STABLE RANK-ONE PERMUTATIONS OF $[n]^2$ OF CYCLE TYPE $(2,\dots,2)$: THE COUNT TABLE, A DEGREE-EIGHT POLYNOMIAL, AND THE THIRTY-SEVEN PATTERNS OF TWO TRANSPOSITIONS

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Abstract. A permutation u of the discrete square $[n]^2$ is *stable of rank one* when $u \otimes 1$ and $1 \otimes u$ commute in $S([n]^3)$; by a theorem of Conti and Szymański the stable permutations are exactly those whose induced endomorphism of the Cuntz algebra $\mathcal{O}_n$ is an automorphism. Brenti, Conti and Nenashev computed that $[4]^2$ carries 60 stable rank-one permutations of cycle type $(2,2)$ and 144 of cycle type $(2,2,2)$ and proposed the classification of the stable rank-one permutations of cycle type $(2,\dots,2)$ as a direction for further research. Let $N(n,k)$ be the number of stable rank-one permutations of $[n]^2$ of cycle type 2^k. We compute $N(n,k)$ for twenty new pairs (n,k) — among them the whole $n=4$ column up to the fixed-point-free case $(294, 180, 324, 72, 249$ for $k=4,\dots,8)$ and the values $N(5,5)=8500$, $N(6,3)=48420$, $N(7,3)=938910$ — and prove that the two-transposition column is the degree-eight polynomial $N(n,2)=(n^8-16n^7+114n^6-468n^5+1179n^4-1780n^3+1466n^2-496n)/8$. The mechanism is that stability depends only on the restriction of u to the indices it moves and is invariant under the diagonal action of S_n, so every stable permutation of cycle type 2^k is the inflation of a full-support pattern on $[m]^2$ with $m \le 4k$; for $k=2$ there are exactly 37 patterns up to that action, and we list them. Every count asserted by a theorem here is the value of a decision procedure checked in the Lean 4 proof assistant, running as compiled code, and the structural statements are proved in Lean on top of Mathlib without any appeal to evaluation; five further values, asserted by no theorem, are marked as such.

1. Introduction

Let $\mathcal{O}_n$ be the Cuntz algebra [6] generated by isometries $S_1,\dots,S_n$, and let $u \in S([n]^t)$ be a permutation of the t-fold cartesian power of $[n]=\{1,\dots,n\}$. Such a u determines a permutative unitary in $\mathcal{O}_n$ and hence a unital endomorphism λ_u of $\mathcal{O}_n$; the permutations for which λ_u is an automorphism are the *stable* ones. Precisely, Conti and Szymański [5, Theorem 3.2] proved — we quote the statement in the form in which Brenti, Conti and Nenashev reproduce it as [4, Theorem 2.2] — that for $n,t \in \mathbb{N}$, $n \ge 2$ and $u \in S([n]^t)$, *u is stable if and only if λ_u is an automorphism of $\mathcal{O}_n$.* This is the only place in the paper where the C^*-algebra appears: everything below is a statement about permutations of the finite set $[n]\times[n]$, and the reader who prefers may take Definition 2.1 as the starting point.

The recent paper [4] of Brenti, Conti and Nenashev describes the maximal finite subgroups of $\mathrm{Out}(\mathcal{O}_4)$ that arise from $S([4]^2)$. Many of the generators of those groups are stable permutations of $[n]^2$ of rank one and cycle type $(2,\dots,2)$, and the paper closes (Section 7, "Outlook", page 48 of the arXiv version) with the following proposal, which we quote verbatim:

> Many of the generators of the groups studied in this work arise from stable permutations of $[n]^2$ of rank 1 and cycle-type $(2,\dots,2)$. The computations that we have carried out for $n=4$ and cycle-types $(2,2)$ and $(2,2,2)$ (see Figures 3 and 4) show that there are 60 and 144 such permutations and they seem to have quite an orderly structure. Thus it could be a promising avenue of research to try to classify all such stable permutations of rank 1 (as was done for stable permutations of rank 1 and cycle-type (r) in [5, Thm. 3.1]).

In the quotation, [5, Thm. 3.1] is Theorem 3.1 of [2]. The same authors say in their introduction that these are figures "for which there is no complete theory yet".

What is known. Write $N(n,k)$ for the number of permutations of $[n]^2$ of cycle type 2^k (exactly k two-cycles, all other points fixed) that are stable of rank one.

The $k=1$ column is published in closed form. Brenti and Conti prove that a transposition of $[n]^2$ is stable — equivalently, stable of rank one — exactly when the set of first coordinates of its two points is disjoint from the set of second coordinates ([1, Theorem 8.1]) and deduce [1, Corollary 8.3]: "In $S([n]^2)$ there are $n(n-1)^2(n-2)/2$ stable transpositions (all of rank 1)"; at $n=4$ this is the $36 = 12+12+12$ quoted in [4, Section 2]. Brenti, Conti and Nenashev extend the classification to every single cycle [2, Theorem 3.1]: the r-cycle $\big((a_1,b_1),\dots,(a_r,b_r)\big)$ of $S([n]^2)$ is stable of rank one if and only if $a_i \neq b_j$ for all $i,j \in [r]$; this is the result the Outlook points to, and [2] turns it into a closed formula for the number of stable rank-one r-cycles, observing that for fixed r that number is a polynomial in n of degree $2r$ once $n \geq r$.

Past $k=1$ the literature prints two entries, both at $n=4$: $N(4,2)=60$ and $N(4,3)=144$ (the Outlook just quoted). Two families of stable rank-one involutions are characterised in closed form in [4]: k horizontal transpositions sharing one column pair, of which there are $\binom{n}{k}\big(\binom{n-k}{2}+\binom{k}{2}\big)$ ([4, Corollary 3.4]), and two horizontal transpositions in distinct rows, of which there are $\binom{n}{2}\big(\binom{n-2}{2}^2+1\big)$ ([4, Theorem 3.7], a proof of Brenti and Conti's Conjecture 12.2; the same characterisation is Theorem 1.2 of Pan [9], which prints no count). At $n=4$ the second family accounts for 12 of the 60 permutations of cycle type $(2,2)$. Pan [10] records that Brenti, Conti and Nenashev [2, 3] "investigated the stable r-cycles in $S([n]^2)$ for $r \leq 5$" and that "even the stability of r-cycles in $S([n]^2)$ is still open for $r \geq 6$", and characterises the stable transpositions of the cube $[n]^3$ [10, Theorem 1.1], a different index set from ours.

What this paper adds. We answer the enumerative half of the Outlook as far as the computation reaches, and the structural half for $k=2$.

(a) *The table.* Theorems 4.2–4.5 and 5.1 establish twenty new entries of $N(n,k)$, listed in Table 1: the whole $n=4$ column down to the fixed-point-free involutions ($k=8$), the $n=3$ column for $k \leq 4$, the $n=5$ column for $k \leq 5$, the $n=6$ and $n=7$ columns for $k \leq 3$, and the two-transposition column for $2 \leq n \leq 10$. The two published entries past $k=1$ are reproduced (Theorem 4.1), as is $N(4,1)=36$.

(b) *A polynomial.* The two-transposition column is a polynomial of degree eight (Theorem 5.1):

$$N(n,2) = 3\binom{n}{2} + 3\binom{n}{3} + 30\binom{n}{4} + 780\binom{n}{5} + 3960\binom{n}{6} + 7560\binom{n}{7} + 5040\binom{n}{8}. \tag{1}$$

The coefficients are determined by the values at $n \leq 8$; the values at $n=9$ and $n=10$ were predicted from (1) before they were computed, and then computed.

(c) *The mechanism.* Stability of rank one depends only on the restriction of u to the indices it moves (Theorem 3.3) and is invariant under the diagonal action of S_n on $[n]^2$ (Proposition 3.1). Hence every stable permutation of cycle type 2^k is the inflation of a full-support *pattern* on $[m]^2$ with $m \leq 4k$, and $N(n,k)$ is a polynomial in n of degree at most $4k$ (Proposition 3.4).

(d) *The classification for $k=2$.* Up to the diagonal action there are exactly 37 stable rank-one permutations of cycle type $(2,2)$: the number of orbits is $3,4,8,21,32,36,37$ for $n=2,\dots,8$ (Theorem 6.1) and is constant from $n=8$ on (Corollary 6.2). Table 2 lists a representative of each orbit. For comparison, transpositions form three patterns — the horizontal, the vertical, and the "disjoint index pairs" type — which is the $12+12+12=36$ of [4] and, uniformly in n, is [1, Theorem 8.1].

Every count asserted by a theorem below is the value of a decision procedure that was executed and checked inside the Lean 4 proof assistant [7] — the five values of Remark 4.7, which no theorem asserts, are the only counts in the paper outside that discipline; the structural statements (Section 3) are proved in Lean on top of Mathlib [8] with no appeal to evaluation. Section 9 says exactly which statements are of which kind, what finite searches were run, and what they cost. The two arguments that are *not* machine-checked — the counting bookkeeping behind Proposition 3.4 and its orbit version

Corollary 3.5 — are written out in full in Section 3; they are the only steps between the Lean theorems and the statements (b) and (d).

Closure of the search. As of 2026-09-07 the source [4] has no citing work in OpenAlex (record W7125135809, `cited_by_count` $= 0$), and the On-Line Encyclopedia of Integer Sequences returns no match for any of the seven sequences 36, 60, 144, 294, 180, 324, 72, 249 (the $n = 4$ column); 3, 12, 60, 990, 9195, 52878 and 60, 990, 9195, 52878, 222432 (the $k = 2$ column, two windows); 1560, 4395, 8500; 48420, 938910; 0, 0, 0, 6, 36, 120, 300, 630 (the $k = 1$ column); and 3, 3, 30, 780, 3960, 7560, 5040 (the coefficients in (1)), while the control query 1, 2, 5, 14, 42, 132 returns A000108. The two papers that continue the line of [4], Pan's [9, 10], treat respectively the sub-case of two horizontal transpositions and the transpositions of $[n]^3$.

The three earlier papers of this line, [1, 2, 3], are not on arXiv. We have read [1] and [2] in the versions posted by their authors and [3] only through its published summary, and neither of the first two contains a count of the permutations of cycle type 2^k with $k \geq 2$: the enumerative results of [2] are for single cycles, and [1] closes (Section 12) with

> Another natural family of permutations to analyze is involutions. This seems to be a much harder problem. Perhaps the most natural class of permutations to consider next would be the product of two commuting transpositions. We leave an exhaustive analysis of this case to future work.

before stating the conjecture on two horizontal transpositions that [4, Theorem 3.7] and [9, Theorem 1.2] settle. The summary of [3] announces a graph criterion for stability and the characterisation and enumeration of "stable quadratic 4 and 5-cycles, as well as a notable class of stable quadratic r-cycles" — again single cycles. So the 2^k table is open in the literature, exactly as the Outlook of [4] says.

2. Preliminaries

Throughout, $[n] = \{1, \dots, n\}$, $S(A)$ is the symmetric group of a finite set A, and permutations are composed as functions, right to left. For $u \in S([n]^2)$ write $u(x, y) = (u_1(x, y), u_2(x, y))$. Following [4, Section 2] define two permutations of the cube $[n]^3$,

$$(u \otimes 1)(x, y, z) = (u_1(x, y), u_2(x, y), z), \qquad (1 \otimes u)(x, y, z) = (x, u_1(y, z), u_2(y, z)). \tag{2}$$

Brenti, Conti and Nenashev say that u is *compatible with* v if $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ in $S([n]^3)$ ([4, equation (5)]) and that u and v are *bicompatible* if each is compatible with the other. Stability and rank are defined in [4, Section 2] through a sequence of permutations $\psi_k(u)$ of $[n]^{2+k}$; for permutations of the square the rank-one case has the following characterisation, which is Proposition 4.5 of Brenti and Conti [1], quoted as [9, Lemma 2.1]: *$v \in S([n]^2)$ is stable of rank 1 if and only if $(v \otimes 1)(1 \otimes v) = (1 \otimes v)(v \otimes 1)$ in $S([n]^3)$*. We take this characterisation as the definition.

Definition 2.1. $u \in S([n]^2)$ is *stable of rank one* if $u \otimes 1$ and $1 \otimes u$ commute in $S([n]^3)$, i.e. if u is bicompatible with itself.

For $k \geq 0$ a permutation has *cycle type* 2^k if it is a product of k disjoint transpositions (so it is an involution with exactly $2k$ moved points). We write

$$N(n, k) = \#\{u \in S([n]^2) : \ u \text{ has cycle type } 2^k \text{ and is stable of rank one}\}.$$

The number of all permutations of cycle type 2^k of an N-element set is $\binom{N}{2k}(2k - 1)!!$; with $N = n^2$ this is the size of the search space behind each entry of Table 1.

Unfolding (2) turns Definition 2.1 into three equations between coordinates.

Proposition 2.2 (coordinate form). *$u \in S([n]^2)$ is stable of rank one if and only if for all $x, y, z \in [n]$*

$$u_1\big(x,\, u_1(y, z)\big) = u_1(x, y), \tag{C1}$$

$$u_2\big(x,\, u_1(y, z)\big) = u_1\big(u_2(x, y),\, z\big), \tag{C2}$$

$$u_2(y, z) = u_2\big(u_2(x, y),\, z\big). \tag{C3}$$

The identity of $[n]^2$ *is stable of rank one, and if* u *is stable of rank one then so is* u^{-1}*.*

Proof. Evaluate both sides of $(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)$ at (x, y, z). With $(a, b) = u(y, z)$ and $(c, d) = u(x, y)$ the left side is $(u_1(x, a), u_2(x, a), b)$ and the right side is $(c, u_1(d, z), u_2(d, z))$; comparing coordinates gives (C1)–(C3). The identity satisfies the three equations trivially. Since $u \mapsto u \otimes 1$ and $u \mapsto 1 \otimes u$ are group homomorphisms $S([n]^2) \to S([n]^3)$, inverting the commutation relation of u gives that of u^{-1}. The Lean names are `stableRank1_iff`, `stableRank1_one` and `stableRank1_inv`; see Appendix A. □

Equations (C1)–(C3) are what every search in this paper evaluates: a permutation of $[n]^2$ is stored as an array of length n^2 in one-line form, and the three equations are tested for all n^3 triples. Writing $A_z(y) = u_1(y, z)$ and $B_x(y) = u_2(x, y)$, they say that $A_{A_z(y)} = A_y$, $B_{B_x(y)} = B_y$, and that every A_z commutes with every B_x.

3. Symmetries, and stability depends only on the index support

The group S_n acts on $[n]^2$ diagonally, $\sigma \cdot (x, y) = (\sigma x, \sigma y)$, and hence on $S([n]^2)$ by conjugation, $u \mapsto (\sigma \times \sigma)\, u\, (\sigma \times \sigma)^{-1}$.

Proposition 3.1 (diagonal invariance)**.** *If* $u \in S([n]^2)$ *is stable of rank one and* $\sigma \in S_n$*, then* $(\sigma \times \sigma)\, u\, (\sigma \times \sigma)^{-1}$ *is stable of rank one.*

Proof. Let $\sigma^{(3)}$ be σ acting on all three coordinates of $[n]^3$. A direct check on triples gives $\big((\sigma\times\sigma)u(\sigma\times\sigma)^{-1}\big)\otimes 1 = \sigma^{(3)}(u\otimes 1)(\sigma^{(3)})^{-1}$ and the same identity for $1 \otimes (\cdot)$. The *single* conjugator $\sigma^{(3)}$ transports both tensor factors, so it transports the commutation relation. (Lean: `stableRank1_conj`.) □

The source uses a different symmetry, the *antitransposition* ${}^{a}u(x, y) = (n + 1 - u_2(n + 1 - y, n + 1 - x),\, n + 1 - u_1(n + 1 - y, n + 1 - x))$ of [4, equation (4)], and records in [4, Proposition 2.3] — there stated as a special case of [1, Proposition 5.13] — that u, v are bicompatible if and only if ${}^{a}u, {}^{a}v$ are. The diagonal case $u = v$ of that proposition is recovered from Proposition 3.1 and a second symmetry.

Proposition 3.2 (transposing the square)**.** *Let* $F(x, y) = (y, x)$ *and let* $w = F \circ (\rho \times \rho)$*, where* $\rho(x) = n + 1 - x$*. If* $u \in S([n]^2)$ *is stable of rank one, then so are* FuF^{-1} *and* $wuw^{-1} = {}^{a}u$*.*

Proof. Let $R(x, y, z) = (z, y, x)$ be the reversal of the cube. Then $(FuF^{-1}) \otimes 1 = R\,(1 \otimes u)\,R^{-1}$ and $1 \otimes (FuF^{-1}) = R\,(u \otimes 1)\,R^{-1}$: the conjugator R *swaps* the two tensor factors, so the commutation relation survives with its two sides exchanged. For the second claim, $wuw^{-1} = F\big((\rho\times\rho)u(\rho\times\rho)^{-1}\big)F^{-1}$ is stable by Proposition 3.1 and the first claim, and unwinding the definition shows $wuw^{-1} = {}^{a}u$. (Lean: `stableRank1_sqFlip`, `stableRank1_antitr`; the antitransposition is defined there as conjugation by w.) □

The *index support* of $u \in S([n]^2)$ is the set of all coordinates of all points moved by u; a permutation of cycle type 2^k has at most $4k$ indices in its support. The next statement is the engine of everything that follows: it says that stability is a property of the restriction of u to its index support, not of the ambient size. Its proof is a case check that anyone would redo in minutes, and we make no claim of novelty for it; we state it because it is what makes the columns of Table 1 polynomial.

Theorem 3.3 (the support theorem)**.** *Let* $m \le N$ *and let* $u \in S([N]^2)$ *be the identity at every point outside* $[m] \times [m]$*. Then* u *maps* $[m] \times [m]$ *into itself, and* u *is stable of rank one if and only if (C1)–(C3) hold for all* $x, y, z \in [m]$*.*

Proof. If $p \in [m]^2$ and $u(p) \notin [m]^2$ then $u(u(p)) = u(p)$, so $u(p) = p$ by injectivity, contradicting $p \in [m]^2$; hence $u([m]^2) \subseteq [m]^2$. For the equivalence, only the "if" direction needs proof, and we check that each triple with a coordinate outside $[m]$ satisfies (C1)–(C3) automatically. If $x \notin [m]$ then $u(x, w) = (x, w)$ for every w, so (C1) reads $x = x$, (C2) reads $a = a$ with $a = u_1(y, z)$, and (C3) reads $u_2(y, z) = u_2(y, z)$. If $x \in [m]$ and $y \notin [m]$ then $u(x, y) = (x, y)$ and $u(y, z) = (y, z)$, and all three equations become identities. If $x, y \in [m]$ and $z \notin [m]$ then $u(y, z) = (y, z)$ and $u(d, z) = (d, z)$

for $d = u_2(x, y)$, and again all three are identities. (Lean: `maps_of_fix`, `stableRank1_iff_restricted`, proved for general m and N.) □

Call a stable rank-one $\bar{u} \in S([m]^2)$ of cycle type 2^k a *pattern* (of size m) if its index support is all of $[m]$, and let $c_m^{(k)}$ be the number of patterns of size m. The following bookkeeping is elementary and is *not* part of the Lean development; it is the only unformalised step between the theorems of Sections 4 and 6 and the statements (b) and (d) of the introduction.

Proposition 3.4 (each column is a polynomial)**.** *Fix $k \geq 1$. Then $c_m^{(k)} = 0$ for $m > 4k$, and for every $n \geq 0$*

$$N(n,k) = \sum_{m=0}^{4k} c_m^{(k)} \binom{n}{m}, \qquad c_m^{(k)} = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} N(j,k). \tag{3}$$

In particular $N(n, k)$ is a polynomial in n of degree at most $4k$, determined by its values at $n \leq 4k$.

Proof. A permutation of cycle type 2^k moves $2k$ points, so its index support S has at most $4k$ elements; this gives $c_m^{(k)} = 0$ for $m > 4k$. Fix n and an m-subset $S \subseteq [n]$, and let $\iota_S \colon [m] \to S$ be the order-preserving bijection. We claim that $\bar{u} \mapsto (\iota_S \times \iota_S)\, \bar{u}\, (\iota_S \times \iota_S)^{-1}$ (extended by the identity outside $S \times S$) is a bijection from the patterns of size m onto the stable rank-one permutations of $[n]^2$ of cycle type 2^k with index support exactly S. Indeed, extend ι_S to some $\sigma \in S_n$; the extension of $\bar{u}$ by the identity to $[n]^2$ is stable by Theorem 3.3 (read right to left, with $N = n$), its conjugate by $\sigma \times \sigma$ is stable by Proposition 3.1, and that conjugate is the displayed permutation, which has cycle type 2^k and index support S. Conversely, if u is stable of cycle type 2^k with index support S, then $(\sigma \times \sigma)^{-1} u (\sigma \times \sigma)$ is stable (Proposition 3.1), is the identity outside $[m]^2$, and maps $[m]^2$ into itself; its restriction $\bar{u}$ to $[m]^2$ is a permutation of cycle type 2^k with index support $[m]$, and the conditions of Theorem 3.3 for the conjugate are exactly (C1)–(C3) for $\bar{u}$, so $\bar{u}$ is a pattern. The two constructions are inverse to each other. Summing over S gives the first formula of (3); the second is binomial inversion. □

The corresponding statement for a *single* cycle is known: Brenti, Conti and Nenashev give a closed formula for the number of stable rank-one r-cycles of $S([n]^2)$ and observe that, for fixed r, it is a polynomial in n of degree $2r$ once $n \geq r$ [2].

The same argument, applied to orbits, gives the following.

Corollary 3.5 (the orbit count stabilises)**.** *Let $O(n, k)$ be the number of orbits of the diagonal S_n-action on the stable rank-one permutations of $[n]^2$ of cycle type 2^k, and let $o_m^{(k)}$ be the number of S_m-orbits of patterns of size m. Then $O(n, k) = \sum_{m \leq n} o_m^{(k)}$; in particular $O(n, k)$ is non-decreasing in n and constant for $n \geq 4k$, where it equals the total number $\sum_m o_m^{(k)}$ of patterns up to the diagonal action.*

Proof. Every orbit contains an element whose index support is an initial segment $[m]$, hence the inflation of a pattern of size m. If two patterns $\bar{u}, \bar{v}$ of size m are conjugate under $\sigma \times \sigma$ with $\sigma \in S_n$, then σ maps the index support $[m]$ of $\bar{u}$ onto that of $\bar{v}$, so σ restricts to an element of S_m and $\bar{u}, \bar{v}$ are S_m-conjugate. Hence the S_n-orbits with support size m are in bijection with the S_m-orbits of patterns of size m, for every $n \geq m$. □

4. The table

Each entry of Table 1 is the value of a Lean function `countStable n k` that enumerates, by a depth-first search over the points of $[n]^2$ in order, every involution of $[n]^2$ with exactly k two-cycles — at each point either leaving it fixed or pairing it with a later free point — and tests each leaf against (C1)–(C3). The completeness of the enumeration is certified numerically: the number of leaves visited equals $\binom{n^2}{2k}(2k-1)!!$ on eleven of the cells (Proposition 7.1(i)), and leaves reached by different choice sequences are different involutions, so every involution with k two-cycles is visited exactly once. The

TABLE 1. $N(n,k)$, the number of stable rank-one permutations of $[n]^2$ of cycle type 2^k. Bold: printed in [4]. Plain: certified in Lean (Theorems 4.1–4.5, 4.6, 5.1). Superscript c: computed by the C prototype only (Remark 4.7), not certified. Blank: not computed.

$n\backslash k$	1	2	3	4	5	6	7	8
2	0	3						
3	6	12	18	9				
4	**36**	**60**	**144**	294	180	324	72	249
5	120	990	1560	4395	8500			
6	300	9195	48420	90300^c				
7	630	52878	938910					
8		222432	10344768^c					
9		752580						
10		2168955						

test at a leaf is a transcription of (C1)–(C3) into array arithmetic; Section 9 discusses the status of that transcription.

Theorem 4.1 (the published entries)**.** *$N(4,1)=36$, $N(4,2)=60$ and $N(4,3)=144$.*

These are the three values printed in [4] (the first one from [1, Corollary 8.3]); only the certification is ours. They are also the control for the new entries: an enumerator that reproduces 36, 60 and 144 from the definition has its definition right. The search spaces are 120, 5460 and 120 120 involutions.

Theorem 4.2 (the $n=4$ column)**.** *$N(4,4)=294$, $N(4,5)=180$, $N(4,6)=324$, $N(4,7)=72$ and $N(4,8)=249$. In particular exactly 249 of the 2 027 025 fixed-point-free involutions of $[4]^2$ are stable of rank one.*

Theorem 4.3 ($n=2$ and $n=3$)**.** *$N(2,2)=3$: all three fixed-point-free involutions of $[2]^2$ are stable of rank one. $N(3,2)=12$, $N(3,3)=18$ and $N(3,4)=9$.*

Theorem 4.4 (the $n=5$ column)**.** *$N(5,2)=990$, $N(5,3)=1560$, $N(5,4)=4395$ and $N(5,5)=8500$.*

The last two entries are searches over 113 565 375 and 3 088 978 200 involutions respectively; the second is the largest single search in the paper.

Theorem 4.5 ($n=6$ and $n=7$)**.** *$N(6,2)=9195$, $N(6,3)=48420$, $N(7,2)=52878$ and $N(7,3)=938910$.*

The $k=3$ entries are searches over 29 216 880 and 209 757 240 involutions.

Proof of Theorems 4.1–4.5. Each equation is a closed term of the form `countStable` $n\ k=v$ and was decided by executing the enumeration described above, compiled to native code, inside Lean (Section 9); the claim rests entirely on that exhaustive computation. Independently, a C implementation of the same search (a different program in a different language, sharing nothing but the definition) returned the same value on every cell, and had returned the three published values and the sub-counts of [4, Corollary 3.4 and Theorem 3.7] — 12 at $(n,k)=(4,3)$ and $12, 100, 555, 2121$ at $n=4,5,6,7$ — before any new entry was recorded. A third implementation, written for this version, which enumerates the k two-cycles in a different order and tests a leaf by comparing the two permutations of $[n]^3$ outright rather than through (C1)–(C3), reproduced every entry of Table 1 — the certified and the uncertified ones alike — and the leaf totals $\binom{n^2}{2k}(2k-1)!!$ on every cell it visited. $\square$

Proposition 4.6 (the transposition column)**.** *For $0\le n\le 7$, $N(n,1)=\frac{1}{2}\,n(n-1)^2(n-2)$; that is, the column reads $0,0,0,6,36,120,300,630$.*

This is published mathematics and we do not claim it as new; only the certification is ours, and it is stated for $n \le 7$ because that is the range that was checked. The closed form for all n is [1, Corollary 8.3], quoted in Section 1; the classification behind it is [1, Theorem 8.1], and its extension to all single cycles is [2, Theorem 3.1], which at $r = 2$ says that the transposition $((a, i), (b, j))$ is stable of rank one if and only if $\{a, b\} \cap \{i, j\} = \emptyset$. That condition splits the stable transpositions into three types — *horizontal*, $a = b \notin \{i, j\}$; *vertical*, $i = j \notin \{a, b\}$; and *disjoint index pairs*, $a \neq b$, $i \neq j$, $\{a, b\} \cap \{i, j\} = \emptyset$ — of sizes $n\binom{n-1}{2}$, $n\binom{n-1}{2}$ and $2\binom{n}{2}\binom{n-2}{2}$, and a direct computation (outside Lean, $n \le 7$) confirms both the split and the sum $\frac{1}{2}n(n-1)^2(n-2)$; at $n = 4$ this is the $12 + 12 + 12$ of [4, Section 2]. In the language of Proposition 3.4, $c_3^{(1)} = 6$, $c_4^{(1)} = 12$ and all other $c_m^{(1)}$ vanish.

Remark 4.7 (entries computed but not certified). The C prototype also gives $N(6, 4) = 90300$ and $N(8, 3) = 10\,344\,768$, and $N(11, 2) = 5\,526\,840$, $N(12, 2) = 12\,774\,828$, $N(13, 2) = 27\,281\,202$. These values are recorded here as computations outside the formal development; they are not part of any theorem of this paper. The last three agree with (1), and each was predicted from (1) before it was computed. All five were recomputed for this version by a further, independently written enumerator — a third implementation of Definition 2.1, which tests a leaf by comparing the two permutations of $[n]^3$ outright — and it returned the same five counts, together with leaf totals equal to $\binom{n^2}{2k}(2k-1)!!$ in each of the five cases.

5. The two-transposition column is a polynomial

Let

$$\begin{aligned} P(n) &= 3\binom{n}{2} + 3\binom{n}{3} + 30\binom{n}{4} + 780\binom{n}{5} + 3960\binom{n}{6} + 7560\binom{n}{7} + 5040\binom{n}{8} \\ &= \frac{n^8 - 16n^7 + 114n^6 - 468n^5 + 1179n^4 - 1780n^3 + 1466n^2 - 496n}{8}. \end{aligned} \tag{4}$$

Theorem 5.1. *$N(n, 2) = P(n)$ for every $n \ge 0$. In particular $N(8, 2) = 222\,432$, $N(9, 2) = 752\,580$ and $N(10, 2) = 2\,168\,955$.*

Proof. By Proposition 3.4 with $k = 2$, $N(n, 2) = \sum_{m \le 8} c_m^{(2)}\binom{n}{m}$ with $c_m^{(2)}$ the binomial transform of the values $N(j, 2)$, $j \le m$. From $N(j, 2) = 0, 0, 3, 12, 60, 990, 9195, 52878, 222432$ for $j = 0, \ldots, 8$ (Theorems 4.1–4.5 and the value at $n = 8$ below) the transform is $c_2^{(2)}, \ldots, c_8^{(2)} = 3, 3, 30, 780, 3960, 7560, 5040$ and $c_0^{(2)} = c_1^{(2)} = 0$, which is (4); the two expressions in (4) agree as polynomials.

The machine-checked content is the statement $\forall n \le 10$: `countStable` n 2 = `poly2` n, where `poly2` is the binomial form of (4) with a Pascal-rule binomial (Lean: `poly2_agrees`), together with the three values $N(8, 2)$, $N(9, 2)$, $N(10, 2)$ (Lean: `count_k2_large`). The values at $n \le 8$ fix the seven coefficients; the checks at $n = 9$ and $n = 10$ are therefore genuine verifications of a prediction, over $4\,991\,220$ and $11\,763\,675$ involutions. The step from $n \le 10$ to all n is Proposition 3.4, which is not formalised. Two controls guard the coefficients: the degree-seven truncation of (4), and (4) with the top coefficient 5040 replaced by 5039, both fail already at $n = 8$ (Proposition 7.1(v)). The transform of the computed column vanishes at $m = 9$ and $m = 10$, exactly the cut-off $4k = 8$ predicted by Proposition 3.4. □

The leading term $n^8/8 = \frac{1}{2}\left(n^4/2\right)^2$ is what one expects: two generic stable transpositions, unordered. The published sub-counts are lower-order pieces of P: the two horizontal transpositions in distinct rows of [4, Theorem 3.7] contribute $\binom{n}{2}\left(\binom{n-2}{2}^2 + 1\right)$, a polynomial of degree six.

Remark 5.2. For $k = 3$, the binomial transform of the certified column $N(3, 3), \ldots, N(7, 3)$ together with the uncertified $N(8, 3)$ of Remark 4.7 is $c_3^{(3)}, \ldots, c_8^{(3)} = 18, 72, 1020, 40860, 628320, 4110960$ — all non-zero and growing. Proposition 3.4 bounds the degree by $4k = 12$, so pinning the $k = 3$ column needs $N(n, 3)$ up to $n = 12$, beyond the searches run for this paper (Section 8).

6. The classification for two transpositions

By Proposition 3.1 the set of stable rank-one permutations of cycle type 2^k is a union of orbits of the diagonal S_n-action, and by Corollary 3.5 the number of orbits $O(n,k)$ counts patterns. The Lean function `countOrbits n k` runs the same enumeration as `countStable` and counts a stable leaf only when it is the lexicographically smallest element of its orbit, comparing it entry by entry with each of its $n!$ diagonal conjugates; every orbit is thus counted exactly once, at its canonical form.

Theorem 6.1 (orbit counts)**.** *The number of orbits of the diagonal S_n-action on the stable rank-one permutations of $[n]^2$ of cycle type 2^2 is*

$$O(n,2) = 3,\ 4,\ 8,\ 21,\ 32,\ 36,\ 37 \qquad \textit{for } n = 2,3,4,5,6,7,8.$$

For transpositions, $O(n,1) = 0,\ 2,\ 3,\ 3,\ 3,\ 3$ for $n = 2,\dots,7$.

Proof. Each equation `countOrbits` $n\ k = v$ was decided by executing the canonical-form enumeration inside Lean (Section 9); the whole sweep $n = 2,\dots,8$ at $k = 2$ visits $\sum_{n\le 8}\binom{n^2}{4}\cdot 3$ involutions, dominated by the 1 906 128 at $n = 8$, and tests up to $8! = 40\,320$ conjugates at each stable leaf. An independent orbit-closure computation in Python reproduced the seven values. □

Corollary 6.2 (exactly 37 patterns)**.** *Up to the diagonal action there are exactly* 37 *stable rank-one permutations of cycle type* $(2,2)$*: $o_m^{(2)} = 3,1,4,13,11,4,1$ patterns of size $m = 2,\dots,8$, and $O(n,2) = 37$ for every $n \ge 8$. For transpositions there are exactly* 3 *patterns ($o_3^{(1)} = 2$, $o_4^{(1)} = 1$) and $O(n,1) = 3$ for every $n \ge 4$.*

Proof. By Corollary 3.5, $o_m^{(k)} = O(m,k) - O(m-1,k)$ and $O(n,k)$ is constant for $n \ge 4k$; read the differences off Theorem 6.1. For $k = 2$ the cut-off is $4k = 8$, which is exactly where the certified sequence reaches; the stabilisation past $n = 8$ is Corollary 3.5, not a Lean theorem. □

Table 2 lists one representative of each of the 37 orbits, as a product of two transpositions of $[m]^2$, together with the size of its S_m-orbit. The list was produced by the orbit-closure script, not by the Lean development; the author of this draft re-checked, by an independent script, that each representative is stable of rank one, has index support all of $[m]$, that the 37 orbits are pairwise distinct, and that the orbit sizes are as printed. Within each size m the orbit sizes sum to the coefficient $c_m^{(2)}$ of (4), as Proposition 3.4 requires: 3, 3, 30, 780, 3960, 7560, 5040. This is a consistency check between Theorem 5.1 and Theorem 6.1 that goes through two independent computations.

The three patterns of size 2 are the two-transposition permutations of $[2]^2$ (all three fixed-point-free involutions of the 2×2 square are stable); the single pattern of size 8, $((4,3)(5,0))((6,2)(7,1))$, is the generic case of two transpositions whose eight indices are all distinct, and its orbit of size $8!/8 = 5040$ is what produces the leading coefficient of (4).

7. Controls

A count obtained by exhaustive search is only as good as the search, so the development records the following checks, each a Lean theorem decided by the same compiled evaluation.

Proposition 7.1 (controls)**.**

(i) *(Completeness.) With the stability test switched off, the enumeration visits exactly $\binom{n^2}{2k}(2k-1)!!$ leaves on the eleven cells* $(3,2)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(4,8)$, $(5,2)$, $(5,3)$, $(6,2)$, $(7,2)$.

(ii) *(Chunking.) Splitting the search by the smallest moved point and adding the pieces recovers $N(4,3)$ and $N(5,2)$.*

(iii) *(Off by one.) $N(4,2) \notin \{59,61\}$, $N(4,3) \notin \{143,145\}$ and $N(5,2) \notin \{989,991\}$.*

(iv) *(Non-vacuity.) $0 < N(4,2) < 5460$ and $0 < N(5,3) < 2\,656\,500$, the right-hand sides being the numbers of all involutions of the respective cycle types; and the identity is stable of rank one (Proposition 2.2).*

TABLE 2. The 37 patterns of cycle type $(2,2)$: one representative per orbit of the diagonal S_m-action on the stable rank-one permutations of $[m]^2$ with full index support, written with 0-based coordinates $0,\dots,m-1$ (the coordinates used by the Lean development), with the orbit size in brackets.

m	representatives [orbit size]
2	$((0,0)(0,1))((1,0)(1,1))$ [1]; $((0,0)(1,0))((0,1)(1,1))$ [1]; $((0,0)(1,1))((0,1)(1,0))$ [1]
3	$((0,2)(1,2))((2,0)(2,1))$ [3]
4	$((2,0)(2,1))((3,0)(3,1))$ [6]; $((2,0)(3,0))((2,1)(3,1))$ [6]; $((2,0)(3,1))((2,1)(3,0))$ [6]; $((1,2)(1,3))((2,0)(3,0))$ [12]
5	$((4,0)(4,1))((4,2)(4,3))$ [15]; $((1,0)(2,0))((3,0)(4,0))$ [15]; $((2,4)(3,4))((4,0)(4,1))$ [30]; $((3,2)(4,2))((4,0)(4,1))$ [60]; $((3,1)(3,2))((4,0)(4,1))$ [60]; $((3,1)(4,0))((3,2)(4,1))$ [60]; $((3,1)(4,0))((3,2)(4,2))$ [60]; $((2,1)(3,0))((4,0)(4,1))$ [60]; $((2,1)(3,0))((3,1)(4,0))$ [60]; $((2,1)(3,1))((4,0)(4,1))$ [60]; $((2,1)(3,1))((3,0)(4,0))$ [60]; $((3,2)(4,0))((4,1)(4,2))$ [120]; $((2,1)(3,0))((3,1)(4,1))$ [120]
6	$((4,2)(4,3))((5,0)(5,1))$ [90]; $((2,1)(3,1))((4,0)(5,0))$ [90]; $((4,2)(5,0))((4,3)(5,1))$ [180]; $((3,2)(4,2))((5,0)(5,1))$ [180]; $((2,1)(3,0))((4,1)(5,0))$ [180]; $((4,3)(5,0))((5,1)(5,2))$ [360]; $((3,2)(4,0))((4,1)(5,2))$ [360]; $((2,1)(3,0))((4,1)(5,1))$ [360]; $((3,2)(4,0))((5,1)(5,2))$ [720]; $((3,2)(4,0))((4,2)(5,1))$ [720]; $((3,2)(4,0))((4,1)(5,1))$ [720]
7	$((4,3)(5,0))((6,1)(6,2))$ [1260]; $((3,2)(4,0))((5,1)(6,1))$ [1260]; $((4,3)(5,0))((5,2)(6,1))$ [2520]; $((3,2)(4,0))((5,2)(6,1))$ [2520]
8	$((4,3)(5,0))((6,2)(7,1))$ [5040]

(v) *(Tightness of* (4)*.)* $N(8,2)$ *differs from the degree-seven truncation of* (4) *and from* (4) *with* 5040 *replaced by* 5039*.*

(vi) *(Orbit sanity.)* $0 < O(4,2) < N(4,2)$*,* $O(4,1) < N(4,1)$*,* $O(3,3) = 6$ *and* $O(3,4) = 3$*.*

Items (i) and (ii) are what make the counts of Section 4 counts of *all* involutions of the given type rather than of the ones some search happened to reach; item (iii) refutes a too-large and a too-small value at the source's own cells, so the evaluation is not satisfied by a misplaced quantifier; item (iv) shows the stability test is neither always true nor always false.

8. Open questions

Question 8.1. Is the $k = 3$ column of Table 1 a polynomial of degree exactly 12? By Proposition 3.4 it is a polynomial of degree at most 12, determined by $N(n,3)$ for $n \le 12$; the transform of the known values gives no sign of an earlier cut-off. The cell $(12,3)$ alone is a search over $\binom{144}{6} \cdot 15 \approx 1.67 \cdot 10^{11}$ involutions.

Question 8.2. Give a proof, uniform in n, that Table 2 is the complete list of stable rank-one permutations of cycle type $(2,2)$ up to the diagonal action — a classification in the sense of [4, Theorem 3.7], which handles the sub-case of two horizontal transpositions in distinct rows. At $k = 1$ the corresponding statement — three patterns, for every n — is already a theorem, [1, Theorem 8.1]; formalising *that* argument, which our development does not, is the natural warm-up.

Question 8.3. Extend the orbit classification to $k = 3$. The canonical-form enumeration of Theorem 6.1 computes $O(n,3)$ as it stands, but a pattern of cycle type 2^3 can have index support of size 12, so the sweep must reach $n = 12$.

9. Verification

Every numbered statement of this paper — except Proposition 3.4, Corollaries 3.5 and 6.2, the passage from $n \le 10$ to all n in Theorem 5.1, and the remarks and tables explicitly marked as computations outside the formal development — has been formalised and checked in Lean 4 (version 4.32.0) [7]; the structural layer uses Mathlib [8]. The repository is private; a repository reference is to be added at submission. Appendix A lists the statements verbatim. Two kinds of checking are involved, and we keep them apart.

Kernel-checked without evaluation. The definitions of $u \otimes 1$, $1 \otimes u$ and stability are stated for `Equiv.Perm (Fin n × Fin n)`, and Proposition 2.2, Propositions 3.1 and 3.2 and Theorem 3.3 are ordinary proofs, for general n (and general $m \le N$), whose axiom footprint is `propext`, `Quot.sound` and, in the proofs of inverse closure and of the two symmetries, `Classical.choice` — the standard Mathlib axioms and nothing else. Nothing in that file is closed by evaluation: neither `native_decide` nor `decide` occurs in it.

Decided by compiled evaluation. Every equation of the form `countStable` $n\ k = v$, `countOrbits` $n\ k = v$ and every control of Proposition 7.1 is closed by the `native_decide` tactic: the Boolean decision procedure is compiled to native code, executed, and the result `true` is admitted through an axiom specific to that theorem (the axiom footprint of each such theorem is `propext` plus its own `native_decide` axiom, and nothing else; there is no `sorry` anywhere in the development). The trust base of these theorems is therefore the Lean compiler and the C compiler, in addition to the kernel; this is the standard trade-off for searches of this size, and the C prototype is the independent check on it. The functions `countStable`, `countCands`, `countStableAt` and `countOrbits` live in modules outside the Mathlib import graph (`Core.lean`, which imports nothing at all, and `Orbits.lean`, which imports only `Core`), for two reasons: their evaluation is then off the Mathlib baseline, and the modules can be compiled into shared libraries (`libCuntzCore.so`, `libCuntzOrbits.so`; each object file and its `.olean` come from one and the same invocation of the compiler, so the compiled code and the checked module provably match) and loaded with `--load-dynlib`, so that `native_decide` runs compiled code rather than Lean's IR interpreter. Every module still checks without the libraries and yields the same values, but the recorded interpreter tax is about 20× over the development as a whole, and more than 100× on the small cell $(4, 4)$ (1 351 350 involutions: 11.6 s interpreted against about 0.1 s compiled); the largest module would take hours instead of 18 minutes.

What is by inspection. The array-level test `stableB` evaluates, for every triple, the three equations (C1)–(C3) on a permutation stored in one-line form; that it computes exactly the conditions of Proposition 2.2 is by inspection of a twenty-line function, not by a proved theorem linking `stableB` to `StableRank1`. Likewise, that the depth-first search enumerates every involution with k two-cycles exactly once is argued in Section 4 and certified numerically by Proposition 7.1(i), not proved as a Lean theorem. Both were cross-checked by the C prototype, and the three-equation form was compared, in a preliminary script, with the direct equality of the two permutations of $[3]^3$ over all 362 880 permutations of $[3]^2$ (no mismatch; 54 of them stable).

Cost. The wall times recorded in the development log, module by module, are 2 s for the definitions, 30 s for the small cells, 433 s for the module holding $N(5, 4)$, $N(6, 3)$, $N(7, 3)$, 505 s for the $k = 2$ column to $n = 10$, 1074 s for $N(5, 5)$, 23 s for the orbit sweep, and 22 s (peak 3.7 GB of memory, the Mathlib import) for the structural layer: about 2090 s of CPU for the Lean checks. The C prototype took about 2640 s, dominated by the uncertified cells of Remark 4.7, and the orbit scripts about 380 s. In total about 1.4 CPU-hours and a peak of 3.7 GB.

Appendix A. The Lean statements

The statements below are quoted verbatim from the development, in the order of the text; `Fin n` is $\{0, \ldots, n-1\}$, so the indices are 0-based. A value `u (x, y)` is a pair whose components are written `.1`, `.2`.

Definitions (Mathlib layer).

```
def tenL (u : Equiv.Perm (Fin n × Fin n)) : Equiv.Perm (Fin n × Fin n × Fin n) where
  toFun p := ((u (p.1, p.2.1)).1, (u (p.1, p.2.1)).2, p.2.2)
  ...
def tenR (u : Equiv.Perm (Fin n × Fin n)) : Equiv.Perm (Fin n × Fin n × Fin n) where
  toFun p := (p.1, u p.2)
  ...
```

```
def StableRank1 (u : Equiv.Perm (Fin n × Fin n)) : Prop :=
  tenL u * tenR u = tenR u * tenL u
def diag (σ : Equiv.Perm (Fin n)) : Equiv.Perm (Fin n × Fin n) where
  toFun p := (σ p.1, σ p.2)
  ...
def sqFlip : Equiv.Perm (Fin n × Fin n) where
  toFun p := (p.2, p.1)
  ...
def revPerm : Equiv.Perm (Fin n) where
  toFun := Fin.rev
  ...
def antitr (u : Equiv.Perm (Fin n × Fin n)) : Equiv.Perm (Fin n × Fin n) :=
  (sqFlip * diag revPerm) * u * (sqFlip * diag revPerm)⁻¹
```

Proposition 2.2.

```
theorem stableRank1_iff (u : Equiv.Perm (Fin n × Fin n)) :
    StableRank1 u ↔ ∀ x y z : Fin n,
      (u (x, (u (y, z)).1)).1 = (u (x, y)).1 ∧
      (u (x, (u (y, z)).1)).2 = (u ((u (x, y)).2, z)).1 ∧
      (u (y, z)).2 = (u ((u (x, y)).2, z)).2
theorem stableRank1_one : StableRank1 (1 : Equiv.Perm (Fin n × Fin n))
theorem stableRank1_inv {u : Equiv.Perm (Fin n × Fin n)} (h : StableRank1 u) :
    StableRank1 u⁻¹
```

Propositions 3.1 and 3.2.

```
theorem stableRank1_conj (σ : Equiv.Perm (Fin n)) {u : Equiv.Perm (Fin n × Fin n)}
    (h : StableRank1 u) : StableRank1 (diag σ * u * (diag σ)⁻¹)
theorem stableRank1_sqFlip {u : Equiv.Perm (Fin n × Fin n)} (h : StableRank1 u) :
    StableRank1 (sqFlip * u * sqFlip⁻¹)
theorem stableRank1_antitr {u : Equiv.Perm (Fin n × Fin n)} (h : StableRank1 u) :
    StableRank1 (antitr u)
```

Theorem 3.3.

```
theorem maps_of_fix {N m : ℕ} {u : Equiv.Perm (Fin N × Fin N)}
    (hfix : ∀ p : Fin N × Fin N, ¬(p.1.val < m ∧ p.2.val < m) → u p = p)
    (p : Fin N × Fin N) (hp : p.1.val < m ∧ p.2.val < m) :
    (u p).1.val < m ∧ (u p).2.val < m
theorem stableRank1_iff_restricted {N : ℕ} (m : ℕ) (u : Equiv.Perm (Fin N × Fin N))
    (hfix : ∀ p : Fin N × Fin N, ¬(p.1.val < m ∧ p.2.val < m) → u p = p) :
    StableRank1 u ↔ ∀ x y z : Fin N, x.val < m → y.val < m → z.val < m →
      ((u (x, (u (y, z)).1)).1 = (u (x, y)).1 ∧
       (u (x, (u (y, z)).1)).2 = (u ((u (x, y)).2, z)).1 ∧
       (u (y, z)).2 = (u ((u (x, y)).2, z)).2)
```

The counting functions (import-free layer). `countStable n k` is the number of stable leaves of the depth-first enumeration of the involutions of $[n]^2$ with k two-cycles; `countCands n k` is the number of all its leaves; `countStableAt n k p` is the number of stable leaves whose smallest moved point is `p`; `involCount N k` is $\binom{N}{2k}(2k-1)!!$ computed as $N!/((N-2k)!\,2^k\,k!)$; `binom` is the Pascal-rule binomial; `countOrbits n k` is the number of stable leaves that are lexicographically minimal in their diagonal S_n-orbit; and

```
def poly2 (n : Nat) : Nat :=
  3 * binom n 2 + 3 * binom n 3 + 30 * binom n 4 + 780 * binom n 5
    + 3960 * binom n 6 + 7560 * binom n 7 + 5040 * binom n 8
```

Theorems 4.1–4.5 and Proposition 4.6.

```
theorem count_n4_k1 : countStable 4 1 = 36
theorem count_n4_k2 : countStable 4 2 = 60
theorem count_n4_k3 : countStable 4 3 = 144
theorem count_n4_k4 : countStable 4 4 = 294
theorem count_n4_k5 : countStable 4 5 = 180
theorem count_n4_k6 : countStable 4 6 = 324
theorem count_n4_k7 : countStable 4 7 = 72
theorem count_n4_k8 : countStable 4 8 = 249
theorem count_small :
    countStable 2 2 = 3 ∧ countStable 3 2 = 12 ∧
    countStable 3 3 = 18 ∧ countStable 3 4 = 9
theorem count_n5_k2 : countStable 5 2 = 990
theorem count_n5_k3 : countStable 5 3 = 1560
theorem count_n5_k4 : countStable 5 4 = 4395
theorem count_n5_k5 : countStable 5 5 = 8500
theorem count_n6_k2 : countStable 6 2 = 9195
theorem count_n6_k3 : countStable 6 3 = 48420
theorem count_n7_k2 : countStable 7 2 = 52878
theorem count_n7_k3 : countStable 7 3 = 938910
theorem count_k1_formula :
    ∀ n ∈ [0, 1, 2, 3, 4, 5, 6, 7],
      countStable n 1 = n * (n - 1) * (n - 1) * (n - 2) / 2
```

Theorem 5.1.

```
theorem poly2_agrees :
    ∀ n ∈ [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10], countStable n 2 = poly2 n
theorem count_k2_large :
    countStable 8 2 = 222432 ∧ countStable 9 2 = 752580 ∧
    countStable 10 2 = 2168955
```

Theorem 6.1.

```
theorem orbits_k2 :
    countOrbits 2 2 = 3 ∧ countOrbits 3 2 = 4 ∧ countOrbits 4 2 = 8 ∧
    countOrbits 5 2 = 21 ∧ countOrbits 6 2 = 32 ∧ countOrbits 7 2 = 36 ∧
    countOrbits 8 2 = 37
theorem orbits_k1 :
    countOrbits 2 1 = 0 ∧ countOrbits 3 1 = 2 ∧ countOrbits 4 1 = 3 ∧
    countOrbits 5 1 = 3 ∧ countOrbits 6 1 = 3 ∧ countOrbits 7 1 = 3
```

Proposition 7.1.

```
theorem enum_complete :
    (countCands 3 2 = involCount 9 2) ∧ (countCands 3 4 = involCount 9 4) ∧
    (countCands 4 1 = involCount 16 1) ∧ (countCands 4 2 = involCount 16 2) ∧
    (countCands 4 3 = involCount 16 3) ∧ (countCands 4 4 = involCount 16 4) ∧
    (countCands 4 8 = involCount 16 8) ∧
    (countCands 5 2 = involCount 25 2) ∧ (countCands 5 3 = involCount 25 3) ∧
    (countCands 6 2 = involCount 36 2) ∧ (countCands 7 2 = involCount 49 2)
theorem chunk_sound :
    ((List.range 16).map (countStableAt 4 3)).foldl (· + ·) 0 = countStable 4 3 ∧
    ((List.range 25).map (countStableAt 5 2)).foldl (· + ·) 0 = countStable 5 2
theorem control_off_by_one :
```

```
    countStable 4 2 ≠ 61 ∧ countStable 4 2 ≠ 59 ∧
    countStable 4 3 ≠ 145 ∧ countStable 4 3 ≠ 143 ∧
    countStable 5 2 ≠ 991 ∧ countStable 5 2 ≠ 989
theorem control_nonvacuous :
    0 < countStable 4 2 ∧ countStable 4 2 < countCands 4 2 ∧
    0 < countStable 5 3 ∧ countStable 5 3 < countCands 5 3
theorem control_poly2_tight :
    countStable 8 2 ≠ 3 * binom 8 2 + 3 * binom 8 3 + 30 * binom 8 4 + 780 * binom 8 5
        + 3960 * binom 8 6 + 7560 * binom 8 7 ∧
    countStable 8 2 ≠ 3 * binom 8 2 + 3 * binom 8 3 + 30 * binom 8 4 + 780 * binom 8 5
        + 3960 * binom 8 6 + 7560 * binom 8 7 + 5039 * binom 8 8
theorem orbits_control :
    0 < countOrbits 4 2 ∧ countOrbits 4 2 < countStable 4 2 ∧
    countOrbits 4 1 < countStable 4 1 ∧
    countOrbits 3 3 = 6 ∧ countOrbits 3 4 = 3
```

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