequence $(n-1) - d$ obtained by complementing the graph, the maximiser set is closed under $d \mapsto d^c$ at every $n \le 12$; it has 2 or 3 elements at $n = 2,3,4,6,7,10,11$ and one self-complementary element at $n = 1,5,8,9,12$. The

number of maximisers, $1, 2, 2, 3, 1, 2, 2, 1, 1, 2, 2, 1$ for $n = 1, \ldots, 12$, is itself absent from [11].

The supporting statements are of three kinds. The reduction to a dynamic program, and the argument that turns the largest weight in a computed table into a maximum over *all* degree sequences, are proved in general (Section 2). The degree-vector companion of Question 1.1 is settled exactly where it needs no multinomial correction, namely at constant vectors, where it reproduces the published counts of regular labelled graphs (Section 5); this is deliberate, since that question — unlike Question 1.1 — does have prior art. Finally the computation is anchored against published data and against independent enumerations (Sections 6 and 7).

Every result below is machine-checked in Lean 4 (Section 9), the exceptions being flagged where they occur; the reasoning layer depends on no axioms beyond the three standard ones, and each numerical answer carries exactly one further axiom, recording that compiled evaluation of the dynamic program at that n returned the stated table.

2. Preliminaries: the distribution and the dynamic program

2.1. **Graphs as staircases.** A labelled graph on $[n]$ is a symmetric loopless 0–1 matrix, and we encode it by the strict upper triangle read one row at a time: a *staircase* $t = (r_1, \ldots, r_n)$ in which the row r_i has length $n - i$ and each entry is one of the two outcomes "no edge" or "edge". The k-th entry of r_i records whether i and $i + k$ are adjacent. This is the same encoding used for round-robin tournaments, with the two-element outcome list

$$\mathcal{O} = \{(0,0),\ (1,1)\}$$

— a non-edge awards nothing to either endpoint, an edge awards one to each — in place of the win/loss list. It is worth noting that this outcome list is not of the shape usually assumed in the tournament setting: the two endpoints' gains sum to 0 or to 2, never to a constant. Nothing in the argument below ever assumed they did.

Proposition 2.1 (the bridge)**.** *For every n, the map sending a staircase to its adjacency relation is a bijection from the set of staircases of height n over $\mathcal{O}$ onto $\mathcal{G}_n$; under it, the vector of accumulated scores of a staircase is the vector of vertex degrees of the corresponding graph. Consequently $|\mathcal{G}_n| = 2^{\binom{n}{2}}$, and $\mathcal{G}_n$ with this notion of degree is the set of simple graphs on an n-element vertex set with its usual degree function.*

Proof sketch. Symmetry, looplessness and support are read off the encoding by induction on n. Injectivity is the same induction: the first row of the staircase is determined by the adjacencies of vertex 1, and the rest by the induction hypothesis applied to the graph with vertex 1 deleted. Surjectivity is the reverse induction. The count follows from the bijection and the fact that the staircase has $\sum_{i<n}(n-i) = \binom{n}{2}$ entries, each with two possible values. That the accumulated scores are the degrees is an induction on the same decomposition. In the formal development the last clause is proved by identifying the Boolean-matrix presentation with the ambient library's type of simple graphs on n vertices, through the library's own constructor, and its degree function with ours; the cardinality statement is a theorem valid at every n, not a check performed at each n separately. □

2.2. **The dynamic program.** Seat the vertices one at a time. If the vertices already seated have degree multiset μ (as a graph on its own), and a newcomer decides an edge with each of them, the resulting multiset of extended degree multisets depends on μ alone — not on which vertex had which degree, and not on the graph that produced μ. Carrying the sorted degree multiset as the state is therefore enough, and the number of states after n vertices is the number of realisable degree sequences on n vertices: 1, 2, 4, 11, 31, 102, 342, 1213, 4361, 16 016, 59 348, 222 117 for $n = 1, \dots, 12$, sequence A004251 of [11], against 2^{66} graphs at $n = 12$.

Write $\mathcal{T}(n)$ for the resulting weighted list of sorted degree sequences: an entry (s, c) says that c labelled graphs on n vertices have sorted degree vector s.

Proposition 2.2 (correctness and transfer)**.** *For every n and every multiset d, the weight that $\mathcal{T}(n)$ carries at d is exactly $N(d)$. More generally, if P is a predicate on degree lists that is invariant under permutation, then*

$$\#\{G \in \mathcal{G}_n : P(\vec{D}(G))\} \;=\; \sum_{(s,c)\in\mathcal{T}(n),\; P(s)} c.$$

In particular the total weight of $\mathcal{T}(n)$ is $2^{\binom{n}{2}}$ at every n.

Proof sketch. Induction on n, with no computation anywhere. The one substantial step is the permutation lemma: if two degree lists agree as multisets, their weighted lists of one-vertex extensions agree; it is proved by induction over the permutation relation, the transposition case being commutativity of multiset insertion together with commutativity of addition. Two bookkeeping facts complete the induction: the enumeration of all ways a newcomer can meet j seated vertices matches the extension operator, and the sorting-and-merging step that makes the program fast is invisible to the represented multiset, so no part of the argument depends on how it is implemented. The transfer statement follows by pushing P through the represented multiset; the total-weight statement is the case $P \equiv$ true together with Proposition 2.1. The general statement is inherited unchanged from the corresponding tournament development [5], where it is proved for an arbitrary outcome list; specialising the outcome list to $\mathcal{O}$ is all that Question 1.1 needs. $\square$

The total-weight clause deserves emphasis: it is an identity, proved once for all n, not a numerical coincidence checked at each n. It is the gate on the whole construction — a state space that lost or double-counted graphs could not have total weight $2^{\binom{n}{2}}$.

2.3. **From the largest weight to the maximum.** The largest weight appearing in $\mathcal{T}(n)$ is $\max_d N(d)$ only if no two of its keys are rearrangements of one another; checking that pairwise would be quadratic in the 222 117 keys at $n = 12$. Instead the computation emits, in the same pass, a certificate: (i) every key is a sorted list of length n with positive weight, and (ii) the base-$(n+1)$ numeral codes of the keys, after sorting, strictly increase. Item (ii) forces the codes, hence the keys, to be pairwise distinct — no injectivity of the encoding is needed, only that distinct codes come from distinct keys — and with (i) two distinct sorted keys cannot be rearrangements of each other.

Proposition 2.3. *Suppose the certificate holds for $\mathcal{T}(n)$ and let B be its largest weight, $B > 0$. Then $N(d) \le B$ for every degree sequence d; the keys of weight B realise $N = B$; every d with $N(d) = B$ is one of them; and $N(d) \neq 0$ if and only if d is a key. In particular the length of $\mathcal{T}(n)$ is the number of realisable degree sequences.*

Proof sketch. By the certificate at most one key rearranges to a given sequence, so Proposition 2.2 evaluates $N(d)$ as the weight of that key, or as 0 if there is none. The four clauses are then immediate. Nothing here is computed: the certificate is a hypothesis, discharged for each $n \le 12$ by the same evaluation that produces the table. $\square$

Proposition 2.4 (the computed tables)**.** *For each $n \le 12$ one evaluation of the dynamic program yields the table $\mathcal{T}(n)$ and, read off it: its total weight, its length, its largest weight and the keys attaining it, the largest value of $N(d)\prod_x m_x(d)!$ and the keys attaining that, the number of graphs with a unique maximum-degree vertex, and the certificate of Section 2.3. In every case the certificate holds and the total weight is $2^{\binom{n}{2}}$.*

Proposition 2.4 is the only place in this paper where a machine evaluates anything large; Theorems 3.1 and 4.2 are what it says about graphs.

3. The most common degree sequence

Theorem 3.1. *For each $n \le 12$, let $M(n)$ be the value and $\mathcal{M}(n)$ the list of sequences in Table 1. Then for every multiset d of n natural numbers,*

$$N(d) \;\le\; M(n),$$

every $d \in \mathcal{M}(n)$ satisfies $N(d) = M(n)$, and every d with $N(d) = M(n)$ lies in $\mathcal{M}(n)$.

n	modal degree sequence(s)	$M(n)$
1	(0)	1
2	$(0,0)$, $(1,1)$	1
3	$(1,1,0)$, $(2,1,1)$	3
4	$(2,1,1,0)$, $(2,2,1,1)$, $(3,2,2,1)$	12
5	$(3,2,2,2,1)$	120
6	$(3,3,3,2,2,1)$, $(4,3,3,2,2,2)$	1 620
7	$(4,3,3,3,2,2,1)$, $(5,4,4,3,3,3,2)$	67 620
8	$(5,4,4,4,3,3,3,2)$	5 812 800
9	$(6,5,5,4,4,4,3,3,2)$	941 371 200
10	$(6,6,5,5,4,4,4,3,3,2)$, $(7,6,6,5,5,5,4,4,3,3)$	214 857 921 600
11	$(7,6,6,6,5,5,5,5,4,4,3)$, $(7,6,6,5,5,5,5,4,4,4,3)$	157 423 180 622 400
12	$(8,7,7,6,6,6,5,5,5,4,4,3)$	225 863 950 348 713 600

Table 1. The mode of the degree-sequence distribution of labelled graphs, $n \le 12$. Sequences are written in decreasing order. $M(n)$ is the maximum of N; the sequences listed are all of its maximisers.

Proof sketch. Fix $n \le 12$. Proposition 2.4 supplies the table $\mathcal{T}(n)$, its largest weight $M(n)$, the list of keys attaining it, and the certificate; the certificate holds and $M(n) > 0$. Proposition 2.3 then gives all three clauses at once, with Proposition 2.2 translating between the table's weights and N, and Proposition 2.1 between the encoded objects and graphs.

The exhaustive part is exactly this: for each $n \le 12$, one evaluation of the dynamic program over its whole state space — 222 117 states at $n = 12$, A004251(n) in general — together with the eight readings taken from the resulting table in the same pass. No search over the $2^{\binom{n}{2}}$ graphs is performed at any $n \ge 5$, and no search over degree sequences is performed at all: the maximum and the maximiser set are read from the table, and it is Proposition 2.3 that makes them statements about every degree sequence. Everything outside those twelve evaluations — the bridge, the correctness of the dynamic program, the certificate argument — is proved for general n. $\square$

Remark 3.2 (why the answer is not the regular sequence). The first guess for Question 1.1 is a regular or near-regular sequence, and it is wrong. At $n = 9$ the 4-regular sequence is realised by 1 024 380 graphs (Proposition 5.2) against 941 371 200 for the modal one, a factor of 918.9...; the ratio grows with n over the computed range, and at $n = 12$ the best regular sequence accounts for 2 977 635 137 862 graphs against 225 863 950 348 713 600 at the mode, a factor of about 7.6×10^4. By (1) the modal d maximises $N^{\mathrm{v}}(v)$ times the number of orderings of d, and the regular sequence, having exactly one ordering, is the worst possible case for the second factor. The two effects pull in opposite directions and the balance point is irregular. This refutation is recorded as a control (Proposition 7.1(3)), because without it the question could be mistaken for the classical problem of counting regular graphs.

Remark 3.3 (complementary pairs). Complementation $G \mapsto \overline{G}$ is an involution of $\mathcal{G}_n$ sending a graph with degree sequence d to one with degree sequence $d^c := (n-1) - d$, so $N(d) = N(d^c)$ for every d and every n; consequently the maximiser set is closed under $d \mapsto d^c$. This is elementary and we state it as a remark rather than as one of our results: it is not part of the formal development, and Table 1 exhibits it rather than depending on it. What is not forced, and is visible only in the data, is which n have a self-complementary unique maximiser ($n = 1, 5, 8, 9, 12$) and which have a complementary pair or triple ($n = 2, 3, 4, 6, 7, 10, 11$); at $n = 4$ the three maximisers are a complementary pair together with one self-complementary sequence.

4. Graphs with a unique vertex of maximum degree

Say $G \in \mathcal{G}_n$ has a *unique maximum* if exactly one vertex attains $\max_i \deg_G(i)$. This property depends only on the degree sequence, so Proposition 2.2 applies.

Proposition 4.1. *For every n, the number of labelled graphs on n vertices with a unique maximum is the sum of the weights of those keys of $\mathcal{T}(n)$ whose largest entry occurs exactly once.*

Theorem 4.2. *For $n = 1, \ldots, 12$ the number $U(n)$ of labelled graphs on n vertices whose maximum degree is attained by exactly one vertex is*

$$1,\ 0,\ 3,\ 28,\ 525,\ 17\,358,\ 1\,143\,009,\ 149\,704\,744,\ 39\,094\,644\,969,$$

$$20\,369\,286\,719\,130, \quad 21\,184\,886\,383\,350\,899, \quad 43\,996\,858\,503\,224\,168\,628,$$

respectively; the corresponding sample-space sizes $2^{\binom{n}{2}}$ are listed in Table 2.

Proof sketch. Proposition 4.1 identifies $U(n)$ with a statistic of the table, and Proposition 2.4 evaluates that statistic in the same pass that produces the mode. The finite computation is the one already described in the proof of Theorem 3.1; no further search is involved. The value at $n = 2$ is 0, both vertices having degree 0 or both degree 1. $\square$

For $2 \leq n \leq 12$ the ratio $U(n)/2^{\binom{n}{2}}$ increases, from 0 at $n = 2$ to 0.5962... at $n = 12$, consistently with the result of Erdős and Wilson [3] — whose proof establishes the labelled form first — that it tends to 1; the approach is slow, which is one reason exact values in the accessible range seem worth having. For comparison, the tournament analogue — the number of round-robin tournaments with a unique maximum score, studied by Malinovsky and Moon [8] — is sequence A013976 of [11] and is known to $n = 20$.

n	$2^{\binom{n}{2}}$	realisable sequences	$U(n)$
1	1	1	1
2	2	2	0
3	8	4	3
4	64	11	28
5	1 024	31	525
6	32 768	102	17 358
7	2 097 152	342	1 143 009
8	268 435 456	1 213	149 704 744
9	68 719 476 736	4 361	39 094 644 969
10	35 184 372 088 832	16 016	20 369 286 719 130
11	36 028 797 018 963 968	59 348	21 184 886 383 350 899
12	73 786 976 294 838 206 464	222 117	43 996 858 503 224 168 628

Table 2. The sample space, the number of realisable degree sequences (Proposition 6.1, sequence A004251) and the number of graphs with a unique vertex of maximum degree (Theorem 4.2).

5. The degree-vector question, and regular graphs

Question 1.1 asks for the most likely degree *multiset*. The companion question — which degree *vector* is most likely — has a different answer and, unlike Question 1.1, has prior art. Krivelevich, Sudakov and Wormald [6] prove (Lemma 2.2 of the e-print) that for *every* degree sequence $\mathbf{d} = (d_1, \ldots, d_k)$,

$$\mathbb{P}\big[G(k, 1/2) \text{ has degree sequence } \mathbf{d}\big] = O(p_k), \qquad p_k = \mathbb{P}\big[G(k, 1/2) \text{ is } \lfloor k/2\rfloor\text{-regular}\big],$$

so that, asymptotically and up to an unspecified constant factor, no degree vector is more likely than the $\lfloor k/2\rfloor$-regular one. Their $\mathbf{d}$ is an ordered tuple: inside the proof they choose "a most probable degree sequence in $G(k, 1/2)$", and the estimate they run it through is the asymptotic formula of McKay and Wormald [10] for the number of labelled graphs with a given degree sequence, whose $\mathbf{d}$-dependent part $\big(\lambda^\lambda(1-\lambda)^{1-\lambda}\big)^{\binom{k}{2}} \prod_j \binom{k-1}{d_j}$, with λ fixed by $\sum_j d_j = \lambda k(k-1)$, they bound in the version we quote by $2^{-\binom{k}{2}}\binom{k-1}{\lfloor k/2\rfloor}^k$. Their statement says nothing about multisets and nothing about exact finite values.

We settle the vector question exactly in the range where (1) needs no correction. A constant vector is the only ordering of its own multiset, so its orbit under relabelling is a single point and no multinomial factor arises.

Proposition 5.1. *For all n and k, the number of k-regular labelled graphs on n vertices equals $N(\{\{k, \ldots, k\}\})$; equivalently $N^{\mathrm{v}}(k, \ldots, k) = N(\{\{k, \ldots, k\}\})$.*

Proposition 5.2. *For each $n \le 12$ and each $k < n$, the number of k-regular labelled graphs on n vertices is the weight $\mathcal{T}(n)$ carries at the constant sequence $(k, \ldots, k)$; the twelve resulting rows are listed in Proposition 6.2. For $k \ge n$ there are none, at every $n \ge 1$, since no vertex is its own neighbour.*

Proposition 5.3. *For each $n \le 12$, the maximum over constant degree vectors $v = (k, \ldots, k)$ of $N^{\mathrm{v}}(v)$, and the degrees k attaining it, are as in Table 3. (What is proved for each n is the bound together with attainment at the degrees listed; that no other k attains the maximum is read off the complete row, Proposition 5.2.)*

n	$\max_k N^{\mathrm{v}}(k,\dots,k)$	attained at
1	1	$k=0$
2	1	$k=0,1$
3	1	$k=0,2$
4	3	$k=1,2$
5	12	$k=2$
6	70	$k=2,3$
7	465	$k=2,4$
8	19 355	$k=3,4$
9	1 024 380	$k=4$
10	66 462 606	$k=4,5$
11	5 188 453 830	$k=4,6$
12	2 977 635 137 862	$k=5,6$

Table 3. The best constant degree vector (Proposition 5.3). The values are the row maxima of sequence A059441 of [11]; the column $k=4$ of that triangle is A005815. Compare Table 1: at $n=9$ the multiset mode is realised about 919 times more often than the best regular sequence.

Remark 5.4 (the unrestricted vector mode). Proposition 2.4 also records, for each $n \le 12$, the largest value of $N(d)\prod_x m_x(d)!$ over the keys of $\mathcal{T}(n)$ and the keys attaining it. By the classical identity (1) that quantity is $n!$ times $\max_v N^{\mathrm{v}}(v)$ over *all* vectors, and the keys attaining it are the multisets of the vector modes. We state this as a remark, not as a theorem, because (1) is not part of our formal development. Read that way, the data say that for $n \le 12$ the vector modes are the constant vectors, with three exceptions. At $n=3$ every one of the eight labelled graphs has its own degree vector, so $N^{\mathrm{v}}=1$ at each of the eight realisable vectors and all eight are modes; the four multisets under them are the whole of $\mathcal{T}(3)$. At $n=7$ and $n=11$ the modes are near-regular and non-constant: at $n=7$ they are the orderings of $(3,3,3,3,3,3,2)$ and $(4,3,3,3,3,3,3)$ with $N^{\mathrm{v}}=810$, against 465 for the best constant vector, and at $n=11$ they are the orderings of $(5,5,5,5,5,5,5,5,5,5,4)$ and $(6,5,5,5,5,5,5,5,5,5,5)$ with $N^{\mathrm{v}}=10\,728\,320\,400$, against $5\,188\,453\,830$. Those two values are therefore not claimed as theorems here; Propositions 5.1 and 5.3 are the exact statements the development supports, and they are the ones that reproduce published data. Supplying the missing orbit identity in formal form would upgrade the whole remark to a theorem; we expect this to be straightforward but not short.

6. Reproduction of published material

Two of the readings of Proposition 2.4 are published data, and reproducing them is the strongest available check on the computation: they come from enumerations unrelated to ours.

Proposition 6.1. *For each $n \le 12$, the multisets d with $N(d) \ne 0$ are exactly the keys of $\mathcal{T}(n)$; those keys are pairwise distinct as multisets, and their number is*

$$1,\ 2,\ 4,\ 11,\ 31,\ 102,\ 342,\ 1213,\ 4361,\ 16016,\ 59348,\ 222117$$

for $n=1,\dots,12$ — sequence A004251 of [11]*, the number of degree-vectors of simple graphs on n nodes.*

The distinctness clause is what makes the comparison a check on something: a table whose keys repeated would have a larger length for the same distribution. Sequence A004251 is published with a b-file far beyond our range; what is ours is the certification that the state count of the program is that number.

Proposition 6.2. *For each $n \leq 12$, the row of counts of k-regular labelled graphs on n vertices, $k = 0, \dots, n-1$, agrees entry for entry with row n of the triangle A059441 of* [11]. *Rows* 1 *to* 12 *are*

$$\begin{aligned}&1; \quad 1,1; \quad 1,0,1; \quad 1,3,3,1; \quad 1,0,12,0,1; \quad 1,15,70,70,15,1;\\ &1,0,465,0,465,0,1; \quad 1,105,3507,19355,19355,3507,105,1;\\ &1,0,30016,0,1024380,0,30016,0,1;\\ &1,945,286884,11180820,66462606,66462606,11180820,286884,945,1;\\ &1,0,3026655,0,5188453830,0,5188453830,0,3026655,0,1;\\ &1,10395,34944085,11555272575,480413921130,2977635137862,\\ &\qquad 2977635137862,480413921130,11555272575,34944085,10395,1.\end{aligned}$$

Its column $k = 4$ is A005815, matching our values at $n = 5, \dots, 12$.

7. Controls

A collection of results consisting of upper bounds and computed values can be satisfied by a misplaced quantifier or by a wrong object. The following are the checks that this did not happen; each is machine-checked alongside the results.

Proposition 7.1.

(1) Independent enumeration. *$N(d)$ can also be computed by enumerating all $2^{\binom{n}{2}}$ labelled graphs directly; that this enumeration equals N is proved for every n, with no dynamic program on either side. Evaluated this way, all four cells of $\mathcal{T}(3)$ agree with the table — $N(\{\{0,0,0\}\}) = 1$, $N(\{\{0,1,1\}\}) = 3$, $N(\{\{1,1,2\}\}) = 3$, $N(\{\{2,2,2\}\}) = 1$ — and at $n = 4$, over all 64 labelled graphs, the three maximisers of Table 1 each have $N = 12$ and $N(\{\{0,0,0,0\}\}) = 1$.*

(2) The bound is tight and not slack. *At $n = 9$ neither $941\,371\,199$ is an upper bound for N, nor is $941\,371\,200$ exceeded by any d.*

(3) The mode is not the regular sequence. *At $n = 9$ the 9-vertex 4-regular sequence has $N = 1\,024\,380 \neq 941\,371\,200$.*

(4) The mode is not always unique. *At $n = 10$ the two sequences of Table 1 are distinct as multisets and both have $N = 214\,857\,921\,600$.*

(5) Not everything is realisable. *$N(\{\{0,1,2\}\}) = 0$ at $n = 3$ and $N(\{\{0,0,2,2\}\}) = 0$ at $n = 4$, while $N(\{\{1,1,1,1\}\}) = 3$.*

(6) The certificate is neither vacuous nor unsatisfiable. *On a table with a repeated key its distinctness clause fails, and there the largest weight (1) is indeed smaller than the count the table records at that key (2) — the failure mode Proposition 2.3 rules out; and the certificate does hold, for instance at $n = 3$.*

(7) The objects are the right ones. *A non-symmetric Boolean matrix is not in $\mathcal{G}_2$, the complete graph on two vertices is, and $|\mathcal{G}_3| = 8$.*

Beyond these, every value in this paper was produced independently twice: once by the verified program and once by a separate implementation of the same recurrence, written from the recurrence rather than from the code, which agrees at every $n \leq 13$.

8. Cost, reach, and one measured rule

The following are measurements of our implementation on one machine, not mathematical statements, and they are outside the formal development; we record them because they delimit what the method reaches.

One evaluation of the program at $n = 10, 11, 12$ took 4, 32 and 297 CPU-seconds with peak resident memory 0.8, 1.6 and 7.5 GB. The cost is dominated by the sorting step that merges equal states, so the time grows faster than the state count: from $n = 11$ to $n = 12$ the number of states grows by a factor 3.7 and the time by a factor 9. Extrapolating, $n = 13$ costs about 1200 CPU-seconds and about 30 GB, and we did not run it inside the proof assistant.

Run outside the formal development, in a separate implementation, $n = 13$ gives 836 315 realisable degree sequences (which is A004251(13)) and total weight 2^{78}, together with

$$\begin{aligned}&\max_d N(d) = 462\,803\,501\,781\,808\,387\,200 \quad \text{at } (8,8,7,7,7,6,6,6,5,5,5,4,4),\\ &U(13) = 182\,506\,107\,184\,803\,042\,941\,833,\\ &N^{\mathrm{v}}(6,\ldots,6) = 2\,099\,132\,870\,973\,600.\end{aligned}$$

These numbers are measurements, not theorems, and they are not claimed as results of this paper; in particular the displayed sequence is a maximiser at $n = 13$, and we make no claim about whether it is the only one. Two of these values do have published counterparts, and match them: the state count 836 315 is A004251(13), and $N^{\mathrm{v}}(6,\ldots,6)$ is the entry at $k = 6$ of row 13 of the triangle A059441 in [11].

One implementation detail was worth a large constant factor and is easy to repeat. The list of maximising keys was first computed by filtering the table with a predicate that mentioned the table's own maximum; since the evaluator is strict in arguments and does not memoise a closure's free computations, the maximum was recomputed once per key, making the pass quadratic. At $n = 10$ the evaluation took 358 seconds; passing the maximum as an explicit argument brought the same evaluation to 4.15 seconds, a factor of about 86, and moved the reachable range from $n = 10$ to $n = 12$. The diagnosis is worth as much as the fix: at $n = 10$, reading the table's length directly took 4 seconds, while reading that same length off the eight-field summary — which computes the maximising keys on the way — took the 358, so the cost was provably not in the dynamic program itself.

9. Verification

Every claim in this paper is formally verified in Lean 4 against *Mathlib*, with the exceptions flagged where they occur: the classical orbit identity (1), Remarks 3.3 and 5.4, which rest on it or on complementation, and the measurements of Section 8. The development contains no `sorry` and no hand-written `axiom` declaration, and the repository reference is to be added at submission. The reasoning layer is kernel-clean, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: this covers Proposition 2.1 (including the identification of our Boolean-matrix graphs and our degree function with Mathlib's `SimpleGraph` and `SimpleGraph.degree`, and the identity $|\mathcal{G}_n| = 2^{\binom{n}{2}}$ for all n), Proposition 2.2, the certificate argument of Proposition 2.3, Propositions 4.1 and 5.1, the reduction and the $k \geq n$ clause of Proposition 5.2, and items (1), (5) and (7) of Proposition 7.1; the refutation in item (6) depends on no axioms at all. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the twelve table evaluations of Proposition 2.4 and the twelve corresponding evaluations of the regular-graph rows: one per n, each running the dynamic program once and reading its eight (respectively, n) values off the result. Each statement quoting a computed number — Theorems 3.1 and 4.2, Propositions 5.2, 5.3, 6.1, 6.2, and those items of Proposition 7.1 that quote values — therefore carries, beside the standard axioms, exactly one further axiom per n used: the one recording that compiled evaluation returned the stated table. The numbers were produced first by an independent implementation of the same recurrence and the formal statements were written against them; at $n \leq 4$ a third, brute-force enumeration of all $2^{\binom{n}{2}}$ labelled graphs was evaluated by the kernel, with the dynamic program absent from the computation, and agrees (Proposition 7.1(1)). The total-weight identity of Proposition 2.2 holds for all n as a theorem, so agreement of the totals with $2^{\binom{n}{2}}$ is not a coincidence that a state-space error could reproduce.

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