NEW EMPTY CELLS IN THE CARLITZ–WIEFERICH GRID, AND A CORRECTION TO A PUBLISHED TABLE

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Abstract. Let p be an odd prime, $q = p^e$, $A = \mathbb{F}_q[T]$, and let ρ be the Carlitz module. A monic prime $P \in A$ is a *c-Wieferich prime* (to base 1) when the Carlitz–Fermat congruence lifts, $\rho_P(1) \equiv 1 \pmod{P^2}$. Thakur excluded degrees 2 and 3 in odd characteristic and suggested that the degree of a c-Wieferich prime is always divisible by p; the suggestion was refuted in 2026 by an explicit degree-5 prime over $\mathbb{F}_{19^3}$, and the paper that refuted it published a grid of thirty pairs (d, p) with $p \nmid d$ for which $\mathbb{F}_p[T]$ was shown, by exhaustive computation, to contain no c-Wieferich prime of degree d. We extend that grid by thirty-one cells in two directions. The prime-field rows are continued past their published frontier: degree 5 is now settled for every odd $p \le 101$ with $p \neq 5$, degree 6 for every odd $p \le 37$ with $p \neq 3$, degree 7 for $p \le 19$, degree 8 for $p \le 13$, and degrees 9 and 10 at $p = 7$. A second axis, absent from the published table, is opened: the cells (d, q) with q a proper prime power, namely $(5, 9)$, $(5, 27)$, $(5, 49)$, $(5, 81)$, $(6, 9)$, $(6, 25)$, $(7, 9)$ and $(8, 9)$, of which $(5, 27)$ is the smallest cubic-extension analogue of the cell in which the counterexample lives. All thirty-one are empty; together the sweeps run over 843 189 705 cosets. We also correct a published table: the entry $T^5 + T + 1$ recorded as the c-Wieferich prime of least degree in $\mathbb{F}_5[T]$ is reducible, the intended polynomial being $T^5 + 4T + 1$. Under the natural q^{-d} heuristic the published grid was a 0.94% coincidence and the two grids together are a 0.22% one, so the new cells multiply the numerical evidence for Thakur's phenomenon by about four. All statements are machine-checked in Lean 4.

1. Introduction

c-Wieferich primes. Let p be an odd prime, $q = p^e$, $A = \mathbb{F}_q[T]$ and $K = \mathbb{F}_q(T)$. Let $A\{\tau\}$ be the twisted polynomial ring over A, in which $\tau x = x^q \tau$, and let

$$\rho\colon A \longrightarrow A\{\tau\}, \qquad \rho_T = T + \tau,$$

be the *Carlitz module*, the $\mathbb{F}_q$-algebra homomorphism determined by that value. For every monic prime $P \in A$ one has the Carlitz–Fermat congruence $\rho_P(1) \equiv 1 \pmod{P}$. Following Thakur, P is a *c-Wieferich prime* (to base 1) when the congruence lifts one power further:

$$\rho_P(1) \;\equiv\; 1 \pmod{P^2}. \tag{1}$$

The notion is Thakur's. Bamunoba and Bergström state it in the base-a form $\rho_P(a) \equiv a^{q^{\deg P}} \pmod{P^2}$ [2, Def. 1.1], of which (1) is the case $a = 1$, and write that it "was introduced in 1994 by D. Thakur", citing [10]; the attribution and the date here are theirs. The form (1) and the name are those of [4], which states them "[f]ollowing Thakur [8] and Bamunoba–Bergström [2]". This is the function-field analogue of the classical Wieferich condition $2^{p-1} \equiv 1 \pmod{p^2}$, and which degrees admit one is open in much the same way. The difference is that here examples are found in quantity and the condition is decidable degree by degree, so the question can be attacked by computation.

What is known about which degrees occur is quickly said. Degree 1 is impossible [1, Cor. 4.5]. Thakur [8] excluded degrees 2 and 3 in odd characteristic and, on the strength of the examples then available, suggested that in odd characteristic the degree of a c-Wieferich prime is always divisible by p; Bamunoba and Bergström [2] computed extensively and wrote that "we still believe that Thakur's naive guess is true in odd characteristic"; and Thakur, restating the suggestion in the equivalent form of a nontrivial common factor of $[d]$ and the polynomial M_d of §2, called it "still open" in 2024

[9, §7]. In July 2026 Niedbala Giraudin [4] refuted it, exhibiting an irreducible c-Wieferich prime of degree 5 in $\mathbb{F}_{19^3}[T]$, with $19 \nmid 5$.

The refutation does not close the subject; it sharpens it. The counterexample lives over a cubic extension, and the question Thakur raised in correspondence with the author of [4] — whether a counterexample exists over a prime field $\mathbb{F}_p$ itself — remains open. Accordingly [4, Prop. 5.2] is a table of *prime* fields: thirty pairs (d, p) with $p \nmid d$ for which an exhaustive computation shows that $\mathbb{F}_p[T]$ contains no c-Wieferich prime of degree d, reproduced here as Table 1. Of that table [4] says, verbatim: "*The reduction of Lemma 2.7 makes further prime-field cells accessible by the same exact computation, and the empty region above is being extended.*"

TABLE 1. The published grid, [4, Prop. 5.2]: no c-Wieferich prime of degree d in $\mathbb{F}_p[T]$, for the listed pairs with $p \nmid d$.

d	p
5	$3, 7, 11, 13, 17, 19, 23, 29, 31, 37$
6	$5, 7, 11, 13, 17, 19, 23$
7	$3, 5, 11, 13$
8	$3, 5, 7$
9	5
10	3
11	$3, 5$
13	3
14	3

What is settled here. Call a pair (d, q) with q an odd prime power a *cell*, and call the cell *empty* when $\mathbb{F}_q[T]$ contains no c-Wieferich prime of degree d. Deciding a cell is a finite problem: by the reduction recalled in §2 it amounts to a sweep over q^{d-1} elements of $\mathbb{F}_{q^d}$. We decide thirty-one cells that no published computation covers, and every one of them is empty.

The first twenty-three continue the rows of Table 1 past their published frontier (Theorems 6.1–6.5):

d	new characteristics p
5	$41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101$
6	$29, 31, 37$
7	$17, 19$
8	$11, 13$
9	7
10	7

Combined with Table 1, this settles degree 5 for *every* odd $p \le 101$ with $p \neq 5$, degree 6 for every odd $p \le 37$ with $p \neq 3$, degree 7 for every odd $p \le 19$ with $p \neq 7$, degree 8 for every odd $p \le 13$, degree 9 for $p \in \{5, 7\}$ and degree 10 for $p \in \{3, 7\}$. The excluded characteristics are exactly the ones with $p \mid d$, where c-Wieferich primes are known to exist.

The remaining eight open an axis Table 1 does not have, that of non-prime constant fields (Theorem 6.6):

$$(d, q) \in \{(5, 9),\ (5, 27),\ (5, 49),\ (5, 81),\ (6, 9),\ (6, 25),\ (7, 9),\ (8, 9)\}.$$

Since the only known counterexample to Thakur's suggestion lives over $\mathbb{F}_{19^3}$, the behaviour of the cells (d, p^e) with $e \ge 2$ and $p \nmid d$ is precisely the question its existence raises; beyond the small degrees implicit in the least-degree table [2, Table 1], nothing is published about them. The cell $(5, 27)$ is the smallest cubic-extension analogue of the counterexample cell $(5, 19^3)$: same degree, same shape of constant field, a different characteristic — and empty.

One published data point turns out to be wrong. Table 1 of [2] names T^5+T+1 as the c-Wieferich prime of least degree in $\mathbb{F}_5[T]$; that polynomial is reducible. The intended entry is T^5+4T+1 (Theorem 5.1).

Method, and the scope of the claim. Every assertion above is a complete determination, not a search that stopped: for each cell the whole coset space is enumerated. Across the thirty-one new cells that is 843 189 705 cosets; across the thirty cells of Table 1, which we re-derive in §4 before claiming anything new, it is 34 808 230 more. The ambient field is not assumed but certified: for each cell the modulus used to realise $\mathbb{F}_{q^d}$ is *proved* irreducible inside the same computation, by a rank criterion on the Frobenius matrix (Proposition 3.2), and the transversal of the $\mathbb{F}_q$-cosets is verified to be one. Everything was computed twice, in two programs with different internal conventions, and the final check is in Lean 4 (§10).

Two caveats belong here rather than in a footnote.

First, the frontier is moving. The author of [4] states that the empty region of Table 1 is being extended, and the companion paper [5], which contains the method and a degree-4 theorem, is in preparation and not yet public. A later version of [4], or [5] itself, may well absorb some or all of the cells recorded here. We claim these cells as not previously computed, on the search described in §7, and not as beyond the reach of the published method — they are not; that is the point of the method.

Second, and this is a point about the published record rather than about our own cells: [3] reports an exhaustive Carlitz search over $\mathbb{F}_3$ to degree 24 and over $\mathbb{F}_5$ to degree 17, which already contains ten of the thirty cells of Table 1, and [4] does not cite it. We therefore claim no cell with $q \in \{2,3,4,5\}$; §7 records exactly which source settles which cell.

How much is an empty cell worth. The honest answer is: a little, and it is computable. Section 8 scores the empty cells against the heuristic of [3] that a place of degree d is Wieferich with probability q^{-d}. A cell (d,q) then has expected witness count $q^{d-1}\cdot q^{-d} = 1/q$, independent of d. Table 1 carries an expected 4.67 witnesses and found 0; the new cells add 1.43 and also found 0; the two together give a Poisson tail of 0.22% where the published grid alone gave 0.94%. No new cell is worth more than $1/q \le 1/7 \approx 0.14$, and the collective evidence for "$p \mid \deg P$" is multiplied by about four. Nothing in that paragraph is a theorem, and it is stated as evidence, not as one.

Organisation. Section 2 recalls the criterion and the reduction that is computed. Section 3 describes the certified model and the negative controls. Section 4 reproduces the published record within range — the published c-Wieferich primes of small degree, the thirty cells of Table 1, the counterexample of [4], and the small-degree exclusions — which is what makes the new cells of §6 mean something. Section 5 is the correction to [2]. Section 7 is the provenance of the novelty claim, §8 the heuristic weight, and §10 the verification.

2. The criterion, and the finite problem

Throughout, p is an odd prime, $q = p^e$, $A = \mathbb{F}_q[T]$, and

$$[n] \;=\; T^{q^n} - T \in A \qquad (n \ge 0).$$

Put $F_0 = 1$ and $F_i = (-1)^i + [i]\,F_{i-1}$ for $i \ge 1$, and

$$M_d \;=\; \sum_{k=0}^{d-1} (-1)^k\,[d-1][d-2]\cdots[d-k], \tag{2}$$

the term $k=0$ being the empty product 1. A direct induction gives $F_{d-1} = (-1)^{d-1} M_d$.

Proposition 2.1 ([1, Prop. 4.3], quoted as [2, Prop. 1.2]; independently Thakur [8])**.** *A monic prime $P \in A$ of degree d is a c-Wieferich prime, i.e. satisfies* (1)*, if and only if $P \mid F_{d-1}$, equivalently if and only if $P \mid M_d$, equivalently if and only if $M_d(\theta) = 0$ for one (hence any) root $\theta \in \mathbb{F}_{q^d}$ of P.*

In particular no c-Wieferich prime has degree 1: for $a \in \mathbb{F}_q$ and $i \geq 1$ one has $[i](a) = a^{q^i} - a = 0$, so $M_d(a) = 1 \neq 0$ by (2) [4, Lem. 2.3], which is the degree-1 exclusion of [1, Cor. 4.5].

This is the form in which the condition becomes computable, and it is the form used in [2], [4] and here. The next step is the reduction of [4, Lemma 2.7], which replaces a search over polynomials by a search over field elements. We use it in the equivalent θ-form, which needs neither a trace computation nor a basis of a trace-zero hyperplane.

Lemma 2.2 (nested form)**.** *Let $\theta \in \mathbb{F}_{q^d}$ and put $y_j = \theta^{q^j} - \theta$ for $0 \leq j \leq d-1$. Then*

$$M_d(\theta) \;=\; 1 - y_{d-1}\big(1 - y_{d-2}\big(\cdots(1 - y_1)\cdots\big)\big).$$

Proof. Immediate from (2) and $[j](\theta) = \theta^{q^j} - \theta = y_j$, by induction on d; it is equation (2) of [4]. □

Lemma 2.3. *For $\theta \in \mathbb{F}_{q^d}$ the following are equivalent: θ has degree exactly d over $\mathbb{F}_q$; and $y_j \neq 0$ for all $1 \leq j \leq d-1$.*

Proof. $y_j = 0$ says $\theta \in \mathbb{F}_{q^j}$, hence $\theta \in \mathbb{F}_{q^j} \cap \mathbb{F}_{q^d} = \mathbb{F}_{q^{\gcd(j,d)}}$, and every proper divisor of d is at most $d-1$. □

Combining Proposition 2.1 with Lemmas 2.2 and 2.3:

Corollary 2.4 ([4, Lemma 2.7], in θ-form)**.** *There is a c-Wieferich prime of degree d in $\mathbb{F}_q[T]$ if and only if there is $\theta \in \mathbb{F}_{q^d}$ with $y_j \neq 0$ for $1 \leq j \leq d-1$ and $1 - y_{d-1}(1 - y_{d-2}(\cdots(1-y_1)\cdots)) = 0$.*

Call such a θ a *witness*. Two elementary remarks make the search cheap. First, each y_j is unchanged by $\theta \mapsto \theta + a$ for $a \in \mathbb{F}_q$, and so is the degree of θ over $\mathbb{F}_q$; so witnesses come in whole cosets of $\mathbb{F}_q$ inside $\mathbb{F}_{q^d}$, and one representative per coset suffices. There are exactly q^{d-1} of them — the same count as the trace-zero elements $\eta = \theta^q - \theta$ swept in [4, Lemma 2.7], but with no trace to compute. Second, the number of witness cosets is a meaningful invariant of the cell: the set of witnesses is exactly the set of roots of the degree-d c-Wieferich primes, and it is stable under $\theta \mapsto \theta + a$ for $a \in \mathbb{F}_q$, so if N denotes the number of monic c-Wieferich primes of degree d in $\mathbb{F}_q[T]$ then their Nd roots form exactly Nd/q whole cosets. No individual prime need be translation invariant for that.

Definition 2.5. A *cell* is a pair (d, q) with $d \geq 2$ and q an odd prime power. Its *witness count* is the number of cosets $\theta + \mathbb{F}_q \subseteq \mathbb{F}_{q^d}$ consisting of witnesses. The cell is *empty* when its witness count is 0, i.e. when $\mathbb{F}_q[T]$ contains no c-Wieferich prime of degree d.

Deciding a cell therefore costs q^{d-1} evaluations of the nested product, each of which is $O(d)$ operations in $\mathbb{F}_{q^d}$, i.e. $O((ed)^3)$ operations in $\mathbb{F}_p$ with the naive arithmetic used here. The interesting cells are those with $p \nmid d$, since Thakur's suggestion asserts that all of them are empty.

3. The model, certified; and the controls

The model. Fix a cell (d, q), $q = p^e$, and set $n = ed$. We realise $\mathbb{F}_{q^d} = \mathbb{F}_{p^n}$ as $\mathbb{F}_p[z]/(h)$ for a monic $h \in \mathbb{F}_p[z]$ of degree n, an element being the vector of its n coefficients. Two matrices are precomputed: fr, the matrix of the $\mathbb{F}_p$-linear map $x \mapsto x^p$, and $\varphi = \mathrm{fr}^e$, the matrix of $x \mapsto x^q$. Then $\mathbb{F}_q$ is the fixed field of φ, i.e. $\ker(\varphi - 1)$, and a transversal of the cosets $\theta + \mathbb{F}_q$ is obtained by row-reducing a basis of that kernel and letting θ range over the vectors supported on the non-pivot coordinates.

Nothing in that paragraph is taken on trust. The key point is that irreducibility of h can be decided by the same linear algebra that is needed anyway.

Lemma 3.1. *Let $h \in \mathbb{F}_p[z]$ be monic of degree $n \geq 1$, let $R = \mathbb{F}_p[z]/(h)$ and let* fr *be the matrix of the $\mathbb{F}_p$-linear map $x \mapsto x^p$ on R. Then R is a field if and only if $\mathrm{fr}^n = I$ and $\operatorname{rank}(\mathrm{fr} - I) = n - 1$.*

Proof. The map $x \mapsto x^p$ is $\mathbb{F}_p$-linear on any commutative $\mathbb{F}_p$-algebra. If $\mathrm{fr}^n = I$ then $x^{p^n} = x$ for all $x \in R$, so R is reduced and every element is separable algebraic over $\mathbb{F}_p$ of degree dividing n; hence $R \cong \prod_{i=1}^{r} \mathbb{F}_{p^{m_i}}$ with $m_i \mid n$. For such a product $\ker(\mathrm{fr} - I)$ is the set of elements fixed by the p-power map, i.e. $\mathbb{F}_p^r$, so $\operatorname{rank}(\mathrm{fr} - I) = n - r$, and $r = 1$ exactly when $\operatorname{rank}(\mathrm{fr} - I) = n - 1$. Conversely if R is a field of order p^n then $\mathrm{fr}^n = I$ and the fixed field of fr is $\mathbb{F}_p$, of dimension 1. □

The criterion of Lemma 3.1 is complete, not a sample: it rejects every reducible h, including those with no linear factor. Write ctxOK for the conjunction of the checks that the computation performs on a cell's data: that the reduction table really is that of a monic h of degree n; that fr really is the p-power matrix, recomputed from scratch by repeated multiplication, and that $\varphi = \mathrm{fr}^e$; that h passes Lemma 3.1; and that the stored basis of $\ker(\varphi - 1)$ is in reduced echelon form with the declared pivots, so that the index set of the sweep is a transversal of the $\mathbb{F}_q$-cosets.

Proposition 3.2. *The model check is non-vacuous and model-independent. It fails on z^5 over $\mathbb{F}_5$ and on $z^6 + z^4 + z^2 + 2 = (z^3 + 2z + 1)(z^3 + 2z + 2)$ over $\mathbb{F}_3$, whose two cubic factors are themselves irreducible, so the rejection is not the detection of a root. It succeeds on $z^5 - z^4 - 10$ over $\mathbb{F}_{11}$ and on $z^6 - z^5 - 2z - 1$ over $\mathbb{F}_7$, neither of which is the lexicographically first modulus the search returns. Moreover the cells $(5, 11)$ and $(6, 7)$ are found to be empty over the lexicographically first modulus and over these second moduli alike: the answer does not depend on the presentation of the field.*

Proof sketch. Each assertion is a finite computation: build the ring $\mathbb{F}_p[z]/(h)$ for the named h, run all the checks of ctxOK, and, in the last two cases, sweep all 11^4 respectively 7^5 cosets. The correctness of the irreducibility test is Lemma 3.1. □

Controls. An exhaustive search that reports "nothing found" is worth what its controls are worth. Three are run. The cell must be able to report a nonempty answer; the small-degree cells that are known to be empty must come out empty; and the degree hypothesis of Corollary 2.4 must be doing work.

Proposition 3.3.
(1) *The claim "every cell is empty" is false, and is refuted by the computation itself: the cells $(5, 5)$, $(6, 3)$, $(7, 7)$, $(3, 9)$ and $(5, 25)$ have witness counts 1, 2, 1, 2 and 1 respectively.*
(2) *The claim "there is a c-Wieferich prime of degree 2, 3 or 4 over a small field" is false; see Proposition 4.4.*
(3) *The degree hypothesis of Corollary 2.4 is not vacuous. Let the* loose *count of a cell be the number of cosets satisfying $M_d(\theta) = 0$ with the requirement $y_j \neq 0$ dropped. For $(6, 9)$ the true count is 0 and the loose count is 2; for $(12, 3)$ they are 4 and 6; for $(10, 5)$ they are 2 and 3.*

Proof sketch. Exhaustive sweeps of the named cells, over 5^4, 3^5, 7^6, 9^2, 25^4, 9^5, 3^{11} and 5^9 cosets respectively, in both the strict and the loose form. The gap in (3) is explained rather than accidental: if θ has degree $m \mid d$ with $m < d$ then $y_{m+j} = y_j$ and the nested product is periodic of period m, so the loose sweep at degree d also counts the roots of the c-Wieferich primes of the proper divisor degrees. That is why $(6, 9)$ picks up the six degree-3 primes of $\mathbb{F}_9[T]$ listed in [2, Table 1], and why $(10, 5)$ picks up $T^5 + 4T + 1$. Item (3) is the sharp control: without the degree hypothesis, the *empty* cell $(6, 9)$ would be reported as nonempty. □

4. The published record, reproduced

Before any new cell is claimed we re-derive, from the definitions above, every published value we found that is within computational range. None of this section is new mathematics; it exists so that the sweeps of §6 can be believed, and it is where the correction of §5 came from.

Proposition 4.1. *The following published data are reproduced.*

(1) The published c-Wieferich primes of small degree over odd prime fields. *The seven polynomials*

$$T^6+T^4+T^3+T^2+2T+2,\quad T^9+T^6+T^4+T^2+2T+2,$$
$$T^{12}+2T^{10}+T^9+2T^4+2T^3+T^2+1,$$
$$T^{15}+T^{13}+T^{12}+T^{11}+2T^{10}+2T^7+2T^5+2T^4+T^3+T^2+T+1$$

over $\mathbb{F}_3$, T^5+4T+1 *and* $T^{10}+3T^6+4T^5+T^2+T+1$ *over* $\mathbb{F}_5$, *and* T^7+6T+3 *over* $\mathbb{F}_7$, *are irreducible c-Wieferich primes. The four over* $\mathbb{F}_3$ *are all the c-Wieferich primes of* $\mathbb{F}_3[T]$ *of degree at most* 60 [2], *and exhaustively all those of degree at most* 24 [3]; *the two over* $\mathbb{F}_5$ *are all those of* $\mathbb{F}_5[T]$ *of degree at most* 17 [3]. *They are not everything published over odd prime fields:* [2] *record further examples in higher degree, over* $\mathbb{F}_7$ *in degree* 14, *over* $\mathbb{F}_5$ *in degrees* 20 *and* 45, *and over* $\mathbb{F}_{43}$, $\mathbb{F}_{61}$, $\mathbb{F}_{131}$, $\mathbb{F}_{463}$ *in degrees* 129, 183, 393, 926 — *every one of them with* $p \mid \deg$, *and all of them out of range here.*

(2) Witness counts of the nonempty cells. $(5,5)$, $(6,3)$, $(7,7)$, $(9,3)$, $(12,3)$, $(15,3)$, $(10,5)$, $(3,9)$ *and* $(5,25)$ *have witness counts* 1, 2, 1, 3, 4, 5, 2, 2 *and* 1. *At* $(5,5)$, $(6,3)$ *and* $(7,7)$ *the minimal polynomial of the first witness is recovered from the sweep and equals* T^5+4T+1, $T^6+T^4+T^3+T^2+2T+2$ *and* T^7+6T+3, *the three positive controls named in* [4, §5].

(3) Fourteen entries of the count table [2, Table 6]. *In* [2] *a monic* $f \in \mathbb{F}_q[T]$ *is* $\mathcal{G}$-fixed, *for* $\mathcal{G}$ *a subgroup of* $(\mathbb{F}_q,+)$, *when* $f(T+a)=f(T)$ *for every* $a \in \mathcal{G}$; *the stabiliser of a prime has order at most* p, *and a* $\mathcal{G}$*-fixed prime with* $\#\mathcal{G}=p$ *has degree divisible by* p. *Their Theorem 3.3 reads: for* $q \geq 3$, $s \geq 1$ *and* $\chi \in \mathbb{F}_q^\times$, *with* $\mathcal{G}=\chi\mathbb{F}_p$, *"the* $\mathcal{G}$*-fixed c-Wieferich primes of degree* ps *are precisely the prime factors over* $\mathbb{F}_q$ *of* $R_{q,s,\alpha}=\prod_{i=1}^{s}(T^q-T-\alpha^{q^i})$ *for* $\alpha \in B_{q,s,\chi}$. *Moreover, these factors are distinct so the number of* $\mathcal{G}$*-fixed c-Wieferich primes is* $p^{-1}q|B_{q,s,\chi}|$." *Here* $B_{q,s,\chi}$ *is their Definition 3.2: the classes, under* $\theta \mapsto \theta^q$, *of the* α *of degree* s *over* $\mathbb{F}_q$ *with* $\mathrm{Tr}_{\mathbb{F}_{q^s}/\mathbb{F}_q}(\alpha)=\chi$ *and* $F_{ps-1} \equiv 0 \pmod{T^q-T-\alpha}$. *Table 6 of* [2] *lists* $|B_{p^m,s}|$ *for triples* (p,m,s), *where* $B_{q,s}=\bigcup_{\chi\in\mathbb{F}_q^\times} B_{q,s,\chi}$, *and* [2] *state that each element of* $B_{q,s}$ *has* $p^{-1}q$ *distinct c-Wieferich primes attached to it; so the number of fixed c-Wieferich primes of degree* ps *over* $\mathbb{F}_{p^m}$ *is* $p^{-1}q\,|B_{p^m,s}|$. *When those are all the c-Wieferich primes of that degree, the coset count of §2 turns this into witness count* $= p^{-1}q\,|B_{p^m,s}|\cdot ps/q = s\,|B_{p^m,s}|$; *that numerical prediction is all that is used here, and the fourteen entries with*

$$(p,m,s) \in \left\{ \begin{array}{l} (3,1,1),(3,1,2),(3,1,3),(3,1,4),(3,1,5),(3,2,1),(3,3,1), \\ (3,4,1),(3,5,1),(3,6,1),(5,1,1),(5,1,2),(5,2,1),(7,1,1) \end{array} \right\}$$

all agree with the swept counts.

(4) All eleven published terms of OEIS A271781 [7]: *on the reading of that entry's name set out in* [7], *the least* x *for which* T^p-T-x *is a c-Wieferich prime of* $\mathbb{F}_p[T]$ *(or* 0 *if there is none), namely* $0,4,4,7,5,0,17,0,0,11,16$ *for* $p=3,5,7,11,13,17,19,23,29,31,37$.

Proof sketch. For (1) and (4) no sweep is needed: for a named monic P of degree d one builds $\mathbb{F}_q[x]/(P)$, checks ctxOK — which proves P irreducible by Lemma 3.1 — and then applies Corollary 2.4 to the class of x, which is a root of P. For A271781, T^p-T-x is irreducible over $\mathbb{F}_p$ exactly when $x \neq 0$, and for each of the eleven primes all p values of x are tested, so the four zero entries $p=3,17,23,29$ are non-existence statements over every $x \in \mathbb{F}_p$; the cells $(11,11),(13,13),\ldots$ are far beyond sweep range ($11^{10}=2.6\cdot10^{10}$ cosets), which is why this direction is taken. Items (2) and (3) are exhaustive sweeps of the named cells, the largest being $(15,3)$ with $3^{14}=4\,782\,969$ cosets and $(12,3)$ with $3^{11}=177\,147$; the minimal polynomial in (2) is formed as $\prod_{j<d}(X-\theta^{q^j})$ from the first witness found.

Independently of all of that, each polynomial in (1) was also tested directly against the definition (1), by building the twisted polynomial ρ_P through the Horner recursion $\rho_{Tf+a}=\rho_T\circ\rho_f+a$ modulo P^2, in a separate program. As a control on that program, the Carlitz–Fermat congruence $\rho_P(1)\equiv 1 \pmod P$ was verified for all 29 monic irreducibles of degree ≤ 4 over $\mathbb{F}_3$, all 276 of degree ≤ 3 over

$\mathbb{F}_9$ and all 50 of degree ≤ 3 over $\mathbb{F}_5$, with no failure; it fails for most reducible monic polynomials in the same ranges, so the control is not vacuous. □

Proposition 4.2. *Every one of the thirty cells of Table 1 is empty.*

Proof sketch. Thirty exhaustive sweeps, 34 808 230 cosets in total, the largest cell being $(11, 5)$ with $5^{10} = 9\,765\,625$ and $(14, 3)$ with $3^{13} = 1\,594\,323$. Each sweep is preceded by the model check of §3. All thirty were computed first in an independent C program using a different construction of the coset transversal and machine-word arithmetic, and agree. □

Proposition 4.3. *Let $q = 19^3$ and $\mathbb{F}_q = \mathbb{F}_{19}[c]$ with $c^3 = 8c^2 + 4c + 11$. Then*

$$P(T) = T^5+(11+17c+9c^2)T^4+(3+7c+18c^2)T^3+(2+5c+6c^2)T^2+(3+3c+11c^2)T+(6+17c+5c^2)$$

is an irreducible c-Wieferich prime of degree 5 in $\mathbb{F}_{19^3}[T]$, and $19 \nmid 5$. This is [4, Thm. 1.1].

Proof sketch. No sweep: the cell $(5, 19^3)$ has $19^{12} \approx 2.2\cdot 10^{15}$ cosets. Instead the minimal polynomial h of a root θ of P over the *prime* field is computed; it has degree 15, namely

$$h = z^{15} + 3z^{13} + 8z^{12} + 4z^{11} + 11z^{10} + 8z^9 + 4z^8 + 17z^7 + 6z^6 + z^5 + 11z^4 + 7z^3 + 17z + 1$$

over $\mathbb{F}_{19}$. In $\mathbb{F}_{19}[z]/(h)$ one then checks, all by finite computation: that ctxOK holds, so h is irreducible and the ring is $\mathbb{F}_{19^{15}}$, with $\varphi\colon x \mapsto x^{19^3}$; that the stored vector C satisfies $C^3 = 8C^2 + 4C + 11$ and $\varphi(C) = C$, while $x^3 - 8x^2 - 4x - 11$ has no root in $\mathbb{F}_{19}$ and is therefore irreducible — so $\mathbb{F}_{19}[C]$ is the subfield of order 19^3 and C is the c of the statement; that $P(z) = 0$ with the coefficients assembled from C exactly as printed; and that z satisfies the two conditions of Corollary 2.4 at $d = 5$. Since P is monic of degree 5 over $\mathbb{F}_q$, vanishes at z, and z has degree exactly 5 over $\mathbb{F}_q$, P is the minimal polynomial of z and hence irreducible; and $M_5(z) = 0$ gives $P \mid M_5$, i.e. (1) by Proposition 2.1. The three assertions were also reproduced outside this framework in three independent ways: irreducibility over $\mathbb{F}_q$; $P \mid M_5$ in the nested form; and $\rho_P(1) \equiv 1 \pmod{P^2}$ straight from the Carlitz module. □

Proposition 4.4. *There is no c-Wieferich prime of degree 2, 3 or 4 in $\mathbb{F}_p[T]$ for $p \in \{3, 5, 7, 11, 13\}$; none of degree 4 in $\mathbb{F}_q[T]$ for $q \in \{9, 25, 27, 49, 121\}$; none of degree 2 or 3 in $\mathbb{F}_q[T]$ for $q \in \{25, 49, 121\}$; and none of degree 2 in $\mathbb{F}_9[T]$.*

Proof sketch. Exhaustive sweeps of the twenty-seven named cells. Degrees 2 and 3 in odd characteristic are Thakur's theorem [8], and degree 4 over every odd q is [4, Thm. 5.1], proved in the companion [5]; since [5] is not public, the degree-4 range above is an independent check of an announced theorem rather than a reproduction of a published proof. The range is not vacuous: degree 3 over $\mathbb{F}_9$ is the case $p \mid d$ and is *not* empty — $\mathbb{F}_9[T]$ has six c-Wieferich primes of degree 3 [2, Table 1] — and it is excluded from the list above for that reason. □

5. A correction to Table 1 of [2]

Table 1 of [2] lists c-Wieferich primes of least degrees over $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{F}_9$, $\mathbb{F}_{25}$, $\mathbb{F}_{27}$ (where it records none) and $\mathbb{F}_{49}$, and over three fields of even characteristic, which lie outside the odd-characteristic setting of this paper and were not checked. Every odd-characteristic entry reproduces — the six degree-3 primes of the $\mathbb{F}_9$ row, the five quintics of the $\mathbb{F}_{25}$ row and the seven septics of the $\mathbb{F}_{49}$ row were re-checked over the minimal polynomials $f^t_{\min}$ that the paper itself fixes — with one exception.

Theorem 5.1. *The $\mathbb{F}_5$ entry of* [2, Table 1] *is misprinted. That row reads, in full, "$T^5 + T + 1$, $T^{10} + 3T^6 + 4T^5 + T^2 + T + 1$", and the first polynomial is reducible: $T = 2$ is a root, since $32 + 2 + 1 = 35$. The c-Wieferich prime of least degree in $\mathbb{F}_5[T]$ is $T^5 + 4T + 1$, which is irreducible.*

Proof sketch. That $T = 2$ is a root is the displayed arithmetic. Machine-checked in both directions: the quotient $\mathbb{F}_5[z]/(T^5+T+1)$ fails the field test of Lemma 3.1, while $\mathbb{F}_5[z]/(T^5+4T+1)$ passes it and the class of z is a witness in the sense of Corollary 2.4. That $T^5 + 4T + 1$ is the prime of *least* degree

in $\mathbb{F}_5[T]$ follows from the exhaustive sweep of $(5,5)$, which has witness count 1 (Proposition 4.1(2)), together with Proposition 4.4 for degrees $2,3,4$ and [1, Cor. 4.5] for degree 1. □

The slip is typographical rather than mathematical, and it is worth recording only because the table is the standard reference for these values. That it is a slip is settled inside [2] itself: Table 3 of the same paper, which lists elements α of subfields of $\overline{\mathbb{F}_5}$ against the c-Wieferich primes they produce, gives T^5+4T+1 at $s=1$, $\alpha \in \{4\}$, and at $s=2$ the same degree-10 prime as Table 1. The corrected value is in the wider literature too: [4] quotes T^5+4T+1 while attributing it to [2], and [3] finds it independently. We checked Table 1 in the arXiv e-print, which is the only arXiv version of [2]; the journal version was not accessible to us, so the correction is recorded against the e-print.

6. New empty cells

All the theorems of this section are complete determinations: for the stated cell the sweep runs over *all* q^{d-1} cosets $\theta+\mathbb{F}_q$ of $\mathbb{F}_{q^d}$, the model having first been certified as in §3. Table 2 lists the thirty-one cells with the size of the search each one required. Every cell was computed twice, in two programs with different transversal constructions and different arithmetic, before being certified. Section 7 records why each is believed to be new.

Table 2. The thirty-one cells determined in §6, with the number q^{d-1} of cosets swept in each. All are empty.

d	q	cosets
5	41, 43, 47, 53, 59	2 825 761, 3 418 801, 4 879 681, 7 890 481, 12 117 361
5	61, 67, 71, 73, 79	13 845 841, 20 151 121, 25 411 681, 28 398 241, 38 950 081
5	83, 89, 97, 101	47 458 321, 62 742 241, 88 529 281, 104 060 401
6	29, 31, 37	20 511 149, 28 629 151, 69 343 957
7	17, 19	24 137 569, 47 045 881
8	11, 13	19 487 171, 62 748 517
9	7	5 764 801
10	7	40 353 607
5	9, 27, 49, 81	6 561, 531 441, 5 764 801, 43 046 721
6	9, 25	59 049, 9 765 625
7	9	531 441
8	9	4 782 969
total		843 189 705

The prime-field rows, continued.

Theorem 6.1. *There is no c-Wieferich prime of degree* 5 *in* $\mathbb{F}_p[T]$ *for*

$$p = 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,\ 101.$$

Together with the $d=5$ *row of Table 1, there is no c-Wieferich prime of degree* 5 *in* $\mathbb{F}_p[T]$ *for any odd prime* $p \le 101$ *with* $p \neq 5$.

Proof sketch. Fourteen exhaustive sweeps, of p^4 cosets each, 460 679 294 in total; the largest is $p=101$ with 104 060 401. The complement statement is Table 1 for $p \le 37$, and $p=5$ is the case $p \mid d$, which is not empty: T^5+4T+1 is a c-Wieferich prime of $\mathbb{F}_5[T]$ (Theorem 5.1). □

Theorem 6.2. *There is no c-Wieferich prime of degree* 6 *in* $\mathbb{F}_p[T]$ *for* $p = 29, 31, 37$. *Together with the* $d=6$ *row of Table 1, there is none for any odd prime* $p \le 37$ *with* $p \neq 3$.

Proof sketch. Three exhaustive sweeps of p^5 cosets: 20 511 149+28 629 151+69 343 957 = 118 484 257. The excluded $p=3$ is the case $p \mid d$ and is not empty (Proposition 4.1). □

Theorem 6.3. *There is no c-Wieferich prime of degree* 7 *in* $\mathbb{F}_{17}[T]$ *or in* $\mathbb{F}_{19}[T]$*. Together with the* $d = 7$ *row of Table 1, there is none for any odd prime* $p \le 19$ *with* $p \ne 7$*; and* $p = 7$ *is not empty,* $T^7 + 6T + 3$ *being a c-Wieferich prime of* $\mathbb{F}_7[T]$*.*

Proof sketch. Two exhaustive sweeps, of $17^6 = 24\,137\,569$ and $19^6 = 47\,045\,881$ cosets. □

Theorem 6.4. *There is no c-Wieferich prime of degree* 8 *in* $\mathbb{F}_{11}[T]$ *or in* $\mathbb{F}_{13}[T]$*, and none of degree* 9 *in* $\mathbb{F}_7[T]$*. Together with Table 1, degree* 8 *is settled for every odd* $p \le 13$ *and degree* 9 *for* $p \in \{5, 7\}$*.*

Proof sketch. Three exhaustive sweeps: $11^7 = 19\,487\,171$, $13^7 = 62\,748\,517$ and $7^8 = 5\,764\,801$ cosets. For $d = 9$ the characteristic $p = 3$ is the case $p \mid d$ and is not empty. □

Theorem 6.5. *There is no c-Wieferich prime of degree* 10 *in* $\mathbb{F}_7[T]$*. Together with the* $d = 10$ *row of Table 1, degree* 10 *is settled for* $p \in \{3, 7\}$*; the omitted* $p = 5$ *is the case* $p \mid d$*, and* $\mathbb{F}_5[T]$ *does contain c-Wieferich primes of degree* 10*.*

Proof sketch. One exhaustive sweep of $7^9 = 40\,353\,607$ cosets. This is the first entry of the $d = 10$ row at a characteristic other than 3. □

A second axis: non-prime constant fields. Table 1 is a table of prime fields because the question it addresses is whether a counterexample exists over a prime field. The counterexample that does exist lives over $\mathbb{F}_{19^3}$, so the cells (d, p^e) with $e \ge 2$ and $p \nmid d$ are exactly the ones its existence puts in question, and beyond the small degrees implicit in [2, Table 1] — reproduced here as Proposition 4.4 — nothing is published about them.

Theorem 6.6. *There is no c-Wieferich prime of degree* 5 *in* $\mathbb{F}_9[T]$*,* $\mathbb{F}_{27}[T]$*,* $\mathbb{F}_{49}[T]$ *or* $\mathbb{F}_{81}[T]$*; none of degree* 6 *in* $\mathbb{F}_9[T]$ *or* $\mathbb{F}_{25}[T]$*; and none of degree* 7 *or* 8 *in* $\mathbb{F}_9[T]$*.*

Proof sketch. Eight exhaustive sweeps, over

$$3^8,\ 3^{12},\ 7^8,\ 3^{16},\ 3^{10},\ 5^{10},\ 3^{12},\ 3^{14}$$

cosets respectively, i.e. 6 561, 531 441, 5 764 801, 43 046 721, 59 049, 9 765 625, 531 441 and 4 782 969; 64 488 608 in all. Here the model check does more work than in the prime-field case: it certifies not only that the degree-ed modulus is irreducible over $\mathbb{F}_p$, but that the fixed field of $x \mapsto x^q$ inside $\mathbb{F}_{p^{ed}}$ is the $\mathbb{F}_q$ used to form the cosets. Every $\mathbb{F}_q$-coset in $\mathbb{F}_{q^d}$ is swept, so each statement is a complete determination, not a search over a subfamily of polynomials. □

Remark 6.7. The cell $(5, 27)$ is the smallest cubic-extension analogue of the counterexample cell $(5, 19^3)$: degree 5, constant field a cubic extension of a prime field, $p \nmid d$. The next such cell, $(5, 7^3)$, needs $343^4 \approx 1.4 \cdot 10^{10}$ cosets and was not attempted.

Remark 6.8. Four further cells were determined in the reference implementation but are not among the theorems above, because they were not carried through the certified pipeline: $(5, 121)$, $(5, 169)$ and $(7, 25)$ are empty, over 214 358 881, 815 730 721 and 244 140 625 cosets. The cell $(5, 125)$ is *not* empty — it has witness count 4 — but there $p \mid d$, so it is no counterexample to anything; it matches the entry $|B_{5^3,1}| = 4$ of the count table of [2]. These four computations lie outside the formal development and are recorded here only so that they are not repeated.

7. Provenance of the new cells

The literature on c-Wieferich primes is small enough to be closed, and we describe the search so that the claim of novelty can be judged rather than taken.

The searches below were run on 28 August 2026. In the arXiv metadata the phrase "c-Wieferich" occurs in exactly two papers, [2] and [4], and the two words "Wieferich" and "Carlitz" occur together in exactly three, [6], [3] and [4]. Two full-text searches were then run over a local mirror holding the LaTeX sources of 880 750 arXiv papers. The literal string "c-Wieferich" occurs in [2] and in no other — a narrower test than it looks, since in their sources [4] sets the letter in mathematics mode and [3]

uses a capital C, so neither matches it. The words "Wieferich" and "Carlitz" occur together in 28 of those sources; six of the 28 are [6], [2], [9], [3], [4] and arXiv:2512.08060, all accounted for elsewhere in this section, and the remaining twenty-two are, by title, about Wieferich, Wilson, Lucas or Genocchi primes of the integers or about subjects with no c-Wieferich content. By Semantic Scholar, [2] has exactly one citing paper, [4], and [3] has two, arXiv:2512.08060 and arXiv:2504.03430; we read the source of both, and neither has a table or a search in it. By OpenAlex, [1], [6] and [8] have 2, 5 and 6 citing works and [2] and [9] none; apart from the papers already named in this section the citing works are, by title, on integer Wieferich primes, on Carlitz–Fermat quotients, on Bang–Zsigmondy analogues, on Mersenne–Horadam identities and a survey of power sums, and the ones we could read tabulate no c-Wieferich primes by degree. On the same date [4] was still at v2 (v1 14 July 2026, v2 22 July 2026), [3] still at v1 and [2] still at v1, and the author of [4] had exactly one paper on the arXiv, so [5] was still not public.

What that leaves published, cell by cell:

- [4, Prop. 5.2]: the thirty cells of Table 1.
- [3]: *all* places of $\mathbb{F}_3[T]$ of degree at most 24 (18 054 379 372 of them) and of $\mathbb{F}_5[T]$ of degree at most 17 (57 005 914 349), exhaustively, together with $\mathbb{F}_4[T]$ to degree 17 (1 376 854 004). Four Wieferich places are found over $\mathbb{F}_3$ and two over $\mathbb{F}_5$, and [3] prints them: over $\mathbb{F}_3$ the four polynomials of Proposition 4.1(1), of degrees 6, 9, 12 and 15; over $\mathbb{F}_5$, T^5+4T+1 and $T^{10}+3T^6+4T^5+T^2+T+1$, of degrees 5 and 10. These are exactly the primes of [2], so the two papers' notions agree where they overlap, and this settles $(d,3)$ for $d \le 24$ and $(d,5)$ for $d \le 17$.
- [2]: the least-degree table for $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_9, \mathbb{F}_{25}, \mathbb{F}_{27}, \mathbb{F}_{49}$; the assertion that the primes listed are the only c-Wieferich primes in $\mathbb{F}_3[T]$ of degree at most 60; and, for everything larger, *fixed* c-Wieferich primes only.
- [8]: degrees 2 and 3 in odd characteristic. [4, Thm. 5.1], proved in the unpublished [5]: degree 4 for every odd q.

Two of these deserve comment.

First, the large-degree machinery of [2] is their Theorem 3.3, which characterises *fixed* c-Wieferich primes of degree ps; a fixed prime has degree divisible by p, and conversely [4, Lemma 2.5] shows that when $p \nmid d$ no c-Wieferich prime of degree d is fixed. Their table of counts by triples (p,m,s) therefore says nothing whatever about a cell with $p \nmid d$, which is every cell claimed new here. Their flat statement about $\mathbb{F}_3[T]$ to degree 60 is the one exception, and we take it at face value: no *prime-field* cell with $p=3$ is claimed new.

Second — and this is a remark about the published record, not about our own cells — [3] settles $(d,3)$ for $d \le 24$ and $(d,5)$ for $d \le 17$, and therefore already contains ten of the thirty cells of Table 1, namely $(5,3)$, $(7,3)$, $(8,3)$, $(10,3)$, $(11,3)$, $(13,3)$, $(14,3)$, $(6,5)$, $(9,5)$ and $(11,5)$. [4] does not cite [3]. We note this because it changes what can be claimed: the whole prime-field range at $p \in \{3,5\}$ is settled in the literature, and accordingly *no cell claimed new in §6 has* $q \in \{2,3,4,5\}$. All fourteen $d=5$ cells have $p \ge 41$; the $d=6$ cells have $p \ge 29$; the smallest characteristic anywhere in §6 is $p=3$, and it occurs only through the non-prime constant fields $\mathbb{F}_9$, $\mathbb{F}_{27}$ and $\mathbb{F}_{81}$, which [3] does not treat.

Finally, the risk that is intrinsic to a claim of this kind. [4] states that the empty region of its Table is being extended by its own author, and [5] is in preparation. The cells of §6 are not out of reach of the published method — they are reached by it — and a later version of [4], or the appearance of [5], may report some of them independently. What is claimed is that they were not in the literature at the time of the search described above, and that they are here determined completely and verified.

8. How much is an empty cell worth

An empty cell is a negative result, and the natural question is how much evidence for Thakur's phenomenon a pile of them constitutes. There is a defensible answer, and it is not large.

Take as null model, in its own words, "our heuristic that a place of degree d is Wieferich with probability q^{-d}" [3]. What [3] *proves* is a statement about a random Drinfeld module rather than about the Carlitz module: its abstract announces a proof "that a place of degree d is Wieferich with the expected probability q^{-d} when we average over large enough sets of Drinfeld modules", and the theorem behind that, [3, Thm. 3.4], fixes $r \geq d + \log_q(2d)$ and a place $\mathfrak{p}$ of degree d and computes the probability that $\mathfrak{p}$ is Wieferich for a uniformly random Drinfeld model of rank r. The Carlitz module is one module, of rank 1, so carrying the figure over to it is an extrapolation; [3] makes it as such, and closes its own Carlitz experiments with "[a]gain, the conclusion is unclear". We use it in the same spirit and claim no more. A cell (d, q) contains q^{d-1} cosets, and a coset is a witness coset when the value $M_d(\theta)$, heuristically uniform in $\mathbb{F}_{q^d}$, vanishes. Hence

$$\mathbb{E}\big[\#\{\text{witness cosets in } (d,q)\}\big] \;=\; q^{d-1} \cdot q^{-d} \;=\; \tfrac{1}{q}, \tag{3}$$

independent of d. The model calibrates against the two exhaustive columns in the literature: over $\mathbb{F}_3$ for $d \leq 24$ it predicts $24/3 = 8$ witness cosets, and the four places [3] found there, of degrees $6, 9, 12, 15$, carry $2 + 3 + 4 + 5 = 14$ of them; over $\mathbb{F}_5$ for $d \leq 17$ it predicts $17/5 = 3.4$, and the two places found there, of degrees 5 and 10, carry $1 + 2 = 3$. The same order of magnitude, which is all a heuristic of this kind can be asked for.

Now score the region $p \nmid d$, where Thakur's suggestion lives, treating the counts as independent Poisson variables with the means (3):

region	cells	expected	observed	P(all 0)
Table 1	30	4.67	0	0.94%
§6	31	1.43	0	24%
both	61	6.10	0	0.22%

So the published grid alone already made "no c-Wieferich prime when $p \nmid d$" a one-in-a-hundred coincidence under the naive model, and the cells of §6 take it to about one in four hundred and fifty. That is the honest size of the contribution: no single new cell is worth more than $1/q \leq 1/7 \approx 0.14$ of an expected witness — and only the two cells at $q = 7$ reach that; the fourteen cells of the $d = 5$ row are worth at most $1/41$ apiece — and collectively they multiply the evidence by a factor of about four.

Two things follow. The evidence is worth extending, but not in every direction: by (3) a cell contributes $1/q$ whatever its degree, while it costs q^{d-1} to decide, so weight accumulates fastest at small q and large d — the direction [3] took — and slowest in the high-characteristic rows, where a cell is cheap but worth at most $1/41$ here. Those rows are extended below anyway, because they are the rows the published question is about. And nothing here is a theorem: (3) is a heuristic, the independence assumption is an assumption, and a single counterexample of the kind found in [4] would end the discussion regardless of the tail probability.

9. Questions, and what the method will not reach

Question 9.1. Is there a c-Wieferich prime of degree d in $\mathbb{F}_p[T]$ for some odd prime p with $p \nmid d$?

This is Thakur's question in the form in which it survives the counterexample of [4], and it is what the grid exists to probe. Nothing in the data of this paper or of [4] points either way. By (3) the evidential weight of a cell depends only on q while its cost is q^{d-1}, so the informative cells are the cheap-per-unit-evidence ones with small q and large d — which is exactly the region [3] exhausted — and no amount of sweeping at large characteristic will settle the matter. A single hit would.

Question 9.2. Is the cell $(5, 7^3)$ empty?

This is the next cubic-extension analogue of the counterexample cell $(5, 19^3)$ after $(5, 27)$, and it needs $343^4 \approx 1.4 \cdot 10^{10}$ cosets — more than a hundred times the largest cell run here, and about sixteen times the whole of Table 2. It is reachable after one structural improvement, which we state because it is the obvious next step and because it changes what has to be proved. The q-power map sends witnesses to witnesses: $y_j(\theta^q) = y_j(\theta)^q$ and $M_d(\theta^q) = M_d(\theta)^q$, so the set of witness cosets is stable under $\theta \mapsto \theta^q$ and only one representative of each orbit of that map on cosets needs the expensive nested product — a saving of a factor close to d. Exploiting it makes the correctness of the sweep depend on that equivariance, which is a short induction on the recursion of Lemma 2.2 and should be proved rather than assumed.

Two things are out of reach of this method and are stated so that they are not attempted. The cell $(5, 19^3)$ itself has $19^{12} \approx 2.2 \cdot 10^{15}$ cosets and cannot be swept; Proposition 4.3 certifies the counterexample as a named object instead. And [4, Conj. 4.2] is out of reach. It asserts that $\gcd([5], M_5) = G$ in $\mathbb{F}_{19^3}[T]$, where

$$G = \mu(T^q - T), \qquad \mu(X) = X^5 + 5X^3 + 3X^2 - 4X - 9 \in \mathbb{F}_{19}[X], \qquad q = 19^3,$$

equivalently that the c-Wieferich primes of degree 5 in $\mathbb{F}_{19^3}[T]$ are exactly the 6859 translates $P(T-a)$, $a \in \mathbb{F}_{19^3}$, of the prime of Proposition 4.3. One inclusion, $G \mid \gcd([5], M_5)$, is [4, Thm. 4.1]; the reverse is what is open. According to [4] the companion [5] reduces it to the single case in which $\eta = \theta^q - \theta$ has degree 15 over $\mathbb{F}_{19}$, and that remaining case "would require a gcd computation between polynomials of degree of order $3 \cdot 10^{11}$ and $2 \cdot 10^{15}$ over $\mathbb{F}_{19}$, or an enumeration of $\mathbb{F}_{19^{15}}$" — four to five orders of magnitude beyond the computations reported here.

10. Verification

Every numbered statement above is machine-checked in Lean 4. The arithmetic engine — the representation of $\mathbb{F}_{p^n}$, the model checks of §3, the nested product of Lemma 2.2 and the sweep — is written in a fragment that imports nothing, in particular no part of Mathlib, so that it can be compiled to a shared library and executed as native code; each cell is then discharged by Lean's `native_decide`, which evaluates the compiled decision procedure and records the result as an axiom of the kernel environment. No statement uses `sorry`: over the whole development the axiom set of every theorem consists of `propext`, `Quot.sound`, the single native-evaluation axiom belonging to that theorem and, in 34 of the 42, `Classical.choice` — and nothing else. The development is 42 theorems in 15 modules; the largest single cell, $(5, 81)$, takes about 45 minutes of native evaluation, and the counterexample of Proposition 4.3 takes 4 seconds because it needs no sweep. Independently, every cell was computed first in a C program that uses a different construction of the coset transversal and machine-word arithmetic, and all results agree; the published primes quoted from [2] and [4] were in addition re-derived from the definition (1) in a separate program, as described in the proof of Proposition 4.1. Repository reference to be added at submission.

It is worth being explicit about where the formal boundary lies. What the machine checks is a statement about the finite structure $\mathbb{F}_p[z]/(h)$: that the model passes every check of ctxOK — hence that h is irreducible over $\mathbb{F}_p$, that $\mathbb{F}_p[z]/(h)$ is a field of order p^{ed}, that φ is the q-power map and that the swept index set is a transversal of the $\mathbb{F}_q$-cosets — and that the sweep over all q^{d-1} cosets returns the stated witness count. The bridge from that to a statement about primes of $\mathbb{F}_q[T]$ is Lemmas 2.2 and 2.3 together with Proposition 2.1; the first two are proved above and the third is quoted from [1]. Proposition 2.1 is the step that is checked rather than reproved here: it was verified against the definition (1) on every monic polynomial of degree ≤ 4 over $\mathbb{F}_3$, ≤ 3 over $\mathbb{F}_9$ and ≤ 3 over $\mathbb{F}_5$, and on each of the seven published primes and the counterexample. This is the same division of labour as in [4] and [2], which likewise compute the finite part and quote the criterion.

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[4] D. Niedbala Giraudin, *A counterexample to a conjecture of Thakur on Carlitz–Wieferich primes*, preprint, arXiv:2607.15305v2 (v1 14 July 2026, v2 22 July 2026, still v2 on 28 August 2026). Numbered references to [4] in this paper are to v2, in which the exactness statement of v1 has become Conjecture 4.2.

[5] D. Niedbala Giraudin, *Effective determination of Carlitz–Wieferich primes of given degree*, in preparation; cited as [NG2] in [4].

[6] Nguyen Ngoc Dong Quan, *Carlitz module analogues of Mersenne primes, Wieferich primes, and certain prime elements in cyclotomic function fields*, J. Number Theory **145** (2014), 181–193; arXiv:1407.7206.

[7] The On-Line Encyclopedia of Integer Sequences, sequence A271781 (F. Fröhlich, 2016), whose name reads, verbatim, "Smallest x such that for p = prime(n), pe is a c-Wieferich prime of 'Artin-Schreier' type, or 0 if no such x exists" (offset 2; keywords `nonn,more,obsc`). Its comments gloss the token "pe" and attribute the values to "section 2.2.5, (vi) of Thakur, 2015", i.e. to [8]; Thakur denotes a prime by $\wp$ ([9, Def. 2.1]). We read the object as the Artin–Schreier polynomial $T^p - T - x$; on that reading all eleven published terms are reproduced.

[8] D. S. Thakur, *Fermat versus Wilson congruences, arithmetic derivatives and zeta values*, Finite Fields Appl. **32** (2015), 192–206.

[9] D. S. Thakur, *Fermat–Wilson supercongruences, arithmetic derivatives and strange factorizations*, J. Théor. Nombres Bordeaux **36** (2024), no. 2, 481–491; arXiv:2211.01076.

[10] D. S. Thakur, *Iwasawa theory and cyclotomic function fields*, in *Arithmetic Geometry* (Tempe, AZ, 1993), Contemp. Math. **174**, Amer. Math. Soc., Providence, RI, 1994, pp. 157–165.