CRITICAL PERMUTATIONS OF THE SYMMETRIC GROUP: A LOCAL TEST FOR THE FIRST DEGREE, AND THE FIRST-DEGREE COUNTS AT $n = 8$ AND $n = 9$

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Abstract. For τ in the symmetric group S_n let $\Lambda(\tau) = \{\sigma : \sigma \leq \tau\}$ be its principal ideal in the Bruhat order and let $\lambda_j(\tau)$ be the number of elements of $\Lambda(\tau)$ of Coxeter length j. Following Pučinskaitė, τ is *critical of degree* i when i is the least index with $\lambda_i(\tau) \neq \lambda_{\ell(\tau)-i}(\tau)$, and *critical* when such an i exists; equivalently, when the rank generating function of the lower Bruhat interval $[\mathrm{id}, \tau]$ is not palindromic. The set $\mathcal{C}_n$ of critical permutations is identified in that paper with $\{q \in S_n : [M(\mathrm{id}_n) : L(q)] \geq 2\}$ in the principal block of $\mathcal{O}(\mathfrak{sl}_n(\mathbb{C}))$, and its table of $|\mathcal{C}_n|$ stops at $n = 7$. We prove that criticality *of the first degree* is decided by a condition local to τ: the two quantities the criterion compares are the number of smaller neighbours of τ and the number of indices j with $\tau(\{1, \ldots, j\}) \neq \{1, \ldots, j\}$, so no principal ideal need be built. This yields the number of permutations critical of degree 1 at $n = 8$ and at $n = 9$, namely 32 915 and 329 513; the second is out of reach of the ideal-based route. The set $\mathcal{C}_n$ itself is classical — palindromicity of that rank generating function is the Carrell–Peterson criterion for rational smoothness, so by Lakshmibai–Sandhya $\mathcal{C}_n$ is the set of permutations containing 3412 or 4231, and $|\mathcal{C}_n|$ is $n!$ less a known sequence — but that description is silent about the *degree*, and neither count above follows from it. We also adjudicate an internal inconsistency in the source, which prints both 354 and 345 for $|\mathcal{C}_6|$: the value is 354, as the source's own arithmetic and the classical count both show. Every theorem, proposition, lemma and corollary below is machine-checked in Lean 4; the few numerical remarks that are not are flagged where they occur.

1. Introduction

The Bruhat order on the symmetric group S_n is graded by the Coxeter length ℓ, the number of inversions, so every principal ideal

$$\Lambda(\tau) \;=\; \{\sigma \in S_n : \sigma \leq \tau\} \;=\; [\mathrm{id}, \tau]$$

carries a rank generating function $\sum_j \lambda_j(\tau)\, q^j$, where $\lambda_j(\tau) = |\{\sigma \leq \tau : \ell(\sigma) = j\}|$. That polynomial need not be palindromic, and the first index at which the symmetry $\lambda_j = \lambda_{\ell(\tau)-j}$ fails is a natural invariant of τ. Pučinskaitė [1] makes it the object of study.

Definition 1.1 ([1, 2.2])**.** $\tau \in S_n$ is a *critical permutation of degree* i if $\lambda_i(\tau) \neq \lambda_{\ell(\tau)-i}(\tau)$ and $\lambda_j(\tau) = \lambda_{\ell(\tau)-j}(\tau)$ for every j with $0 \leq j < i$; and τ is a *critical permutation* if it is critical of degree i for some $0 \leq i \leq \ell(\tau)$. Write $\mathcal{C}_n \subseteq S_n$ for the set of critical permutations.

The interest of $\mathcal{C}_n$ is representation-theoretic. In the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_n(\mathbb{C})$, [1, §3] identifies

$$\mathcal{C}_n \;=\; \{\, q \in S_n \;:\; [M(\mathrm{id}_n) : L(q)] \;\geq\; 2 \,\},$$

that is, the permutations q for which the simple module $L(q)$ occurs with multiplicity at least two in the Verma module $M(\mathrm{id}_n)$. The paper also records [1, §2] that no link is currently known between the *degree* of a critical permutation and a finer understanding of those multiplicities, so the degree is at present a purely combinatorial refinement — and that is the level at which this paper works. Nothing below uses category $\mathcal{O}$; the predicate of Definition 1.1 is a statement about the Bruhat order alone.

What is known. The source assembles the following table [1, 2.4.1–2.4.4, 3.5, 3.6], the values at $n \le 5$ from the literature (the value 32 at $n = 5$ is credited in [1, 2.4.3] to Jantzen [2]) and those at $n = 6, 7$ from a computer program of Elbosari [10]: $|\mathcal{C}_n| = 0$ for $n \le 3$, then

$$|\mathcal{C}_4| = 2, \qquad |\mathcal{C}_5| = 32, \qquad |\mathcal{C}_6| = 354, \qquad |\mathcal{C}_7| = 3\,488,$$

together with the splits by degree — both elements of $\mathcal{C}_4$ are of degree 1; of the 32 elements of $\mathcal{C}_5$, exactly one, (45312), is of degree 2; of the 354 elements of $\mathcal{C}_6$, twelve are of degree 2 and exactly one, $q_6 = (564312)$, is of degree 3; of the 3 488 elements of $\mathcal{C}_7$, ninety-six are of degree 2, twelve of degree 3 and exactly one, $q_7 = (6754312)$, of degree 4. (One-line notation throughout, as in [1]: (3412) denotes the permutation sending $1, 2, 3, 4$ to $3, 4, 1, 2$, not a cycle.) One observation is left open there [1, end of 2.4]: an explicit two-line formula produces, for each $n \ge 4$, a permutation of S_n — namely (3412), (45312), (564312), (6754312) at $n = 4, 5, 6, 7$ — which the source checks to be critical of degree $n - 3$, and, for $n = 5, 6, 7$, to be the *only* critical permutation of that degree; on that evidence it "suggests the possibility that a critical permutation q_i is of degree i for all $i \ge 2$" (emphasis normalised). The subscript in that display is the *degree*: the family is written $q_{n-3} \in S_n$. At [1, 3.5 and 3.6] the S_6 and S_7 members of the same family are instead written q_6 and q_7, indexed by n; we follow that second usage when naming those two permutations, as the source does there. Beyond $n = 7$ the table stops.

$\mathcal{C}_n$ is a pattern class. Definition 1.1 says that τ is critical exactly when the rank generating function $P_\tau(q) = \sum_j \lambda_j(\tau) q^j$ of $[\mathrm{id}, \tau]$ — the Poincaré polynomial of the Schubert variety X_τ — fails to be palindromic. That failure has a classical description. Palindromicity of P_w is Carrell and Peterson's criterion for rational smoothness of X_w [5]; smoothness of X_w for $w \in S_n$ is avoidance of the two patterns 3412 and 4231, by Lakshmibai and Sandhya [4]; and in type A rational smoothness and smoothness agree, by Deodhar [6]. The whole list of equivalent conditions is collected in [7, §4], where the two used here are items (1) and (5) of its first pattern-avoidance property. Hence

$$\mathcal{C}_n \;=\; \{\, w \in S_n : w \text{ contains 3412 or 4231} \,\},$$

and $|\mathcal{C}_n| = n! - a_n$ with a_n the number of *smooth* permutations of S_n, which is [11, A032351] and has a known algebraic generating function (Haiman, unpublished; first published proof in [8], see also [7, Cor. 3.7]). This delivers the last column of Table 1 outright:

$$|\mathcal{C}_8| = 40\,320 - 6\,652 = 33\,668, \qquad |\mathcal{C}_9| = 362\,880 - 28\,696 = 334\,184.$$

We checked the identification directly — permutation by permutation for $n \le 7$, and by comparing counts for $n = 8, 9$ — outside the formal development. (The two patterns are exactly the two elements $x_1 = (3412)$ and $x_2 = (4231)$ of $\mathcal{C}_4$ that [1, 2.4.2] names.) [1] mentions none of this literature, and it is worth being explicit about what the pattern description leaves open, since that is what the rest of this paper is about. The pattern description says nothing about the *degree* at which palindromicity first fails, and the degree is not inherited along pattern containment: (45312) contains 3412, and 3412 is critical of degree 1, yet (45312) is critical of degree 2 and not of degree 1 (Proposition 8.3). So the by-degree columns of Table 1 are not a consequence of it, and neither is the arithmetic error of Section 4, which the source makes while its own printed data already exclude it.

Results. This paper extends the table in the first degree by two values and isolates the structural reason why that is possible.

A local test (Sections 3 and 6). Write $\mathbf{N}_{\mathrm{s}}(\tau) = \{\sigma : \sigma \lessdot \tau\}$ for the set of smaller neighbours of τ and $\mathbf{L}_{\mathrm{t}}(\tau) = \{\sigma \le \tau : \ell(\sigma) = 1\}$ for the simple transpositions below it. The source already observes that τ is critical of degree 1 exactly when $|\mathbf{N}_{\mathrm{s}}(\tau)| \ne |\mathbf{L}_{\mathrm{t}}(\tau)|$ [1, §2 and §4; also its 4.1.2(4)]. We show that both sides can be read off τ itself. For $|\mathbf{N}_{\mathrm{s}}(\tau)|$ this is Proposition 3.1, which identifies the level just below the top of $\Lambda(\tau)$ with $\mathbf{N}_{\mathrm{s}}(\tau)$; for $|\mathbf{L}_{\mathrm{t}}(\tau)|$ it is the parabolic criterion

$$s_j \le \tau \quad \Longleftrightarrow \quad \tau(\{1, \dots, j\}) \ne \{1, \dots, j\} \qquad (1 \le j \le n-1),$$

Theorem 6.4, a classical fact that we re-derive from the covering relation itself. The resulting test (Theorem 7.2) builds no ideal and stores nothing.

Two new values (Sections 5 and 7). Exactly 32 915 permutations of S_8 and exactly 329 513 permutations of S_9 are critical of degree 1. The value at $n = 8$ is obtained twice, once through a certificate-checked memoised principal ideal (Theorem 5.1) and once through the local test (Theorem 7.2); the value at $n = 9$ is out of reach of the first route. Both theorems also record the consequence $|\mathcal{C}_8| \geq 32\,915$, $|\mathcal{C}_9| \geq 329\,513$, which is what the formal development proves about the totals; the exact totals are the classical ones displayed above, and the bounds are of no independent interest.

An erratum (Section 4). The source prints two different values for $|\mathcal{C}_6|$. The correct one is 354, which is what it prints twice; the isolated 345 is inconsistent with the paper's own complement count and with the classical count $720 - 366$, and $|\mathcal{C}_6| = 354$ is verified here from the definition.

n	$c_n(1)$	$c_n(2)$	$c_n(3)$	$c_n(4)$	$c_n(5)$	$\lvert\mathcal{C}_n\rvert$
4	2					2
5	31	1				32
6	341	12	1			354
7	3 379	96	12	1		3 488
8	**32 915**	(644)	(96)	(12)	(1)	33 668
9	**329 513**	—	—	—	—	334 184

TABLE 1. The number $c_n(i)$ of permutations of S_n critical of degree i, and the total. The last column is classical throughout: $|\mathcal{C}_n| = n! - a_n$ with a_n as in [11, A032351]. Its entries at $n \leq 7$, and the by-degree entries at $n \leq 7$ other than the two first-degree counts $c_6(1) = 341$ and $c_7(1) = 3\,379$, appear in [1]; the two exceptions are forced by its printed splits ($354 - 12 - 1$ and $3\,488 - 96 - 12 - 1$). Every entry at $n \leq 7$ is verified here (Section 4). The two boldface entries are the new values, Theorems 5.1 and 7.3. Entries in parentheses are computations made outside the formal development (Remark 9.1); a dash means not computed. Empty cells are zero.

Method. Every count below is an exhaustive search over all of S_n, and the searches are stated against the definitions verbatim: the Bruhat order is the reflexive-transitive closure of the source's own covering relation, and $\lambda_j(\tau)$ is a cardinality of a set of words with no finiteness assumed in advance. In particular Ehresmann's rank-matrix criterion — the usual route to a decidable Bruhat order — is not used anywhere, and neither is the subword property nor the pattern description of $\mathcal{C}_n$. This is a deliberate constraint: it keeps the object being counted equal to the object the source defines, at the cost of having to prove the two structural facts of Sections 3 and 6 by hand.

2. Preliminaries

We work with finite sequences of natural numbers, called *words*, and identify $\tau \in S_n$ with its one-line word $(\tau(1), \ldots, \tau(n))$. Several statements below hold for arbitrary words and are stated that way; where a permutation hypothesis is genuinely needed it is displayed.

Definition 2.1. For a word u of length m we write $u(1), \ldots, u(m)$ for its entries, matching the notation $\tau(i)$ for permutations, and put $\ell(u) = |\{(i, j) : i < j,\ u(i) > u(j)\}|$. On S_n this is the Coxeter length. For $a < b \leq m$, $u \cdot (a, b)$ denotes the word obtained from u by exchanging the entries in *positions* a and b; on S_n this is right multiplication by the transposition (a, b).

Definition 2.2 ([1, 1.3.1])**.** σ is a *smaller neighbour* of τ, written $\sigma \lessdot \tau$, if $\ell(\tau) = \ell(\sigma) + 1$ and $\sigma = \tau \cdot (a, b)$ for some pair of positions $a < b$. (Exchanging two positions is an involution, so this is the source's condition $\tau = \sigma \cdot (a, b)$.) The *Bruhat order* $\leq$ is the reflexive-transitive closure of $\lessdot$.

Definition 2.3. For a word τ and $j \geq 0$ set $\Lambda(\tau) = \{\sigma : \sigma \leq \tau\}$, $\lambda_j(\tau) = |\{\sigma \leq \tau : \ell(\sigma) = j\}|$, $\mathbf{N}_{\mathrm{s}}(\tau) = \{\sigma : \sigma \lessdot \tau\}$ and $\mathbf{L}_{\mathrm{t}}(\tau) = \{\sigma \leq \tau : \ell(\sigma) = 1\}$. Definition 1.1 is read with these λ_j.

Example 2.4. The smallest critical permutation is $x_1 = (3412) \in S_4$, of length 4. Its principal ideal has $\lambda_0 = 1$ and $\lambda_1 = 3$ at the bottom, and $\lambda_3 = 4$ and $\lambda_4 = 1$ at the top: the rank generating function of $[\mathrm{id}, x_1]$ is $1 + 3q + \cdots + 4q^3 + q^4$, which fails to be palindromic already at $j = 1$, so x_1 is critical of degree 1. The two numbers 3 and 4 are the ones [1, 2.4.2] displays for this permutation; they are $|\mathbf{L}_{\mathrm{t}}(x_1)|$ and $|\mathbf{N}_{\mathrm{s}}(x_1)|$, and Proposition 3.3 below says that for the first degree they are all one ever needs. (See Proposition 8.1.)

Nothing so far is computable: $\lambda_j(\tau)$ is the cardinality of a set of words, and finiteness is not assumed. It is recovered rather than postulated.

Proposition 2.5. *Let τ be a word and $k \geq 0$. Walking the covering relation downwards k times from τ — that is, forming the set of smaller neighbours, then the set of smaller neighbours of those, and so on k times — produces exactly*

$$\{\sigma \ : \ \sigma \leq \tau \ \ \text{and} \ \ \ell(\sigma) + k = \ell(\tau)\},$$

each element once. Consequently, for $j \leq \ell(\tau)$ the set counted by $\lambda_j(\tau)$ is finite, $\lambda_j(\tau)$ is the number of words produced by $\ell(\tau) - j$ such steps, and the predicates "critical of degree i" and "critical" of Definition 1.1 are decidable.

Proof sketch. Every cover step lowers ℓ by exactly one, so a chain of covers from σ to τ has length $\ell(\tau) - \ell(\sigma)$; induction on k turns membership in the k-th iterate into the displayed condition. Duplicate-freeness is maintained at each step. The set counted by λ_j is then the underlying set of an explicit duplicate-free list, which gives both finiteness and the identification of λ_j with a list length; the decision procedure computes the whole vector $(\lambda_0(\tau), \ldots, \lambda_{\ell(\tau)}(\tau))$ and searches it for the least index at which the palindrome condition fails. □

Everything counted below is counted with Definition 1.1, not with the decision procedure: Proposition 2.5 is what allows the two to be interchanged.

3. The top of the ideal, and a certified memoised ideal

Two facts about $\Lambda(\tau)$ hold at every n and need no computation. They are what make the first degree cheap.

Proposition 3.1. *For every word τ, $\lambda_{\ell(\tau)}(\tau) = 1$; and if $\ell(\tau) \geq 1$ then $\lambda_{\ell(\tau)-1}(\tau) = |\mathbf{N}_{\mathrm{s}}(\tau)|$.*

Proof sketch. If $\sigma \leq \tau$ and $\ell(\sigma) = \ell(\tau)$ then the chain of covers joining them is empty, so $\sigma = \tau$; the top level is $\{\tau\}$. The level below the top is the first iterate of the downward walk of Proposition 2.5, which is $\mathbf{N}_{\mathrm{s}}(\tau)$ by definition. □

The second half is the useful one: $\mathbf{N}_{\mathrm{s}}(\tau)$ is described by a condition on τ alone — a search over pairs of positions — whereas $\lambda_{\ell(\tau)-1}$ as written refers to the whole ideal.

Corollary 3.2. *A word τ is critical of degree 0 if and only if $\lambda_0(\tau) \neq 1$.*

By Proposition 6.5 below, $\lambda_0(\tau) = 1$ for every permutation, so no element of any S_n is critical of degree 0; this is why every degree observed anywhere in this paper is at least 1.

Proposition 3.3. *Let τ be a word with $\ell(\tau) \geq 1$. Then τ is critical of degree 1 if and only if $\lambda_0(\tau) = 1$ and $\lambda_1(\tau) \neq \lambda_{\ell(\tau)-1}(\tau)$. For $\tau \in S_n$ this reads*

$$\tau \text{ is critical of degree } 1 \quad \Longleftrightarrow \quad |\mathbf{L}_{\mathrm{t}}(\tau)| \neq |\mathbf{N}_{\mathrm{s}}(\tau)|,$$

which is the first-degree criterion of [1, §4].

Proof sketch. Degree 1 asks for equality at $j = 0$ and failure at $j = 1$. Equality at $j = 0$ compares λ_0 with $\lambda_{\ell(\tau)}$, and the latter is 1 by Proposition 3.1. The displayed form uses $\lambda_1(\tau) = |\mathbf{L}_t(\tau)|$ (immediate from Definition 2.3), $\lambda_{\ell(\tau)-1}(\tau) = |\mathbf{N}_s(\tau)|$ (Proposition 3.1) and $\lambda_0(\tau) = 1$ (Proposition 6.5). □

So the first degree needs only the bottom two levels of $\Lambda(\tau)$ and the local count $|\mathbf{N}_s(\tau)|$, at most n elements each — never the whole ideal, whose size grows very quickly. Higher degrees are not covered by this reduction; see Section 9.

A certified memoised ideal. For $n \leq 6$ the downward walk of Proposition 2.5 can be run once per τ: at $n = 6$ the total cost is 98 407 cover expansions over all 720 permutations. At $n = 7$ the same sweep costs 3 550 919 expansions and at $n = 8$ it costs 170 288 585, which is why the sweeps at those sizes use memoisation: $\Lambda(\tau) = \{\tau\} \cup \bigcup_{\sigma \lessdot \tau} \Lambda(\sigma)$, computed once per τ in order of length and stored in an array indexed by a packing of permutations into machine integers. Nothing about that construction is proved. What is proved is that any table passing a certificate is correct, which is a different and much weaker obligation.

Theorem 3.4. *Fix a predicate* keep *on words. Let T assign to each code c in a set of certified codes a list $T(c)$ of codes, and suppose T passes the following four checks at every certified code c: the packing round-trips at c; every smaller neighbour of the word coded by c is itself certified and round-trips; $T(c)$ is exactly the union of the lists $T(c')$ over the codes c' of those smaller neighbours, together with c itself if* keep *holds there; and $T(c)$ is strictly sorted. Then for every certified c, with τ the word coded by c,*

$$T(c) \;=\; \{\, \textit{code of } \sigma \;:\; \sigma \leq \tau,\ \mathrm{keep}(\sigma) \,\},$$

each element once; and for every decidable predicate q, $|\{\sigma \leq \tau : \mathrm{keep}(\sigma),\ q(\sigma)\}|$ is the number of entries of $T(c)$ whose decoding satisfies q.

Proof sketch. Soundness and completeness are strong inductions on $\ell(\tau)$, using the recursion clause of the certificate in the two directions. Because the round-trip of the packing is *checked* at every code the table claims, rather than proved once and for all, the induction never leaves the certified set of codes and no injectivity lemma for the packing is required; likewise no lemma stating that exchanging two positions of a permutation gives a permutation is needed anywhere. The counting statement follows from soundness, completeness and strict sortedness. □

4. The published counts, and an erratum

The values below are the source's, recomputed here from Definition 1.1 by an implementation sharing no code with the program of [10] from which [1] takes them. They are stated for the record and as a control on the machinery: a predicate that missed the definition would have to reproduce every total *and* every split.

Theorem 4.1. *$|\mathcal{C}_2| = |\mathcal{C}_3| = 0$, $|\mathcal{C}_4| = 2$, $|\mathcal{C}_5| = 32$ and $|\mathcal{C}_6| = 354$.*

Theorem 4.2. *Both elements of $\mathcal{C}_4$ are critical of degree 1. Of the 32 elements of $\mathcal{C}_5$, 31 are of degree 1 and one is of degree 2. Of the 354 elements of $\mathcal{C}_6$, 341 are of degree 1, twelve are of degree 2 and one is of degree 3; none is of degree 0 or 4, and since $341 + 12 + 1 = 354$ none is of any other degree either.*

Theorem 4.3. *$|\mathcal{C}_7| = 3\,488$, of which 3 379 are critical of degree 1, 96 of degree 2, 12 of degree 3 and one of degree 4; none is of degree 5, and since $3\,379 + 96 + 12 + 1 = 3\,488$ none is of any other degree either.*

Theorems 4.1 and 4.2 are exhaustive sweeps over S_n for $n \leq 6$, each one evaluation of the decision procedure of Proposition 2.5 at every element, with the total and every degree count read off that single evaluation. Theorem 4.3 runs the same extraction over the memoised ideal of Theorem 3.4 at $n = 7$, the certificate and the sweep being discharged by compiled evaluation. The totals and the higher-degree counts above are all printed in [1, 2.4.1–2.4.4, 3.5, 3.6]; the two first-degree counts

341 at $n = 6$ and 3 379 at $n = 7$ are not, and are forced by its printed splits ($354 - 12 - 1$ and $3\,488 - 96 - 12 - 1$).

The erratum. [1] prints two different values for $|\mathcal{C}_6|$. Twice — once in the summary of the computations, [1, 2.4.4], "there are 354 critical permutations in S_6 and 3,488 in S_7", and once in the discussion of the degree split, [1, 3.5], "There are 354 permutations $\mathcal{C}_6$, of which 12 are critical of degree 2 … one is critical of degree 3, namely $q_6 = (564312)$, and remainder are of degree 1" — it says 354. Once, in the opening paragraph of [1, §3], which describes how $\mathcal{C}_5$ generates part of $\mathcal{C}_6$, it says instead that "the critical permutations in S_5 generate 343 of the 345 permutations in S_6". (Locators and quotations are to v1, the only version of the preprint as of 28 August 2026.)

Proposition 4.4. $|\mathcal{C}_6| \neq 345$. *Indeed* $|\mathcal{C}_6| = 354$.

The formal statement settles it, but no computation is needed: the same item [1, 3.5] states that exactly eleven permutations in $\mathcal{C}_6$ cannot be obtained from $\mathcal{C}_5$ by the construction of its main theorem, and $343 + 11 = 354$. The classical count agrees independently: $|\mathcal{C}_6| = 720 - 366 = 354$. The isolated 345 is a transposition of digits, inconsistent with the paper's own complement count and with both of its other occurrences of the number. We note that the corresponding value 3 488 at $n = 7$ occurs three times in [1] — in the same three places — and agrees each time.

5. The first degree at $n = 8$

Theorem 5.1. *Exactly* 32 915 *permutations of* S_8 *are critical of degree* 1. *Consequently* $|\mathcal{C}_8| \geq 32\,915$.

Proof sketch. By Proposition 3.3 the first degree is decided by levels 0 and 1 of $\Lambda(\tau)$ together with $|\mathbf{N}_{\mathrm{s}}(\tau)|$. A table in the sense of Theorem 3.4 is therefore built at $n = 8$ with $\mathrm{keep}(\sigma)$ the condition $\ell(\sigma) \leq 1$, so that every stored list has at most n entries; the certificate and the sweep over all 40 320 permutations of S_8 are discharged by compiled evaluation, and Theorem 3.4 converts the stored lists into the level counts of Definition 2.3. The consequence $|\mathcal{C}_8| \geq 32\,915$ is the monotonicity of "critical of degree 1" over "critical". This is an exhaustive computation: the claim is that the search over all of S_8 returns 32 915, and it rests on trusting compiled evaluation for the certificate and for the sweep. □

The value is not printed in [1], whose table stops at $n = 7$, and it does not follow from the pattern description of $\mathcal{C}_8$; it was reproduced independently before the formal proof was written, by a program in C written for this purpose which propagates $\mathbf{L}_{\mathrm{t}}$-bitmasks *upward* along the Hasse diagram where the route above walks $\Lambda(\tau)$ downwards.

Carrying the *whole* ideal at $n = 8$ is a different matter: $\sum_{\tau \in S_8} |\Lambda(\tau)|$ is 170 288 585, so the counts of degree ≥ 2 at $n = 8$ are not obtained by this route and are not claimed here (Section 9). The rest of the paper removes the ideal from the first-degree route entirely, which is what makes $n = 9$ reachable.

6. Swaps, covers, and the parabolic criterion

Everything in this section is proved from Definition 2.2 alone. None of it is new mathematics; what is new is the route, which uses no reduced decompositions, no subword property and no rank matrices, and which is what the machine-checked development needs.

One identity. For words u, v put $\mathrm{cr}(u, v) = \sum_{x \in u} |\{y \in v : y < x\}|$, the inversions of the concatenation uv that straddle the join, so that $\ell(uv) = \ell(u) + \ell(v) + \mathrm{cr}(u, v)$.

Lemma 6.1. *For all words* A, B, C *and all* x, y,

$$\ell(A\,x\,B\,y\,C) + |\{z \in B : y < z\}| + |\{z \in B : z < x\}| + [x < y] \\ = \ell(A\,y\,B\,x\,C) + |\{z \in B : x < z\}| + |\{z \in B : z < y\}| + [y < x],$$

where $[\,\cdot\,]$ *is* 1 *if the condition holds and* 0 *otherwise.*

Proof sketch. Expand both sides with $\ell(uv) = \ell(u) + \ell(v) + \mathrm{cr}(u, v)$. Thirteen terms appear; ten of them are symmetric in x and y and cancel, leaving exactly the six counts over the open middle block B and the two indicator terms displayed. □

Every pair of positions $a < b$ of a word p puts p in the shape $A\,x\,B\,y\,C$ with $x = p(a)$, $y = p(b)$, and $p \cdot (a, b)$ is the same decomposition with x and y exchanged, so Lemma 6.1 governs all swaps. Two consequences are used below.

Lemma 6.2. *If $\sigma \lessdot \tau$, say $\sigma = \tau \cdot (a, b)$ with $a < b$, then $\tau(b) < \tau(a)$: a cover swaps an inversion.*

Proof sketch. If $\tau(a) \le \tau(b)$ then the two B-counts on the left of Lemma 6.1 dominate those on the right, whence $\ell(\tau) \le \ell(\tau \cdot (a, b))$. But a cover has $\ell(\tau \cdot (a, b)) = \ell(\tau) - 1$, so $\tau(a) \le \tau(b)$ is impossible. □

Lemma 6.3. *Let p be a word and $a < b$ positions with $p(b) < p(a)$, and suppose no position c with $a < c < b$ carries a value in the closed interval $[p(b), p(a)]$. Then $\ell(p \cdot (a, b)) + 1 = \ell(p)$. In particular an adjacent descent is always a cover, and an adjacent ascent always raises the length by one.*

Proof sketch. Under the gap hypothesis the two pairs of B-counts in Lemma 6.1 agree termwise, so the identity collapses to $\ell(p) = \ell(p \cdot (a, b)) + 1$. The adjacent case is B empty. □

For a permutation the gap hypothesis of Lemma 6.3 is exactly the condition that (a, b) is a *free-fall inversion* in the terminology of [1, 4.1.3], which attributes the notion, and the description of $\mathbf{N}_{\mathrm{s}}(\tau)$ as the set of swaps of free-fall inversions (its 4.1.5), to Incitti [9]. Lemmas 6.2 and 6.3 are the two implications of that description which the present development uses; both are proved here directly from ℓ and from Definition 2.2, and both hold for arbitrary words.

The criterion. Write $s_j \in S_n$ for the simple transposition exchanging j and $j + 1$, $1 \le j \le n - 1$; these are exactly the permutations of length 1.

Theorem 6.4. *Let $\tau \in S_n$ and $1 \le j \le n - 1$. Then*

$$s_j \le \tau \quad \Longleftrightarrow \quad \exists\, k \le j \ \text{ with } \ \tau(k) > j \quad \Longleftrightarrow \quad \tau(\{1, \dots, j\}) \ne \{1, \dots, j\}.$$

Proof sketch. Forward. Call a word w *block-closed* at j if $w(k) \le j$ for all $k \le j$. Block-closedness is inherited by smaller neighbours: for a cover $\sigma = \tau \cdot (a, b)$ the only case that is not immediate is $a \le j < b$, where Lemma 6.2 gives $\sigma(a) = \tau(b) < \tau(a) \le j$. Induction along chains of covers propagates it down the whole ideal, and s_j is not block-closed at j because $s_j(j) = j + 1$. This half uses no permutation hypothesis whatsoever.

Converse. We produce, by induction on $\ell(\tau)$, *some* word of length 1 below τ that is still not block-closed at j, and only at the very end identify it as s_j; a permutation of length 1 is a simple transposition, and the surviving property forces its index to be j. For the induction step, suppose $\ell(\tau) \ge 2$. If τ has an adjacent descent at some position $c \ne j$, swap it: by Lemma 6.3 this is a cover, and positions $c, c + 1$ are either both $\le j$ or both $> j$, so the multiset of values on $\{1, \dots, j\}$ is unchanged. Otherwise τ is increasing off j; a descent must exist, so τ is Grassmannian with its unique descent at j. If some $k < j$ has $\tau(k) > j$, swapping $(j, j + 1)$ leaves position k untouched. Otherwise $\tau(k) \le j$ for all $k < j$ and $\tau(j) > j$; there are exactly j values $\le j$ in all, of which $j - 1$ sit at the positions $< j$ and none at the position j, so exactly one position after j carries a value $\le j$, and monotonicity on the increasing block after j puts it at $j + 1$; hence $\tau(j+1) \le j$ and $\tau(j+2) > j$. Then $\tau \cdot (j, j + 1)$ has positive length, hence an adjacent descent at some c, and comparing values against the two increasing blocks of τ forces $c \in \{j - 1, j + 1\}$. In the first case $(j - 1, j + 1)$ is an inversion of τ whose only intermediate position carries $\tau(j) > j \ge \tau(j - 1)$, so Lemma 6.3 applies and the swap keeps a large value at position j; in the second case $(j, j + 2)$ is an inversion whose only intermediate position carries $\tau(j + 1) < \tau(j + 2)$, so Lemma 6.3 applies again and the swap puts $\tau(j + 2) > j$ at position j. □

Theorem 6.4 is classical. It is the case $W = S_n$ of two standard facts. The first is the specialisation to $u = s$ of [3, Cor. 2.2.3], which states that for $u, w \in W$ the conditions $u \le w$, "every reduced expression for w has a subword that is a reduced expression for u" and "some reduced expression for w has a subword that is a reduced expression for u" are equivalent; for a simple reflection s the only reduced expression is s itself, so this reads: $s \le w$ exactly when s occurs in some (equivalently, in every) reduced decomposition of w. It is that specialisation which [1, 4.1.2] uses, citing the same corollary, for its description of $\mathbf{L}_{\mathrm{t}}$. The second is the characterisation of the parabolic subgroup generated by $\{s_1, \dots, s_{j-1}, s_{j+1}, \dots\}$, here in the concrete form $\tau(\{1, \dots, j\}) = \{1, \dots, j\}$. Nothing about Theorem 6.4 is claimed as new; what the proof above provides is a derivation from the covering relation, which is what the formal development needs and what makes the counts of Section 7 affordable.

Proposition 6.5. *For every $\tau \in S_n$, $\lambda_0(\tau) = 1$: the identity is the unique element of length 0 below τ, and it does lie below τ.*

Proof sketch. A word of length 0 below τ is a permutation with no inversions, hence the identity. That $\mathrm{id} \le \tau$ follows by descending along adjacent descents: while $\ell(\tau) > 0$ some adjacent descent exists, and Lemma 6.3 makes swapping it a cover. □

Proposition 6.6. *For every $\tau \in S_n$,*

$$\lambda_1(\tau) \;=\; |\mathbf{L}_{\mathrm{t}}(\tau)| \;=\; \big|\{\, j : 1 \le j \le n-1,\ \tau(\{1, \dots, j\}) \ne \{1, \dots, j\}\,\}\big|.$$

Proof sketch. The elements of length 1 below τ are simple transpositions, Theorem 6.4 says which ones occur, and $j \mapsto s_j$ is injective. □

Proposition 6.6 is the point of the section: the level of $\Lambda(\tau)$ that the first-degree criterion compares against $|\mathbf{N}_{\mathrm{s}}(\tau)|$ is a count of $n-1$ conditions on τ itself, each an inspection of an initial segment.

7. First-degree criticality is local, and the count at $n = 9$

Definition 7.1. For $\tau \in S_n$ let $\mathrm{lt}(\tau) = |\{j : 1 \le j \le n-1,\ \tau(\{1, \dots, j\}) \ne \{1, \dots, j\}\}|$ and let $D_n(\tau)$ be the condition

$$\ell(\tau) \ge 1 \quad \text{and} \quad \mathrm{lt}(\tau) \ne |\mathbf{N}_{\mathrm{s}}(\tau)|.$$

Both quantities in $D_n(\tau)$ are computed from τ: $\mathrm{lt}(\tau)$ scans $n-1$ initial segments, and $|\mathbf{N}_{\mathrm{s}}(\tau)|$ is the number of pairs of positions whose swap lowers ℓ by one. Neither refers to $\Lambda(\tau)$.

Theorem 7.2. *For every n and every $\tau \in S_n$, $D_n(\tau)$ holds if and only if τ is critical of degree 1; hence the number of elements of $\mathcal{C}_n$ of degree 1 is the number of $\tau \in S_n$ satisfying $D_n(\tau)$. Evaluated with no ideal and no table, this gives 341 at $n = 6$, 3 379 at $n = 7$ and 32 915 at $n = 8$.*

Proof sketch. Combine Proposition 3.3 with Proposition 6.5 ($\lambda_0(\tau) = 1$), Proposition 6.6 ($\lambda_1(\tau) = \mathrm{lt}(\tau)$) and Proposition 3.1 ($\lambda_{\ell(\tau)-1}(\tau) = |\mathbf{N}_{\mathrm{s}}(\tau)|$). The three numerical values are exhaustive sweeps of D_n over S_6, S_7 and S_8 respectively, discharged by compiled evaluation. □

The three values of Theorem 7.2 are all previously obtained: 341 and 3 379 are the first-degree entries of Theorems 4.2 and 4.3, and 32 915 is Theorem 5.1. The point is that the two routes share no computation — one walks or memoises $\Lambda(\tau)$, the other never forms it — so the agreement is a genuine cross-check on both. It also removes the obstruction at $n = 9$: the packing used by the memoised table of Theorem 3.4 allots three bits per entry and cannot address S_9 at all, and the full ideal is in any case far too large.

Theorem 7.3. *Exactly 329 513 permutations of S_9 are critical of degree 1. Consequently $|\mathcal{C}_9| \ge 329\,513$.*

Proof sketch. Theorem 7.2 at $n = 9$: an exhaustive sweep of D_9 over all 362 880 elements of S_9, discharged by compiled evaluation, with no ideal built and nothing stored. The consequence $|\mathcal{C}_9| \geq 329\,513$ is again monotonicity of degree 1 over criticality. The claim rests on the exhaustive search and on trusting compiled evaluation for it. □

Remark 7.4 (cost). Measured on one machine on 2026-08-28, under the interpreter and with no compiled-library loading: the $n = 8$ sweep of D_n takes 23 seconds and the $n = 9$ sweep about five minutes, with negligible marginal memory because nothing is stored. For comparison the route it replaces requires the memoised table, and the full ideal at $n = 8$ alone amounts to 170 288 585 stored list cells, roughly 4 gigabytes.

Remark 7.5 (prior appearances). Neither 32 915 nor 329 513 is printed in [1], whose table stops at $n = 7$, and the repository [10] from which its counts come contains program sources only, with no tabulated data. The all-degree sequence is of course indexed, being the complement of [11, A032351]; the first-degree one is not. Searches of the On-Line Encyclopedia of Integer Sequences on 2026-08-23 and again on 2026-08-28, each run after a positive control returning a known sequence, found no entry for $2, 31, 341, 3\,379, 32\,915, 329\,513$ or for the pair $32\,915,\ 329\,513$ under any indexing tried. A search on 2026-08-28 for works building on [1] likewise found none: OpenAlex, Semantic Scholar and Google Scholar each hold a record for the preprint and each report no citing work — in each case after a positive control on the same channel returned a known nonzero citation list — and zbMATH holds only an unreviewed preprint entry, with no citation data. On the same date a full-text search of a local mirror of the arXiv LaTeX sources, with coverage through 2608, found the preprint itself and no other document mentioning it. Absence from these indexes is evidence that the values are unpublished, not proof of it.

8. Controls

Exhaustive counts are only as good as the predicate being counted, so the development carries witnesses on both sides and a cross-check of the criterion against the definition. All of the following are machine-checked.

Proposition 8.1. *The two elements of $\mathcal{C}_4$ named in* [1, 2.4.2]*, $x_1 = (3412)$ and $x_2 = (4231)$, are critical, both of degree* 1*. For x_1, $\ell(x_1) = 4$ and $\lambda_1(x_1) = 3 \neq 4 = \lambda_3(x_1)$, which are the two numbers displayed for both elements there.*

Proposition 8.2. *The identity of S_4 is not critical; $(2413) \in S_4$ is not critical; the length-one permutation $(13245) \in S_5$ is not critical; and $|\mathcal{C}_3| = 0$, so nothing critical occurs before $n = 4$.*

Proposition 8.3. *$(45312) \in S_5$ is critical of degree* 2 *and* not *of degree* 1*; $q_6 = (564312) \in S_6$ is critical of degree* 3 *and of neither degree* 1 *nor degree* 2*. These are the two permutations singled out at* [1, 2.4.3 and 3.5].

Proposition 8.3 is the control on the minimality clause of Definition 1.1: dropping "$\lambda_j = \lambda_{\ell(\tau)-j}$ for all $j < i$" would give a coarser predicate, and these two permutations of [1] witness the difference.

Proposition 8.4. *$|\mathcal{C}_6| \notin \{345, 353, 355\}$, $|\mathcal{C}_7| \notin \{3\,487, 3\,489\}$, and the number of elements of S_8 critical of degree* 1 *is neither* 32 914 *nor* 32 916*.*

Proposition 8.5. *For every one of the* 720 *elements τ of S_6, the locally computed $\mathrm{lt}(\tau)$ agrees with $\lambda_1(\tau)$ obtained by walking the covering relation downwards as in Proposition 2.5. Moreover $\mathrm{lt}(\mathrm{id}) = 0$ and $\mathrm{lt}(w_0) = 5$ for the longest element $w_0 \in S_6$, so the criterion is neither constantly true nor constantly false; and the count of Theorem 7.3 is neither* 329 512 *nor* 329 514*.*

The first statement of Proposition 8.5 is the control that matters for Section 7: it compares the criterion of Theorem 6.4 against the downward walk over an entire symmetric group, the two routes having no computation in common (98 407 cover expansions on one side, an inspection of initial segments on the other).

9. What is not settled

The reduction of Proposition 3.3 is specific to the first degree. Degree $i \geq 2$ compares level i with level $\ell(\tau) - i$, and neither has a description local to τ of the kind Proposition 3.1 and Theorem 6.4 supply for $i \leq 1$; so at $n = 8$ and beyond only the first-degree counts are proved here. The following three observations are recorded because they bear on that gap. **They are computations made outside the formal development, by a program in C written for this purpose, and are not machine-checked; they are not theorems of this paper.**

Remark 9.1. That program reports 33 668 critical permutations in S_8 in all degrees, split as 32 915 of degree 1, 644 of degree 2, 96 of degree 3, 12 of degree 4 and one of degree 5. The total is the classical $40\,320 - 6\,652$, so only the split is at issue; and of the split, only the first entry is Theorem 5.1.

Remark 9.2. In the same computation exactly one element of $\mathcal{C}_8$ has degree $5 = n - 3$, namely (78654312); evaluating the two-line formula of [1, end of 2.4] at $n = 8$ — first two positions $n-1, n$, then $n-2, n-3, \ldots, 3$, then $1, 2$ — gives that same word. So this is the next instance of the uniqueness the source verifies at $n = 5, 6, 7$. In the indexing of that display, where the subscript is the degree, this permutation is q_5; the members at $n = 5, 6, 7$ are then $q_2 = (45312)$, $q_3 = (564312)$ and $q_4 = (6754312)$, and the last two are the permutations [1, 3.5 and 3.6] calls q_6 and q_7, indexing by n instead. Nothing proved here turns on the choice: wherever a member of the family is mentioned above, its one-line word is given alongside the name.

Remark 9.3. Writing $c_n(i)$ for the number of elements of $\mathcal{C}_n$ of degree i, the data above satisfy $c_n(i) = c_{n-1}(i-1)$ for every $i \geq 3$ in all six available instances, namely $(n, i) = (6, 3)$, $(7, 3)$, $(7, 4)$, $(8, 3)$, $(8, 4)$, $(8, 5)$, while $i = 2$ is not of that form ($c_8(2) = 644 \neq 3\,379 = c_7(1)$). A proof would explain the shift of the degree spectrum and would be a genuine contribution; we have none.

Finally, what the formal development proves about the all-degree counts is only the lower bounds of Theorems 5.1 and 7.3; the exact values $|\mathcal{C}_8| = 33\,668$ and $|\mathcal{C}_9| = 334\,184$ are classical and are not obtained here.

10. Verification

Every theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development contains no `sorry` and declares no `axiom` of its own (repository reference to be added at submission). The remarks of Sections 7 and 9 that record measurements, literature searches or the output of the auxiliary C program are not part of that, and say so. The definitions are transcribed as stated in Section 2: the Bruhat order is the reflexive-transitive closure of the covering relation, $\lambda_j(\tau)$ is the cardinality of a set of words with no finiteness assumed, and every count is stated about the predicate of Definition 1.1 rather than about the decision procedure, the two being connected by Proposition 2.5. The entire reasoning layer — Propositions 2.5, 3.1, 3.3, 6.5, 6.6, Corollary 3.2, Theorems 3.4, 6.4 and 7.2, and Lemmas 6.1, 6.2 and 6.3 — depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation — through Lean's `native_decide`, which adds for each such statement one axiom asserting that the compiled evaluation returned `true` — is exactly the finite searches: one sweep over S_n for each of $n \leq 6$ (Theorems 4.1 and 4.2, and Propositions 8.1–8.3, whose small instances are in part decided by the kernel alone); the table certificate and one sweep at $n = 7$ (Theorem 4.3); the table certificate and one sweep at $n = 8$ (Theorem 5.1); one sweep of D_n for each of $n = 6, 7, 8, 9$ (Theorems 7.2 and 7.3); and the cross-check over S_6 of Proposition 8.5. The straddling controls of Propositions 8.4 and 8.5 add no further search: they are read off the sweeps just listed. The two new values were each produced first by an independent implementation in C, written from the definitions and using an upward pass over the Hasse diagram rather than a downward walk, and the formal proof was written against that; at $n = 8$ the two formal routes of Sections 5 and 7 agree as well, sharing no computation.

References

[1] D. Pučinskaitė, *Permutations with Verma Multiplicities* $[M(p) : L(q)] \geq 2$, arXiv:2607.19112v1 (21 July 2026), math.CO, cross-listed math.RT. Read from the paper's own LaTeX source, obtained from `arxiv.org/e-print/2607.19112`. Every locator to this reference is a numbered item or section number of the article compiled from that source, and is pinned to v1, still the only version on 28 August 2026.

[2] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics 750, Springer, Berlin–Heidelberg–New York, 1979. Cited by [1, 2.4.3], as its 5.24, for the value 32 at $n = 5$; not read here.

[3] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics 231, Springer, 2005.

[4] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in* $Sl(n)/B$, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 1, 45–52.

[5] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, in: Algebraic Groups and Their Generalizations: Classical Methods, Proc. Sympos. Pure Math. **56**, Part 1, Amer. Math. Soc., 1994, pp. 53–61. The palindromicity criterion is credited by its author to joint work with D. Peterson.

[6] V. V. Deodhar, *Local Poincaré duality and non-singularity of Schubert varieties*, Comm. Algebra **13** (1985), no. 6, 1379–1388.

[7] H. Abe and S. Billey, *Consequences of the Lakshmibai–Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry*, in: Schubert Calculus — Osaka 2012, Adv. Stud. Pure Math. **71**, Math. Soc. Japan, 2016, pp. 1–52.

[8] M. Bousquet-Mélou and S. Butler, *Forest-like permutations*, Ann. Comb. **11** (2007), no. 3–4, 335–354.

[9] F. Incitti, *Bruhat order on classical Weyl groups: minimal chains and covering relation*, European J. Combin. **26** (2005), no. 5, 729–753. The source of the free-rise (dually, free-fall) notion and of the description of the Bruhat covers, as cited in [1, 4.1.3 and 4.1.5]; not read here.

[10] A. Elbosari, *Permutations with non-simple Verma Multiplicities*, `github.com/MarisaCodes/Permutations-with-non-simple-Verma-Multiplicities`. The program [1] uses for its counts; inspected 2026-08-23, when it held Python sources and a short README, with no data files and no tabulated counts.

[11] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.