THE MAXIMUM NUMBER OF ONCE-CROSSED EDGES IN A RECTILINEAR DRAWING OF K_n

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Abstract. For a drawing $\mathcal{D}$ of a graph, $e_k(\mathcal{D})$ is the number of edges involved in exactly k crossings, and the sequence $(e_0(\mathcal{D}), e_1(\mathcal{D}), \dots)$ is the *crossing profile* of $\mathcal{D}$. Chen and Solé-Pi, who introduced the profile, determined $\overline{\max} e_0(K_n) = 2n-2$ and asked, as their Problem 1, for the value of $\overline{\max} e_1(K_n)$, the maximum of e_1 over rectilinear drawings of K_n; for $n \geq 8$ they could only place it between $\lceil 3n/2 \rceil - 7$ and $2n + \lfloor (n-1)/2 \rfloor + 7$. We determine $\overline{\max} e_1(K_n)$ exactly for $4 \leq n \leq 11$, where its values are $2, 2, 6, 7, 10, 12, 14, 14$, and we prove $16 \leq \overline{\max} e_1(K_{12}) \leq 17$. In particular the pattern $\overline{\max} e_1(K_n) = 2n-6$, which holds at $n = 8, 9, 10$, fails at $n = 11$, so this route yields no candidate answer to the limit half of Problem 1. We also exhibit drawings giving $\overline{\max} e_1(K_n) \geq 17, 19, 19, 21$ for $n = 13, \dots, 16$, each above the published lower bound. The upper bounds come from an exhaustion over uniform rank-3 chirotopes rather than over order types — dropping realizability enlarges the search space and so preserves an upper bound — made finite by a deletion bound that lets an n-element search be driven by the $(n-1)$- or $(n-2)$-element objects with large e_1. The supporting statements are machine-checked in Lean 4.

1. Introduction

A *drawing* $\mathcal{D}$ of a graph G maps the vertices to distinct points of the plane and the edges to simple curves joining their endpoints, subject to the usual non-degeneracy conventions: no edge passes through a vertex other than its endpoints, two edges share at most one point and are never tangent, and no three edges share an interior point. A *crossing* is a shared interior point of two distinct edges. The drawing is *rectilinear* if every edge is a straight line segment; a rectilinear drawing of K_n is the same thing as a set of n points in general position in the plane.

Chen and Solé-Pi [3] introduced the following statistics. For $k \geq 0$ the *k-crossing set* $E_k(\mathcal{D})$ is the set of edges taking part in exactly k crossings, $e_k(\mathcal{D}) = |E_k(\mathcal{D})|$, and the *crossing profile* of $\mathcal{D}$ is the sequence $(e_0(\mathcal{D}), e_1(\mathcal{D}), \dots)$. The *$k$-crossing sum* $S_k(\mathcal{D})$ is the number of edges taking part in at most k crossings. Two families of extremal parameters are attached to these: $\max e_k(G)$ and $\min e_k(G)$ range over *all* drawings of G, while $\overline{\max} e_k(G)$ and $\overline{\min} e_k(G)$ range over the *rectilinear* drawings only, and likewise for S_k. Everything below concerns the rectilinear parameters.

The e_0 column is settled. Ringel proved $\max e_0(K_n) = 2n-2$, and, as recorded in [3], Károlyi and Welzl observed that the rectilinear parameter is the same: $\overline{\max} e_0(K_n) = 2n-2$ for $n \geq 4$. Harborth and Thürmann [5] determined the minimum, $\overline{\min} e_0(K_n) = 3, 4, 5, 5, 6$ for $3 \leq n \leq 7$ and $\overline{\min} e_0(K_n) = 5$ for $n \geq 8$.

The next column is not settled. Chen and Solé-Pi write, immediately before stating their bounds, that they “do not know the exact value of $\overline{\max} e_1(K_n)$”, and pose:

Problem 1.1 ([3, Problem 1]). Determine $\overline{\max} e_1(K_n)$ for all n. Less ambitiously, does the limit $\lim_{n\to\infty} \overline{\max} e_1(K_n)/n$ exist? If so, what is its value?

What was known is a pair of bounds, valid for $n \geq 8$:

$$\left\lceil \frac{3n}{2} \right\rceil - 7 \;\leq\; \overline{\max} e_1(K_n) \;\leq\; 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 7. \tag{1}$$

Even the linear constant is open: (1) pins it only to $[1.5, 2.5]$. The upper bound is not specific to e_1; it is inherited from Harborth and Mengersen’s bound [4] on $\max S_1(K_n)$ for general drawings.

Results. This paper determines $\overline{\max}e_1(K_n)$ for every n up to 11 and narrows $n = 12$ to two values.

n	4	5	6	7	8	9	10	11	12
$\overline{\max}e_1(K_n)$	2	2	6	7	10	12	14	14	16 or 17
$\lceil 3n/2\rceil - 7$	—	—	—	—	5	7	8	10	11
$2n + \lfloor (n-1)/2\rfloor + 7$	—	—	—	—	26	29	31	34	36

Theorem 5.1 gives the values for $4 \le n \le 10$, Theorem 5.3 the value at $n = 11$, and Theorems 6.1 and 6.2 the bracket at $n = 12$. Beyond $n = 12$ we give lower bounds only (Theorems 7.1 and 7.2): $\overline{\max}e_1(K_n) \ge 17, 19, 19, 21$ for $n = 13, 14, 15, 16$. Every one of these lower bounds is strictly larger than $\lceil 3n/2\rceil - 7$.

Two remarks on what the table does and does not say. At $n = 8, 9, 10$ the values are $2n - 6$; had that continued, the limit in Problem 1.1 would have had the candidate value 2. It does not continue: $\overline{\max}e_1(K_{11}) = 14$ while $2n - 6 = 16$. So the pattern breaks at the first n beyond the range in which it was observed, and Problem 1.1 gains no candidate limit here; the constant remains pinned to $[1.5, 2.5]$ by (1). Second, the small values are irregular — e_1 is not monotone in n, since it is 14 at both $n = 10$ and $n = 11$ and 19 at both $n = 14$ and $n = 15$ — and no closed form is proposed.

Method. The quantity e_k of a rectilinear drawing depends only on the *order type* of its point set, so an upper bound on $\overline{\max}e_k(K_n)$ is a statement about a finite set of combinatorial objects. Order types are usually obtained from Aichholzer's database [1], which is complete to $n = 10$. We do not use it. An order type is a *realizable* uniform rank-3 chirotope, and realizability may simply be dropped, because dropping it makes the search space *larger*:

$$\overline{\max}e_k(K_n) \;\le\; \max\bigl\{\, e_k(\chi) : \chi \text{ a uniform rank-3 chirotope on } [n] \,\bigr\},$$

and where a point-set witness attains the right-hand side the value is exact. The chirotopes are generated from the 3-term Grassmann–Plücker relations, which together with alternation characterize uniform rank-3 chirotopes [2, Thm. 3.6.2]. For $n \le 8$ nothing at all is lost, since every uniform rank-3 chirotope on at most eight elements is realizable, the smallest exception being the non-Pappus configuration on nine.

Flat enumeration reaches $n = 10$ and no further: there are 112 018 190 normalized uniform rank-3 chirotopes on ten elements and about a hundred times as many on eleven. What carries the search past that point is a counting bound (Lemma 4.1): an edge counted by e_1 has a unique crossing partner disjoint from it, so it survives the deletion of any vertex outside the four endpoints of the two, and therefore

$$\max_{w \ne 0} e_1(\chi - w) \;\ge\; \left\lceil \tfrac{(n-5)\, e_1(\chi)}{n-1} \right\rceil.$$

Consequently an n-element chirotope with large e_1 can be reached by extending an $(n-1)$-element chirotope with large e_1, and only those need be extended. At $n = 11$ the relevant threshold selects 278 620 of the 112 018 190 ten-element chirotopes, that is 0.249% of them, and the resulting pass *computes* the maximum rather than merely bounding it. At $n = 12$ the bound is applied twice in succession.

Organization. Section 2 fixes the model. Section 3 describes the exhaustion and its one unformalized step. Section 4 proves the deletion bound and the completeness identities that validate the extension search. Sections 5 and 6 contain the upper bounds, Section 7 the constructions. Section 8 records the published values that the same machinery reproduces, and Section 9 the negative controls. Section 10 describes the machine verification.

2. The model

We use the source's definitions throughout, in the form in which they are checked.

Definition 2.1. A *rectilinear drawing of K_n* is a set P of $n \ge 3$ points of the plane in general position, i.e. pairwise distinct with no three collinear; its edges are the $\binom{n}{2}$ closed segments joining pairs of points of P. Two edges *cross* if they share exactly one point and that point is interior to both. The drawing is *generic* if no three of its edges share an interior point. For $k \geq 0$,

$$e_k(P) = \#\{\text{edges of } P \text{ lying on exactly } k \text{ crossings}\},$$
$$S_k(P) = \#\{\text{edges of } P \text{ lying on at most } k \text{ crossings}\}.$$

Finally $\overline{\max} e_k(K_n)$ is the maximum of $e_k(P)$ over all such P with $|P| = n$.

Genericity is part of the source's definition of a drawing, not an extra hypothesis, so every witness below is verified to be generic and not merely in general position. On a generic drawing the number of crossings an edge lies on equals the number of other edges crossing it, so e_k may be computed edge-pairwise; the two readings differ only off the admitted class, and Proposition 9.2 records that they agree at every witness in this paper. Every witness has integer coordinates, which costs nothing in the direction claimed: an integer point set is a point set, so every profile computed is attained and every witness theorem is a genuine lower bound on $\overline{\max} e_k(K_n)$.

3. Exhaustion over chirotopes

A *uniform rank-3 chirotope* on $[n] = \{0, \ldots, n-1\}$ is an alternating map χ from ordered triples of distinct elements of $[n]$ to $\{+, -\}$ satisfying the 3*-term Grassmann–Plücker relations*: for every e and every $a < b < c < d$ in $[n] \setminus \{e\}$, the three signs

$$\chi(e,a,b)\chi(e,c,d), \qquad -\chi(e,a,c)\chi(e,b,d), \qquad \chi(e,a,d)\chi(e,b,c)$$

are not all equal. Alternation together with these relations characterizes uniform rank-3 chirotopes [2, Thm. 3.6.2]; realizable ones are exactly the order types of planar point sets in general position. Whether two edges ab and cd cross is decided by the four signs of χ on the triples of $\{a, b, c, d\}$, so e_0, e_1 and S_1 are defined on chirotopes and agree with Definition 2.1 on realizable ones. We call χ *normalized* if $\chi(0, i, j) = +$ for all $1 \leq i < j < n$.

Lemma 3.1 (Normalization)**.** *Every set P of $n \geq 3$ points in general position has a labelling whose chirotope is normalized, and relabelling does not change e_k.*

Argument. Let $p_0 \in P$ have least y-coordinate, ties broken by least x, and label it 0. Every other point p has $p - p_0$ in the half-plane of directions of angle in $[0, \pi)$, and no two of these directions coincide, since two equal directions would place p_0 and two further points on a line. Label the remaining points $1, \ldots, n-1$ by increasing direction angle. Then $(p_i - p_0) \times (p_j - p_0) > 0$ for $i < j$, which is $\chi(0, i, j) = +$. Relabelling permutes the edges of the drawing and preserves which pairs cross, hence preserves e_k. □

Lemma 3.1 is the one step in this paper that is argued rather than formally verified. It is used only in the direction that says every drawing's order type occurs among the normalized chirotopes, i.e. only for upper bounds; every lower bound below is independent of it. It is also exercised: the relabelling is implemented and checked at all 25 point sets appearing in this paper to produce a normalized chirotope whose e_0 and e_1 agree with the values computed from the coordinates, by two computations sharing no code — one intersects segments pairwise, the other reads four orientation signs per 4-subset.

Normalization is worth much more than the factor $n!$ it appears to be worth, because most labellings of one order type give the same normalized chirotope.

Proposition 3.2. *The number of normalized uniform rank-3 chirotopes on n elements is*

$$1,\ 2,\ 8,\ 62,\ 908,\ 24\,698,\ 1\,232\,944,\ 112\,018\,190 \qquad \textit{for } n = 3, \ldots, 10,$$

against $1, 2, 3, 16, 135, 3315, 158\,817, 14\,309\,547$ *order types. The values for $n \leq 8$ are verified in Lean; the values at $n = 9, 10$ are external computations reproduced by two independently written enumerators.*

One subtlety in the crossing count deserves to be isolated, because it is invisible in the realizable case and the enumeration is not confined to it. A 4-subset $\{a, b, c, d\}$ admits three pairings, and a crossing counter that returns after the first pairing found to cross is exact only when at most one of the three can cross.

Lemma 3.3. *Write $p = \chi(a,b,c)$, $q = \chi(a,b,d)$, $r = \chi(a,c,d)$, $s = \chi(b,c,d)$. All three pairings of $\{a,b,c,d\}$ cross precisely when (p,q,r,s) is alternating, i.e. equal to $(+,-,+,-)$ or $(-,+,-,+)$. The Grassmann–Plücker relations do not exclude this: of the 384 sign vectors on the ten triples of a five-element ground set satisfying all five relations, 120 contain such a 4-subset. Normalization does exclude it: exactly 8 five-element sign vectors are both normalized and Grassmann–Plücker valid, none has an alternating 4-subset, and restricting an arbitrary 4-subset $\{a,b,c,d\}$ of a normalized χ to $\{0,a,b,c,d\}$ reduces every n to that case (for $a = 0$ the pattern is $(+,+,+,s)$, which is not alternating).*

So on the normalized chirotopes the enumeration ranges over, the first-match crossing counter agrees with the counter that inspects all three pairings, and the counts feeding every bound below are the intended ones. The hypothesis is load-bearing: there is a Grassmann–Plücker valid, non-normalized sign vector with an alternating 4-subset, on which the first-match counter records two crossed edges where all six are crossed.

4. The deletion bound

Write $\chi - w$ for the restriction of χ to $[n] \setminus \{w\}$, relabelled order-preservingly. Deleting $w \neq 0$ from a normalized chirotope leaves a normalized chirotope, and conversely every normalized n-element chirotope is a single-element insertion into a normalized $(n-1)$-element one at one of $n-1$ positions. Both directions are exact, and that is what makes a search by extension a complete enumeration.

Lemma 4.1 (Deletion bound)**.** *Let χ be a uniform rank-3 chirotope on $[n]$ with $e_1(\chi) = v$. Then*

$$\sum_{w\neq 0} e_1(\chi - w) \;\geq\; (n-5)\,v, \qquad \text{so} \qquad \max_{w\neq 0} e_1(\chi - w) \;\geq\; \left\lceil \tfrac{(n-5)v}{n-1} \right\rceil \;=:\; \delta(n,v).$$

Proof. Let $e \in E_1(\chi)$ and let f be its unique crossing partner. By Lemma 3.3, at most one pairing of a 4-subset crosses, so e and f are disjoint and $|e \cup f| = 4$. Let $w \notin e \cup f$. Both e and f survive in $\chi - w$, and whether two pairs cross depends only on χ restricted to their four elements, so e still crosses f there; anything crossing e in $\chi - w$ also crosses it in χ and hence equals f. So $e \in E_1(\chi - w)$. Each $e \in E_1(\chi)$ therefore forbids at most four of the $n-1$ choices $w \neq 0$ and survives at least $n-5$ of them, which is the sum. The maximum bound follows. □

The bound is applied through the predicate

$$\mathrm{Cov}(n,t,V) \iff \big(t \leq \delta(n,v) \text{ for every } v \text{ with } V \leq v \leq \tbinom{n}{2}\big),$$

which says: extending only the $(n-1)$-element chirotopes with $e_1 \geq t$ reaches every n-element chirotope with $e_1 \geq V$. Every threshold used below is licensed by deciding Cov over the whole range of v, not by evaluating δ at one endpoint.

Lemma 4.1 is elementary, but it is what makes the searches finite, so it was checked exhaustively rather than trusted, and its constant was shown to be the right one.

Proposition 4.2 (The bound, checked and sharp)**.** *Over all normalized uniform rank-3 chirotopes on n elements, the inequality $(n-1)\max_{w\neq 0} e_1(\chi - w) \geq (n-5)e_1(\chi)$ has no violations for $n = 6,7,8$ (verified in Lean) or $n = 9$ (external, 1 232 944 chirotopes). It is attained with equality at 99 of the 24 698 chirotopes on eight elements and at 583 of the nine-element ones, so no constant can be added to it; and the same inequality with $n-4$ in place of $n-5$ is false at 52 of the eight-element chirotopes, so the constant produced by the counting argument is not a safe over-estimate.*

Two completeness identities validate the extension search itself. With the e_1 filter switched off, every child is produced exactly once per deletion position, so the leaf count is forced.

Proposition 4.3 (One-level completeness)**.** *Let C_m denote the number of normalized uniform rank-3 chirotopes on m elements. The unfiltered single-element extension of all m-element chirotopes produces exactly $m \cdot C_{m+1}$ leaves:*

$$32 = 4\cdot 8, \quad 310 = 5\cdot 62, \quad 5\,448 = 6\cdot 908, \quad 172\,886 = 7\cdot 24\,698, \quad 9\,863\,552 = 8\cdot 1\,232\,944,$$

the first four verified in Lean and the last externally. Dividing the last row's child e_1 *histogram by* 8 *reproduces the independently computed nine-element tail counts* 12 866*,* 2 808*,* 594*,* 56*,* 8 *for* $e_1 \geq 8, \dots, 12$*.*

Proposition 4.4 (Two-level completeness)**.** *With both filters off, the two-level extension of all* m*-element chirotopes produces exactly* $m(m+1)C_{m+2}$ *leaves:*

$$1\,240 = 4 \cdot 5 \cdot 62, \quad 27\,240 = 5 \cdot 6 \cdot 908, \quad 1\,037\,316 = 6 \cdot 7 \cdot 24\,698, \quad 69\,044\,864 = 7 \cdot 8 \cdot 1\,232\,944,$$

the first three verified in Lean and the last externally. These unfiltered passes also see every chirotope of the target size, and none on eight elements has $e_1 = 11$*.*

Too few leaves would mean the search misses chirotopes; too many, that it repeats them or admits objects that are not chirotopes.

Lemma 4.5 (Chaining is the stronger argument)**.** *Deleting two elements at once counts* $\binom{n-5}{2}$ *good pairs out of* $\binom{n-1}{2}$*; at* $n = 12$ *that ratio is* $21/55$*, which is exactly* $(7/11) \cdot (6/10)$*. Applying Lemma 4.1 twice in succession rounds up twice and therefore dominates the one-shot double count at every value of* e_1*, strictly at* $e_1 = 13, 18, 26, \dots$*.*

Finally, a remark on splitting a two-level pass into pieces, which the $n = 12$ computation of Section 6 needed.

Proposition 4.6 (Splitting the pass)**.** *Restricting a two-level pass to a range of the enumeration order of its starting chirotopes is exact at the outer two levels and on the answer: the kept counts, the first-level leaf counts and the histograms add across a partition of the range, and the maximum over pieces is the maximum. It is not exact on the amount of work at the innermost level. Intermediate chirotopes are produced once per (parent, position) pair reaching them — measured multiplicity between* 7.6 *and* 7.8 *— and de-duplication is per piece, so an intermediate reachable from several pieces is extended once per piece. Measured at a smaller size, four pieces multiply the distinct intermediates and the innermost leaf count by* 2.60*, and nine pieces by* 3.4*.*

5. Exact values for $4 \leq n \leq 11$

Theorem 5.1. $\overline{\max} e_1(K_n) = 2,\ 2,\ 6,\ 7,\ 10,\ 12,\ 14$ *for* $n = 4, 5, 6, 7, 8, 9, 10$*.*

Proof sketch. Each value is an exhaustion upper bound meeting an explicit lower bound.

Lower halves. For each n we exhibit an explicit set D_n of n integer points, verified to be in general position and generic, with the stated e_1. Their full profiles are listed in Table 1. These are unconditional and use nothing from Section 3.

Upper halves. Enumerate all normalized uniform rank-3 chirotopes on n elements by depth-first search over the free signs, testing each Grassmann–Plücker relation as soon as its last triple is assigned, and record $\max e_1$. By Lemma 3.1 every drawing's order type occurs in the enumeration, and by the superset argument of Section 3 the maximum bounds $\overline{\max} e_1(K_n)$ from above. The searches visit $2, 8, 62, 908, 24\,698$ chirotopes for $n = 4, \dots, 8$ and return $2, 2, 6, 7, 10$; the $n = 9$ search visits 1 232 944 chirotopes and returns 12. These are verified in Lean. The $n = 10$ search visits 112 018 190 chirotopes and returns 14; it is an external computation of about ten minutes, not a machine-checked theorem, and is labelled as such wherever it is used.

Upper and lower meet at each of the seven values. □

Remark 5.2. All eight values of Theorems 5.1 and 5.3 lie inside the published bracket (1) at every $n \geq 8$ where it is stated; a witness outside it would have meant the model implemented is not the source's.

The value at $n = 11$ is out of reach of a flat enumeration — there are of the order of 10^{10} normalized chirotopes on eleven elements — and is obtained by extension.

Theorem 5.3. $\overline{\max} e_1(K_{11}) = 14$*. In particular* $\overline{\max} e_1(K_n) \neq 2n - 6$ *at* $n = 11$*, so the pattern observed at* $n = 8, 9, 10$ *does not continue.*

Proof sketch. The lower half is unconditional: D_{11} is a set of eleven integer points, in general position and generic, with $e_1 = 14$ (Table 1).

For the upper half, $\delta(11, v) \geq 9$ for every $v \geq 14$, i.e. $\mathrm{Cov}(11, 9, 14)$ holds; this is decided over the whole range of v and depends on no axioms. So by Lemma 4.1 every eleven-element chirotope with $e_1 \geq 14$ arises as a single-element extension of a ten-element chirotope with $e_1 \geq 9$. There are 278 620 such, namely 0.249% of the 112 018 190; extending each of them at all ten insertion positions, with Grassmann–Plücker pruning, produces 447 977 762 leaves in 49 minutes, and the largest e_1 among them is 14. Note that the threshold is exactly right and not merely safe: $\mathrm{Cov}(11, 10, 14)$ is false, so 9 cannot be raised. Because $\mathrm{Cov}(11, 9, 14)$ covers the value 14 itself and not only everything above it, the pass *computes* the maximum rather than only bounding it. Hence no eleven-element uniform rank-3 chirotope has $e_1 > 14$, and with Lemma 3.1 and the superset argument, $\overline{\max} e_1(K_{11}) \leq 14$. □

The eleven-element search is an external computation; what supports it is that its machinery is checked wherever an independent answer exists. The following two propositions record those checks.

Proposition 5.4 (Two-sided gates)**.** *At every size where the answer is already proved, the same filtered pass, at the threshold the same arithmetic licenses, returns that answer:* $\overline{\max} e_1(K_8) = 10$ *from* 260 *of the* 908 *seven-element chirotopes (*49 280 *leaves),* $\overline{\max} e_1(K_9) = 12$ *from* 2 166 *of the* 24 698 *eight-element ones (*853 766 *leaves), both in Lean, and* $\overline{\max} e_1(K_{10}) = 14$ *from* 12 866 *of the* 1 232 944 *nine-element ones (*10 284 179 *leaves) externally in* 41 *seconds. Each gate is two-sided: at those thresholds the pass reaches every chirotope at or above the optimum, so it must find the known value and not merely fail to beat it.*

Proposition 5.5 (Controls at $n = 11$)**.** *Let W_{11} be the chirotope of D_{11} after the relabelling of Lemma 3.1, computed in exact integer arithmetic, and X_{11} the maximizing chirotope returned by the eleven-element pass.*

(1) *W_{11} is realizable and has $e_0 = 6$, $e_1 = 14$ and* 142 *crossings, reproducing the values computed from the coordinates of D_{11} by a computation sharing no code with them.*
(2) *Of the* 165 *single sign flips of W_{11},* 141 *violate a Grassmann–Plücker relation — so the validity test is a real constraint and not one that accepts arbitrary sign vectors — and all* 24 *flips that give another chirotope have $e_1 \leq 13$, so W_{11} is a strict local maximum in the flip neighbourhood.*
(3) *X_{11} is a normalized chirotope satisfying every Grassmann–Plücker relation, with $e_0 = 6$, $e_1 = 14$ and* 130 *crossings. Since the number of crossings is a relabelling invariant, X_{11} is not a relabelling of W_{11}: the search does not return its own input.*
(4) *No nine-element chirotope has $e_1 = 13$ (too-large refuted); the filter discards, keeping* 2 166 *of* 24 698*; and past the threshold the deletion bound licenses, the filtered pass keeps nothing at all, so the threshold is load-bearing.*

Two by-products of the eleven-element pass: exactly six normalized ten-element chirotopes attain $\overline{\max} e_1(K_{10}) = 14$, and exactly 96 eleven-element ones attain 14.

6. The cell $n = 12$

At $n = 12$ the deletion bound is chained: to see every twelve-element chirotope with $e_1 \geq V$ it suffices to keep the eleven-element chirotopes with $e_1 \geq t_1$ and, for those, the ten-element ones with $e_1 \geq t_0$, where $\mathrm{Cov}(12, t_1, V)$ and $\mathrm{Cov}(11, t_0, t_1)$ both hold. Writing $\mathrm{Chain}(V) = (t_1, t_0)$ for the licensed thresholds, the price list at $n = 12$ is

$$\mathrm{Chain}(15, \ldots, 22) = (10, 6),\ (11, 7),\ (11, 7),\ (12, 8),\ (13, 8),\ (13, 8),\ (14, 9),\ (14, 9),$$

and the number of ten-element chirotopes with $e_1 \geq 7, 8, 9$ is 4 241 700, 1 225 796, 278 620 respectively. Two rungs of this ladder were run.

Theorem 6.1. $\overline{\max} e_1(K_{12}) \leq 20$.

Proof sketch. $\mathrm{Cov}(12, 14, 21)$ and $\mathrm{Cov}(11, 9, 14)$ both hold, decided over the whole range of e_1, so extending the 278 620 ten-element chirotopes with $e_1 \geq 9$, keeping the eleven-element children with

$e_1 \geq 14$, and extending those, reaches every twelve-element chirotope with $e_1 \geq 21$. The pass produced 447 977 762 first-level leaves — the same number as the eleven-element run of Theorem 5.3, reproduced by a different program — 96 distinct eleven-element intermediates with $e_1 = 14$, and 302 756 second-level leaves, in 45 minutes; the largest e_1 of any of them is 15. So no twelve-element uniform rank-3 chirotope has $e_1 \geq 21$, whence the bound by Lemma 3.1 and the superset argument. The chain does not cover $e_1 = 20$, so ≤ 20 is exactly what this rung licenses. □

Theorem 6.2. $\overline{\max} e_1(K_{12}) \leq 17$. *Together with the twelve-point drawing D_{12} of Theorem 7.1 this gives* $16 \leq \overline{\max} e_1(K_{12}) \leq 17$, *where* (1) *gives only the bracket* $[11, 36]$.

Proof sketch. Here $\delta(12, 18) = 12$ and $\delta(11, 12) = 8$, so $\mathrm{Cov}(12, 12, 18)$ and $\mathrm{Cov}(11, 8, 12)$ hold and the chain reaches every twelve-element chirotope with $e_1 \geq 18$ from the 1 225 796 ten-element chirotopes with $e_1 \geq 8$, which are 1.094% of all of them and 4.40 times as many as the previous rung used. Both thresholds are sharp — 13 fails at the upper level and 9 at the lower — and the chain does not cover $e_1 = 17$, so this rung licenses ≤ 17 and not the exact value. The pass was run in nine pieces of the enumeration range, at most two at a time, the longest 42.5 minutes, for 5.49 CPU-hours in total; it produced 1 979 273 436 eleven-element and 946 310 408 twelve-element leaves, and the largest e_1 of any of them is 16, in every piece. By Proposition 4.6 the maximum over pieces is the maximum, and the nine index ranges are verified to tile the whole enumeration range consecutively, so every starting chirotope was seen exactly once; the pieces' filtered counts sum to 1 225 796, which is independently the number of ten-element chirotopes with $e_1 \geq 8$. Hence no twelve-element uniform rank-3 chirotope has $e_1 \geq 18$. □

Proposition 6.3 (The rung shape, validated)**.** *The rung of Theorem 6.2 has the shape: target* $V =$ (*known maximum*) $+ 1$, *so that the chain covers everything strictly above the witness and finding nothing there proves* $\leq V - 1$. *At every size where the maximum is already proved this shape is two-sided and returns that maximum:* 10 *at* $n = 8$ *and* 12 *at* $n = 9$ *entirely inside Lean (the* $n = 9$ *gate keeps* 496 *of* 908 *seven-element chirotopes, produces* 94 030 *first-level and* 1 825 944 *second-level leaves),* 14 *at* $n = 10$ *externally in* 63 *seconds, and* 14 *at* $n = 11$. *The* $n = 12$ *run carries its own gate: its first level reaches every eleven-element chirotope with* $e_1 \geq 14$, *so its first-level maximum had to be exactly* $\overline{\max} e_1(K_{11}) = 14$, *and it was, in all nine pieces. Two de-duplication counts land on independently computed chirotope counts:* 686 *at the* $n = 9$ *gate (the* $e_1 \geq 7$ *tail of the eight-element histogram, with* 269 466 *second-level leaves, matching a one-level computation leaf for leaf) and* 2 808 *at the* $n = 10$ *gate (the* $e_1 \geq 9$ *tail of the nine-element histogram). Controls: no nine-element chirotope has* $e_1 = 13$; *the filter discards at both levels; and past the licensed threshold the second level keeps nothing, so the threshold is load-bearing.*

Proposition 6.4 (The price of the exact cell)**.** *Deciding between* 16 *and* 17 *requires the target* $V = 17$, *which forces* $\delta(12, 17) = 11$ *and then* $\delta(11, 11) = 7$: *the chain must start from the* 4 241 700 *ten-element chirotopes with* $e_1 \geq 7$, *not from the* 278 620 *with* $e_1 \geq 9$, *a factor of* 15.2. *Indeed* $\mathrm{Cov}(11, 9, 11)$ *is false, so no extension of the smaller starting set can license the exact value. The cost is about* 14 *CPU-hours as a single process and, by Proposition 4.6, about* 20 *CPU-hours split into pieces.*

Proposition 6.5 (Chirotopes attaining 16)**.** *Let* W_{12} *be the chirotope of* D_{12} *after the relabelling of Lemma 3.1. It is realizable, with* $e_0 = 6$, $e_1 = 16$ *and* 240 *crossings, reproducing the values computed from the coordinates of* D_{12} *by a computation sharing no code with them. Of its* 220 *single sign flips,* 191 *violate a Grassmann–Plücker relation and all* 29 *that give another chirotope have* $e_1 \leq 15$, *so* W_{12} *is a strict local maximum in the flip neighbourhood. The two-level filter discards at both levels — at the eight-element gate,* 39 *of* 62 *and then* 784 *of* 3 422 *— and past the threshold the deletion bound licenses, the second level keeps nothing at all and the pass returns nothing.*

Proposition 6.6. *The pass of Theorem 6.2 reaches* 486 *twelve-element leaves with* $e_1 = 16$. *One of them,* Z_{12}, *is exhibited: a normalized chirotope satisfying all* $12\binom{11}{4}$ *Grassmann–Plücker relations, with* $e_0 = 6$, $e_1 = 16$ *and* 237 *crossings. Since* W_{12} *has* 240 *crossings and that count is a relabelling invariant,* Z_{12} *is not a relabelling of* W_{12}. *We make no claim that* Z_{12} *is realizable, so it does not move the lower bound; its role is to exhibit an object attaining the run's maximum.*

7. Drawings: lower bounds for $11 \le n \le 16$

Theorem 7.1. $\overline{\max}e_1(K_n) \geq$ 14, 16, 17, 19, 19 *for* $n = 11, 12, 13, 14, 15$*. Each of these exceeds the published lower bound* $\lceil 3n/2 \rceil - 7$*, whose values there are* $10, 11, 13, 14, 16$.

Theorem 7.2. $\overline{\max}e_1(K_{16}) \geq 21$*, against* $\lceil 3 \cdot 16/2 \rceil - 7 = 17$.

Both theorems are witnessed by explicit sets of integer points verified to be in general position and generic; they use neither Lemma 3.1 nor any exhaustion. The full profiles are:

Table 1. The witnesses. Every one is verified to be in general position and generic.

drawing	e_0	e_1	e_2	e_3	S_1	crossings
D_4	4	2	0	0	6	1
D_5	6	2	2	0	8	3
D_6	9	6	0	0	15	3
D_7	9	7	4	1	16	9
D_8	8	10	2	4	18	22
D_9	6	12	5	2	18	42
D_{10}	6	14	0	8	20	84
D_{11}	6	14	3	3	20	142
D_{12}	6	16	2	4	22	240
D_{13}	6	17	2	3	23	361
D_{14}	8	19	0	7	27	609
D_{15}	8	19	2	6	27	857
N_{16}	8	21	2	5	29	1 134

The drawings for $n \le 15$ were found by simulated annealing over integer coordinates on a 48×48 or 64×64 grid with restarts and reheating, followed for $n \ge 11$ by an incremental pass seeded from the best $(n-1)$-point set and extended by an exhaustive scan for the added point. All of them were found to be generic as produced.

The sixteen-point drawing needed a different instrument, and the reason is visible in its coordinates: fourteen of its points have coordinates in the low hundreds and one is at $(-7916, -6828)$. A drawing spanning two scales like that cannot be represented on a 64×64 grid at all, and the source's own construction asks for exactly such a spread ("far away", "sufficiently close", "very close" occur in one paragraph of it). What replaces the grid is the observation that the position of a single point is a *finite* problem: the order type of $P \cup \{v\}$ depends only on which cell of the arrangement of the $\binom{n}{2}$ lines through pairs of P contains v, and every bounded cell has a vertex of that arrangement on its boundary. Probing $v = x + \varepsilon u$ for each arrangement vertex x and one direction u per sector of the line directions, with ε symbolic so that no rounding enters, together with the unbounded wedges as points at infinity, enumerates every cell. The winning cell is then realized as an integer point by scaling up until a lattice point lands in it and halving back while the order type is unchanged, so the output is a genuine point set. Applying this to D_{15} gives a sixteen-point drawing with $e_1 = 20$ — optimal given D_{15}, not merely the best found — and then repeatedly removing one point and reinserting it at the best position in the plane reaches 21.

Remark 7.3. An earlier sixteen-point set with $e_1 = 20$ was found by annealing but was *not* generic, and perturbation could not repair it without losing the value; it is therefore not a drawing in the sense of Definition 2.1. The witness of Theorem 7.2 is generic and one better.

The same exhaustive one-point move was applied at $n = 11$ without improving on 14, which Theorem 5.3 explains after the fact.

Proposition 7.4. *There is a second eleven-point drawing* N_{11}*, in general position and generic, with* $e_1 = 14$*, found independently of* D_{11}*. Its profile agrees with* D_{11}*'s in* e_0*,* e_1*,* e_2*,* e_3 *and* S_1*, but it has* 144 *crossings against* D_{11}*'s* 142*; since the number of crossings is an order-type invariant, the two drawings have different order types.*

The searches that led to N_{11} — exhaustive one-point extension, ruin-and-recreate, beam search over exhaustive extensions of pools of 657 and 2 500 ten-point drawings, and a two-point deep sweep — are external computations and are not part of any proof here. They are worth one sentence only because their outcome was informative: D_{10}, the ten-point optimum, admits no extension above $e_1 = 14$ over every position in the plane, and neither did any drawing in the pools. On the control side, the exhaustive one-point move strictly beats a ± 120 to ± 150 integer grid scan of the same objective at $n = 7, 8, 9, 10$ (10 against 9, 11 against 10, 12 against 11, 14 against 13), and a beam search built only from that move rediscovers $\overline{\max} e_1(K_{10}) = 14$ from random six-point sets.

8. Published values reproduced

Nothing new was read off the enumeration until it reproduced what is known. The following two propositions claim no new mathematics; they are the evidence that the predicate implemented is the one in the definitions.

Proposition 8.1. *The enumeration of Section 3 reproduces three published columns, with nothing to tune.*

1. $\overline{\max} e_0(K_n) = 2n - 2$ *(Károlyi–Welzl, via* [3]*): the enumerated maximum of* e_0 *is* $6, 8, 10, 12, 14$ *at* $n = 4, \dots, 8$ *and* 16 *at* $n = 9$*, stated against the formula* $2n - 2$ *rather than as five separate coincidences.*
2. $\overline{\min} e_0(K_n)$ *(Harborth–Thürmann* [5]*): the enumerated minimum of* e_0 *is* $3, 4, 5, 5, 6$ *at* $n = 3, \dots, 7$*, matching the published table entry for entry, and* 5 *at* $n = 8$ *and* $n = 9$*, matching the published value for* $n \geq 8$*. This is the more discriminating of the two, because a minimum over the same space cannot be reproduced by a lucky construction the way a maximum can.*
3. *Aichholzer's order-type census* [1]*: collapsing the enumerated chirotopes modulo relabelling and reorientation gives* $1, 2, 3, 16, 135$ *classes at* $n = 3, \dots, 7$*, exactly the published counts. This is the sharpest of the three, because it tests the Grassmann–Plücker implementation directly rather than through a crossing count; it is an external computation. The count* 3315 *at* $n = 8$ *was not recomputed.*

A fourth, weaker check: rectilinear drawings are a subset of all drawings, so $\overline{\max} S_1(K_n) \leq \max S_1(K_n)$*, whose values* $18, 20, 22$ *at* $n = 7, 8, 9$ *are Harborth–Mengersen's* [4]*. The enumeration gives* $16, 18, 19$*, and gives* $\overline{\max} S_1(K_n) = \binom{n}{2}$ *exactly at* $n = 4, 5, 6$*, as it must.*

Proposition 8.2. *The value* $\overline{\max} e_0(K_n) = 2n - 2$ *is attained at every* n *with* $4 \leq n \leq 16$ *by the following drawing: a hub at the origin together with the points* $q_i = (i, N + i^4)$*,* $1 \leq i \leq n - 1$*, where* $N > 3(n-1)^4$*. Its uncrossed edges are the* $n - 1$ *hub edges, the* $n - 2$ *consecutive edges* $q_i q_{i+1}$ *and the long chord* $q_1 q_{n-1}$ *— a wheel, with* $(n-1) + (n-1) = 2n - 2$ *edges.*

The construction was not read off the figure in [3]. It was recovered from the optimizer: annealing for e_0 at $n = 4, 6, 8, 10, 12$ and then examining the optima gave the same shape every time — convex hull a triangle, uncrossed edges a wheel with one hub of degree $n - 1$ and every other vertex of degree 3 — and a wheel has $2n - 2$ edges, so the extremal graph explains the formula. Both constants are forced rather than tuned. The condition $N > 3(n-1)^4$ makes the angular order of the cup around the hub equal the index order, without which the count drops. The exponent must be 4 and not 2: the natural parabolic cup $q_i = (i, N + i^2)$ gives the right *count* at every n but stops being generic at $n = 10$, because three chords of a parabola through integer abscissae become concurrent, which Definition 2.1 forbids. That failure is invariant under changing N, since translating the cup translates its chords, so no search over N could have repaired it; the curve itself had to change.

9. Negative controls

A search that proves an extremal value can be satisfied by a misplaced quantifier, in which case every computation still succeeds while proving nothing. The following are the checks that exclude the standard failures.

Proposition 9.1. (1) The predicate is not vacuous. *Four points, three of them on the line $y = x$, are rejected as not in general position — and their* e_0 *is still computed as* 6*, so it is the predicate and not the counter that excludes them.*

(2) Too weak, in two independent directions. *Points in convex position give* $e_0 = n$ *exactly, at* $n = 6, 7, 8$*, so* $2n - 2$ *does not fall out of an arbitrary drawing and the construction of Proposition 8.2 is carrying the result. And the two optimization problems are genuinely different: the* e_0*-optimal wheel has* $e_1 = 0$*, while the* e_1*-optimal* D_8 *has* $e_0 = 8$*, six short of the maximum.*

(3) Too strong. *No eight-point drawing has* $e_0 = 15$ *or* $e_1 = 11$*, one more than each claimed value, which is what stops the values being lower bounds quoted as maxima; and* $e_0 \leq 13$ *is refuted at* $n = 8$*, so the maximum is not vacuously small either. The same pair of refutations is repeated at* $n = 7$*.*

(4) Perturbation. *Moving one coordinate of* D_8 *or of* D_{10} *by a single unit drops* e_1 *from* 10 *to* 9 *and from* 14 *to* 12*: the values are properties of the configuration, not of the grid it was found on.*

(5) The construction control. *The parabolic cup at* $n = 10$ *attains the same* $e_0 = 18$ *as the quartic one and is in general position, but is not generic. This is the content of the choice of exponent in Proposition 8.2, kept as a theorem so that the near miss cannot be quietly undone.*

Proposition 9.2. *The source defines a crossing as a* point *shared by two edges, so* E_1 *is the set of edges lying on exactly one such point, whereas counting edge-pairwise gives the edges crossed by exactly one other edge. On a generic drawing — which is what Definition 2.1 admits — the two readings agree. The point-based reading was computed directly, from the exact homogeneous intersection points of the arrangement, at all twelve drawings* $D_4, \ldots, D_{15}$*, giving* $2, 2, 6, 7, 10, 12, 14, 14, 16, 17, 19, 19$*, and at* N_{11} *and* N_{16}*, giving* 14 *and* 21*, together with* $e_0 = 30$ *for the sixteen-point wheel. So every value in this paper holds with* e_1 *read as in* [3]*, checked rather than inferred.*

Proposition 9.3. *Genericity is load-bearing, and in the direction one would expect. There is a set of eight points in general position, three of whose edges pass through one interior point, for which the edge-pairwise count is* 8 *and the point-based count is* 9*: the two readings of* e_1 *come apart exactly off the admitted class, and dropping the predicate makes the count larger. We did not find a non-generic configuration whose point-based* e_1 *exceeds a proved maximum; four minutes of annealing directly on that count at* $n = 8$ *never exceeded* $\overline{\max} e_1(K_8) = 10$*. This is recorded as a gap and not as a theorem. At* $n = 16$ *the family's standard controls hold as well: convex position and the* e_0*-optimal wheel both give* $e_1 = 0$ *against the witness's* 21*, one unit of movement drops* e_1*, and a collinear triple is rejected.*

10. Verification

Every statement above other than the external computations named as such is machine-checked in Lean 4 with Mathlib, sorry-free; the sources are available in a repository that is currently private, and a repository reference is to be added at submission. The threshold arithmetic (the predicate Cov and every chained licence, Sections 4 and 6) is decided over the whole range of e_1 and depends on no axioms at all; the finite searches (the flat enumerations to $n = 9$, the completeness identities, the gates, the deletion-bound checks, the explicit chirotopes and every witness report) each carry one per-declaration compiled-evaluation axiom, together with the standard propext, Quot.sound and, where a chirotope is parsed from a string, Classical.choice. The computations too large for the Lean evaluator were performed in C and are labelled external wherever they are used; the four that carry results are the flat enumeration at $n = 10$, the eleven-element extension of Theorem 5.3 (49 minutes), the two-level passes of Theorems 6.1 and 6.2 (45 minutes and 5.49 CPU-hours), and the order-type census of Proposition 8.1(3). For each of them a second, independently written implementation of the same search is checked in Lean at every size it can reach and agrees; the completeness identities, the two-sided gates and the de-duplication counts of Propositions 4.3, 4.4, 5.4 and 6.3 are the tests those implementations pass. No order-type database was used at any point: the chirotopes are generated from the Grassmann–Plücker relations.

Besides the external searches, the results depend on exactly two steps that are argued rather than machine-checked: Lemma 3.1, and the characterization of uniform rank-3 chirotopes by alternation and the 3-term Grassmann–Plücker relations, which is quoted from [2]. Both are used only for upper bounds. All lower bounds — Theorems 7.1 and 7.2, and the lower halves of Theorems 5.1, 5.3 and 6.2 — are witnesses and are unconditional.

11. Open questions

(1) $n = 12$. Is $\overline{\max}e_1(K_{12})$ equal to 16 or to 17? By Proposition 6.4 the exhaustion that decides this is about 20 CPU-hours in the form used here. A single twelve-point drawing with $e_1 = 17$ would settle it at once, and the search that has not been tried is local search in *chirotope* space — moves that flip one triple and repair the Grassmann–Plücker relations — rather than in coordinate space, where the landscape past $n \approx 10$ is visibly hard to move around in.

(2) *The limit.* The limit half of Problem 1.1 is untouched. It remains pinned to $[1.5, 2.5]$ by (1); the pattern $2n-6$ that would have answered it with 2 is dead at $n = 11$, and the values computed here are too few and too irregular to suggest a replacement.

(3) *Lemma 3.1.* A formal proof would remove the last unformalized step from every upper bound above. It is the standard sort-by-angle argument; the work is in sorting lemmas rather than in geometry.

(4) *Beyond $n = 12$.* A third level of the deletion chain is available in principle, but the thresholds it forces grow the starting set faster than the chain prunes; a better pruning, not more hours, is what $n \geq 13$ needs.

References

[1] O. Aichholzer, F. Aurenhammer and H. Krasser, *Enumerating order types for small point sets with applications*, Order **19** (2002), no. 3, 265–281. This is the reference given for the associated order type database, `http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/ordertypes/`, which supplies realizing point sets for all order types of $3 \leq n \leq 10$ points; the extension of the census to $n = 11$ is not used here. The published counts $1, 2, 3, 16, 135, 3315$ for $n = 3, \dots, 8$ are used for validation; the database itself is not used.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999. Theorem 3.6.2 is the local characterization: an alternating sign map satisfies the Grassmann–Plücker relations as soon as its support is the set of bases of a matroid and it satisfies the 3-term relations. In the uniform rank-3 case the support is every triple, so the matroid condition is automatic and the theorem reads: alternation together with the 3-term relations characterizes uniform rank-3 chirotopes.

[3] I. Chen and O. Solé-Pi, *On the crossing profile of rectilinear drawings of K_n*, arXiv:2501.04980 (9 January 2025).

[4] H. Harborth and I. Mengersen, *Edges with at most one crossing in drawings of the complete graph*, in: R. Bodendiek and R. Henn (eds.), Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel, Physica-Verlag, Heidelberg, 1990, pp. 757–763. Source of $\max S_1(K_n)$ for general drawings — $18, 20, 22$ at $n = 7, 8, 9$ — and hence of the upper bound in (1). Print-only; quoted through [3] and not consulted directly.

[5] H. Harborth and C. Thürmann, *Numbers of edges without crossings in rectilinear drawings of the complete graph*, Congressus Numerantium **119** (1996). Source of $\overline{\min}e_0(K_n) = 3, 4, 5, 5, 6$ for $3 \leq n \leq 7$ and $= 5$ for $n \geq 8$. Print-only; quoted through [3] and not consulted directly. Secondary sources disagree on the page range, so none is given here.

[6] Gy. Károlyi and E. Welzl, *Crossing-free segments and triangles in point configurations*, Discrete Applied Mathematics **115** (2001), no. 1–3, 77–88. Cited in [3] for $\overline{\max}e_0(K_n) = 2n - 2$ for rectilinear drawings.

[7] G. Ringel, *Extremal problems in the theory of graphs*, in: Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., Prague, 1964, pp. 85–90. Cited in [3] for $\max e_0(K_n) = 2n - 2$ over all drawings.