SPRAGUE-GRUNDY VALUES OF CRIM, THE ROW/COLUMN-DELETION GAME ON INTEGER PARTITIONS

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Abstract. Crim is the impartial game whose positions are integer partitions and whose moves delete one row or one column of the Young diagram, the surviving cells reattaching into a partition again. Bašić, Gottlieb and Krnc determined the Sprague–Grundy values of several families of positions and closed with nine conjectures. We show that one of them is false, and false on the paper's own terms. The conjecture predicts $\mathcal{G}(R^k_{r,r-1})$ for $r \geq 7$; on the line $k = r-2$ the rectair $R^{r-2}_{r,r-1}$ *is* the padded staircase PS_{r-1}, an identity recorded in the source itself, so that conjecture and the source's conjecture on padded staircases speak about the same positions and predict different values at every $r \geq 7$. Computation decides between them: $\mathcal{G}(\mathrm{PS}_6) = 1$, $\mathcal{G}(\mathrm{PS}_7) = 3$ and $\mathcal{G}(\mathrm{PS}_8) = 1$, where the first conjecture predicts 3, 1 and 3. Its parity condition is inverted. We record four further failures at degenerate parameters, one of them a genuine counterexample to a published theorem: the characterisation of losing three-part partitions fails at $[1,1,1]$, which the statement places in $\mathcal{P}$ although the single column may be removed at once. We then verify finite instances of the remaining conjectures, among them the main conjecture (every losing partition has even Dyson rank) for all 18 460 partitions of size at most 28. All statements are machine-checked in Lean 4.

1. Introduction

Let $\lambda = [\lambda_1, \ldots, \lambda_r]$ with $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$ be an integer partition, drawn as a Young diagram. In the game Crim (Column-Row Impartial Merge) a move deletes one entire row or one entire column of the diagram; if the deleted line is neither the first nor the last, the two remaining pieces are pushed together, so that the result is again a Young diagram. Both players have the same moves, the empty partition is the only terminal position, and under normal play the player who cannot move loses. The game was introduced by Bašić [8], and it belongs to a family of impartial games on partitions that includes Lctr, with which it coincides on rectangles [7]. Bašić, Gottlieb and Krnc [1] analyse it in detail: they determine the $\mathcal{P}/\mathcal{N}$-status of partitions with two and three parts, of a new family they call *hook-squares*, and of the *OAE* and *EAO* partitions; they introduce *rectairs*, a common generalisation of rectangles and staircases, and the *meld* operation, and they locate Crim in the Conway–Gurvich–Ho hierarchy [2, 3].

Their results decide a little over half of all partitions of size at most 50, and the paper closes with a section of nine conjectures and six questions. Almost every conjecture names an explicit Sprague–Grundy value for an explicit family, so each of its instances is a finite computation, and that is what this paper exploits. Our findings are of three kinds.

A conjecture that is false. Conjecture 3 of [1] predicts the Sprague–Grundy values of the rectairs $R^k_{r,r-1}$ for $r \geq 7$, with the exceptional value 3 exactly when $k = r-2$ and r is odd. But $R^{r-2}_{r,r-1}$ is the padded staircase PS_{r-1} — one of the identities listed in Section 2 of [1] — so Conjecture 3 and Conjecture 5 of the same paper, which predicts $\mathcal{G}(\mathrm{PS}_n)$, describe one and the same family of positions. They disagree at every $r \geq 7$, and the computation sides with Conjecture 5:

r	$R^{r-2}_{r,r-1}$	Conjecture 3	Conjecture 5	true value
7	$\mathrm{PS}_6 = [6,6,5,4,3,2,1]$	3	1	1
8	$\mathrm{PS}_7 = [7,7,6,5,4,3,2,1]$	1	3	3
9	$\mathrm{PS}_8 = [8,8,7,6,5,4,3,2,1]$	3	1	1

The three points $r = 7, 8, 9$ are all the points of the line that our tables reach, and all three refute Conjecture 3 (Theorem 4.3). The evidence is that the parity condition in Conjecture 3 is inverted and that the intended statement reads "$k = r - 2$ and r even" (Remark 4.4). Because the contradiction is internal to [1], it does not rest on our computation at all: one of the two conjectures had to fall, and the computation only says which.

Errata at degenerate parameters. Four published statements of [1] fail at small parameters that their authors plainly did not have in mind, and each of the four contradicts another statement of the same paper. The characterisation of losing three-part partitions fails at $[1, 1, 1]$, which it places in $\mathcal{P}$ although $\mathcal{G}([1, 1, 1]) = 1$ (Theorem 5.1); the catch-all clause of Conjecture 3 fails at $R^0_{3,2} = [2, 2, 2]$ and at $R^0_{2,1} = [1, 1]$, both of value 2 and both settled by the paper's own rectangle formula; and Conjecture 2 fails at $R^0_{1,1} = [1]$, where it predicts 2 while the paper's Conjecture 4 predicts, correctly, 1 (Theorem 4.6).

Verified instances. For the conjectures we could not decide we report what the computation does establish: the main conjecture for every partition of size at most 28, the parity self-conjugate classification for every partition of size at most 28, the Conway-pair characterisation of even-by-even hooks for every nonempty partition of size at most 26, the rectair, staircase and padded-staircase values in the ranges $2 \le r \le 9$, $n \le 11$ and $n \le 8$, and four instances of the meld conjecture at sizes 51 to 83 (Section 6). For two of these the source reports its own check to size 50, so our sweeps confirm rather than extend them; we say in each case exactly which finite search was performed.

All computations are certified: no Grundy value in this paper is obtained by an unchecked program. Section 3 describes the mechanism — a table of positions and values which a proved-sound checker validates entry by entry against the recursion — and Section 8 records the machine verification.

2. Preliminaries

We follow the notation of [1].

Partitions. A partition $\lambda = [\lambda_1, \dots, \lambda_r]$ is a weakly decreasing finite sequence of positive integers; $r = \ell(\lambda)$ is its number of parts and $|\lambda| = \sum_i \lambda_i$ its size. Exponential notation abbreviates repetitions, so $[5^2, 4, 1^3] = [5, 5, 4, 1, 1, 1]$. The empty partition is written $[\,]$. The *conjugate* λ' has $\lambda'_k = \#\{i : \lambda_i \ge k\}$, and the *(Dyson) rank* of λ is $|\ell(\lambda) - \lambda_1|$. The *meld* of $\mu = [\mu_1, \dots, \mu_k]$ and $\lambda = [\lambda_1, \dots, \lambda_m]$ is

$$\mu \diamond \lambda = [\mu_1 + \lambda_1, \dots, \mu_1 + \lambda_m, \mu_1, \dots, \mu_k],$$

an associative operation with unit $[\,]$.

The families of [1] used below are, for $r, c \ge 1$: the *rectangle* $R_{r,c} = [c^r]$; the *hook* $\Gamma_{r,c} = [c, 1^{r-1}]$; the *thick hook* $\Gamma^{a,b}_{r,c} = [c^a, b^{r-a}]$ with $r > a \ge 2$ and $c > b \ge 2$; the *almost-hook* $\Gamma^{1,b}_{r,c} = [c, b^{r-1}]$ with $c \ge b$; the *hook-square* $[c, 2, 1^{r-2}]$; the *staircase* $S_n = [n, n-1, \dots, 1]$; the *padded staircase* $\mathrm{PS}_n = [n, n, n-1, \dots, 1]$; the *doubly padded staircase* $\mathrm{DPS}_n = [n+1, n+1, n, \dots, 2]$; and, for $0 \le k < \min(r, c)$, the *rectair*

$$R^k_{r,c} = [c^{\,r-k}, c-1, c-2, \dots, c-k].$$

Rectairs specialise to all of the above staircase families [1, Section 2.2]:

$$R^0_{r,c} = R_{r,c}, \qquad R^{k-1}_{k,k} = S_k, \qquad R^{k-1}_{k+1,k} = \mathrm{PS}_k, \qquad R^{k-2}_{k,k} = \mathrm{DPS}_k. \tag{1}$$

Finally, λ is *parity self-conjugate* if $\lambda_k \equiv \lambda'_k \pmod 2$ for every k (parts beyond the length being read as 0), and the *Durfee length* $d(\lambda)$ is the largest d with $\lambda_d \ge d$.

The game. A *row move* on λ deletes a part λ_i; a *column move* is the conjugate of a row move on λ', that is, it decreases by one every part of size at least j for a fixed $j \le \lambda_1$ and discards the parts that become 0. Write $\mathrm{opt}(\lambda)$ for the set of positions so reachable. Both kinds of move strictly decrease $2|\lambda| + \ell(\lambda)$, which is the induction measure used throughout.

Under normal play the Sprague–Grundy value [5, 6] is

$$\mathcal{G}([\,]) = 0, \qquad \mathcal{G}(\lambda) = \mathrm{mex}\{\mathcal{G}(\mu) : \mu \in \mathrm{opt}(\lambda)\},$$

and λ is a $\mathcal{P}$-position (losing for the player to move) if and only if $\mathcal{G}(\lambda) = 0$. The misère Grundy value $\mathcal{G}^-$ is given by the same recursion with the terminal value 1 instead of 0, and the *Conway pair* of λ is $\mathrm{CP}(\lambda) = (\mathcal{G}(\lambda), \mathcal{G}^-(\lambda))$; it is the datum on which the Conway–Gurvich–Ho classification of impartial games rests [2, 3, 4].

3. Certified Grundy tables

Every value quoted in this paper is obtained the same way, and the point of the construction is that the program which computes the values is not trusted.

A *table* is a finite list T of pairs (λ, v). The checker verifies, for each entry of T, that either $\lambda = [\,]$ and v is the terminal value, or else every $\mu \in \mathrm{opt}(\lambda)$ occurs in T and v equals the mex of the values that T assigns to those μ. If the checker accepts T, then every entry of T records the true Grundy value; the proof is a strong induction on $2|\lambda| + \ell(\lambda)$, which decreases along every move, and it is completely independent of how T was produced. Tables are produced by a memoised closure of the move relation from a list of roots, an ordinary unverified program: a table that is wrong or incomplete simply fails the checker. Running the checker once on the finished table is the only computation in this paper that is trusted, and the same construction with terminal value 1 certifies misère values, hence Conway pairs.

Statements quantified over *all* partitions of a given size need one further ingredient: an enumeration of the partitions of size at most n, together with a proof that it omits nothing. Sweeps then range over genuine partitions rather than over a hand-written list, and a claim such as Theorem 6.1 below is a universally quantified statement, not a finite conjunction of examples.

The tables used here are the following; the size is the number of positions in the closure, so it is also the number of individual Grundy values certified.

table	positions
published value grids, normal play	2 525
published value grids, misère play	128
Appendix A positions and the two large examples of [1, Section 5]	4 279
rectairs $R^k_{r,c}$, $c \in \{r-1, r\}$, $r \le 9$, with $\mathrm{PS}_n, \mathrm{DPS}_n$ $(n \le 8)$, S_{10}, S_{11}	$\approx 33\,000$
melds of thick hooks	7 403
all partitions of size ≤ 28, normal play	18 460
all partitions of size ≤ 26, misère play	11 732

Remark 3.1. What limits the reach is the closure of a position, not its size. A partition with r rows and c columns has at most 2^{r+c} descendants, so $r + c$, and not $|\lambda|$, is the cost parameter: the 91-cell position $[7^5, 5^7, 3^7]$ of [1, Section 5] certifies through a table of only 4 096 positions, while the 66-cell staircase S_{11} needs 28 657. For every $n \le 11$, the closure of S_n turned out to have exactly $F(2n+1)$ positions, F the Fibonacci sequence — 610 for S_7, 28 657 for S_{11}. This is an observation over the computed range $n \le 11$ only; we do not prove it for all n.

4. The line $k = r - 2$, and Conjecture 3

Conjecture 3 of [1] reads: for $r \ge 7$,

$$\mathcal{G}(R^k_{r,r-1}) = \begin{cases} 3 & k = r-2 \text{ and } r \text{ odd}, \\ 1 & \text{otherwise}, \end{cases}$$

together with the four exceptional values $\mathcal{G}(R^1_{3,2}) = \mathcal{G}(R^3_{5,4}) = 3$, $\mathcal{G}(R^2_{4,3}) = 2$, $\mathcal{G}(R^4_{6,5}) = 5$, and a catch-all clause assigning the value 1 to all remaining rectairs. Conjecture 5 of the same paper reads

$$\mathcal{G}(\mathrm{PS}_n) = \begin{cases} 2 & n \in \{1,3\}, \\ 5 & n = 5, \\ 1 & n \geq 6 \text{ even}, \\ 3 & \text{otherwise}. \end{cases}$$

The two conjectures are not independent. Putting $c = r-1$ and $k = r-2$ in the definition of a rectair gives, since $c - k = 1$ and $r - k = 2$,

$$R^{r-2}_{r,r-1} = [(r-1)^2, r-2, r-3, \ldots, 1] = \mathrm{PS}_{r-1} \qquad (r \geq 2), \tag{2}$$

which is the third identity of (1) with $k = r-1$. So the two conjectures assign values to the same positions.

Remark 4.1. They never agree there. For $r \geq 7$ write $n = r - 1 \geq 6$. If r is odd then n is even and $n \geq 6$, so Conjecture 3 predicts 3 and Conjecture 5 predicts 1; if r is even then n is odd and $n \geq 7$, so Conjecture 3 predicts 1 and Conjecture 5 predicts 3. Hence at every $r \geq 7$ the two conjectures of [1] contradict each other, and at least one of them is false at every point of the line $k = r-2$.

The computation says which.

Proposition 4.2. *For $3 \leq r \leq 9$ the Sprague–Grundy value of $R^{r-2}_{r,r-1}$ is the value that Conjecture 5 of* [1] *predicts for* PS_{r-1}. *In particular*

$$\mathcal{G}(R^5_{7,6}) = \mathcal{G}(\mathrm{PS}_6) = 1, \qquad \mathcal{G}(R^6_{8,7}) = \mathcal{G}(\mathrm{PS}_7) = 3, \qquad \mathcal{G}(R^7_{9,8}) = \mathcal{G}(\mathrm{PS}_8) = 1.$$

Proof. The identity $R^{r-2}_{r,r-1} = \mathrm{PS}_{r-1}$ is (2), an unwinding of the definition. The values are read from the rectair table of Section 3: its roots are all rectairs $R^k_{r,c}$ with $c \in \{r-1, r\}$ and $2 \leq r \leq 9$, all PS_n and DPS_n for $n \leq 8$, and the staircases S_{10} and S_{11}; its closure under row and column moves has about 33 000 positions, and the checker of Section 3 accepts it, so each of those 33 000 values is the true Grundy value. Comparing the seven entries $r = 3, \ldots, 9$ with the formula of Conjecture 5 is then a finite check. ☐

Theorem 4.3. *Conjecture 3 of* [1] *is false. It fails at each of the three points of the line $k = r-2$ that lie in the range $7 \leq r \leq 9$:*

$$\mathcal{G}(R^5_{7,6}) = 1 \neq 3, \qquad \mathcal{G}(R^6_{8,7}) = 3 \neq 1, \qquad \mathcal{G}(R^7_{9,8}) = 1 \neq 3,$$

the second value in each display being the one the conjecture predicts.

Proof. Immediate from Proposition 4.2 and the statement of the conjecture: at $r = 7$ and $r = 9$ the parameter r is odd and $k = r-2$, so the conjecture predicts 3 while the true value is 1; at $r = 8$ it predicts 1 while the true value is 3. ☐

Remark 4.4. The three counterexamples are consistent with a single misprint. Replacing the condition "$k = r-2$ and r odd" by "$k = r-2$ and r even" turns Conjecture 3 into the assertion that $\mathcal{G}(R^{r-2}_{r,r-1})$ is 3 for even $r \geq 8$ and 1 for odd $r \geq 7$, which by (2) is exactly what Conjecture 5 says about PS_{r-1}, and which agrees with the computed values at $r = 7, 8, 9$. We therefore read the failure as an inverted parity condition rather than as evidence against the shape of the conjecture.

Off the line the conjecture is in good order as far as we can see.

Proposition 4.5. *For $4 \leq r \leq 9$ and $0 \leq k < r-2$ we have $\mathcal{G}(R^k_{r,r-1}) = 1$, as Conjecture 3 of* [1] *predicts.*

Proof. A sweep over the 27 pairs (r, k) in range against the same rectair table. ☐

We turn to the degenerate parameters. Both conjectures of [1] that assign values to rectairs have a clause that, read literally, covers positions so small that no interesting statement about them was intended; we record the failures because the statements as printed include them, and because each failure is visible from another statement of [1] alone.

Theorem 4.6. (1) *$\mathcal{G}(R^0_{3,2}) = \mathcal{G}([2,2,2]) = 2$ and $\mathcal{G}(R^0_{2,1}) = \mathcal{G}([1,1]) = 2$, whereas the catch-all clause of Conjecture 3 of* [1] *assigns the value* 1 *to both.*
(2) *$\mathcal{G}(R^0_{1,1}) = \mathcal{G}([1]) = 1$, whereas Conjecture 2 of* [1] *assigns the value* 2 *to it.*

Proof. The three values are entries of the rectair table. For the comparison: $R^0_{3,2} = [2,2,2]$ and $R^0_{2,1} = [1,1]$ both belong to the family $c = r-1$ to which Conjecture 3 is addressed, and neither is among its four listed exceptions, so the clause "all other rectairs have Sprague–Grundy value 1" applies to them. For (2), Conjecture 2 predicts $\mathcal{G}(R^k_{r,r}) = 0$ when r is even or $k < r-1$, the value 1 when $r \in \{3,5\}$ and $k = r-1$, and 2 otherwise; at $r = 1$, $k = 0$ neither of the first two clauses applies, so the prediction is 2. □

Remark 4.7. Each of these three points is refuted by [1] itself, independently of our tables. $[2,2,2] = R_{3,2}$ and $[1,1] = R_{2,1}$ are rectangles, and the rectangle formula of [1] gives the value 2 in both cases, since $r+c$ is odd and $\min(r,c) \leq 2$; and $[1] = S_1$, for which Conjecture 4 of [1] predicts the value 1. The catch-all clause of Conjecture 3 is in any case false in a wider sense if read as a statement about all rectairs whatsoever: every even staircase $S_{2m} = R^{2m-1}_{2m,2m}$ is a losing position by the staircase proposition of [1, Section 3] and so has Grundy value 0, not 1. The narrower reading, under which the clause speaks only of the rectairs $R^k_{r,r-1}$ with $r < 7$, is the one refuted by Theorem 4.6(1).

5. Three-part partitions

Theorem 5.1 below concerns the following statement of [1]: a partition $\lambda = [\lambda_1, \lambda_2, \lambda_3]$ on three parts is a $\mathcal{P}$-position if and only if λ_1 and λ_2 are odd and either λ is a hook or $\lambda_3 > 1$.

Theorem 5.1. *The characterisation of losing three-part partitions in* [1] *fails at $\lambda = [1,1,1]$: the stated condition holds there, since $\lambda_1 = \lambda_2 = 1$ are odd and $[1,1,1] = \Gamma_{3,1}$ is a hook, but $\mathcal{G}([1,1,1]) = 1$, so $[1,1,1]$ is a winning position. At every other point of the range $1 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq 9$ the stated equivalence holds.*

Proof. That $[1,1,1]$ is winning needs no computation: the diagram is a single column, its removal is a legal move, and it leads to the terminal position $[\,]$; hence $\mathcal{G}([1,1,1]) = \operatorname{mex}\{0, \dots\} \neq 0$, and the table gives the exact value 1. The remaining assertion is a sweep over the 164 triples with $1 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq 9$ other than $[1,1,1]$, comparing, for each, the certified value $\mathcal{G}([\lambda_1, \lambda_2, \lambda_3])$ against the stated condition; the values come from the normal-play validation table. □

Remark 5.2. The proof in [1] splits the sufficiency into a hook case and a case $\lambda_3 > 1$, and the hook case is settled by an appeal to the paper's own formula for $\mathcal{G}(\Gamma_{r,c})$. That formula has three branches, and the branch giving the value 0 requires $\min(r,c) \geq 2$; when $\min(r,c) = 1$ it gives $(r+c) \bmod 2+1$ instead. For $\lambda = [1,1,1] = \Gamma_{3,1}$ the second branch is the applicable one and returns 1, in agreement with the elementary argument above. The appeal therefore silently assumes $\lambda_1 \geq 2$, and $[1,1,1]$ is the only three-part partition on which the assumption fails: for odd $\lambda_1 \geq 3$ the hook $[\lambda_1, 1, 1] = \Gamma_{3,\lambda_1}$ has $\min(3, \lambda_1) \geq 2$ and equal parities, so the same formula returns 0. Adding the hypothesis $\lambda_1 \geq 2$ to the hook clause repairs the statement.

6. Instances of the remaining conjectures

None of the remaining conjectures of [1] is finitely checkable as it stands; what a computation can do is verify them on an initial segment, and this section records exactly which segment. Two of them — Conjectures 1 and 9 — are reported in [1] as checked by the authors up to size 50, so the statements below confirm published computational evidence rather than add to it; we include them because they are verified sweeps over a proved-complete enumeration of partitions rather than over a list.

Proposition 6.1. *Every partition λ with $|\lambda| \le 28$ and $\mathcal{G}(\lambda) = 0$ has even Dyson rank. This is Conjecture 1 of* [1] *restricted to $|\lambda| \le 28$.*

Proof. The normal-play table over all 18 460 partitions of size at most 28 is accepted by the checker, so it records the true value of each of them; the sweep then tests, for every partition in the certified enumeration, that the value is nonzero or the rank $|\ell(\lambda) - \lambda_1|$ is even. Because the enumeration is proved to contain every partition of size at most 28, the conclusion is a statement about all such partitions. $\square$

Proposition 6.2. *Let λ be a nonempty parity self-conjugate partition with $|\lambda| \le 28$, and write $d = d(\lambda)$. Then λ is a winning position if and only if $\lambda_i = \lambda'_i$ and $\lambda_i \equiv i \pmod 2$ for all $1 \le i \le d$, and $\lambda_d = \lambda'_d = d$ — that is, if and only if λ is a stack of odd self-conjugate hooks whose innermost hook is* [1]*. This is Conjecture 9 of* [1] *restricted to $|\lambda| \le 28$.*

Moreover, a nonempty partition λ with $|\lambda| \le 26$ has Conway pair $\mathrm{CP}(\lambda) = (0,1)$ if and only if $\lambda = \Gamma_{r,c}$ with $r, c \ge 2$ both even. This is Conjecture 6 of [1] *restricted to nonempty partitions of size at most* 26.

Proof. The first sweep runs over the 18 460 partitions of size at most 28 against the normal-play table, testing the equivalence at each parity self-conjugate position. The second needs misère values as well: a second table, over the 11 732 partitions of size at most 26, is checked against the same recursion with terminal value 1, and the sweep compares the pair of certified values with the even-by-even hook test at every nonempty partition of that size. $\square$

Remark 6.3. The restriction to nonempty partitions in the second half is necessary as printed: $\mathrm{CP}([\,]) = (0,1)$ directly from the two recursions, while $[\,]$ is not a hook.

Proposition 6.4. *Conjectures 2, 4 and 5 of* [1] *hold in the following ranges: $\mathcal{G}(R^k_{r,r})$ agrees with Conjecture 2 for all $2 \le r \le 9$ and $0 \le k < r$; $\mathcal{G}(S_n)$ agrees with Conjecture 4 for all $1 \le n \le 11$; and $\mathcal{G}(\mathrm{PS}_n)$ agrees with Conjecture 5 for all $1 \le n \le 8$.*

Proof. Three sweeps against the rectair table, of 44, 11 and 8 points respectively. $\square$

The conjectural content of Proposition 6.4 sits in the odd cases, which is where the published theorems of [1] stop: the even staircases are losing by the results of [1], whereas the values

$$\mathcal{G}(S_7) = \mathcal{G}(S_9) = \mathcal{G}(S_{11}) = 2, \qquad \mathcal{G}(\mathrm{PS}_7) = 3, \qquad \mathcal{G}(\mathrm{PS}_8) = 1$$

are the points where the conjectures go beyond what [1] proves. By (1) the first three are also the values of the rectairs $R^{n-1}_{n,n}$ at $n = 7, 9, 11$, so they are simultaneously instances of Conjecture 2. The staircase S_{11} is the largest single computation of this kind that we made: 66 cells and a closure of 28 657 positions.

Proposition 6.5. *Conjecture 7 of* [1] *— a meld of losing thick hooks is losing — holds at the following four melds. Write $A = \Gamma^{2,3}_{5,5} = [5^2, 3^3]$, $B = \Gamma^{2,5}_{5,7} = [7^2, 5^3]$ and $C = \Gamma^{2,2}_{4,4} = [4^2, 2^2]$, each of which is a losing thick hook. Then*

$$A \diamond A = [10^2, 8^3, 5^2, 3^3], \quad A \diamond B = [12^2, 10^3, 5^2, 3^3], \quad B \diamond A = [12^2, 10^3, 7^2, 5^3], \quad C \diamond A = [9^2, 7^3, 4^2, 2^2]$$

are all losing, of sizes 63*,* 73*,* 83 *and* 51 *respectively.*

Proof. One table, the closure of the four melds under row and column moves, 7 403 positions, accepted by the checker; the four roots carry the value 0. The thick hooks A, B, C are losing by the thick-hook formula of [1], and this is re-verified in Proposition 7.1. None of the four melds is an OAE-partition — in each case λ_i or λ'_i is odd for some odd i — so the theorem of [1] that OAE-partitions are losing does not settle any of them; this is why the thick hooks were chosen with an odd number of rows and of columns. $\square$

7. Reproduction of the published values, and controls

A computation about a game is only as good as its move relation, and the move relation of Crim admits several plausible misreadings. We guarded against them in two ways.

Proposition 7.1. *Every value formula proved or quoted in* [1] *was recomputed and agrees with the certified tables on the following grids: the rectangle and hook formulas for* $1 \le r, c \le 10$*; the two-part formula for* $3 \le \lambda_2 \le \lambda_1 \le 12$*; the three-part characterisation for* $1 \le \lambda_3 \le \lambda_2 \le \lambda_1 \le 9$ *(with the single exception of Theorem 5.1); the almost-hook formula for* $4 \le r \le 9$ *and* $3 \le b \le c \le 8$*; the thick-hook formula for* $2 \le a < r \le 8$ *and* $2 \le b < c \le 8$*; the hook-square characterisation for* $2 \le r, c \le 9$*; the Conway-pair formulas for rows (*$c \le 12$*), hooks and rectangles (*$1 \le r, c \le 8$*); the staircase, padded and doubly padded staircase values of* [1, Section 3]*; and the meld, padding and Conway–Gurvich–Ho examples quoted in the text, including* $\mathrm{CP}([4,3]) = (0,4)$ *and* $\mathrm{CP}([3,2,2,2,1]) = (1,0)$*.*

Proposition 7.2. *All* 38 *partitions drawn in Appendix A of* [1] *("some unexplained losing partitions") are losing. Moreover* $\mathcal{G}([5^5, 3^7, 1^7]) = 2$ *and* $\mathcal{G}([7^5, 5^7, 3^7]) = 0$*, confirming the two positions quoted in* [1, Section 5] *as a winning and a losing example.*

Proof. The 38 positions were transcribed from the diagram commands of the published source, and the values are entries of a single table of 4 279 positions; the 91-cell position $[7^5, 5^7, 3^7]$ has a closure of 4 096 positions. □

Roughly 1 100 published values are covered by Propositions 7.1 and 7.2. This is the guard we rely on: each of the plausible misreadings of the move relation fails one of those grids. The second guard is explicit.

Proposition 7.3. (1) $\mathcal{G}([3,3]) = 2$*, so neither* 1 *nor* 3 *is its value; and* $\mathcal{G}(S_7) = 2$*, so the value* 1 *suggested by* $\mathcal{G}(S_1) = \mathcal{G}(S_3) = \mathcal{G}(S_5) = 1$ *is not correct.*

(2) *If column moves are dropped, the resulting game has value* 1 *at* $[2]$*, whereas* $\mathcal{G}([2]) = 2$*; if only the first row may be removed, the resulting game has value* 0 *at* $[3,1]$*, whereas* $\mathcal{G}([3,1]) = 3$*. The values* 2 *and* 3 *are published in* [1]*, so either misreading would have been detected by Proposition 7.1.*

(3) *The direct implementation of column moves agrees, move list against move list, with the definition of* [1] *as conjugation followed by a row move followed by conjugation, at every one of the* 272 *partitions of size at most* 12*.*

(4) *The checker rejects a table in which a single value has been altered, and rejects a normal-play table offered as a misère one.*

Proof. Items (1) and (2) are lookups in certified tables, the one for S_7 being the closure of S_7 alone, 610 positions; the two altered games are defined by the same recursion over a restricted move set and evaluated directly. Item (3) is a sweep over the certified enumeration of the 272 partitions of size at most 12, comparing the two move lists entrywise. For item (4) the checker is run on the S_7 table with one value increased by one, and on a normal-play table presented with the misère terminal value; it rejects both. □

The first two items are the customary too-small and too-large controls: a claim proved by exhaustive search is worth little unless neighbouring false claims are seen to fail. The fourth item checks the checker, which is the one component whose failure would invalidate every value above.

8. Verification

All statements in this paper are machine-checked in Lean 4 [9] with Mathlib [10]. The game, the Grundy recursion, the table checker and its soundness proof, the partition enumeration and its completeness proof are all formalised, and every numerical claim above is derived inside the proof assistant from a checker run on an explicit table: the tables of Section 3, the sweeps of Sections 4–6 and the controls of Section 7 are all kernel-checked, with the single exception of the compiled evaluation of the checker on each table, which is the one trusted computational step and is recorded per statement. Repository reference to be added at submission.

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