THE STABILITY INDEX OF THE CARLOTTO–SCHULZ MINIMAL HYPERTORUS X^n_{CS} IN $\mathbb{S}^{2n}$: WHAT PERDOMO'S CONJECTURE $(n^3+9n^2+11n+3)/3$, AND 27 AT $n=2$, ACTUALLY RESTS ON — WITH AN ERRATUM FOR THE PRINTED SHAPE OPERATOR

KOREA SUPERINTELLIGENCE LABS

Abstract. For every integer $n \geq 2$ Carlotto and Schulz constructed an embedded minimal hypersurface $X^n_{CS} : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1 \to \mathbb{S}^{2n}$, generated by a closed profile curve whose initial radius r_0 and period T are known only numerically. Perdomo (arXiv:2508.09104, arXiv:2606.16980) proved that its stability index is at least n^2+4n+3 and conjectured, on numerical evidence for $2 \leq n \leq 100$, that it equals $(n^3+9n^2+11n+3)/3$ for $n > 2$ and 27 for $n = 2$. We take that argument apart and record, with machine-checked proofs, exactly what it rests on. (1) An erratum: the principal curvature κ_t in the profile direction printed in both papers is $1/(n-1)$ times the true one; the printed triple of principal curvatures has trace $(2n-4)(\cot r \sin\alpha - \csc r \cos\alpha \cot 2\theta) \neq 0$, so it cannot belong to a minimal hypersurface once $n > 2$; the correct value is $\kappa_t = \alpha' + \cot r \sin\alpha$, the two agree at $n = 2$, and the papers' own numerics must have used the correct one. (2) The eigenfunction $f = (\csc^2 r \sec\theta \csc\theta)^{n-1}$ of Perdomo's "unexpected multiplicity" lemma is exactly e^{-b} with $b = \frac{n-1}{2}\log(EG)$, the reciprocal of the Riemannian volume density, and the lemma collapses to the single identity $b'' = |A|^2 - (n-1)(1/E + 1/G)$, true for every n with the corrected κ_t and false from $n = 3$ with the printed one. (3) Writing the stability spectrum as the union of the spectra of Sturm–Liouville blocks S_{ij} with multiplicities $m_i m_j$, and taking the number $\operatorname{neg}(i,j)$ of negative eigenvalues of S_{ij} as an abstract function antitone in each index, the sign facts the papers verify numerically pin the index to the conjectured polynomial, with both hypothesis sets satisfiable and the values ± 1 refuted; but the itemised list for $n > 2$ is one fact short: a second antitone table satisfies every item of it with index larger by $n(n^2+n-2)$, and the fact that closes the gap, $\lambda^1_{32} > 0$, is supplied by the source outside the list. The analytic core — the sign facts themselves — is not proved here, and we say why. All results are verified in Lean 4 against *Mathlib*; the source's printed numbers were recomputed independently, including its whole $2 \leq n \leq 100$ index sweep, and we report the two places where its tables do not match that recomputation.

1. Introduction

Let $M^N \subset \mathbb{S}^{N+1}$ be a closed minimal hypersurface with Gauss map ν and shape operator A. Its stability (Jacobi) operator is $J = -\Delta - |A|^2 - N$, and its *stability index* $\operatorname{ind}(M)$ is the number of negative eigenvalues of J, counted with multiplicity [7]. The coordinate functions ν_i of the Gauss map satisfy $J\nu_i = -N\nu_i$, so $-N$ is an eigenvalue of J of multiplicity at least $N+2$ when M is not totally geodesic, and since the first eigenvalue is simple, $\operatorname{ind}(M) \geq N+3$ for every non-totally-geodesic M; the Clifford hypersurfaces have index exactly $N+3$, and for $N = 2$ Urbano proved that they are the only ones [8]. Perdomo studied the low-index question in higher dimensions [9]. Beyond spheres, Clifford hypersurfaces and a few isoparametric examples, closed minimal hypersurfaces of spheres with a known index are rare.

For every integer $n \geq 2$, Carlotto and Schulz [5] constructed an embedded minimal hypersurface

$$X^n_{CS} : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1 \to \mathbb{S}^{2n}, \qquad X(y,z,t) = \big(\sin r(t)\cos\theta(t)\, y,\ \sin r(t)\sin\theta(t)\, z,\ \cos r(t)\big),$$

of cohomogeneity one, generated by a closed profile curve $\gamma(t) = (\sin r\cos\theta, \sin r\sin\theta, \cos r)$ in $\mathbb{S}^2$; for $n = 2$ it is a minimal hypertorus in $\mathbb{S}^4$. Perdomo, in two papers we call part I [2] and part II [1], separated variables on this hypersurface: the stability spectrum is the union, over pairs (i,j) of positive integers, of the spectra of one-dimensional operators S_{ij} on the profile circle, each eigenvalue of

S_{ij} counting $m_i m_j$ times, where α_i, m_i are the eigenvalues and multiplicities of the Laplacian of $\mathbb{S}^{n-1}$ (part I, Theorem 3.3). Part I proves $\operatorname{ind}(X^n_{CS}) \geq n^2+4n+3$ (its Theorem 4.3); for $n=2$ this improves the bound ind ≥ 8 that Carlotto, Schulz and Wiygul derived from [9] ([6], where numerical simulations are also reported to give $O(2)\times O(2)$-equivariant index 3). Part II then states

> **Conjecture 1.1 of** [1]**.** If $n>2$, then the stability index of X^n_{CS} is $\frac{1}{3}(n^3+9n^2+11n+3)$ and for the hypertorus in $\mathbb{S}^4$ (case $n=2$) the stability index is 27.

Conjecture 4.1 of [1] repeats this word for word, without the "then". Part II adds: "We numerically show the conjecture for the first 100 values of n." The evidence is a floating-point scan of Floquet discriminants; the later paper [3] restates the conjecture and does not prove it. As of 2026-09-07 the best proved statement in the literature is $\operatorname{ind}(X^n_{CS}) \geq n^2+4n+3$, and the exact value is open at every $n \geq 2$. (Statement numbers of [2, 1] throughout are those of the compiled e-prints, which we compiled from the author's own TEX sources; part II exists only in version 1 and part I is cited in its live version 2.)

This paper does not prove the conjecture. It takes the argument of [1] apart into its algebraic and its analytic components, proves the algebraic ones with machine-checked proofs, corrects one of them, and identifies precisely which numerically verified facts the conjectured value follows from. Concretely:

- **An erratum** (Section 3, Theorem 3.2). The principal curvature κ_t in the profile direction, printed in Section 2 of [1] and, unchanged, in Section 2 of [2], is $1/(n-1)$ times the true one. The printed triple has trace $(2n-4)(\cot r \sin\alpha - \csc r\cos\alpha\cot 2\theta)$, which is not identically zero for $n>2$, so it cannot be the shape operator of a minimal hypersurface; the correct value is $\kappa_t = \alpha' + \cot r \sin\alpha$, which turns minimality into exactly the papers' own minimality equation. At $n=2$ the two agree, so every $n=2$ statement of both papers stands. The error is in the write-up, not in the computation: the index sweep repeated with the printed $|A|^2$ gives 27 at $n=2$ but $19, 11, 13$ at $n=3,4,5$ instead of the conjectured $48, 85, 136$, so the papers' own numerics must have used the correct formula (Remark 3.3).
- **Lemma 3.1 of** [1] **in one identity** (Section 4, Theorem 4.1). The eigenfunction $f = (\csc^2 r \sec\theta\csc\theta)^{n-1}$ of that lemma is exactly e^{-b} with $b = \frac{n-1}{2}\log(EG)$, the reciprocal of the Riemannian volume density of the immersion, and the three-page derivative computation of the source reduces to the single scalar identity $b'' = |A|^2 - (n-1)(1/E + 1/G)$, a five-variable polynomial identity modulo $\sin^2\alpha + \cos^2\alpha = 1$. It holds for every n with the corrected κ_t and fails from $n=3$ with the printed one; this is what exposed the erratum.
- **What the conjecture follows from** (Section 5, Theorems 5.1, 5.2, 5.3). With $\operatorname{neg}(i,j)$ an abstract function antitone in each index (the Rayleigh characterisation of part I), the sign facts of [1] pin $\operatorname{ind}(X^n_{CS}) = \sum_{i,j} m_i m_j \operatorname{neg}(i,j)$ to the conjectured polynomial for $n>2$ and to 27 for $n=2$; both hypothesis sets are satisfiable and the values one more and one less are refuted. But the *itemised* sign list of [1] for $n>2$ does not determine the index: a second antitone table satisfies all of it with index larger by $n(n^2+n-2)$. The fact that closes the gap is $\lambda^1_{32} > 0$, which [1] does supply — through its Figure 8 at $n=2$ and a sentence of its Section 5.2 for $n>2$ — outside the list.

Everything above is algebra, and it is verified in Lean 4 (Section 8, Appendix A). What is *not* here is the analytic content: the sign facts ($\lambda^4_{11} > 0$, $\lambda^2_{22} = 0$, $\lambda^1_{31} < 0$, …) are statements about the periodic eigenvalues of Hill equations whose coefficients are themselves the solution of a nonlinear shooting problem, and certifying even one of them needs validated enclosures of two nested ODE problems. Section 7 says what that would take. Section 6 records the independent reproduction of the numbers [1] prints, including the whole $2 \leq n \leq 100$ sweep, and two observations about its two tables: three r_0 entries of Table 1 disagree with the author's own published data set, and five rows of Table 2 carry the value belonging to a different n.

What is new and what is not. The construction is [5]; the separation of variables, the Rayleigh monotonicity, the Floquet reformulation and the lower bound n^2+4n+3 are [2]; the conjecture, the extra eigenfunctions $f\,y_i z_j$, the sign facts and the arithmetic $n^2+4n+3+2m_3+2m_4 = (n^3+9n^2+11n+3)/3$ are [1]. What we claim is: the erratum and its evidence; the identification $f = e^{-b}$ and the one-line form of Lemma 3.1, with the analytic layer showing that the identity really is a statement about derivatives along a solution; the exact sufficient hypothesis set for the conjectured value and the fact that the itemised list is one fact short; and the formal verification of all of it. A full-text sweep of the local arXiv corpus (376 shards, 2026-09-07) for the surface's name and for the index polynomial found the polynomial only in [1, 3] and in three unrelated coincidences, and found no paper other than the author's own using this parametrisation; Semantic Scholar lists exactly one paper citing [1], namely [3], which restates the conjecture, writes the minimality relation for the same family correctly as $k\kappa_y + \ell\kappa_z + \kappa_t = 0$, and flags a typo in a third paper of the author rather than this one. The erratum has not been communicated to the author at the time of writing.

2. Preliminaries

2.1. **The profile system.** Fix an integer $n \geq 2$ and put $N = 2n-1 = \dim X^n_{CS}$. The profile curve γ is traversed at unit speed, so there is a function $\alpha(t)$ with

$$r' = \cos\alpha, \qquad \theta' = \frac{\sin\alpha}{\sin r} \tag{1}$$

(equation (4) of [1]), and the Gauss map of X is $\nu(t,y,z) = (\nu_1 y, \nu_2 z, \nu_3)$ with

$$\nu_1 = \cos r\sin\alpha\cos\theta + \cos\alpha\sin\theta, \quad \nu_2 = \cos r\sin\alpha\sin\theta - \cos\alpha\cos\theta, \quad \nu_3 = -\sin r\sin\alpha \tag{2}$$

(equation (5)). By [5], as restated in [1], the immersion is minimal if and only if

$$\alpha' = (2n-2)\csc r\cos\alpha\cot(2\theta) - (2n-1)\cot r\sin\alpha \tag{3}$$

(equation (6)). Theorem 2.1 of [1] (the existence theorem of [5]) says that there are $r_0 \in (0,\pi)$ and $s^* = T/4$ such that the solution of (1)–(3) with $\theta(0) = \pi/4$, $r(0) = r_0$, $\alpha(0) = -\pi/2$ satisfies $r(s^*) = \pi/2$, $\alpha(s^*) = 0$; by the symmetries of Remark 2.2 there, r and θ are then T-periodic and $\alpha(t+T) = \alpha(t)+2\pi$, so γ is a closed embedded curve. The numbers r_0 and T are known only numerically; they are tabulated to ten decimals in Table 1 of [1]. The metric coefficients along the sphere factors are

$$E = \sin^2 r\cos^2\theta, \qquad G = \sin^2 r\sin^2\theta,$$

and by Lemma 2.4 of [1] (Lemma 3.1 of [2]) the Laplacian of a function $\zeta(t,y,z)$ on X^n_{CS} is

$$\Delta\zeta = \zeta_{tt} + \tfrac{n-1}{2}\big(\ln(EG)\big)'\zeta_t + \tfrac{1}{E}\Delta^y_{\mathbb{S}^{n-1}}\zeta + \tfrac{1}{G}\Delta^z_{\mathbb{S}^{n-1}}\zeta. \tag{4}$$

We write

$$b := \tfrac{n-1}{2}\log(EG) = (n-1)\big(2\log\sin r + \log\sin\theta + \log\cos\theta\big) \tag{5}$$

on the region $\sin r, \sin\theta, \cos\theta > 0$ where the profile lives, so that the coefficient in (4) is b', and $e^b = (EG)^{(n-1)/2}$ is the Riemannian volume density of the immersion in the coordinates (t,y,z).

2.2. **The shape operator.** Immediately after (3), in an unnumbered display, [1] prints the principal curvatures

$$\begin{aligned} \kappa_u &= \csc r\cos\alpha\tan\theta + \cot r\sin\alpha, \\ \kappa_v &= \cot r\sin\alpha - \csc r\cos\alpha\cot\theta, \\ \kappa_t^{\mathrm{pr}} &= 2\csc r\cos\alpha\cot(2\theta) - 2\cot r\sin\alpha, \end{aligned} \tag{6}$$

and then (its equation (7)) $|A|^2 = (n-1)\kappa_u^2 + (n-1)\kappa_v^2 + \kappa_t^2$, i.e. κ_u and κ_v have multiplicity $n-1$ each (the principal directions tangent to the two sphere factors) and κ_t multiplicity one (the profile direction). We write κ_t^{pr} for the printed value because Section 3 shows it is not the principal curvature; the same three lines, with the same equation for $|A|^2$, are printed in Section 2 of [2].

2.3. **Algebraic notation.** All identities of Sections 3 and 4 are rational identities in the six trigonometric values of the profile functions and the parameter n, and we write them as such:

$$s = \sin r, \quad c = \cos r, \quad p = \sin\theta, \quad q = \cos\theta, \quad S = \sin\alpha, \quad C = \cos\alpha, \qquad K := \cot(2\theta) = \frac{q^2 - p^2}{2pq}.$$

In this notation (6) reads $\kappa_u = cS/s + Cp/(sq)$, $\kappa_v = cS/s - Cq/(sp)$, $\kappa_t^{\mathrm{pr}} = 2CK/s - 2cS/s$, the right-hand side of (3) is

$$\alpha'_{\mathrm{rhs}}(n; s, c, p, q, S, C) := \frac{(2n-2)\,C\,K - (2n-1)\,c\,S}{s},$$

and the corrected principal curvature of Section 3 is $\kappa_t := \alpha'_{\mathrm{rhs}} + cS/s$. We put $|A|^2 := (n-1)\kappa_u^2 + (n-1)\kappa_v^2 + \kappa_t^2$ and $|A|^2_{\mathrm{pr}} := (n-1)\kappa_u^2 + (n-1)\kappa_v^2 + (\kappa_t^{\mathrm{pr}})^2$. In the formal development (Appendix A) n and the six values are real variables, and only the hypotheses actually used are assumed: $s, p, q \neq 0$ throughout, and $p^2 + q^2 = 1$, $S^2 + C^2 = 1$ where stated; in particular $s^2 + c^2 = 1$ is never needed.

2.4. **The stability spectrum.** By Theorem 3.3 of [2], restated in Section 4 of [1], the spectrum of J is the union over pairs (i, j) of positive integers of the spectra of the operators

$$S_{ij}[\eta] = -\eta'' - b'\eta' + \Big(\frac{\alpha_i}{E} + \frac{\alpha_j}{G} - |A|^2 - (2n-1)\Big)\eta \tag{7}$$

on T-periodic functions of t, each eigenvalue of S_{ij} counted $m_i m_j$ times, where $\alpha_1 < \alpha_2 < \cdots$ are the eigenvalues of $-\Delta$ on $\mathbb{S}^{n-1}$ and m_i their multiplicities (Theorem 2.3 of [1]):

$$\alpha_i = (i-1)(n+i-3), \qquad m_1 = 1, \quad m_2 = n, \quad m_i = \binom{n+i-2}{i-1} - \binom{n+i-4}{i-3}\ (i \geq 3). \tag{8}$$

Write $\lambda_{ij}^1 < \lambda_{ij}^2 \leq \lambda_{ij}^3 \leq \cdots$ for the eigenvalues of S_{ij} and $\mathrm{neg}(i,j)$ for the number of negative ones. Then

$$\mathrm{ind}(X^n_{CS}) = \sum_{i,j \geq 1} m_i m_j\ \mathrm{neg}(i,j). \tag{9}$$

Since $S_{i'j} - S_{ij} = (\alpha_{i'} - \alpha_i)/E \geq 0$ pointwise for $i' \geq i$, and likewise in the second index through G, the Rayleigh characterisation (Theorem 3.9 of [2]; the form $\lambda_{ji}^\ell > \lambda_{ki}^\ell$ for $j > k$ is stated in Section 5 of [1]) gives

$$i \leq i',\ j \leq j' \implies \mathrm{neg}(i', j') \leq \mathrm{neg}(i,j), \tag{10}$$

which is the only property of neg used in Section 5. Each S_{ij} is a Hill operator on the circle: Lemma 3.4 of [2] substitutes $\eta = e^{-b/2}v$ and Section 5 of [1] characterises its periodic eigenvalues as the zeros of a Floquet discriminant $\delta_{ij}(\lambda) = z_1(T) + z_2'(T) - 2$ built from two solutions of the eigenvalue equation; see [11] for the general theory.

The Laplace data (8) are classical, but the formal development needs the specialisations that [1] prints in its Section 4, so we record them.

Proposition 2.1 (the printed Laplace data)**.** *For every integer $n \geq 2$, $\alpha_1, \ldots, \alpha_5 = 0,\ n-1,\ 2n,\ 3(n+1),\ 4(n+2)$, and for every $n \geq 1$*

$$m_3 = \frac{n^2+n-2}{2}, \qquad m_4 = \frac{n(n+4)(n-1)}{6}, \qquad m_5 = \frac{n(n-1)(n^2+7n+6)}{24}.$$

Proof. The α_i are immediate from (8). For the m_i the formal statements are the subtraction-free forms $2m_3 + 2 = n^2 + n$, $6m_4 + 6n = n^3 + 3n^2 + 2n$ and $24m_5 + 12(n^2+n) = n^4 + 6n^3 + 11n^2 + 6n$, proved from the three binomial identities $2\binom{k+2}{2} = (k+2)(k+1)$, $6\binom{k+3}{3} = (k+3)(k+2)(k+1)$, $24\binom{k+4}{4} = (k+4)\cdots(k+1)$, each by induction from Pascal's rule (formal: `sphEig_values`, `sphMult_three`, `sphMult_four`, `sphMult_five`). This is Theorem 2.3 of [1] specialised to $k = n-1$; the mathematics is classical and only the certification is ours. ∎

3. The erratum

3.1. **The principal curvature in the profile direction.** The hypersurface X^n_{CS} is invariant under $O(n)\times O(n)$ acting on the two sphere factors, and the profile curve γ lives in the orbit space, the quarter of $\mathbb{S}^2$ with $\gamma_1,\gamma_2>0$, with its round metric $dr^2+\sin^2 r\,d\theta^2$ in the coordinates (r,θ). For a cohomogeneity-one hypersurface the principal curvature in the direction of the profile curve is the geodesic curvature of the profile curve in the orbit space, and for a unit-speed curve in geodesic polar coordinates making the angle α with ∂_r — which is what (1) says — Liouville's formula gives the geodesic curvature $\alpha'+\cot r\sin\alpha$: the r-lines are geodesics and the θ-circles have geodesic curvature $\cot r$. Hence

Lemma 3.1 (paper step, not formalised)**.** *With the sign convention in which the curvatures along the sphere factors are the κ_u,κ_v of* (6)*, the principal curvature of X^n_{CS} in the profile direction is*

$$\kappa_t=\alpha'+\cot r\sin\alpha. \tag{11}$$

Proof. Liouville's formula as above. We also verified (11) directly from the source's own data: with $T=X_t=(g_1y,g_2z,g_3)$, $g_1=\cos r\cos\alpha\cos\theta-\sin\alpha\sin\theta$, $g_2=\cos r\cos\alpha\sin\theta+\sin\alpha\cos\theta$, $g_3=-\sin r\cos\alpha$ (from (1)), the second fundamental form in the profile direction is $\langle X_{tt},\nu\rangle=g_1'\nu_1+g_2'\nu_2+g_3'\nu_3$ with ν from (2), and for a unit vector e tangent to the first sphere factor it is $\langle X_{ee},\nu\rangle/|X_e|^2=-\nu_1/(\sin r\cos\theta)$. Treating α' as a *free* variable, a floating-point evaluation at 2000 random frames gives $\langle X_{ee},\nu\rangle/|X_e|^2=-\kappa_u$, its z-analogue $=-\kappa_v$, and $\langle X_{tt},\nu\rangle=-(\alpha'+\cot r\sin\alpha)$, all below 10^{-14}; so relative to the normal $-\nu$ the three principal curvatures are κ_u,κ_v and (11). A third check uses neither (2) nor (6): writing X out in $\mathbb{R}^{2n+1}$ with the two sphere factors in polar coordinates, taking ν to be the unit vector orthogonal to X and to the $2n-1$ coordinate directions obtained by Gram–Schmidt, and computing the first and second fundamental forms by central differences, the eigenvalues of the shape operator at $n=2,3,4$ and α' free are κ_u with multiplicity $n-1$, κ_v with multiplicity $n-1$ and $\alpha'+\cot r\sin\alpha$ once, to 10^{-8} (the accuracy of the differences), up to the overall sign of ν. Nothing in the formal development depends on this lemma: there $\kappa_t:=\alpha'_{\mathrm{rhs}}+cS/s$ is a definition, i.e. (11) with α' replaced by the right-hand side of (3). □

Substituting (11) into $(n-1)(\kappa_u+\kappa_v)+\kappa_t=0$ and using $\tan\theta-\cot\theta=-2\cot2\theta$ gives

$$\alpha'=-(2n-1)\cot r\sin\alpha+(2n-2)\csc r\cos\alpha\cot2\theta,$$

which is exactly (3): the corrected κ_t reproduces the papers' minimality equation, and along a minimal profile

$$\kappa_t=(n-1)\big(2\csc r\cos\alpha\cot2\theta-2\cot r\sin\alpha\big)=(n-1)\,\kappa_t^{\mathrm{pr}}. \tag{12}$$

3.2. **The statement.**

Theorem 3.2 (erratum for the printed κ_t)**.** *Let n, s, c, p, q, S, C be real numbers, and let κ_u, κ_v, κ_t^{pr}, α'_{rhs} and κ_t be as in Section 2.*

(i) *$\kappa_t=(n-1)\,\kappa_t^{\mathrm{pr}}$, identically.*

(ii) *If $s,p,q\neq0$ then $(n-1)\kappa_u+(n-1)\kappa_v+\kappa_t=0$: with the corrected κ_t the trace of the shape operator vanishes identically along the profile system, for every n.*

(iii) *If $s,p,q\neq0$ then $(n-1)\kappa_u+(n-1)\kappa_v+\kappa_t^{\mathrm{pr}}=(2n-4)\big(cS/s-CK/s\big)$, i.e. the printed triple has trace $(2n-4)(\cot r\sin\alpha-\csc r\cos\alpha\cot2\theta)$.*

(iv) *At $n=3$ and $(s,c,p,q,S,C)=(1,0,\frac35,\frac45,0,1)$ — the frame $r=\pi/2$, $\sin\theta=3/5$, $\alpha=0$, which satisfies the three Pythagorean relations and the three nonvanishing conditions — the trace in (iii) equals $-7/12\neq0$. So the printed triple cannot be the shape operator of a minimal hypersurface once $n>2$.*

(v) *At $n=2$, $|A|^2_{\mathrm{pr}}=|A|^2$ identically: the printed and the corrected κ_t agree, and every $n=2$ computation of* [2, 1] *is unaffected.*

Proof. (i) is the definition of κ_t rearranged (formal: `kapT_eq_printed_smul`); (ii) and (iii) are the rational identities obtained by clearing denominators (formal: `mean_curvature_zero`, `mean_curvature_printed`);

(iv) is an evaluation (`mean_curvature_printed_ne_zero`, with the admissibility of the frame as `control_frame_admissible`); (v) is (i) at $n = 2$ (`normA2Printed_dim_two`). We reproduced (i)–(v) in exact rational arithmetic at 200 random rational frames and at the frame of (iv). □

Remark 3.3 (the error is in the write-up, not in the computation). Three pieces of evidence, none of them formal. (a) The conjectured values are what the *corrected* formula produces: re-running the $2 \le n \le 100$ index computation of Section 6 with $|A|^2_{\mathrm{pr}}$ in place of $|A|^2$ gives 27 at $n = 2$ (as it must, by (v)) but $19, 11, 13, 15, 16, 18, 20, 22$ at $n = 3, \ldots, 10$ in place of $48, 85, 136, 203, 288, 393, 520, 671$; since the numerics of [1] do produce the conjectured values, the code behind them used the correct $|A|^2$. (b) Part I states in its introduction that minimality means $\kappa_1 + \cdots + \kappa_N = 0$, which the printed triple violates. (c) The later paper [3] of the same author writes, for the more general family $\mathbb{S}^k \times \mathbb{S}^\ell \times \mathbb{S}^1 \to \mathbb{S}^{k+\ell+2}$, the relation $k\kappa_y + \ell\kappa_z + \kappa_t = 0$ and a κ_t of a different form. It does flag a typo of the author's — "there is a typo in the last formula in [P1], where a_1 appears as $\frac{h^2}{f^2}$ instead of $\frac{f^2}{h^2}$" — but the [P1] there is neither [2] nor [1]: by its own bibliography it is a third paper of the author, *Spectrum of the Laplacian and the Jacobi operator on generalized rotational minimal hypersurfaces of spheres* (arXiv:2403.04223; J. Geom. Anal. **35** (2025), Paper No. 390). Nothing in [2, 1] is corrected in [3]. We found no correction of the display (6) anywhere: the only papers in the corpus using this parametrisation are the author's own ([2, 1, 3] and arXiv:2403.04223), and [2] still prints it in its live version 2. A separate observation, floating-point only and deliberately not made a claim. In the proof of Lemma 3.1 of [1] the printed df/dt is correct, but the printed d^2f/dt^2 — a display of the shape $(n-1)\tan^3\theta\,\csc^{2n} r\,\csc^n\theta\sec^n\theta\big(\frac{1}{8}\cos^2\alpha\cot^2\theta\csc^2\theta\,w_1 + \cot^3\theta\,w_2 + \cot^3\theta\,w_3\big)$ with three auxiliary polynomials w_1, w_2, w_3 — is not the second derivative of f along the profile. We read that display character by character off the compiled e-print before saying so, so it is not a transcription slip of ours. The mismatch is confined to part of it: the outer factor and the w_1 term are right, and the printed expression agrees with f'' identically on the locus $\sin\alpha = 0$ (to 10^{-13} at every frame we tried); with $\sin\alpha \ne 0$ the two differ, with no constant ratio (we saw 0.80 to 1.41 at four random frames), so what is off are the two $\cot^3\theta$ terms. None of this touches the lemma: what its proof needs is the equation (14) printed immediately after, and that equation is true — it is Theorem 4.1. We record the display for the author and assert nothing about it.

4. Lemma 3.1 of the source in one identity

Lemma 3.1 of [1] asserts that for $i, j \in \{1, \ldots, n\}$ the functions $\zeta_{ij} = f(t)\,y_i z_j$, with y_i, z_j the coordinate functions of the two sphere factors and

$$f = \big(\csc^2 r\,\sec\theta\,\csc\theta\big)^{n-1}, \tag{13}$$

satisfy $J\zeta_{ij} = -(2n-1)\zeta_{ij}$; its Corollary 3.2 concludes that $-(2n-1)$ has multiplicity at least n^2+2n+1 (the n^2 functions ζ_{ij} together with the $2n+1 = N+2$ components of the Gauss map), which is the "unexpected multiplicity" of its title. By (4) and $\Delta_{\mathbb{S}^{n-1}} y_i = -(n-1)y_i$, the lemma is equivalent to the ordinary differential equation

$$f'' + b'f' + |A|^2 f - (n-1)\Big(\frac{1}{E} + \frac{1}{G}\Big)f = 0 \tag{14}$$

along the profile, and [1] proves it "by a direct computation", printing df/dt and a display-long d^2f/dt^2. The observation that collapses it: by (5),

$$f = \big(\sin^2 r\,\sin\theta\cos\theta\big)^{-(n-1)} = e^{-b},$$

the reciprocal of the volume density. Substituting $f = e^{-b}$ into (14), $f' = -b'f$ and $f'' = (b'^2 - b'')f$, the b'^2 terms cancel, and (14) becomes

$$b'' = |A|^2 - (n-1)\Big(\frac{1}{E} + \frac{1}{G}\Big). \tag{15}$$

Along a solution of (1), (3), $b' = 2(n-1)(cC + KS)/s$ and

$$b'' = \frac{2(n-1)}{s^2}\Big(-C^2 - 2(1+K^2)S^2 - cCKS - \alpha'_{\mathrm{rhs}}\, s\,(cS - CK)\Big) =: \beta(n; s, c, p, q, S, C), \tag{16}$$

and (15) is a rational identity. Its proof needs $|A|^2$ in closed form,

$$|A|^2 = 2(n-1)(2n-1)\Big(\frac{cS - CK}{s}\Big)^2 + (n-1)\,\frac{C^2(p^2+q^2)^2}{2p^2q^2s^2}, \qquad 1 + K^2 = \frac{1}{4p^2q^2}\ \ (p^2+q^2=1), \tag{17}$$

after which everything reduces to a polynomial identity in (n, c, S, C, K) modulo $S^2 + C^2 = 1$ alone.

Theorem 4.1 (Lemma 3.1 of [1], corrected and verified)**.** (i) *(the identity) For all real n, s, c, p, q, S, C with $s, p, q \neq 0$, $p^2 + q^2 = 1$ and $S^2 + C^2 = 1$,*

$$\beta(n; s, c, p, q, S, C) = |A|^2 - (n-1)\Big(\frac{1}{s^2q^2} + \frac{1}{s^2p^2}\Big),$$

with $|A|^2$ built from the corrected κ_t.

(ii) *(the analytic layer) Let $n \in \mathbb{R}$ and let $r, \theta, \alpha : \mathbb{R} \to \mathbb{R}$ be differentiable with $r' = \cos\alpha$, $\theta' = \sin\alpha/\sin r$, $\alpha' = \alpha'_{\mathrm{rhs}}(n; \sin r, \cos r, \sin\theta, \cos\theta, \sin\alpha, \cos\alpha)$ at every t, and with $\sin r, \sin\theta, \cos\theta$ nowhere zero. Put $b = (n-1)(2\log\sin r + \log\sin\theta + \log\cos\theta)$ and $f = e^{-b}$. Then at every t: b has derivative $b' = 2(n-1)(cC + KS)/s$; b' has derivative β evaluated at the frame of t; and*

$$(b'^2 - b'')f + b'\,(-b'f) + |A|^2 f - (n-1)\Big(\frac{1}{E} + \frac{1}{G}\Big)f = 0,$$

which, since $f' = -b'f$ and $f'' = (b'^2 - b'')f$, is (14)*.*

(iii) *(identification with the printed f) If $n = m+1$ with $m \in \mathbb{N}$ and $\sin r, \sin\theta, \cos\theta > 0$ at t, then $e^{-b} = \big(1/(\sin^2 r\, \sin\theta\cos\theta)\big)^m$, i.e. f is* (13)*.*

(iv) *(non-vacuity) At $n = 3$ and the frame of Theorem 3.2(iv), the hypotheses of (i) hold and both sides equal $-95/36$.*

(v) *(negative control) At the same n and frame, with $|A|^2_{\mathrm{pr}}$ in place of $|A|^2$ the identity of (i) is* false*: the left side is $-95/36 = -380/144$ and the right side is $-527/144$.*

Proof. (i) The core is the polynomial identity

$$-C^2 - 2(1+K^2)S^2 - cCKS - \big((2n-2)CK - (2n-1)cS\big)(cS - CK) = (2n-1)(cS - CK)^2 + (1+K^2)(C^2 - 2),$$

which holds modulo $S^2 + C^2 = 1$ (it is $-2(1+K^2)$ times that relation; formal: `bpp_core`, one `linear_combination`); combined with (17) (formal: `normA2_closed`, `one_add_cot2_sq`) and $p^2 + q^2 = 1$ it gives the statement (`bppExpr_eq`). (ii) is a chain-rule computation from the hypotheses, treated as `HasDerivAt` facts: the derivative of b (`hasDerivAt_bFun`), then of b', using $(\cot 2\theta)' = -2(1 + \cot^2 2\theta)\theta'$ and the derivative of α supplied by (3) (`hasDerivAt_bpFun`), then of $f = e^{-b}$ and of $f' = -b'f$, and finally (i) at the frame of t (`lemma31`). The last step is where the analytic layer earns its place: it shows that β really is b'' along a solution, so what is verified is the lemma and not a relabelled expression. (iii) is log of a product of positives (`fFun_eq_pow`); the formal b uses the real logarithm of *Mathlib*, which is $\log|x|$ for $x < 0$, so (iii) is stated with the positivity hypotheses under which b is (5). (iv) and (v) are evaluations (`bppExpr_sample`, `bppExpr_eq_sample`, `lemma31_fails_printed_dim_three`). We reproduced (i) in exact rational arithmetic at 300 random rational frames with (p, q) and (S, C) on the unit circle, and (iv), (v) exactly; along the numerically computed closed profiles the residual of (15) is below 10^{-11} for every $n \leq 200$ tried, and with $|A|^2_{\mathrm{pr}}$ it is of order 10 to 10^3 from $n = 3$ on. □

Remark 4.2. (ii) is stated for an arbitrary solution of the profile system on all of $\mathbb{R}$ with the three trigonometric values nowhere zero; the Carlotto–Schulz profile is such a solution (Theorem 2.1 and Remark 2.2 of [1] give $\sin r > 0$ and $0 < \theta < \pi/2$ along it). The conclusion $J\zeta_{ij} = -(2n-1)\zeta_{ij}$ then follows from (14) by (4), exactly as in [1]; that last step is not formalised, since the formal development contains no manifold, no Laplacian and no spectrum — only the algebra and the one-variable calculus that the source's proof consists of.

5. What the index conjecture follows from

Section 4 of [1] lists what is known about the $\mathrm{neg}(i,j)$ and what it checks numerically. Known, from [2] and Lemma 3.1: $\lambda^3_{11} = -(2n-1)$, so $\mathrm{neg}(1,1) \geq 3$; $\lambda^2_{21} = -(2n-1)$ and $\lambda^3_{21} = 0$, so $\mathrm{neg}(2,1) = \mathrm{neg}(1,2) = 2$ exactly; $\lambda^1_{22} = -(2n-1)$, so $\mathrm{neg}(2,2) \geq 1$. These three items give the lower bound n^2+4n+3 of [2]. Numerically checked, in the bullet for $n = 3,\dots,100$: $\lambda^4_{11} > 0$, $\lambda^2_{22} = 0$, $\lambda^1_{31}, \lambda^1_{13}, \lambda^1_{41}, \lambda^1_{14} < 0$, and $\lambda^2_{31}, \lambda^2_{13}, \lambda^2_{41}, \lambda^2_{14}, \lambda^1_{15}, \lambda^1_{51} > 0$; and the bullet concludes that the index is $n^2+4n+3+2m_3+2m_4 = \frac{1}{3}(n^3+9n^2+11n+3)$. In the bullet for $n=2$ the list is longer: $\lambda^1_{51}, \lambda^1_{15} < 0$ as well, and $\lambda^2_{51}, \lambda^2_{15}, \lambda^1_{16}, \lambda^1_{61} > 0$, giving $n^2+4n+3+2m_3+2m_4+2m_5 = 27$ at $n=2$.

We formalise the bookkeeping with neg abstract. For $\mathrm{neg} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and $n, M \in \mathbb{N}$ put

$$\Sigma_M(n,\mathrm{neg}) := \sum_{1\leq i,j\leq M} m_i m_j \ \mathrm{neg}(i,j), \tag{18}$$

with m_i as in (8) (and $m_0 := 0$), so that $\mathrm{ind}(X^n_{CS}) = \lim_M \Sigma_M(n,\mathrm{neg})$ by (9). Three hypothesis sets:

- $\mathcal{F}(\mathrm{neg})$, *the itemised facts for* $n > 2$: (10), and $\mathrm{neg}(1,1) = 3$, $\mathrm{neg}(2,1) = \mathrm{neg}(1,2) = 2$, $\mathrm{neg}(2,2) = 1$, $\mathrm{neg}(3,1) = \mathrm{neg}(1,3) = 1$, $\mathrm{neg}(4,1) = \mathrm{neg}(1,4) = 1$, $\mathrm{neg}(5,1) = \mathrm{neg}(1,5) = 0$. These are exactly the three known items and the $n > 2$ bullet: $\mathrm{neg}(1,1) = 3$ combines $\lambda^3_{11} < 0 < \lambda^4_{11}$, $\mathrm{neg}(2,2) = 1$ combines $\lambda^1_{22} < 0 = \lambda^2_{22}$, and the rest are the listed signs.
- $\mathcal{F}^+(\mathrm{neg})$: $\mathcal{F}(\mathrm{neg})$ together with $\mathrm{neg}(3,2) = \mathrm{neg}(2,3) = 0$, i.e. $\lambda^1_{32}, \lambda^1_{23} > 0$.
- $\mathcal{F}_2(\mathrm{neg})$, *the facts for* $n = 2$: (10), $\mathrm{neg}(1,1) = 3$, $\mathrm{neg}(2,1) = \mathrm{neg}(1,2) = 2$, $\mathrm{neg}(2,2) = 1$, $\mathrm{neg}(3,1) = \mathrm{neg}(1,3) = 1$, $\mathrm{neg}(4,1) = \mathrm{neg}(1,4) = 1$, $\mathrm{neg}(5,1) = \mathrm{neg}(1,5) = 1$, $\mathrm{neg}(6,1) = \mathrm{neg}(1,6) = 0$, $\mathrm{neg}(3,2) = \mathrm{neg}(2,3) = 0$.

Where each item comes from matters, because it is what Theorem 5.3 is about. The three known items together with the $n > 2$ bullet are exactly $\mathcal{F}$; the two further items of $\mathcal{F}^+$ come from the Section 5.2 sentence of [1] quoted in Remark 5.4. For $\mathcal{F}_2$ the $n=2$ bullet and Section 5.1 of [1] give every item except $\mathrm{neg}(2,3) = 0$: what the source states there is $\lambda^1_{32} > 0$ (its Figure 8), and $\mathrm{neg}(2,3) = 0$ follows from that through the symmetry $\delta_{ij} = \delta_{ji}$, which [1] observes numerically and which we discuss in Section 6.3. Three explicit tables, each a sum of indicator functions of down-closed staircases (hence antitone in each index):

$$\mathrm{neg}_{\mathrm{big}} = \mathbf{1}_{[1,4]\times[1,1]\,\cup\,[1,2]\times[1,2]\,\cup\,[1,1]\times[1,4]} + \mathbf{1}_{[1,2]\times[1,1]\,\cup\,[1,1]\times[1,2]} + \mathbf{1}_{\{(1,1)\}},$$

$\mathrm{neg}_{\mathrm{two}}$ the same with 4 replaced by 5 in the first indicator, and $\mathrm{neg}_{\mathrm{gap}}$ the same as $\mathrm{neg}_{\mathrm{big}}$ with the first staircase enlarged by the corners $(2,3)$ and $(3,2)$. In rows $i = 1,\dots,6$ and columns $j = 1,\dots,6$:

$$\mathrm{neg}_{\mathrm{big}} = \begin{pmatrix} 3&2&1&1&0&0\\ 2&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \mathrm{neg}_{\mathrm{two}} = \begin{pmatrix} 3&2&1&1&1&0\\ 2&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \mathrm{neg}_{\mathrm{gap}} = \begin{pmatrix} 3&2&1&1&0&0\\ 2&1&1&0&0&0\\ 1&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.$$

Theorem 5.1 (the conditional index theorem for $n > 2$)**.** (i) *(truncation) If* neg *satisfies* $\mathcal{F}$, *then* $\mathrm{neg}(i,j) = 0$ *whenever* $i \geq 5$ *or* $j \geq 5$, *and* $\Sigma_M(n,\mathrm{neg}) = \Sigma_4(n,\mathrm{neg})$ *for every* n *and every* $M \geq 4$.

(ii) *(the value) If* neg *satisfies* $\mathcal{F}^+$, *then for every* $n \geq 1$

$$3\,\Sigma_4(n,\mathrm{neg}) = n^3+9n^2+11n+3.$$

(iii) *(non-vacuity)* $\mathrm{neg}_{\mathrm{big}}$ *satisfies* $\mathcal{F}^+$, *and* $3\,\Sigma_4(n,\mathrm{neg}_{\mathrm{big}}) = n^3+9n^2+11n+3$ *for every* $n \geq 1$.

(iv) *(controls) If* neg *satisfies* $\mathcal{F}^+$ *and* $n \geq 1$, *then* $3(\Sigma_4(n,\mathrm{neg})+1) \neq n^3+9n^2+11n+3$ *and* $3\,\Sigma_4(n,\mathrm{neg}) \neq n^3+9n^2+11n+6$*: under the same hypotheses the index is neither one less nor one more than the conjectured value.*

Consequently, if the Carlotto–Schulz profile satisfies the sign facts $\mathcal{F}^+$ *for some* $n > 2$ *— which* [1] *asserts numerically for* $3 \leq n \leq 100$ *— then* $\mathrm{ind}(X^n_{CS}) = \frac{1}{3}(n^3+9n^2+11n+3)$.

Proof. (i) $\mathrm{neg}(5,1) = 0$ and antitonicity kill every (i,j) with $i \geq 5$, and $\mathrm{neg}(1,5) = 0$ every (i,j) with $j \geq 5$ (formal: `neg_eq_zero_of_five`, `idxSum_truncate`). (ii) Antitonicity from $\mathrm{neg}(3,2) = \mathrm{neg}(2,3) = 0$ gives $\mathrm{neg}(i,j) = 0$ for the six remaining cells $(3,3),(3,4),(4,2),(4,3),(4,4),(2,4)$; expanding the 4×4 sum with $m_1 = 1$, $m_2 = n$ and Proposition 2.1 for m_3, m_4 gives $3\big(3 + 4n + n^2 + 2m_3 + 2m_4\big) = n^3 + 9n^2 + 11n + 3$ (`index_value`). (iii) The staircase indicators are antitone (`stair_antitone`) and the finite facts are decided by evaluation (`negBig_facts`, `index_value_negBig`). (iv) is arithmetic from (ii) (`index_controls`). The argument is the source's own; the theorem records exactly which hypotheses it consumes. We checked (ii)–(iii) exactly for $1 \leq n \leq 300$. ☐

Theorem 5.2 (the $n = 2$ case)**.** *If* neg *satisfies* $\mathcal{F}_2$*, then* $\Sigma_5(2, \mathrm{neg}) = 27$*. The table* $\mathrm{neg}_{\mathrm{two}}$ *satisfies* $\mathcal{F}_2$*, and* $\Sigma_5(2, \mathrm{neg}_{\mathrm{two}}) = 27$*,* $\neq 26$*,* $\neq 28$*. Consequently, if the hypertorus profile satisfies* $\mathcal{F}_2$ *— which* [1] *asserts numerically in its Section 4 bullet for n = 2 and its Section 5.1, one item of it only through the symmetry recorded above — then its stability index is* 27*.*

Proof. As for Theorem 5.1(ii)–(iv), with the 5×5 sum and $m_1, \ldots, m_5 = 1,2,2,2,2$ at $n = 2$: $3 + 4 \cdot 2 + 4 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 = 27$ (formal: `index_value_two`, `negTwo_facts`, `index_two_controls`). Two remarks on exactness. The formal statement is about the sum truncated at $M = 5$; that the blocks with $i \geq 6$ or $j \geq 6$ contribute nothing follows from $\mathrm{neg}(6,1) = \mathrm{neg}(1,6) = 0$ and antitonicity exactly as in Theorem 5.1(i), but that truncation lemma is not a separate Lean statement for this hypothesis set. And $n = 2$ is genuinely a separate case, not a specialisation: the $n > 2$ polynomial gives 23 at $n = 2$, and the difference $27 - 23 = 4 = 2m_5$ is the contribution of S_{51} and S_{15}, whose first eigenvalue is negative at $n = 2$ (Figure 6 of [1], $-2.5 < \lambda^1_{51} < -1.5$) and positive for $3 \leq n \leq 100$ (Section 5.2 of [1]). ☐

Theorem 5.3 (the itemised list is one fact short)**.** *Let* $n \geq 3$*. Both* $\mathrm{neg}_{\mathrm{big}}$ *and* $\mathrm{neg}_{\mathrm{gap}}$ *satisfy every item of* $\mathcal{F}$*, and* $\Sigma_4(n, \mathrm{neg}_{\mathrm{big}}) \neq \Sigma_4(n, \mathrm{neg}_{\mathrm{gap}})$*; in fact*

$$\Sigma_4(n, \mathrm{neg}_{\mathrm{gap}}) - \Sigma_4(n, \mathrm{neg}_{\mathrm{big}}) = 2n\, m_3 = n(n^2 + n - 2) \qquad (= 30,\ 72,\ 140 \textit{ at } n = 3,4,5),$$

and $\mathrm{neg}_{\mathrm{gap}}(3,2) = 1$*. So the itemised sign list of the* $n > 2$ *bullet of* [1]*, together with the three known items and the Rayleigh antitonicity, does not determine the index: it leaves* $\mathrm{neg}(3,2) = \mathrm{neg}(2,3) \in \{0,1\}$ *free, and the fact that closes the gap is* $\lambda^1_{32} > 0$ *(equivalently* $\lambda^1_{23} > 0$*).*

Proof. Antitonicity only gives $\mathrm{neg}(3,2) \leq \min(\mathrm{neg}(3,1), \mathrm{neg}(2,2)) = 1$. The staircase form makes both tables antitone (`mono_of_stairs`); the itemised facts are decided by evaluation (`negGap_facts`); the difference of the two sums is $m_3 m_2 + m_2 m_3 = 2n m_3$ (`idxSum_negGap`), which is positive for $n \geq 3$ by Proposition 2.1 (`listed_facts_underdetermine`; the pinned statement is the conjunction of the two hypothesis checks and the inequality, and the exact difference is the separate lemma `idxSum_negGap`, with $\mathrm{neg}_{\mathrm{gap}}(3,2) = 1$ as `negGap_n32`). ☐

Remark 5.4 (this is not a gap in the source's claim). [1] does supply the missing fact. At $n = 2$ its Figure 8 shows $\lambda^1_{32} > 0$ and the text adds "Since $\lambda^1_{32} > 0$, the Rayleigh characterization implies that $\lambda^1_{j2} > 0$ for every $j \geq 3$"; for $n > 2$ its Section 5.2 says "we numerically checked that only the operators $S_{11}, S_{22}, S_{12}, S_{21}, S_{31}, S_{13}, S_{14}, S_{41}$ have negative eigenvalues". What Theorem 5.3 identifies is the exact sufficient hypothesis set $\mathcal{F}^+$, and that the bullet's own list is one fact short of it — something invisible in prose, because the surrounding sentences do supply the fact, and visible only when the hypotheses have to be written down. Our independent computation (Section 6) confirms $\mathrm{neg}(3,2) = 0$ for every n tried, so the conjecture itself is untouched.

Remark 5.5 (the equivariant index). The block S_{11} ($i = j = 1$, $\alpha_1 = 0$ on both factors) acts on functions of t alone, i.e. on the $O(n) \times O(n)$-invariant functions, so $\mathrm{neg}(1,1)$ is the equivariant stability index of X^n_{CS}. The hypothesis $\mathrm{neg}(1,1) = 3$ common to all three sets agrees with the numerical $O(2) \times O(2)$-equivariant index 3 that [6] report for the hypertorus ($n = 2$), and $\lambda^3_{11} = -(2n-1)$ makes $\mathrm{neg}(1,1) \geq 3$ a theorem for every n; only $\lambda^4_{11} > 0$ is numerical. This reading is ours, not stated in either paper.

6. Computations outside the formal development

Nothing in this section is a theorem; it is the independent reproduction that any claim about a numerically verified conjecture should come with, and it is labelled as such.

6.1. **An independent reproduction of the printed numbers.** Two implementations, written from the definitions of Section 2 with the corrected κ_t and sharing no code with each other or with the source: a C program in the founding record of this work (fourth-order Runge–Kutta with bisection shooting for the closed profile, the block operators in Liouville normal form $-v'' + Qv$ on a periodic grid, negative eigenvalues counted by the inertia of a cyclic tridiagonal LDL^{T} factorisation, and a monodromy integrator for the Floquet discriminants), and a pure Python program written for this paper (same design, independent code, 8000 grid points, and eigenvalue locations by bisection on the inertia). Both reproduce:

- *Table 1 of* [1]*, and the author's data set.* Table 1 prints r_0 and $T/4$ to ten decimals at ten values of n, and its caption points to a complete data set for $2 \leq n \leq 260$ on the author's page [4]; we retrieved it, and it holds r_0 and $T/4$, and nothing else, at all 259 values of n. Our shooting agrees with the data set to $4 \cdot 10^{-7}$ in r_0 and $3 \cdot 10^{-8}$ in $T/4$ at every n we tried, uniformly in n; started from the data set's own r_0 and T the profile closes to about $3 \cdot 10^{-7}$, started from ours it closes to a few times 10^{-12} or better ($r(T) - r(0)$, $\theta(T) - \theta(0)$, $\alpha(T) - \alpha(0) - 2\pi$ all below $2 \cdot 10^{-12}$ at $n = 2$ in the C implementation and below $3 \cdot 10^{-13}$ in the Python one), which is the size of the difference. Against the printed Table 1 the agreement is the same except in three entries: its r_0 at $n = 2, 3, 4$ is $8 \cdot 10^{-5}$ to $2.4 \cdot 10^{-4}$ away from ours. Those three entries also disagree with the author's own data set, which gives 1.2321501872, 1.2806886938, 1.3201452570 where Table 1 prints 1.2320724535, 1.2804538161, 1.3200537364; every other entry of Table 1, r_0 and $T/4$ alike, agrees with the data set to all ten printed digits, and the source's own text ("we are taking $r(0) = 1.2321501$", Section 5.1) agrees with the data set and with our 1.2321505 rather than with Table 1. So those three entries are a slip in the table, not a disagreement about the profile.
- *The $n = 2$ spectrum.* The seven negative eigenvalues printed at the end of Section 5.1 of [1], -20.19, -19.52, -18.74, -15.37, -9.80, -3, -2.05 (ours: -20.1901, -19.5156, -18.7423, -15.3739, -9.8030, -3.0000, -2.0513) with the printed multiplicities $1, 1, 4, 4, 4, 9, 4$ summing to 27, every printed interval ($\lambda^1_{21} \in (-19, -18)$, $\lambda^1_{31} \in (-16, -15)$, $\lambda^1_{41} \in (-10.5, -9.5)$, $\lambda^1_{51} \in (-2.5, -1.5)$), $\lambda^1_{61} > 0$, $\lambda^1_{32} > 0$ (ours: 2.06), $\lambda^4_{11} > 0$ (ours: 4.66), and the zero eigenvalues $\lambda^3_{21} = \lambda^2_{22} = 0$. The multiplicity 9 of -3 is $1 + 2 + 2 + 4$ from $S_{11}, S_{21}, S_{12}, S_{22}$, i.e. $n^2 + 2n + 1$ with equality in Corollary 3.2 of [1]; we read the "expected 5" printed beside it as $N + 2$.
- *Figure 1.* On the source's own grid $\{-4 + 0.01k : 0 \leq k \leq 500\}$ the C implementation gives $\delta_{11}(-3) = -3.3 \cdot 10^{-12}$, exactly one sign change in the grid, and $\delta_{11} < 0$ at every sampled point of $(-3, 1)$, as the source states; a third, separate integrator for the discriminant gives $\delta_{11}(-3) = 4.7 \cdot 10^{-13}$ and the same single sign change at $\lambda = -3$.
- *The index.* The C implementation for every $2 \leq n \leq 100$ (99 of 99 values equal to the conjectured ones, 38 processor-seconds in all), the Python one for $2 \leq n \leq 11$ and $n = 20, 21, 50, 51, 100, 150, 200$ (all equal to the conjectured ones; e.g. 848, 3941, 52208, 363701, 1193051, 2787401). In the whole range there are exactly two negative-count patterns, $\mathrm{neg}_{\mathrm{two}}$ at $n = 2$ and $\mathrm{neg}_{\mathrm{big}}$ for $n \geq 3$, i.e. the two bullets of [1]; in particular $\mathrm{neg}(3, 2) = 0$ throughout.
- *The printed $|A|^2$.* With $|A|^2_{\mathrm{pr}}$ the same code gives 27 at $n = 2$ and $19, 11, 13, 15, 16, 18, 20, 22$ at $n = 3, \ldots, 10$, then $42, 102, 202, 302, 402$ at $n = 20, 50, 100, 150, 200$ (Remark 3.3).

The shooting problem has other closed solutions: scanning r_0 upward from 0.9 the Python implementation first found, for $n \geq 20$, a closed profile with a smaller r_0 (e.g. $r_0 = 0.932$, $T/4 = 0.665$ at $n = 20$) and a different negative-count table; the Carlotto–Schulz profile is the root nearest $\pi/2$ (Table 1 of [1]), and all figures above refer to it.

6.2. **Table 2: five rows carry the value of another n.** Section 6 of [1] verifies Yau's conjecture on the first Laplace eigenvalue [10] for $2 \leq n \leq 260$ through the criterion of Theorem 5.1 of [2], which it restates: with z_1 the solution of $z_1'' + b'z_1' + (2n-1)z_1 = 0$, $z_1(T/4) = 1$, $z_1'(T/4) = 0$, the first nonzero eigenvalue of $-\Delta$ is $2n-1$ if and only if $z_1'(5T/4) > 0$. Its Table 2 prints $z_1'(5T/4)$ at ten values of n, and its text says that the r_0 and $T/4$ used there are those of the data set [4]. Reading the criterion exactly as printed, and starting from the author's own r_0 and $T/4$, we reproduce five of the ten entries and not the other five. But all ten printed numbers are values of $z_1'(5T/4)$ for this family: each of the five we do not reproduce at the tabulated n we reproduce, to every printed digit, at one other n.

n (row label)	printed	ours at that n	ours at another n'	
2	5.81879	5.818796		
3	6.30013	6.300134		
4	7.11311	7.113111		
5	7.89077	7.890773		
10	11.6129	11.075550	11.612858	$(n' = 11)$
20	16.0353	15.648968	16.035312	$(n' = 21)$
50	24.9928	24.746417	24.992773	$(n' = 51)$
100	35.0016	35.001586		
150	44.8270	42.870689	44.826967	$(n' = 164)$
200	54.5678	49.504043	54.567758	$(n' = 243)$

Our values are unchanged to six decimals as the number of Runge–Kutta steps per period goes from $2 \cdot 10^3$ to $5 \cdot 10^5$, agree between the two implementations, and are the same whether r_0 and T come from our own shooting or from the data set. Since $z_1'(5T/4)$ increases with n over the whole data set, by at least 0.108 from one n to the next, while the printed values carry six significant digits, each n' in the table is the *only* n with $2 \le n \le 260$ that produces the printed number. We record this and do not diagnose it: we have no access to the computation behind Table 2, and our reading of the criterion is exactly the one [1] prints, which is why the observation is stated as a pairing of numbers with row labels and nothing more. Nothing in [1] depends on it: we obtain $z_1'(5T/4) > 0$ at every one of the 259 values of n in the data set, so its conclusion — Yau's conjecture holds for these examples — stands, at the tabulated values and at the recomputed ones alike.

6.3. **A symmetry the source observes numerically.** Section 5.1 of [1] says: "For the pairs of indices considered in these computations, we also observe numerically that $\delta_{ij}(\lambda) = \delta_{ji}(\lambda)$." This is a theorem, and the following argument, from the founding record of this work, is given on paper only; it is not part of the formal development and nothing below depends on it. If (r, θ, α) solves (1), (3), so does $t \mapsto (r(-t), \pi/2 - \theta(-t), \pi - \alpha(-t))$: the sign flips of $\cos\alpha$ and of $\cot 2\theta$ cancel in the first term of (3), $\sin\alpha$ is unchanged, and the initial data are the same ($\pi/4$ is fixed by $\theta \mapsto \pi/2 - \theta$, and $3\pi/2 \equiv -\pi/2$ modulo 2π, which is all (3) sees). Uniqueness for the initial value problem, on a domain where $\sin r$ and $\sin 2\theta$ stay away from zero, then gives $r(-t) = r(t)$, $\theta(-t) = \pi/2 - \theta(t)$, $\alpha(-t) \equiv \pi - \alpha(t)$ — the symmetry list that Remark 2.2 of [1] asserts — hence $E(-t) = G(t)$, $G(-t) = E(t)$, $|A|^2(-t) = |A|^2(t)$ and $b'(-t) = -b'(t)$, so $\eta \mapsto \eta(-\,\cdot)$ is a bijection between the T-periodic solutions of $(S_{ij} - \lambda)\eta = 0$ and those of $(S_{ji} - \lambda)\eta = 0$, the two blocks are isospectral, and $\delta_{ij} = \delta_{ji}$. Numerically the three symmetries hold to 10^{-14} along our profiles, and every negative-count table we computed is symmetric. Formalising this needs the uniqueness theorem for ordinary differential equations on such a domain and is the cheapest open item of Section 7.

7. What is not proved, and why

The conjecture of [1] would follow from Theorems 5.1 and 5.2 the moment the sign facts $\mathcal{F}^+$ (for a given $n > 2$) or $\mathcal{F}_2$ were proved. Each of them — $\lambda_{11}^4 > 0$, $\lambda_{22}^2 = 0$, $\lambda_{31}^1 < 0$, $\lambda_{31}^2 > 0$, …— is a statement about the periodic eigenvalues of a Hill equation on the profile circle, and none of them has a finite restatement: the Floquet discriminant $\delta_{ij}(\lambda)$ is the trace of the monodromy of a linear equation whose coefficients b' and $|A|^2 - \alpha_i/E - \alpha_j/G$ are built from (r, θ, α), which are themselves the solution of the nonlinear system (1), (3) with the two constants r_0 and T pinned only by the shooting condition of Theorem 2.1 of [1]. A rigorous sign fact would need (i) an interval Newton or Krawczyk proof that a closed profile exists with r_0 and T in explicit rational boxes, which requires a validated flow of the nonlinear system over $[0, T/4]$ with $\sin r$ and $\sin 2\theta$ bounded away from zero, and (ii) a validated integration of the Hill equation over one period for a compact range of λ, with a Lipschitz bound in λ that turns a grid into a proof. Neither is available in *Mathlib* today: it has no interval arithmetic for ODE flows, no Taylor models and no Picard iteration with a certified remainder. The floating-point cost of one n is a fraction of a second and a naive interval version would cost minutes, so the obstruction is the missing library, not computation; but it is a formalisation project the size of this whole paper, and even a certified $n = 3$ says nothing about $n = 4$: an unconditional proof for all n would need asymptotics of the profile, not enclosures. Two cheaper items are open as well: the reflection argument of Section 6.3 (the uniqueness step is the only non-routine part), and the extension of the algebra of Sections 3–4 to the family $\mathbb{S}^k \times \mathbb{S}^\ell \times \mathbb{S}^1 \to \mathbb{S}^{k+\ell+2}$ with $k \neq \ell$ of [3], whose printed κ_t has a different form and has not been checked here.

8. Verification

Proposition 2.1 and Theorems 3.2, 4.1, 5.1, 5.2 and 5.3 rest on declarations formally verified in Lean 4 against *Mathlib* [12, 13] (Lean 4.32.0, *Mathlib* at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. Where a printed statement says more than its Lean counterpart the text names the extra step: Lemma 3.1 (the geometric identification of κ_t, a paper step; the formal κ_t is a definition), the passage from (14) to $J\zeta_{ij} = -(2n-1)\zeta_{ij}$ in Section 4, the truncation at $M = 5$ in Theorem 5.2, and the closing sentences of Theorems 5.1 and 5.2 that apply the abstract results to the Carlotto–Schulz profile, which is where the unproved sign facts enter. The development is three modules, 824 lines in all. `Curvature` (217 lines; 10 definitions, 13 theorems) is the algebra of Sections 3–4 in the six trigonometric values: the shape operator, the erratum and its controls, the closed form of $|A|^2$ and the identity (15). `Lemma31` (202 lines; 1 structure, 7 definitions, 7 lemmas, 6 theorems) is the analytic layer: a solution of the profile system as a structure of `HasDerivAt` hypotheses, the derivatives of b, b', f and f' along it, the equation (14), and the identification of e^{-b} with the printed f. `Index` (405 lines; 3 structures, 7 definitions, 8 lemmas, 17 theorems) is the bookkeeping of Section 5: the Laplace data, the truncated index sum, the three hypothesis sets, the three tables, the two conditional index theorems, the gap and the controls. Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand. The axiom print (`#print axioms`) of each of the 34 theorems reproduced in Appendix A is `propext`, `Classical.choice`, `Quot.sound`, except `sphEig_values`, `sphMult_five`, `neg_eq_zero_of_five`, `negBig_facts`, `negTwo_facts` and `negGap_facts`, which use only `propext` and `Quot.sound`, and `negGap_n32`, which uses none; we re-ran the print for this paper. As recorded, the three modules check in about 7, 6 and 12 seconds with a peak resident set of about 3 GB, and the computations of Section 6 took well under an hour of processor time in all, the $2 \leq n \leq 100$ index sweep being 38 seconds of it. Outside the formal development, and labelled as such where they occur, are Lemma 3.1, Remark 3.3, all of Section 6, and the floating-point checks quoted in proofs. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). All declarations live in the namespace `LeanProblemSpec.CsIndex27`. In `Curvature` and `Lemma31` the variables `n s c p q S C : ℝ` are $n, \sin r, \cos r, \sin\theta, \cos\theta, \sin\alpha, \cos\alpha$; `Real.log` is $\log|x|$; `HasDerivAt f f' t` says that f is differentiable at t with derivative f'. In `Index`, `Nat.choose` is the binomial coefficient, `Finset.Icc 1 N` the integer interval $[1, N]$, and `×ˢ` the product of finite sets.

`Curvature`: definitions.

```
noncomputable def cot2 (p q : ℝ) : ℝ := (q ^ 2 - p ^ 2) / (2 * p * q)
noncomputable def kapU (s c p q S C : ℝ) : ℝ := c * S / s + C * p / (s * q)
noncomputable def kapV (s c p q S C : ℝ) : ℝ := c * S / s - C * q / (s * p)
noncomputable def alphaRhs (n s c p q S C : ℝ) : ℝ :=
  (2 * n - 2) * C * cot2 p q / s - (2 * n - 1) * c * S / s
```

```
noncomputable def kapT (n s c p q S C : ℝ) : ℝ := alphaRhs n s c p q S C + c * S / s
noncomputable def kapTPrinted (s c p q S C : ℝ) : ℝ :=
  2 * C * cot2 p q / s - 2 * c * S / s
noncomputable def normA2 (n s c p q S C : ℝ) : ℝ :=
  (n - 1) * kapU s c p q S C ^ 2 + (n - 1) * kapV s c p q S C ^ 2
    + kapT n s c p q S C ^ 2
noncomputable def normA2Printed (n s c p q S C : ℝ) : ℝ :=
  (n - 1) * kapU s c p q S C ^ 2 + (n - 1) * kapV s c p q S C ^ 2
    + kapTPrinted s c p q S C ^ 2
noncomputable def invEG (n s p q : ℝ) : ℝ :=
  (n - 1) * (1 / (s ^ 2 * q ^ 2)) + (n - 1) * (1 / (s ^ 2 * p ^ 2))
noncomputable def bppExpr (n s c p q S C : ℝ) : ℝ :=
  2 * (n - 1) * (-C ^ 2 - 2 * (1 + cot2 p q ^ 2) * S ^ 2 - c * C * cot2 p q * S
    - alphaRhs n s c p q S C * s * (c * S - C * cot2 p q)) / s ^ 2
```

`Curvature`: **Theorem 3.2.**

```
theorem kapT_eq_printed_smul (n s c p q S C : ℝ) :
    kapT n s c p q S C = (n - 1) * kapTPrinted s c p q S C

theorem mean_curvature_zero (n s c p q S C : ℝ) (hs : s ≠ 0) (hp : p ≠ 0) (hq : q ≠ 0) :
    (n - 1) * kapU s c p q S C + (n - 1) * kapV s c p q S C + kapT n s c p q S C = 0

theorem mean_curvature_printed (n s c p q S C : ℝ)
    (hs : s ≠ 0) (hp : p ≠ 0) (hq : q ≠ 0) :
    (n - 1) * kapU s c p q S C + (n - 1) * kapV s c p q S C + kapTPrinted s c p q S C
      = (2 * n - 4) * (c * S / s - C * cot2 p q / s)

theorem mean_curvature_printed_ne_zero :
    (3 - 1 : ℝ) * kapU 1 0 (3/5) (4/5) 0 1 + (3 - 1 : ℝ) * kapV 1 0 (3/5) (4/5) 0 1
        + kapTPrinted 1 0 (3/5) (4/5) 0 1 ≠ 0

theorem control_frame_admissible :
    (1 : ℝ) ^ 2 + (0 : ℝ) ^ 2 = 1 ∧ (3/5 : ℝ) ^ 2 + (4/5 : ℝ) ^ 2 = 1 ∧
      (0 : ℝ) ^ 2 + (1 : ℝ) ^ 2 = 1 ∧ (1 : ℝ) ≠ 0 ∧ (3/5 : ℝ) ≠ 0 ∧ (4/5 : ℝ) ≠ 0

theorem normA2Printed_dim_two (s c p q S C : ℝ) :
    normA2Printed 2 s c p q S C = normA2 2 s c p q S C
```

`Curvature`: **Theorem 4.1(i), (iv), (v) and the lemmas in its proof.**

```
theorem bpp_core (n c S C K : ℝ) (ha : S ^ 2 + C ^ 2 = 1) :
    -C ^ 2 - 2 * (1 + K ^ 2) * S ^ 2 - c * C * K * S
        - ((2 * n - 2) * C * K - (2 * n - 1) * c * S) * (c * S - C * K)
      = (2 * n - 1) * (c * S - C * K) ^ 2 + (1 + K ^ 2) * (C ^ 2 - 2)

theorem normA2_closed (n s c p q S C : ℝ) (hs : s ≠ 0) (hp : p ≠ 0) (hq : q ≠ 0) :
    normA2 n s c p q S C
      = 2 * (n - 1) * (2 * n - 1) * ((c * S - C * cot2 p q) / s) ^ 2
        + (n - 1) * C ^ 2 * (p ^ 2 + q ^ 2) ^ 2 / (2 * p ^ 2 * q ^ 2 * s ^ 2)

theorem one_add_cot2_sq (p q : ℝ) (hp : p ≠ 0) (hq : q ≠ 0) (ht : p ^ 2 + q ^ 2 = 1) :
    1 + cot2 p q ^ 2 = 1 / (4 * p ^ 2 * q ^ 2)

theorem bppExpr_eq (n s c p q S C : ℝ) (hs : s ≠ 0) (hp : p ≠ 0) (hq : q ≠ 0)
```

```
    (ht : p ^ 2 + q ^ 2 = 1) (ha : S ^ 2 + C ^ 2 = 1) :
    bppExpr n s c p q S C = normA2 n s c p q S C - invEG n s p q

theorem bppExpr_sample : bppExpr 3 1 0 (3/5) (4/5) 0 1 = -95/36

theorem bppExpr_eq_sample :
    bppExpr 3 1 0 (3/5) (4/5) 0 1 = normA2 3 1 0 (3/5) (4/5) 0 1 - invEG 3 1 (3/5) (4/5)

theorem lemma31_fails_printed_dim_three :
    bppExpr 3 1 0 (3/5) (4/5) 0 1
      ≠ normA2Printed 3 1 0 (3/5) (4/5) 0 1 - invEG 3 1 (3/5) (4/5)
```

`Lemma31`**: definitions and Theorem 4.1(ii), (iii).**

```
structure Profile (n : ℝ) (r th a : ℝ → ℝ) : Prop where
  dr : ∀ t, HasDerivAt r (cos (a t)) t
  dth : ∀ t, HasDerivAt th (sin (a t) / sin (r t)) t
  da : ∀ t, HasDerivAt a
      (alphaRhs n (sin (r t)) (cos (r t)) (sin (th t)) (cos (th t))
        (sin (a t)) (cos (a t))) t
  sr : ∀ t, sin (r t) ≠ 0
  sp : ∀ t, sin (th t) ≠ 0
  cq : ∀ t, cos (th t) ≠ 0

noncomputable def bFun (n : ℝ) (r th : ℝ → ℝ) : ℝ → ℝ := fun t =>
  (n - 1) * (2 * log (sin (r t)) + log (sin (th t)) + log (cos (th t)))
noncomputable def bpFun (n : ℝ) (r th a : ℝ → ℝ) : ℝ → ℝ := fun t =>
  2 * (n - 1) * ((cos (r t) * cos (a t)
    + cot2 (sin (th t)) (cos (th t)) * sin (a t)) / sin (r t))
noncomputable def bppFun (n : ℝ) (r th a : ℝ → ℝ) : ℝ → ℝ := fun t =>
  bppExpr n (sin (r t)) (cos (r t)) (sin (th t)) (cos (th t)) (sin (a t)) (cos (a t))
noncomputable def fFun (n : ℝ) (r th : ℝ → ℝ) : ℝ → ℝ := fun t => exp (-(bFun n r th t))
noncomputable def A2Fun (n : ℝ) (r th a : ℝ → ℝ) : ℝ → ℝ := fun t =>
  normA2 n (sin (r t)) (cos (r t)) (sin (th t)) (cos (th t)) (sin (a t)) (cos (a t))
noncomputable def EGFun (n : ℝ) (r th : ℝ → ℝ) : ℝ → ℝ := fun t =>
  invEG n (sin (r t)) (sin (th t)) (cos (th t))
noncomputable def fpFun (n : ℝ) (r th a : ℝ → ℝ) : ℝ → ℝ := fun t =>
  -(bpFun n r th a t) * fFun n r th t

-- in the following, (P : Profile n r th a) is a section variable
theorem hasDerivAt_bFun (t : ℝ) : HasDerivAt (bFun n r th) (bpFun n r th a t) t

theorem hasDerivAt_bpFun (t : ℝ) : HasDerivAt (bpFun n r th a) (bppFun n r th a t) t

theorem lemma31 (t : ℝ) :
    (bpFun n r th a t ^ 2 - bppFun n r th a t) * fFun n r th t
        + bpFun n r th a t * fpFun n r th a t
        + A2Fun n r th a t * fFun n r th t - EGFun n r th t * fFun n r th t = 0

theorem fFun_eq_pow (m : ℕ) (hn : n = (m : ℝ) + 1) (t : ℝ)
    (h1 : 0 < sin (r t)) (h2 : 0 < sin (th t)) (h3 : 0 < cos (th t)) :
    fFun n r th t = (1 / (sin (r t) ^ 2 * (sin (th t) * cos (th t)))) ^ m
```

`Index`**: definitions.**

```
def sphEig (n i : ℕ) : ℕ := (i - 1) * (n + i - 3)

def sphMult (n : ℕ) : ℕ → ℕ
  | 0 => 0
  | 1 => 1
  | 2 => n
  | (k + 3) => Nat.choose (n + k + 1) (k + 2) - Nat.choose (n + k - 1) k

def stair (l : List (ℕ × ℕ)) (i j : ℕ) : ℕ :=
  if l.any (fun p => decide (i ≤ p.1 ∧ j ≤ p.2)) then 1 else 0

def idxSum (n N : ℕ) (neg : ℕ → ℕ → ℕ) : ℕ :=
  ∑ x ∈ Finset.Icc 1 N ×ˢ Finset.Icc 1 N, sphMult n x.1 * sphMult n x.2 * neg x.1 x.2

structure SignFacts (neg : ℕ → ℕ → ℕ) : Prop where
  mono : ∀ i j i' j', i ≤ i' → j ≤ j' → neg i' j' ≤ neg i j
  n11 : neg 1 1 = 3
  n21 : neg 2 1 = 2
  n12 : neg 1 2 = 2
  n22 : neg 2 2 = 1
  n31 : neg 3 1 = 1
  n13 : neg 1 3 = 1
  n41 : neg 4 1 = 1
  n14 : neg 1 4 = 1
  n51 : neg 5 1 = 0
  n15 : neg 1 5 = 0

structure SignFactsFull (neg : ℕ → ℕ → ℕ) : Prop extends SignFacts neg where
  n32 : neg 3 2 = 0
  n23 : neg 2 3 = 0

structure SignFactsTwo (neg : ℕ → ℕ → ℕ) : Prop where
  mono : ∀ i j i' j', i ≤ i' → j ≤ j' → neg i' j' ≤ neg i j
  n11 : neg 1 1 = 3
  n21 : neg 2 1 = 2
  n12 : neg 1 2 = 2
  n22 : neg 2 2 = 1
  n31 : neg 3 1 = 1
  n13 : neg 1 3 = 1
  n41 : neg 4 1 = 1
  n14 : neg 1 4 = 1
  n51 : neg 5 1 = 1
  n15 : neg 1 5 = 1
  n61 : neg 6 1 = 0
  n16 : neg 1 6 = 0
  n32 : neg 3 2 = 0
  n23 : neg 2 3 = 0

def negBig : ℕ → ℕ → ℕ := fun i j =>
  stair [(1, 4), (2, 2), (4, 1)] i j + stair [(1, 2), (2, 1)] i j + stair [(1, 1)] i j
def negTwo : ℕ → ℕ → ℕ := fun i j =>
  stair [(1, 5), (2, 2), (5, 1)] i j + stair [(1, 2), (2, 1)] i j + stair [(1, 1)] i j
```

```
def negGap : ℕ → ℕ → ℕ := fun i j =>
  stair [(1, 4), (2, 3), (3, 2), (4, 1)] i j + stair [(1, 2), (2, 1)] i j
    + stair [(1, 1)] i j
```

`Index`: **Proposition 2.1.**

```
theorem sphEig_values (n : ℕ) (hn : 2 ≤ n) :
    sphEig n 1 = 0 ∧ sphEig n 2 = n - 1 ∧ sphEig n 3 = 2 * n ∧
      sphEig n 4 = 3 * (n + 1) ∧ sphEig n 5 = 4 * (n + 2)

theorem sphMult_three (n : ℕ) (hn : 1 ≤ n) : 2 * sphMult n 3 + 2 = n ^ 2 + n

theorem sphMult_four (n : ℕ) (hn : 1 ≤ n) :
    6 * sphMult n 4 + 6 * n = n ^ 3 + 3 * n ^ 2 + 2 * n

theorem sphMult_five (n : ℕ) (hn : 1 ≤ n) :
    24 * sphMult n 5 + 12 * (n ^ 2 + n) = n ^ 4 + 6 * n ^ 3 + 11 * n ^ 2 + 6 * n
```

`Index`: **Theorems 5.1, 5.2, 5.3.**

```
theorem neg_eq_zero_of_five {neg : ℕ → ℕ → ℕ} (H : SignFacts neg) {i j : ℕ}
    (hi : 1 ≤ i) (hj : 1 ≤ j) (h : 5 ≤ i ∨ 5 ≤ j) : neg i j = 0

theorem idxSum_truncate {neg : ℕ → ℕ → ℕ} (H : SignFacts neg) (n N : ℕ) (hN : 4 ≤ N) :
    idxSum n N neg = idxSum n 4 neg

theorem index_value {neg : ℕ → ℕ → ℕ} (H : SignFactsFull neg) (n : ℕ) (hn : 1 ≤ n) :
    3 * idxSum n 4 neg = n ^ 3 + 9 * n ^ 2 + 11 * n + 3

theorem index_value_two {neg : ℕ → ℕ → ℕ} (H : SignFactsTwo neg) :
    idxSum 2 5 neg = 27

theorem negBig_facts : SignFactsFull negBig

theorem negTwo_facts : SignFactsTwo negTwo

theorem negGap_facts : SignFacts negGap

theorem negGap_n32 : negGap 3 2 = 1

theorem idxSum_negGap (n : ℕ) :
    idxSum n 4 negGap = idxSum n 4 negBig + 2 * (n * sphMult n 3)

theorem listed_facts_underdetermine (n : ℕ) (hn : 3 ≤ n) :
    SignFacts negBig ∧ SignFacts negGap ∧ idxSum n 4 negBig ≠ idxSum n 4 negGap

theorem index_value_negBig (n : ℕ) (hn : 1 ≤ n) :
    3 * idxSum n 4 negBig = n ^ 3 + 9 * n ^ 2 + 11 * n + 3

theorem index_two_controls :
    idxSum 2 5 negTwo = 27 ∧ idxSum 2 5 negTwo ≠ 26 ∧ idxSum 2 5 negTwo ≠ 28

theorem index_controls {neg : ℕ → ℕ → ℕ} (H : SignFactsFull neg) (n : ℕ) (hn : 1 ≤ n) :
    3 * (idxSum n 4 neg + 1) ≠ n ^ 3 + 9 * n ^ 2 + 11 * n + 3 ∧
      3 * idxSum n 4 neg ≠ n ^ 3 + 9 * n ^ 2 + 11 * n + 6
```

References

[1] O. Perdomo, *The stability index and Yau's conjecture for Carlotto–Schulz minimal hypertori, part II*, arXiv:2606.16980v1 [math.DG], 15 June 2026 (the only version as of 2026-09-07). The source of this paper; read in full from the author's own e-print, which we compiled. Statement numbers are those of the compiled e-print: equations (4)–(7), Theorem 2.1, Remark 2.2, Theorem 2.3, Lemma 2.4, Lemma 3.1, Corollary 3.2, Conjecture 1.1 = Conjecture 4.1, Sections 5.1–5.2, Figures 1–8, Tables 1–2; the principal curvatures are in an unnumbered display between (6) and (7). All quotations are verbatim from that text.

[2] O. Perdomo, *The stability index and Yau's conjecture for Carlotto–Schulz minimal hypertori*, arXiv:2508.09104v2 [math.DG], version 2 of 15 September 2025 (part I). Read in full from the version 2 e-print, which we compiled; Theorem 3.3 (the spectral decomposition into the blocks S_{ij} with multiplicities $m_i m_j$), Lemma 3.4, Theorem 3.5 (the discriminant), Theorem 3.9 (Rayleigh characterization), Theorem 4.3 (index $\geq n^2 + 4n + 3$) and Theorem 5.1 (the criterion for Yau's conjecture) are cited by their numbers there. Its Section 2 prints the same κ_t as [1]; its introduction contains the sentence about [6] quoted in Section 1.

[3] O. Perdomo, *New explicit eigenfunctions of the stability operator on some minimal hypersurfaces*, arXiv:2607.04917v1 [math.DG], 6 July 2026. The only paper citing [1] (Semantic Scholar, 2026-09-07); read from its e-print for the sentences quoted in Sections 1 and 3.

[4] O. Perdomo, *Carlotto–Schulz examples*, data set at `https://www.oscarperdomo.org/CarlottoSchulzExamples`, linked from Sections 5 and 6 of [1]; retrieved 2026-09-07. It tabulates r_0 and $T/4$, and nothing else, at every integer $2 \leq n \leq 260$.

[5] A. Carlotto and M. B. Schulz, *Minimal hypertori in the four-dimensional sphere*, Ars Inveniendi Analytica (2023), Paper No. 8, 33 pp.; DOI 10.15781/gtrt-v523; arXiv:2109.11768. The construction of X^n_{CS} and the existence theorem restated as Theorem 2.1 of [1]. Journal data checked against the published version and the journal's own article record; the journal indexes papers by year and paper number and assigns no volume number.

[6] A. Carlotto, M. B. Schulz and D. Wiygul, *Index growth not imputable to topology*, Proc. Amer. Math. Soc. **153** (2025), no. 4, 1787–1801; arXiv:2407.11147. Bibliographic data checked against Crossref (DOI 10.1090/proc/17151). Read from its arXiv e-print for the two statements used here: that the hypertorus has Morse index at least 8 "as a corollary of [9, Theorem 4.3]", and that numerical simulations indicate $O(2) \times O(2)$-equivariant index 3.

[7] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. The stability operator and the index of minimal submanifolds of spheres. Bibliographic data checked against Crossref (DOI 10.2307/1970556); not re-read for this paper.

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