EXACT $(2,0)$-PARTITION NUMBERS FOR TWO BINARY COVERING CODES OF COVERING RADIUS TWO

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let $\boldsymbol{H}$ be a parity-check matrix of a binary linear code of covering radius 2. A $(2,0)$-*partition* of its column set is a partition into nonempty blocks such that every syndrome, the zero syndrome included, is a sum of at most two columns lying in distinct blocks. The number of blocks of such a partition governs the q^m-concatenating constructions that produce infinite families of covering codes from a single short seed, and the quantity has been named in the literature since 2009; it has only ever been bounded above, by exhibiting a partition. We determine it exactly, for the two codimension-10 matrices that currently seed the covering-radius-2 families: the 10×50 matrix $\boldsymbol{H}$ of arXiv:2608.27494v1, and the 10×51 Kaikkonen–Rosendahl matrix $\boldsymbol{H}_{\mathrm{KR}}$ of 2003. In both cases the minimum number of blocks is exactly 10. For $\boldsymbol{H}$ this shows that the published ten-block partition is of minimum size, which its source leaves open in so many words; for $\boldsymbol{H}_{\mathrm{KR}}$ it produces a ten-block partition where the published one has eleven, and shows that ten cannot be improved. A consequence is negative and is the point of having a lower bound at all: neither matrix admits a partition into $2^3 = 8$ blocks, so the hypothesis $n_0 \geq 2^m \geq p(\boldsymbol{H}_0)$ of Construction QM_2^2 fails at $m = 3$ for both seeds, and the route that would have given a length-407 (respectively length-415) code at codimension 16, against the standing 431, is closed — not by a search that failed, but by an impossibility. Both impossibility halves are refutations of propositional formulas, replayed inside Lean 4 by a formally verified checker against formulas that are built, not read from a file; the soundness of the symmetry breaking used to make the refutations affordable is itself a theorem, with no appeal to compiled code.

1. INTRODUCTION

Let $C \leq \mathbb{F}_2^n$ be a binary linear code of codimension r and let $\boldsymbol{H} \in \mathbb{F}_2^{r \times n}$ be a parity-check matrix of C, with column set $S \subset \mathbb{F}_2^r$. The *covering radius* of C is the least R such that every vector of $\mathbb{F}_2^n$ lies within Hamming distance R of a codeword; equivalently, the least R such that every syndrome is a sum of at most R columns of $\boldsymbol{H}$. Write $\ell_2(r, R)$ for the least length of a binary linear code of codimension r and covering radius R. For $R = 2$ the covering condition reads

$$\{0\} \cup S \cup (S + S) = \mathbb{F}_2^r, \qquad S + S = \{\, s + s' : s, s' \in S \,\}. \tag{1}$$

Almost every upper bound on $\ell_2(r, 2)$ in the tables is produced by a propagation mechanism rather than by a direct search. The q^m-concatenating constructions, introduced by Davydov [3] and developed in a long line of work, take one short seed code and output an infinite family of covering codes of growing codimension; their hypotheses are not about the seed code alone but about a *partition* of its parity-check matrix.

Definition 1.1 ([6, Definition 3.2(i)], in the notation used here)**.** Let $\boldsymbol{H} \in \mathbb{F}_2^{r \times n}$ be a parity-check matrix of an $[n, n-r]_2 R$ code and let $0 \leq \ell \leq R$. A partition of the column set of $\boldsymbol{H}$ into nonempty subsets is called an (R, ℓ)-*partition* if every column of $\mathbb{F}_2^r$ (including the zero column) is equal to a sum of at least ℓ and at most R columns of $\boldsymbol{H}$ belonging to *distinct* subsets. For an $(R, 0)$-partition one formally treats the zero column as the sum of 0 columns.

At $(R, \ell) = (2, 0)$, which is the only case treated here, the condition says: the zero syndrome and the n single columns are free, and every remaining syndrome must be the sum of two columns lying in *different* blocks. Following [6] we write $p(\boldsymbol{H}; \Psi)$ for the number of blocks of a given $(2,0)$-partition Ψ of $\boldsymbol{H}$.

Two representative hypotheses show what the block count does. Construction QM_2^2 [6, Theorem 4.1, equations (4.2), (4.4)] takes an $[n_0, n_0 - r_0]_2 2$ seed whose parity-check matrix $\boldsymbol{H}_0$ carries a $(2,0)$-partition into $p(\boldsymbol{H}_0)$ blocks, together with an integer m subject to

$$n_0 \ \geq \ 2^m \ \geq \ p(\boldsymbol{H}_0), \tag{2}$$

and outputs an $[n, n-r]_2 2$ code with

$$n = 2^m(n_0+1) - 1, \qquad r = r_0 + 2m, \qquad p(\boldsymbol{H}_C) \leq 2^{m+1} + 1. \tag{3}$$

Construction QM_6^2 [6, Theorem 5.6] asks instead, of a seed of one specific length, for $p(\boldsymbol{H}_0) \leq 2^{\lambda_0} + 2$ with $t_0 \geq m \geq \lambda_0 + 1$. In both, a *smaller* block count widens the admissible range of m, hence the set of codimensions the seed can reach. Every published value of p is therefore an upper bound, obtained by exhibiting a partition; [6] obtains its own by unqualified "computer search".

The quantity that has never been determined. The minimum has a name. Davydov, Giulietti, Marcugini and Pambianco [5] — one of the two papers [6] cites as the source of Definition 1.1 — write, verbatim,

> "We denote by $p^{(\ell)}(V)$ the upper bound of the minimal possible cardinality of an (R,ℓ)-partition for a parity-check matrix of an $[n, n-r]_q R, \ell$ code V."

So the minimal cardinality has been named since 2009. It is only ever bounded above. The sentence quoted defines $p^{(\ell)}$ as an upper bound, and both places where [5] instantiates it record the cardinality of a partition that has been exhibited — once the *trivial* partition into singletons, which is the largest one, and once an explicit $(3,3)$-partition, into $3 \cdot 9 = 27$ subsets, of the columns of the code attached to a union of nine planes. The sentence and both instantiations stand in the section on codes of covering radius 3: in the version read here, arXiv:0904.3835v1 of 24 April 2009, that is §VI, the sentence between Theorems 6.1 and 6.2 and the instantiations inside the proofs of Theorems 6.2 and 6.4. The notation of [6] is explicit about the same thing: the text following Definition 1.1 there states that "the notation $p(\boldsymbol{H}, \ell)$ means that the matrix $\boldsymbol{H}$ *admits* an (R,ℓ)-partition into $p(\boldsymbol{H}, \ell)$ subsets". As far as the search recorded in §9 reached, no value of the minimum is determined anywhere in this literature, for any matrix. This paper determines two of them.

We write

$$p_{\min}(\boldsymbol{H}) \ \triangleq \ \min\{\, p(\boldsymbol{H}; \Psi) \ : \ \Psi \text{ a } (2,0)\text{-partition of } \boldsymbol{H} \,\},$$

a quantity invariant under $GL(r,2)$ and under permutation of the columns. It is defined whenever $\boldsymbol{H}$ has distinct columns and covering radius at most 2, since the partition into singletons is then a $(2,0)$-partition: a syndrome that is a sum of two *equal* columns is 0, which the definition treats as a sum of no columns at all. So the minimum is taken over a nonempty set, and $p_{\min}(\boldsymbol{H}) \leq n$.

The two matrices. Both matrices sit at codimension $r = 10$, which is the seed row of the whole covering-radius-2 table.

$\boldsymbol{H}_{\mathrm{KR}}$ is the 10×51 parity-check matrix $[\, I_{10} \mid M_{\mathrm{KR}} \,]$ of the $[51, 41]_2 2$ code of Kaikkonen and Rosendahl [7], printed in hexadecimal in [6, Theorem 4.3]. It gives $\ell_2(10,2) \leq 51$, the entry that stood in row $r = 10$ of the tables from 2003, and it seeds the infinite family of [6]. Theorem 5.2 of [6] exhibits an eleven-block $(2,0)$-partition Ψ_{KR} of it, obtained by computer search, and records $p(\boldsymbol{H}_{\mathrm{KR}}; \Psi_{\mathrm{KR}}) = 11$ in Table 5.1 as an important new result.

$\boldsymbol{H}$ is the 10×50 matrix of [10, Theorem 1], found by simulated annealing on column sets, which gives $\ell_2(10,2) \leq 50$ — one column below the Kaikkonen–Rosendahl length. That paper also exhibits a $(2,0)$-partition of $\boldsymbol{H}$ into ten blocks, and closes the subsection that contains it with one sentence:

> "Minimality of the 10-block partition remains open."

Its authoring repository is more explicit still: the script that searched for a nine- or eight-block partition records in its own documentation that the failure to find one is "an incomplete search, not a lower bound on $p(\boldsymbol{H})$".

What is settled here.

- $p_{\min}(\boldsymbol{H}) = 10$ (Theorem 4.1). No $(2,0)$-partition of the fifty columns of $\boldsymbol{H}$ into nine or fewer blocks exists, so the ten-block partition of [10, Table 2] is of minimum size. This settles the sentence quoted above.
- $p_{\min}(\boldsymbol{H}_{\mathrm{KR}}) = 10$ (Theorem 4.2). We exhibit a ten-block $(2,0)$-partition of the Kaikkonen–Rosendahl matrix, one block below the published eleven, and show that nine is impossible.
- Neither matrix admits a $(2,0)$-partition into $2^3 = 8$ blocks (Theorem 5.1), so the hypothesis (2) of Construction QM_2^2 fails at $m = 3$ for both. Had it held, (3) would have given a covering code of length $8 \cdot 51 - 1 = 407$ from $\boldsymbol{H}$, or $8 \cdot 52 - 1 = 415$ from $\boldsymbol{H}_{\mathrm{KR}}$, at codimension $r = 16$, against the standing entry $431 = \phi(16)$ of [6, Table 5.1], due to Davydov and Drozhzhina-Labinskaya [4] — and $r = 16$ is one of the codimensions that the iteration of [10] explicitly leaves as a gap. That route is now closed.

What is not claimed. Theorem 4.2 does not improve any published bound on a length function. The hypothesis (2) at $n_0 = 51$ forces $2^m \geq 11$ at the published block count and $2^m \geq 10$ at ours, and both give $m \geq 4$; the drop from eleven to ten changes nothing there. Its content is the *exactness*, and the ten-block partition it exhibits. Whether the drop crosses a power-of-two boundary in the hypotheses of QM_3^2 or QM_6^2 is a question about those constructions that we have not worked out and do not assert.

Nothing here is a correction to [6]. Its quantity $p(\boldsymbol{H}_{\mathrm{KR}}; \Psi_{\mathrm{KR}}) = 11$ is the cardinality of the partition Ψ_{KR} that its Theorem 5.2 exhibits, not a minimum; the paper claims minimality nowhere, and we recompute both its partition and its dependent-triple statement and find them correct as printed (§3). See Remark 4.3.

Nor do we claim $\ell_2(10,2) \leq 50$, or the matrix, or the ten-block partition of $\boldsymbol{H}$: those are the results of [10], which we recompute (§3) but do not extend. On the other side, we prove nothing about $\ell_2(10,2) \geq 46$; the sphere-covering bound $\ell_2(10,2) \geq 45$ (from $1 + n + \binom{n}{2} = 991 < 1024$ at $n = 44$) is all that is available, and whether a covering 49-set exists is open, as [10] says.

Finally, all novelty claims in this paper are "not found" claims, and §9 records exactly which searches were run, what each returned, and where the coverage stops. The gap that matters is the pre-arXiv one: the proceedings literature of the 1990s, in which a partition minimum is a natural thing to have computed in passing, was not read.

2. The partition number

Throughout, $r = 10$ and a column of $\mathbb{F}_2^{10}$ is identified with the integer whose bit i (counting from the least significant) is row $i+1$ of the matrix. Addition of columns is bitwise exclusive-or, written $\oplus$. We write $S = (s_0, \ldots, s_{n-1})$ for the column list of a matrix, keeping the left-to-right order of the printed matrix, and we index columns by their position.

It is convenient to replace a partition by a labelling.

Definition 2.1. Let S be a list of n columns and $k \geq 1$. A map $c\colon \{0, \ldots, n-1\} \to \{0, \ldots, k-1\}$ is a $(2,0)$-*labelling of S into k blocks* if for every $g \in \mathbb{F}_2^{10}$ either $g = 0$, or $g = s_i$ for some i, or there are i, j with $c(i) \neq c(j)$ and $s_i \oplus s_j = g$.

A $(2,0)$-labelling into k blocks and a $(2,0)$-partition into at most k nonempty blocks are the same datum: the nonempty fibres of c are the blocks, and conversely a partition into $m \leq k$ blocks is a labelling after any injective naming of the blocks by $\{0, \ldots, k-1\}$. Two consequences are used constantly and are immediate from the definition.

Lemma 2.2. *If c is a $(2,0)$-labelling of S into k blocks and $k \leq k'$, then c is a $(2,0)$-labelling into k' blocks. Hence if no $(2,0)$-labelling into k blocks exists, none into $k' \leq k$ blocks exists either, and $p_{\min}(S) > k$.*

Lemma 2.3. *Let c be a $(2,0)$-labelling of S into k blocks and let σ be any map $\{0, \ldots, k-1\} \to \{0, \ldots, k-1\}$ that is injective. Then $\sigma \circ c$ is again a $(2,0)$-labelling into k blocks.*

Proof. The defining condition mentions block labels only through the relation $c(i) \neq c(j)$, which an injective σ preserves in both directions. □

Lemma 2.2 is why a single refutation at $k = 9$ settles every $k \leq 9$ at once, and Lemma 2.3 is half of the symmetry-breaking argument of §6. The other half is a forcing observation.

Definition 2.4. A syndrome $g \in \mathbb{F}_2^{10}$ *needs a pair* if $g \neq 0$ and g is not a column of S. A pair of column positions $\{a, b\}$ is a *forced edge* of S if the syndrome $s_a \oplus s_b$ needs a pair and $\{a, b\}$ is its *only* covering pair. The *forced-edge graph* of S has the column positions as vertices and the forced edges as edges.

Lemma 2.5. *Every* $(2, 0)$*-labelling of* S *gives the two endpoints of a forced edge distinct labels. Consequently every* $(2, 0)$*-labelling is a proper colouring of the forced-edge graph, and a clique of that graph receives pairwise distinct labels.*

Proof. Let $\{a, b\}$ be a forced edge and $g = s_a \oplus s_b$. Since g needs a pair, the labelling condition at g supplies i, j with $c(i) \neq c(j)$ and $s_i \oplus s_j = g$; since $\{a, b\}$ is the only pair with that sum, $\{i, j\} = \{a, b\}$. □

The converse of Lemma 2.5 is false, and by a wide margin: see Proposition 5.2.

3. The two matrices, and their published data

Every number quoted from [10] and [6] in this section was recomputed from the column integers alone before any of the new results were attempted, and each recomputation is a machine-checked statement (§8). None of it is new; it is recorded because the theorems of §4 are statements about these specific matrices, and a transcription error would make them statements about nothing.

The fifty-column matrix. $\boldsymbol{H}$ is the 10×50 matrix whose columns, as LSB-first integers in the order of [10, Table 1], are

1	2	4	15	16	32	65	86	128	173
183	202	212	247	256	297	320	329	341	366
373	381	391	403	438	460	479	491	502	559
576	608	653	734	742	754	771	777	789	821
846	855	869	881	893	897	927	981	1003	1004

Proposition 3.1. *The fifty integers above are distinct, nonzero, and less than* 2^{10}*. Moreover:*

1. *the* $1 + 50 + \binom{50}{2} = 1276$ *sums of at most two columns hit all* 1024 *syndromes, with multiplicity histogram* {1:859, 2:129, 4:24, 5:9, 6:3}*; so* (1) *holds and the covering density is* $\mu = 1276/1024 = 319/256$*;*
2. *exactly* 973 *syndromes need a pair, so the covering radius is exactly* 2*, and their pair-multiplicity histogram is* {1:821, 2:123, 4:19, 5:8, 6:2}*; in particular the forced-edge graph of* $\boldsymbol{H}$ *has* 821 *edges;*
3. $\boldsymbol{H}$ *has exactly* 10 *linearly dependent triples of columns, so its minimum distance is* 3*, and exactly one of them,* $(491, 734, 821)$*, meets three distinct blocks of the partition of* [10, Table 2]*;*
4. *every single-column deletion leaves at least* 9 *of the* 1024 *syndromes uncovered, and the minimum* 9 *is attained exactly at the columns* 381*,* 479*,* 927*;*
5. *the partition of* [10, Table 2] *is a* $(2, 0)$*-partition of* $\boldsymbol{H}$ *into ten nonempty blocks.*

Items (2) and (3) pin each other: the ten dependent triples consume $3 \cdot 10 = 30$ of the $\binom{50}{2} = 1225$ pairs, and $1225 - 30 = 1195$ is exactly the sum $821 + 2 \cdot 123 + 4 \cdot 19 + 5 \cdot 8 + 6 \cdot 2$ of the pair-multiplicity histogram, so the triple count and the histogram determine each other.

One further check on the transcription is deliberately outside the formal development, because it is a statement about a printed page rather than about the matrix: the fifty integers above are exactly what the bit matrix of Appendix A of [10] encodes LSB-first, row by row. That was verified separately, as was the agreement of both column lists and of the ten-block partition with the files in the artifact repository of [10] (§9).

Proposition 3.2. *Let $\boldsymbol{H}$ be read as a matrix over $\mathbb{Z}/2$, let $\sigma_{\boldsymbol{H}}(x) = \boldsymbol{H}x$ be its syndrome map on $(\mathbb{Z}/2)^{50}$, and let $C = \ker \sigma_{\boldsymbol{H}}$. Then $\sigma_{\boldsymbol{H}}$ is surjective, so C is a $[50, 40]_2$ code; every $x \in (\mathbb{Z}/2)^{50}$ satisfies $d(x, C) \leq 2$; and some x satisfies $d(x, C) \geq 2$. Hence C has covering radius exactly 2 and $\ell_2(10, 2) \leq 50$.*

This is [10, Theorem 1]. We restate it because the statement we prove is about the linear-algebraic objects — the matrix over $\mathbb{Z}/2$, its kernel, and Hamming distance — rather than about the bitmask condition (1), and because the “exactly 2” matters: $\ell_2(r, R)$ is the least length *at* covering radius R, so a definition with “at most R” would enlarge the set the infimum runs over and weaken the bound. Surjectivity is not a separate check: the covering condition exhibits every syndrome as $\sigma_{\boldsymbol{H}}(e)$ for some e of weight at most two. The witness for the third clause is the syndrome 3, which is neither 0 nor a column of $\boldsymbol{H}$.

The Kaikkonen–Rosendahl matrix. $\boldsymbol{H}_{\mathrm{KR}} = [\, I_{10} \mid M_{\mathrm{KR}} \,]$ is reconstructed from the hexadecimal listing of [6, Theorem 4.3], which is MSB-first (row 1 is the high bit), by reversing the ten bits of each entry. In LSB-first integers its columns are

1	2	4	8	16	32	64	128	256	512
438	806	206	902	446	956	474	10	240	421
977	771	519	667	738	790	185	317	169	424
960	520	939	65	176	595	709	739	687	594
943	948	223	285	241	724	459	682	413	571
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Proposition 3.3. *The 51 integers above are distinct, nonzero and less than 2^{10}; they satisfy (1), so the Kaikkonen–Rosendahl code has covering radius at most 2 (*[6, Theorem 4.3] *states that it is exactly 2); the columns in positions 5, 27, 29 sum to zero; and the eleven-block partition Ψ_{KR} of* [6, Theorem 5.2]*, read as a partition of the column positions, is a $(2,0)$-partition into eleven nonempty blocks in which those three positions lie in three distinct blocks.*

The reconstruction is determined only up to $GL(10, 2)$, which is exactly the equivalence under which $p_{\min}$ is invariant, so nothing below depends on the choice; and it is pinned by three independent statements of [6] at once — the covering property, the dependent triple of its Theorem 5.2(ii), and the validity of Ψ_{KR} indexed by column position. A misread reconstruction that failed any of the three would be visible; see Remark 7.2 for exactly how much of that is true.

4. The minimum is ten

Theorem 4.1. $p_{\min}(\boldsymbol{H}) = 10$. *That is: the ten blocks of* [10, Table 2] *form a $(2,0)$-partition of the fifty columns of $\boldsymbol{H}$, and there is no $(2,0)$-partition of those columns into nine or fewer blocks.*

Theorem 4.2. $p_{\min}(\boldsymbol{H}_{\mathrm{KR}}) = 10$. *A $(2,0)$-partition of the fifty-one columns of $\boldsymbol{H}_{\mathrm{KR}}$ into ten blocks is, by column position in the numbering of* [6, Theorem 5.2]*,*

$$\{1,3,9,22,23,27,29,30,33,39\},\quad \{2,4,10,11,15,16,18,20,32,44\},$$
$$\{5,6,8,12,14,24,26,28,41,49\},\quad \{7,19,25,34,35,36,38,40,48\},$$
$$\{13,17,21,31,37,43,46\},\quad \{42\},\ \{45\},\ \{47\},\ \{50\},\ \{51\},$$

and there is no $(2,0)$-partition of those columns into nine or fewer blocks.

Proof of Theorems 4.1 and 4.2. Each theorem has an easy half and a hard half, and only the hard half rests on exhaustive computation.

Upper half. For $\boldsymbol{H}$, the ten-block labelling of [10, Table 2] is checked against Definition 2.1 by one pass over the 1024 syndromes, each syndrome requiring a scan of the $\binom{50}{2} = 1225$ pairs in the worst case; the check passes, and the labelling uses ten distinct labels. This is item (5) of Proposition 3.1. For $\boldsymbol{H}_{\mathrm{KR}}$ the same check is run on the ten-block labelling displayed above, over 1024 syndromes and $\binom{51}{2} = 1275$ pairs. That labelling was located outside the formal development by a satisfiability solver, in 0.03 s; how it was found is irrelevant to the theorem, since the labelling is exhibited and re-verified from scratch.

Lower half. By Lemma 2.2 it suffices to refute $k = 9$. For each matrix we build a propositional formula $\Phi(S, 9, V)$ whose satisfying assignments include the assignment read off any $(2, 0)$-labelling of S into nine blocks that sends the columns listed in V to the blocks $0, 1, \ldots$ in order (§6, Theorem 6.1), and we refute it. The set V is a maximum clique of the forced-edge graph: six columns for $\boldsymbol{H}$, five for $\boldsymbol{H}_{\mathrm{KR}}$. By Lemma 2.5 and Lemma 2.3 the restriction to labellings that pin V costs nothing — this is Theorem 6.2 — so an unsatisfiable Φ refutes *every* nine-block labelling. The formula for $\boldsymbol{H}$ has 2950 variables and 13,854 clauses; the one for $\boldsymbol{H}_{\mathrm{KR}}$ has 3060 variables and 14,339 clauses. Their refutations are LRAT proofs of 11,713 and 69,590 lines (1.13 MB and 10.33 MB), produced by `cadical` [1] version 2.1.2 in 0.10 s and 1.40 s of wall clock on a shared machine, and each is replayed against the corresponding formula by a formally verified checker (§8). No enumeration over labellings takes place at any point: there are $9^{50} \approx 5.2 \cdot 10^{47}$ maps from the columns of $\boldsymbol{H}$ to nine blocks, and $9^{51} \approx 4.6 \cdot 10^{48}$ for $\boldsymbol{H}_{\mathrm{KR}}$.

The two halves together give the value of the minimum. □

Remark 4.3. Theorem 4.2 is not in tension with [6, Theorem 5.2], and we want to be exact about why, because the numbers 10 and 11 invite the wrong reading. The quantity that paper states is $p(\boldsymbol{H}_{\mathrm{KR}}; \Psi_{\mathrm{KR}}) = 11$: the second argument is the exhibited partition, and the quantity is its cardinality. The text of [6] that follows Definition 1.1 is explicit that the Ψ-free notation $p(\boldsymbol{H}, \ell)$ likewise "means that the matrix $\boldsymbol{H}$ admits an (R, ℓ)-partition into $p(\boldsymbol{H}, \ell)$ subsets". Neither notation is a minimum, and no minimality is claimed there or anywhere else in that paper. We recomputed Ψ_{KR} and found it valid with eleven blocks, exactly as printed (Proposition 3.3). What Theorem 4.2 adds is the value of $p_{\min}(\boldsymbol{H}_{\mathrm{KR}})$, which is a different quantity and was not previously known for any matrix. All statements about [6] in this paper — every theorem, equation and table number included — refer to arXiv:2511.02542v1 as posted on 4 November 2025, and were resolved by compiling that source; the numbering of the journal version listed in the bibliography was not checked.

5. Consequences

Construction QM_2^2 **at** $m = 3$**.**

Theorem 5.1. *Neither* $\boldsymbol{H}$ *nor* $\boldsymbol{H}_{\mathrm{KR}}$ *admits a* $(2, 0)$*-partition into* $2^3 = 8$ *blocks.*

Proof. Immediate from the lower halves of Theorems 4.1 and 4.2 by Lemma 2.2, since $8 \le 9$. □

The interest is in what the hypothesis (2) does with it. With $n_0 = 50$ and $p(\boldsymbol{H}_0) = 10$, (2) permits exactly $m = 4$ and $m = 5$, as [10, §2.5] observes; the next value down, $m = 3$, would need a seed partition into at most $2^3 = 8$ blocks. By (3) such a seed would produce

$$n = 2^3(50 + 1) - 1 = 407 \quad \text{from } \boldsymbol{H}, \qquad n = 2^3(51 + 1) - 1 = 415 \quad \text{from } \boldsymbol{H}_{\mathrm{KR}},$$

both at codimension $r = 10 + 2 \cdot 3 = 16$, where the standing table entry is $\phi(16) = 27 \cdot 2^4 - 1 = 431$ [6, Table 5.1], [4]. A dent of 24 or 16 columns at a codimension that the iteration of [10] lists among its own gaps ($r = 12, 14, 16, 22, 24, 26, 28, 42, 44$) would have been worth having. Theorem 5.1 says it is not available from either seed — and the distinction from the pre-existing state of knowledge is the whole point: it is not that a search for an eight-block partition failed, it is that no eight-block partition exists.

Two boundaries of this consequence should be stated plainly. It closes $m = 3$ for QM_2^2 *from these two seeds only*; some other $[n_0, n_0 - 10]_2 2$ code at $r_0 = 10$, if one exists with $n_0 \ge 8$ and an eight-block partition, is untouched by it. And the arithmetic in the display above, together with the reading of the hypothesis and output of QM_2^2, is taken from [6, equations (4.2), (4.4)] and is not part of the machine-checked development; what is machine-checked is Theorem 5.1.

The obstruction is not a colouring bound. By Lemma 2.5 every $(2,0)$-partition properly colours the forced-edge graph, so its chromatic number is a lower bound for $p_{\min}$. It is a weak one.

Proposition 5.2. *The forced-edge graph of* $\boldsymbol{H}$ *has* 821 *edges and chromatic number exactly* 6.

Proof. For the upper bound, an explicit map from the fifty columns to six colours is exhibited and checked against all 821 edges. For the lower bound, the six columns $1, 2, 4, 15, 32, 893$ — positions $0, 1, 2, 3, 5, 44$ — are pairwise joined by forced edges, which is checked pair by pair from Definition 2.4; by Lemma 2.5 any proper colouring gives them six distinct colours. Both halves are finite computations over the column list; the clique was located by a Bron–Kerbosch search and the colouring by a satisfiability solver, and both are re-verified from scratch. □

Since $6 < 10$, no colouring bound on the forced edges can reach Theorem 4.1: the four extra blocks are forced entirely by the 152 syndromes with two to six covering pairs, each of which demands only that *at least one* of its pairs be split. That disjunction is why the problem is stated as a $(2,0)$-partition rather than as a graph colouring, and why the lower half of Theorem 4.1 is a propositional refutation rather than a clique argument. Proposition 5.2 is a routine computation about a 50-vertex graph; we record it because it is the natural first attempt at a lower bound, and it fails by four.

6. The refutation, and why it may be trusted

The two lower halves of §4 are the only places where an exhaustive machine search enters, and this section says exactly what is being trusted there. Two obligations stand between a solver's verdict of unsatisfiability and a theorem about partitions. Both are discharged as ordinary proofs, with no appeal to computation.

The formula. Fix a column list S of length n, a block count K, and a list V of column positions. The variables are $x_{i,b}$ for $i < n$, $b < K$ ("column i sits in block b"), laid out at index $iK + b$, and $y_{i,j}$ for $i < j < n$ ("columns i and j sit in distinct blocks"), laid out at index $nK + in + j$. The clauses of $\Phi(S, K, V)$ are, in this order:

(1) for each $i < n$, the clause $\bigvee_{b<K} x_{i,b}$;
(2) for each $i < n$ and each $b < b' < K$, the clause $\neg x_{i,b} \vee \neg x_{i,b'}$;
(3) for each $i < j < n$ and each $b < K$, the clause $\neg y_{i,j} \vee \neg x_{i,b} \vee \neg x_{j,b}$;
(4) for each syndrome g that needs a pair, the clause $\bigvee y_{i,j}$ over the pairs $i < j$ with $s_i \oplus s_j = g$;
(5) for each position t in V, the unit clause $x_{V_t,t}$.

Group (5) is the symmetry break. The order of the clause list is load-bearing, because an LRAT certificate names input clauses by position.

Theorem 6.1 (Encoding faithfulness)**.** *Let c be a $(2,0)$-labelling of S into K blocks with $c(V_t) = t$ for every position t of V, and let A be the assignment with $A(x_{i,b})$ true iff $c(i) = b$ and $A(y_{i,j})$ true iff $c(i) \neq c(j)$. Then A satisfies $\Phi(S, K, V)$. Consequently, if $\Phi(S, K, V)$ is unsatisfiable then no such labelling exists.*

Only that direction is proved and only that direction is used. The converse — that a satisfying assignment yields a labelling — is never needed, which is why the presence of extra clauses costs nothing: an unsatisfiable superset of what a labelling satisfies still refutes the labelling. In particular the variables $y_{i,j}$ for the 30 pairs consumed by dependent triples never occur positively, and are harmless.

Soundness of the symmetry break. Pinning columns to blocks is *not* an implied constraint, and an unsound narrowing of the search space looks, from outside, exactly like a good optimisation. It is therefore proved.

Theorem 6.2 (Normalisation)**.** *Let V be a list of column positions that are pairwise joined by forced edges of S, with $|V| \le K$. Then every $(2,0)$-labelling c of S into K blocks can be replaced by a $(2,0)$-labelling d into K blocks with $d(V_t) = t$ for every t.*

Proof. Induction on t. Suppose $c(V_s) = s$ for all $s < t$. Let $a = c(V_t)$ and let τ be the transposition of the labels a and t; then $\tau \circ c$ is a $(2,0)$-labelling into K blocks by Lemma 2.3, and it sends V_t to t. It still sends V_s to s for $s < t$: the label s is moved by τ only if $s = a$ or $s = t$, the second is excluded by $s < t$, and the first would say $c(V_s) = c(V_t)$, contradicting Lemma 2.5 at the forced edge $\{V_s, V_t\}$. After $|V|$ steps the labelling is normalised. □

Combining Theorems 6.1 and 6.2: if $\Phi(S, K, V)$ is unsatisfiable and the columns of V are pairwise forced apart, then S has no $(2,0)$-labelling into K blocks at all, with no residual hypothesis about pinning. The forced-edge conditions on the pairs of V are themselves finite computations over the column list, not assumptions.

The searches that were run. The finite computations behind the results fall into three kinds, and it is worth separating them by what each establishes.

Exhaustive, and load-bearing. Two: the refutations of $\Phi(\boldsymbol{H}, 9, V_{\boldsymbol{H}})$ and $\Phi(\boldsymbol{H}_{\mathrm{KR}}, 9, V_{\boldsymbol{H}_{\mathrm{KR}}})$, with $|V_{\boldsymbol{H}}| = 6$ and $|V_{\boldsymbol{H}_{\mathrm{KR}}}| = 5$. These are the lower halves of Theorems 4.1 and 4.2 and hence of Theorem 5.1, and their sizes are in the proof in §4.

Non-exhaustive, and not load-bearing. The ten-block labelling of $\boldsymbol{H}_{\mathrm{KR}}$, the six-colouring of the forced-edge graph, and the maximum-clique computation were all located by search, and none of them is trusted: the first two are exhibited and re-verified, and the clique is re-verified edge by edge. A better or worse search would change nothing.

Finite verification of published data. The statements of §3 are complete sweeps: over the 1024 syndromes, over the $\binom{n}{2}$ pairs, over the 50 single-column deletions, over the column triples.

The formula built inside the proof assistant was also dumped to DIMACS and compared byte for byte with the files the solver read, for the two refuted instances and for the satisfiable instances at $k = 10$; and the verdicts were reproduced with a deliberately different encoding, in which the auxiliary variables are numbered first rather than last, the block variables are laid out block-major rather than column-major, the 821 unique-pair syndromes get direct binary clauses instead of auxiliary variables, and the symmetry break is value precedence instead of a clique. That second encoding agrees on all eight cells $(\boldsymbol{H}, \boldsymbol{H}_{\mathrm{KR}}) \times (k = 8, 9, 10, 11)$: unsatisfiable at $k = 8, 9$ and satisfiable at $k = 10, 11$ in both rows. The cross-check is corroboration; the theorems rest on the clique break, which is proved.

Remark 6.3 (Outside the formal development). The symmetry break is what makes the computation affordable, by three orders of magnitude, and the measurements that show it are worth recording even though they are not theorems. On the nine-block formula for $\boldsymbol{H}$, with the same encoding and the number of pinned columns varied, `cadical` produced: with none pinned, a 9.09 GB LRAT proof in 28 minutes without terminating, at which point it was killed; with three, unsatisfiable in 13.2 s and 63.5 MB; with four, 1.44 s and 10.1 MB; with five, 0.47 s and 2.6 MB; with six, 0.10 s and 1.13 MB. These are wall-clock measurements on a shared machine and should be read to within perhaps 25%. They are the reason Theorem 6.2 is proved rather than the symmetry break dropped: without it the certificate is far past the size at which a verified checker can replay it, and the run does not terminate in an hour.

7. Controls

A development in which the $(2,0)$ condition were accidentally vacuous, or in which the checker accepted anything, would produce every theorem above while proving nothing. Seven failure modes are ruled out explicitly.

Proposition 7.1. (1) The condition is not vacuous. *The one-block labelling of the fifty columns of $\boldsymbol{H}$ is not a $(2,0)$-labelling. This is checked directly, not deduced from Theorem 4.1.*

(2) The minimum is a minimum from below. *Nine blocks is not attained (Theorem 4.1), so* $p_{\min}(\boldsymbol{H}) \neq 9$.

(3) And from above. *Eleven blocks is attained but is not the minimum, since ten is; so* $p_{\min}(\boldsymbol{H}) \neq 11$.

(4) Nine blocks fails narrowly, and concretely. *Blocks* $\{86\}$ *and* $\{403\}$ *of* [10, Table 2] *are singletons, so merging them is the cheapest route to nine blocks. The result has nine labels and leaves exactly one syndrome with no bichromatic covering pair, where the published partition leaves none.*

(5) The covering condition is not vacuous. *Deleting the column* 381 *from* $\boldsymbol{H}$ *leaves exactly nine syndromes uncovered.*

(6) The forced-edge hypothesis is not vacuous. *The columns* 1 *and* 65 *are not a forced edge; of the* 1225 *pairs of columns of* $\boldsymbol{H}$*,* 821 *are forced and* 404 *are not. So Theorem 6.2 was applied to a real hypothesis.*

(7) The certificate replay is not vacuous. *The nine-block certificate for* $\boldsymbol{H}$ *verifies against* $\Phi(\boldsymbol{H}, 9, V_{\boldsymbol{H}})$*, and is* rejected *against the satisfiable formula* $\Phi(\boldsymbol{H}, 10, V_{\boldsymbol{H}})$*, and rejected against the nine-block formula of the other matrix; that formula's own certificate is in turn rejected against* $\Phi(\boldsymbol{H}, 9, V_{\boldsymbol{H}})$*. Only the intended pairings verify.*

Item (7) is the one that matters most, because a checker that returned "true" unconditionally would validate both lower halves and be invisible everywhere else.

Remark 7.2 (A correction to one sentence of [10])*.* Section 4.3 of [10] says of the LSB-first/MSB-first convention: "A verifier that misses the reversal fails on the Kaikkonen–Rosendahl baseline and only there, which makes the mistake visible." The *coverage* check does not see the mistake. Reading the hexadecimal listing of [6, Theorem 4.3] without the reversal produces a 10×51 matrix with 51 distinct nonzero columns that still satisfies (1) on all 1024 syndromes. It cannot see it: bit reversal is a linear automorphism of $\mathbb{F}_2^{10}$ under which the identity block $\{1, 2, 4, \ldots, 512\}$ is stable as a set, so the misread matrix has the same column set as the image of the correct one under an element of $GL(10,2)$, and (1) is $GL(10,2)$-invariant. What does detect the misreading is either check indexed by column *position*: the guard of [6, Theorem 5.2(ii)] becomes $h_5 \oplus h_{27} \oplus h_{29} = 48 \neq 0$, and Ψ_{KR} leaves 27 syndromes unsplit. No result of [10] is affected — its own verifiers do check the partition, and the guard it names is exactly one of the two checks that work — but a reader who implemented only the coverage check would not catch the error the sentence promises to make visible.

8. Formal verification

Every lemma, proposition and theorem above is proved in Lean 4 [9] (toolchain `v4.32.0`), the code-theoretic Proposition 3.2 against Mathlib [8] and the rest against the core library alone; the repository reference is to be added at submission. The remarks are the exception, and are marked as such where they stand: Remark 4.3 reads the sources, Remark 6.3 reports measurements, and Remark 7.2 mixes a quotation with two computational claims — that the misread reconstruction still satisfies (1), and that the two position-indexed checks fail on it — which are proved with the rest. Two further things are checks made outside Lean and are labelled as such where they stand: the transcription of Appendix A of [10] (§3) and the comparisons against that paper's artifact repository (§9); and where a proposition draws a one-line consequence from checked numbers — the covering density in Proposition 3.1(1), the minimum distance in its item (3) — the consequence is arithmetic from the machine-checked facts beside it, not a further check. The division of trust is the point of the arrangement. Lemmas 2.2, 2.3 and 2.5, Theorem 6.1, Theorem 6.2, the passage from an unsatisfiable formula to the non-existence of a labelling, the assembly of the two halves into the value of $p_{\min}$, and the bridge of Proposition 3.2 from bitmasks to vectors over $\mathbb{Z}/2$ are ordinary proofs: their axiom sets contain nothing beyond `propext`, `Classical.choice` and `Quot.sound`, the three axioms of Lean's logical foundation, and no compiled code enters them. In particular the entire soundness layer — encoding faithfulness *and* symmetry-break soundness — is kernel-checked, so what is certified is the minimality of the partition itself and not merely the unsatisfiability of a formula. Native evaluation of compiled code, which adds the axiom `Lean.ofReduceBool` and places the Lean compiler in the trusted base, appears in exactly two roles: evaluating finite Boolean sweeps on concrete data (the histograms and validity checks of §3, the forced-edge conditions, the controls), and running Lean core's formally verified LRAT checker on the two certificates. The checker itself is not trusted, being verified inside Lean; the solver that produced the certificates is not trusted at all, since a wrong certificate makes the checker return false — and Proposition 7.1(7) shows that it does. Nor is the DIMACS file trusted: the formula is built by a function inside the proof assistant, and any disagreement in clause order, literal order or clause content would make the checker reject rather than produce a wrong theorem. Checking the whole development, including both certificate replays, takes about 160 s on one core of a shared machine; the module holding the two certificate replays accounts for about 15 s of that, at a peak resident set of 1.7 GB, and the Mathlib-importing module of Proposition 3.2 is the memory peak, at about 7 GB. There is no `sorry` anywhere in it.

9. Provenance of the novelty claims

Three claims above are claims of novelty — Theorems 4.1, 4.2 and 5.1 — and each is a "not found" claim resting on a search rather than on a proof. The search was run on 2026-09-02 and 2026-09-03 and covered: a local full-text arXiv corpus (369 shards, coverage through the 2026-09-02 announcement window), from which thirteen core covering-code papers were extracted whole and read; the live arXiv API with a positive control; OpenAlex; Semantic Scholar; and web search. What it found is stated in §1: the minimal cardinality of an (R, ℓ)-partition is named in [5] and only ever bounded above, that sentence is the sole minimality phrase applied to a partition in the thirteen papers, and no exact-method paper on covering-code partitions surfaced in any channel — every published partition in this literature is the output of an unqualified "computer search". Only 22 papers in the corpus mention Kaikkonen at all, and only [6] and [10] in the covering-code sense; [6] is the only source giving a number for $p(\boldsymbol{H}_{\mathrm{KR}})$.

The search is incomplete in three specific ways, and we prefer to say so.

(1) *The citation graph is thin, and one channel of it is provably wrong.* OpenAlex resolves both [10] and [6] with a citation count of zero, which is a demonstrated false negative: [10] cites [6], and OpenAlex does not know it. Semantic Scholar, the channel that corrects this, was unavailable when the search first ran (HTTP 429 on every call including its control) and was re-run successfully on 2026-09-03, control included: it records exactly one citation of [6], namely [10], and none of [10]. So the two channels now agree on everything except the one edge OpenAlex misses, and no third party has cited either paper. That is a small graph on two fresh preprints, and it is evidence of absence only to the extent that indexing has caught up.

(2) *Pre-arXiv literature is invisible to all of these channels.* The 1990 and 1994 papers of Davydov and Drozhzhina-Labinskaya, Davydov's 1995 work, §5.4 of the Cohen–Honkala –Litsyn–Lobstein book [2], and the ACCT and WCC proceedings of the 1990s are not in the corpus and were not read. [10] raises the same caveat about its own bound, writing that a proof of $\ell_2(10, 2) \leq 50$ "may exist in older proceedings"; the caveat applies at least as strongly to a partition minimum, which is a natural thing to have computed in passing. What can be said from inside the arXiv era is only indirect: [6] attributes Definition 1.1

to [4, Definition 2.1] and to [5, Definition 2.1], and lists [2, §5.4] among the places the q^m-concatenating constructions were developed; neither [6] nor [5] attributes to any of them a determination of a minimum. A reader with library access should still check those sources, and Giulietti's 2013 survey, directly.

(3) *Both sources are at their first arXiv version, and one has since been published.* On 2026-09-03 the arXiv metadata showed [10] at arXiv:2608.27494v1 of 26 August 2026 and [6] at arXiv:2511.02542v1 of 4 November 2025, with no later version of either; so neither has answered its own question in a revision. [6] has however appeared in *IEEE Transactions on Information Theory* (early access, 2026), with the same abstract as the preprint; that version is behind a paywall and was not read, and everything this paper says about [6] is pinned to the arXiv version (Remark 4.3).

One further piece of evidence bears on Theorem 4.1 specifically, and it comes from the author of [10]. That paper names a public artifact repository and a commit; the matrix, the ten-block partition and an independent reconstruction of $\boldsymbol{H}_{\mathrm{KR}}$ in it agree element for element with the data used here, which were obtained separately, from the printed tables and from the hexadecimal listing of [6]. The repository was read again on 2026-09-03 and had not been pushed to since 2026-08-30, four days after the arXiv posting. Its main branch carries further results not in the paper, each with an explicit disclaimer of minimality attached — "no minimality is claimed", "no minimality of 64 is claimed", "no partition minimality is claimed" — and none of them concerns codimension 16 or the minimum of p for either matrix here. Its partition-search script, identical at the pinned commit and on the main branch, records that the failure to find a nine- or eight-block partition was "an incomplete search, not a lower bound". For $\boldsymbol{H}_{\mathrm{KR}}$ the repository reconstructs the fifty-one-column set but contains no partition of it at all, and its own notes treat $p(\boldsymbol{H}_{\mathrm{KR}}) = 11$ as the incumbent to beat.

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