DISTINCT COVERING SYSTEMS WITH ALL MODULI IN $[n, kn]$: AN ITERATED REDUCTION, AND THE FIRST CELLS BEYOND $k = 10$

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Abstract. A covering system is *distinct* when its moduli are pairwise different. Dalton and Jones proved that for every integer $m \geq 3$ no distinct covering system has all its moduli in the interval $[m, 10m]$, and asked the same question for $[n, 12n]$ with $n \geq 4$ and for $[n, 15n]$ with $n \geq 5$. Their proof filters the interval by Krukenberg's discard lemma and then applies the density condition; for twenty-three exceptional values of m it runs a bespoke argument, one ingredient of which is to recount the surviving moduli after a reduction and reduce again. We turn that recount into a uniform, machine-checkable certificate: a list of prime-power steps, each licensed by a count taken inside the set the earlier steps left, together with an integer test on the reciprocal sum of the survivors. A single step of the certificate closes twenty cells with $k > 10$, namely $[m, km]$ for $k = 11$ and $m \in \{13, 30, 85\}$, for $k = 12$ and $m \in \{12, 28, 78, 79, 127, 128, 129, 154, 155, 156, 169, 170, 171, 173, 174, 175\}$, and for $k = 13$ and $m = 145$; no cell with $k > 10$ appears to have been recorded before. We also compute the one-pass filter tables for $11 \leq k \leq 15$, note five values of m that are missing from the exceptional list of Dalton and Jones, and prove a fixpoint theorem which shows that no certificate at all exists for four named cells, among them the interval $[6, 167]$ behind a conjecture of Klein. All statements are formally verified in Lean 4.

1. Introduction

A *covering system* is a finite set of congruences

$$x \equiv a_1 \pmod{n_1}, \quad \ldots, \quad x \equiv a_r \pmod{n_r}$$

such that every integer satisfies at least one of them. It is *distinct* when the moduli $n_1, \ldots, n_r$ are pairwise different. Distinct covering systems have been studied since Erdős introduced them [4], and the constraints on where their moduli can live are still not understood. The smallest modulus of a distinct covering system is bounded above by an absolute constant: Hough [5] solved Erdős's minimum modulus problem with the bound 10^{16}, which Balister, Bollobás, Morris, Sahasrabudhe and Tiba [1] improved to 616000. The moduli also cannot all be squeezed into a short interval, and it is the exact shape of this second obstruction that is open.

Dalton and Jones [2] study the interval question directly.[1] For integers $c \leq d$ write

$$\mathcal{N}(c, d): \quad \text{there is no distinct covering system all of whose moduli lie in } [c, d].$$

Their main theorem is that $\mathcal{N}(m, 10m)$ holds for every $m \geq 3$, and their closing section asks two questions, the first attributed to a private communication with M. Filaseta:

Question 1.1 ([2, §4])**.** Does $\mathcal{N}(n, 12n)$ hold for every $n \geq 4$?

Question 1.2 ([2, §4])**.** Does $\mathcal{N}(n, 15n)$ hold for every $n \geq 5$?

The restrictions $n \geq 4$ and $n \geq 5$ are forced: $[3, 36]$ and $[4, 60]$ do carry distinct covering systems, and both facts go back to Krukenberg's 1971 dissertation [7], as recorded by Klein [6] and, for the lower bound at minimum modulus 4, by Dalton and Trifonov [3]. Klein [6] has since settled the next frontier at minimum modulus 5 and left two conjectures open. The first is that a distinct

[1] Only one version of [2] exists; its numbered lemmas, tables and equations are cited below by that version's numbering.

covering system with smallest modulus 6 has largest modulus at least 168, which would follow from $\mathcal{N}(6, 167)$; we found nothing addressing it. The second, on the least common multiple at smallest modulus 7, has since been refuted [10].

The engine of [2] is a two-step filter. Krukenberg's discard lemma says that congruences whose moduli are divisible by a prime power p^α may be deleted from a covering system whenever fewer than p of them exist; counting the multiples of p^b that the interval $[c, d]$ contains therefore rules out whole classes of moduli in advance. What is left is the set $\mathcal{A}(c, d)$ of *admissible* moduli, and if $S(c, d) = \sum_{n \in \mathcal{A}(c,d)} 1/n < 1$ then the density condition — the reciprocals of the moduli of a covering system sum to at least 1 — gives $\mathcal{N}(c, d)$ with no search at all. At $k = 10$ this settles every $m \in [3, 116]$ except thirty-one values, of which [2] treats twenty-three by hand; the range $117 \le m \le 616000$ is handled by a further computation, tabulated as Table 3.1 of [2], and every $m > 616000$ at once by the bound of [1], which says that the smallest modulus of a distinct covering system never exceeds 616000. Nothing below depends on that bound: every statement in this paper concerns a single interval.

Results. This paper is about what happens when the filter is applied *again*, to its own output. The counting test of the previous paragraph asks how many multiples of p^b lie in the ambient interval $[c, d]$. Once moduli have been discarded, the honest question is how many lie in the set that survived, and that set is smaller. Section 5 makes the recount into a checkable object: a step is a pair (p, a), it is *licensed* for a finite set A of moduli when for every exponent $\alpha \ge a$ with $p^\alpha \le d$ the set A contains fewer than p elements of p-adic valuation exactly α, and taking the step deletes every multiple of p^a from A. Theorem 5.2 is that a licensed step preserves the invariant "every distinct covering system with moduli in $[c, d]$ has a sub-system with all moduli in A", so that a list of steps followed by a reciprocal-sum test is a proof of $\mathcal{N}(c, d)$.

The recount is not our idea: [2, §3] counts *permissible* multiples of a prime power inside the surviving set, and for two of its exceptional cells does so twice in a row. What is new is reach. Where the tables of [2] stop, at $k = 10$, one licensed step is enough to close twenty cells:

Theorem 1.3 (Theorem 6.1)**.** *$\mathcal{N}(m, km)$ holds for $k = 11$ and $m \in \{13, 30, 85\}$; for $k = 12$ and $m \in \{12, 28, 78, 79, 127, 128, 129, 154, 155, 156, 169, 170, 171, 173, 174, 175\}$; and for $k = 13$ and $m = 145$.*

Every one of these cells is invisible to the one-pass filter, and $m = 145$ at $k = 13$ is the first cell at $k = 13$ that any method in this paper reaches. We could find no computation at all with $k > 10$ in the literature; §6 records the search. These are sporadic cells and not an answer to Question 1.1 or Question 1.2, which ask for *all* n in a range; a sweep found nothing whatsoever at $k = 14$ or $k = 15$.

Section 4 carries the one-pass tables themselves, which have not been computed past $k = 10$ either. At $k = 11$ the filter settles exactly twelve values of $m \le 70$ and at $k = 12$ exactly sixteen values of $m \le 120$ — twenty-eight further cells with $k > 10$, obtained with no iteration at all — while at $k = 13, 14, 15$ it settles none with $m \le 80$ (Theorem 4.4), a quantitative form of the remark in [2] that for larger k the number of values with reciprocal sum above 1 "would be quite larger". The same section records a small gap in [2]: five further values, $m = 60, \dots, 64$, have admissible reciprocal sum greater than 1 and so are missing from the exceptional list that paper treats by hand (Theorem 4.2). This is a gap in a proof and not a counterexample; nothing here suggests that $\mathcal{N}(m, 10m)$ fails for any $m \ge 3$.

Section 7 is the negative half, and it is what distinguishes a certificate from a search that happened to fail. A set on which *no* licensed step deletes anything is a fixpoint of every licensed step list (Theorem 7.2); combined with a one-pass sum of at least 1 this yields, for a given cell, that no certificate exists at all — for every denominator and every step list. We instantiate it at four cells: the two intervals $[3, 36]$ and $[4, 60]$ that do carry distinct covering systems, so that the method provably never claims a false cell; the interval $[6, 167]$ of Klein's conjecture, which is therefore out of reach of this method and needs a different one; and $[60, 600]$, the largest of the five cells that [2] omits, which stays open here as well.

Section 3 collects the complementary, known side: explicit distinct covering systems in $[3, 36]$, $[4, 60]$ and $[2, 12]$, and four refutations, obtained from a SAT solver and replayed through a verified proof checker, which together pin the exact threshold at minimum modulus 3: $[3, d]$ carries a distinct covering system if and only if $d \geq 36$. Both halves of that statement are Krukenberg's; what is added is that a machine now checks them from the definition.

Every statement below has been formally verified in Lean 4 against *Mathlib*; §8 says precisely which claims rest on exhaustive computation and what was computed.

2. Preliminaries

2.1. **Covering systems.** Because a distinct covering system uses each modulus at most once, we present it as a pair: a finite set $M \subset \mathbb{Z}_{>0}$ of moduli together with a residue $a(n)$ for each $n \in M$. Thus distinctness is structural rather than a side condition.

Definition 2.1. Let $M \subset \mathbb{Z}_{>0}$ be finite and $a : M \to \mathbb{Z}$. The pair (M, a) *covers* $\mathbb{Z}$ if for every $x \in \mathbb{Z}$ there is $n \in M$ with $n \mid x - a(n)$. For $c \leq d$ we say (M, a) is a distinct covering system *with moduli in* $[c, d]$ if it covers $\mathbb{Z}$ and $c \leq n \leq d$ for every $n \in M$, and we write $\mathcal{N}(c, d)$ for the assertion that no such pair exists.

The first tool is classical and is used without attribution in [2, (3.1)]; Erdős already argues this way in [4].

Theorem 2.2 (Density condition)**.** *Let (M, a) be a pair as in Definition 2.1 which covers $\mathbb{Z}$, with all moduli positive. Then $\sum_{n \in M} 1/n \geq 1$. In particular, if $c \geq 1$ and the reciprocals of the integers of $[c, d]$ sum to less than 1, then $\mathcal{N}(c, d)$.*

Proof sketch. Put $L = \operatorname{lcm}(M)$. The residues below L lying in the class $a(n) \bmod n$ number exactly L/n, because $n \mid L$; the classes together exhaust $\{0, 1, \ldots, L-1\}$, so $L \leq \sum_{n \in M} L/n$, and dividing by L gives the claim. The consequence follows because $M \subseteq [c, d] \cap \mathbb{Z}$ and every term is positive. □

We also record the finite reformulation that makes an *existence* claim checkable: if every $n \in M$ divides L and every residue y with $0 \leq y < L$ is covered, then (M, a) covers $\mathbb{Z}$. This is the only reduction needed in §3, and it too is verified formally.

Already at $k = 2$ the density condition alone is decisive, and — unusually for this subject — it settles infinitely many cells at once, with no computation.

Proposition 2.3. *$\mathcal{N}(n, 2n)$ holds for every $n \geq 3$.*

Proof sketch. Let $T(n) = \sum_{m=n}^{2n} 1/m$. A two-line identity gives $T(n+1) - T(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n} = -\frac{3n+2}{n(2n+1)(2n+2)} < 0$ for $n \geq 1$, so T is strictly decreasing, and $T(3) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20} < 1$. Apply Theorem 2.2. This result is subsumed by $\mathcal{N}(m, 10m)$ of [2]; it is included because it is the only statement here covering infinitely many intervals at once, and it costs no computation. □

2.2. **Krukenberg's discard lemma and admissible moduli.** For $q \geq 1$ let

$$\nu_q(c, d) \;=\; \left\lfloor \tfrac{d}{q} \right\rfloor - \left\lfloor \tfrac{c-1}{q} \right\rfloor$$

be the number of multiples of q in $[c, d]$. The following is due to Krukenberg [7]; it appears as Corollary 7 of [3] and as Lemma 2.1 of [2]. We have not consulted [7], which was never published, nor [3]; the statement and the proof below are ours, and no novelty is claimed for either.

Lemma 2.4 (Discard lemma)**.** *Let (M, a) cover $\mathbb{Z}$ with all moduli positive, let p be prime and $\alpha \geq 1$. If fewer than p elements of M are divisible by p^α, then $\big(\{n \in M : p^\alpha \nmid n\},\, a\big)$ still covers $\mathbb{Z}$.*

Proof sketch. Let $\Lambda = p^{\alpha-1} \prod_{n \in M,\ p^\alpha \nmid n} n / p^{v_p(n)}$, so that every *kept* modulus divides Λ while $v_p(\Lambda) = \alpha - 1$. Given x, consider the p translates $x, x + \Lambda, \ldots, x + (p-1)\Lambda$. If some translate is covered by a kept congruence, then so is x, since kept moduli divide Λ. Otherwise all p translates are covered

by the fewer than p discarded congruences, so one discarded modulus covers two of them, whence it divides $(t-s)\Lambda$ for some $0 < |t-s| < p$; but p^α divides that modulus while $p^\alpha \nmid (t-s)\Lambda$, a contradiction. □

Definition 2.5. An integer n is *admissible for* $[c,d]$ if for every prime p and every $b \geq 1$ with $p^b \mid n$ one has $\nu_{p^b}(c,d) \geq p+1$. Write $\mathcal{A}(c,d)$ for the set of admissible n in $[c,d]$ and $S(c,d) = \sum_{n\in\mathcal{A}(c,d)} 1/n$.

Proposition 2.6 (Reduction to admissible moduli)**.** *Let $c \geq 1$. Every distinct covering system with moduli in $[c,d]$ contains a sub-system, with the same residues, which still covers $\mathbb{Z}$ and all of whose moduli are admissible for $[c,d]$. Consequently $S(c,d) < 1$ implies $\mathcal{N}(c,d)$.*

Proof sketch. Induct on $|M|$. If some $n_0 \in M$ fails admissibility, witnessed by $p^b \mid n_0$ with $\nu_{p^b}(c,d) \leq p$, let $\alpha = \max_{n\in M} v_p(n) \geq b$. No modulus is divisible by $p^{\alpha+1}$, and the multiples of p^α in $[c,d]$ are p^α times a block of $\nu_{p^\alpha}(c,d) \leq \nu_{p^b}(c,d) \leq p$ consecutive integers, one of which is divisible by p; that one is excluded, so fewer than p moduli are divisible by p^α. Lemma 2.4 deletes them and $|M|$ drops. The consequence is Theorem 2.2 applied to the sub-system, whose moduli lie in $\mathcal{A}(c,d)$. □

Remark 2.7. The counting hypothesis in Definition 2.5 is taken over the interval $[c,d]$, not over $[1,d]$. Lemma 2.2 of [2] is stated in the second form, as $p^a(p+1) > d$; the first form is what its proof supports, since every modulus is at least c, and it is strictly stronger. The difference is not cosmetic. For each of 78 values of m, [2, §3] prints an integer L_m which every modulus of a distinct covering system in $[m,10m]$ is asserted to divide. Recomputing $\operatorname{lcm}\{n \in [m,10m] : n \text{ admissible for } [m,10m]\}$ with the interval count of Definition 2.5 reproduces all 78 printed values, whereas the $[1,d]$ count gives a strictly larger answer at exactly eleven of them, namely $m = 14, 19, 38, 39, 40, 41, 56, 59, 95, 107, 108$, in each case by one further prime factor. So the printed table is right and the lemma quoted for it is weaker than the table needs. In particular $L_m = \operatorname{lcm}\mathcal{A}(m,10m)$ at each of the 78 values, a fact used once below.

3. Which intervals do carry a distinct covering system

The two questions of [2] begin at $n \geq 4$ and $n \geq 5$ because the cells below are false. We record explicit systems. Since a covering of the residues below $L = \operatorname{lcm}(M)$ is a covering of $\mathbb{Z}$, each is a finite check.

Theorem 3.1. *The* 14 *congruences with (n,a) running over*

$$(3,0),(4,1),(5,2),(6,4),(8,0),(9,2),(10,4),(12,7),(15,8),(18,17),(20,6),(24,20),(30,20),(36,23)$$

form a distinct covering system with all moduli in $[3,36]$. Hence $\mathcal{N}(3,36)$ is false, and so are $\mathcal{N}(3,39)$, $\mathcal{N}(3,42)$ and $\mathcal{N}(3,45)$.

Theorem 3.2. *The* 25 *congruences with (n,a) running over*

$$(4,1),(5,4),(6,4),(7,6),(8,7),(9,5),(10,8),(12,3),(14,10),(15,6),(16,3),(18,8),(20,10),$$
$$(21,5),(24,11),(28,18),(30,12),(35,16),(36,20),(40,6),(42,14),(45,2),(48,43),(56,50),(60,0)$$

form a distinct covering system with all moduli in $[4,60]$, using all 25 *admissible moduli of that interval. Hence $\mathcal{N}(4,60)$ is false.*

Verification is one evaluation over a single period: $L = 360$ for Theorem 3.1 and $L = 5040$ for Theorem 3.2. Both existence facts are Krukenberg's — Klein [6] records that he "gave an example of a distinct covering system with minimum modulus 3 and largest modulus 36", and gave examples showing his bounds best possible at minimum modulus 4, the matching lower bound 60 there being [3]. The particular systems above were found here by a direct search and by a SAT solver respectively and are almost certainly not his.

For the non-existence side at $c = 3$ the filter of Proposition 2.6 is useless: $S(3,d) = 29/20$ for every d from 30 to 35, and $S(3,36) = 74/45$. We therefore encode the question as a Boolean formula. After Proposition 2.6 the moduli may be assumed to lie in $\mathcal{A}(c,d)$, each is used at most once, and

its residue is one of finitely many, so the variable $x_{n,r}$ ("the congruence r mod n is in the system", $n \in \mathcal{A}(c,d)$, $0 \le r < n$) suffices, with clauses saying that every residue below $\operatorname{lcm} \mathcal{A}(c,d)$ is caught, that each modulus carries at least one residue, and that it carries at most one.

Proposition 3.3. *Let $c \ge 1$. If the formula just described, together with the unit clause fixing the residue of the largest admissible modulus to 0, is unsatisfiable, then $\mathcal{N}(c,d)$.*

The two narrowings are the reason the solver terminates, and each is proved rather than asserted: the at-least-one clauses are sound because a covering system with moduli in $\mathcal{A}(c,d)$ extends to one using *every* modulus of that set — extra congruences never destroy a covering — and the unit clause is sound because translating all residues by a common t preserves covering, so some translate sends any chosen residue to 0. Without them a solver does not finish $[4, 44]$ in five minutes; with them it takes 45 seconds.

Theorem 3.4. *$\mathcal{N}(3,30)$, $\mathcal{N}(3,33)$, $\mathcal{N}(3,34)$ and $\mathcal{N}(3,35)$ all hold. With Theorem 3.1: $[3,d]$ carries a distinct covering system if and only if $d \ge 36$.*

Proof sketch. Each of the four cells is a refutation of the formula of Proposition 3.3, produced by the SAT solver `cadical` in LRAT format and replayed inside the proof assistant by a verified checker. The formula is rebuilt from the definitions before the certificate is checked, so a mismatch between the file that was solved and the formula that is refuted would make the check fail. For $[3,30]$ the formula declares 961 variables and 1311 clauses; each of the four certificates is about 2 MB. Both halves of the threshold are known: the existence half is Krukenberg's, and for $\mathcal{N}(3,35)$ his theorem that minimum modulus 3 forces largest modulus at least 36 [6] handles the systems of smallest modulus 3, while $\mathcal{N}(m,10m)$ of [2] handles those of smallest modulus $m \ge 4$, as $[m,35] \subseteq [m,10m]$. What is new is only that a machine checks the pair from Definition 2.1. □

Proposition 3.5 (Controls)**.** *Four checks guard against a vacuous or misplaced quantifier. The classical system* 0 (2), 0 (3), 1 (4), 5 (6), 7 (12) *shows that $\mathcal{N}(2,12)$ is false, so covering systems in the sense of Definition 2.1 do exist and the hypothesis $m \ge 3$ in* [2] *is load-bearing. Three congruences modulo* 3 *cover $\mathbb{Z}$ while the single modulus* 3 *covers nothing, so distinctness is what the whole question rests on. $\mathcal{N}(3,36)$ is false, so the statement "$\mathcal{N}(m,km)$ for every $m \ge 3$" fails for every $k \ge 12$, and the threshold of Theorem 3.4 cannot be pushed further. Finally the filter of Proposition 2.6 is blind at both $d = 35$ and $d = 36$, whose admissible sums are* $29/20$ *and* $74/45$*; since one of those cells has a covering system and the other does not, the search layer is not silently re-deriving the filter layer.*

4. The one-pass filter and its tables

We now evaluate $S(m,km)$. For a set I of integers write

$$H(k,I) = \{m \in I : S(m,km) \ge 1\}, \qquad E(k,I) = \{m \in I : S(m,km) < 1\},$$

so that E lists the cells Proposition 2.6 settles and H those it leaves open.

Theorem 4.1 (The $k = 10$ table)**.** *$H(10,[3,116])$ is*

$$\{3,4,5,6,7,8,9,15,16,18,20,21,22,23,24,25,33,42,43,44,45,46,47,48,49,50,60,61,62,63,64\},$$

a set of 31 *values, and $E(10,[3,116])$ is its complement, of size* 83*. In particular $\mathcal{N}(m,10m)$ holds for each of those* 83 *values of m.*

The content of Theorem 4.1 is that of [2, §3]; the values are theirs and the point is that they are here recomputed from Definition 2.5 rather than transcribed. The recomputation does turn up one discrepancy.

Theorem 4.2. *$S(m,10m) \ge 1$ for $m = 60,61,62,63,64$; exactly,*

$$S(60,600) = \frac{982657138571}{963761198400} = 1.01960645\ldots$$

Remark 4.3. After proving that $S(m, 10m) < 1$ outside an exceptional list, [2] treats that list,

$$\{6, 7, 8, 9, 15, 16, 18, 20, 21, 22, 23, 24, 25, 33, 42, 43, 44, 45, 46, 47, 48, 49, 50\},$$

by three case analyses, having settled $m \le 5$ separately by Krukenberg's theorem. The inequality that produces the list is its (3.2), asserting $\sum_{d \mid L_m,\, m \le d \le 10m} 1/d < 1$ off the list; every admissible modulus divides L_m (Remark 2.7), so that sum is at least $S(m, 10m)$, which by Theorem 4.2 is at least 1 at $m = 60, \dots, 64$. Those five cells are thus excluded by neither (3.2) nor the list, and receive none of the three case analyses. We emphasise that this is a gap in a proof and not a counterexample: nothing computed here suggests that $\mathcal{N}(m, 10m)$ is false for any $m \ge 3$, and the repair is presumably one further case analysis of the kind [2] already runs. We did not attempt it; the least common multiple of the admissible moduli of $[60, 600]$ is about 1.35×10^{13}, which puts the cell out of search range, and §7 shows that the method of this paper does not reach it either.

Past $k = 10$ nothing has been tabulated, and the tables thin out fast.

Theorem 4.4 (Tables for $11 \le k \le 15$)**.**

$$E(11, [3, 70]) = \{11, 12, 27, 28, 29, 53, 54, 66, 67, 68, 69, 70\},$$
$$E(12, [3, 120]) = \{71, 73, 74, 75, 76, 77, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120\},$$
$$E(13, [3, 80]) = E(14, [3, 80]) = E(15, [3, 80]) = \varnothing.$$

In particular $\mathcal{N}(m, 11m)$ holds for the twelve listed m, and $\mathcal{N}(m, 12m)$ for the sixteen listed m.

The last line is the quantitative form of the remark in [2] that for larger k the number of values with reciprocal sum above 1 "would be quite larger": past $k = 12$ the one-pass filter settles nothing at all below $m = 80$, so every small m there needs a genuinely new ingredient. Supplying one is the subject of the next two sections.

Each of the tables above is a single exhaustive evaluation over the stated range of m: for each m the set $\mathcal{A}(m, km)$ is built from Definition 2.5 and its reciprocal sum is compared with 1 in exact rational arithmetic. The individual cell statements $\mathcal{N}(m, km)$ are then decided against the table.

5. The iterated reduction

5.1. **The step.** Fix an interval $[c, d]$ and let A be a finite set of positive integers, all at most d, with the property

$$\text{every distinct covering system with moduli in } [c, d] \text{ has a sub-system covering } \mathbb{Z} \text{ with all moduli in } A. \tag{1}$$

Proposition 2.6 says that $A = \mathcal{A}(c, d)$ has this property. For a prime p and $\alpha \ge 0$ write

$$E_{p,\alpha}(A) \;=\; \#\{n \in A : v_p(n) = \alpha\}.$$

Definition 5.1. A pair (p, a) with p prime and $a \ge 1$ is a *licensed step* for A (relative to d) if $E_{p,\alpha}(A) < p$ for every $\alpha \ge a$ with $p^\alpha \le d$. Taking the step replaces A by $A \setminus p^a\mathbb{Z}$.

Theorem 5.2 (Soundness of a step)**.** *Let $c \ge 1$, let A satisfy* (1) *with* $\max A \le d$*, and let (p, a) be a licensed step for A. Then $A \setminus p^a\mathbb{Z}$ satisfies* (1)*. Consequently a list of steps, each licensed for the set the previous steps left, produces a set that still satisfies* (1)*.*

Proof sketch. Let (M, ρ) be a distinct covering system with moduli in $[c, d]$ and, by (1), assume $M \subseteq A$; the letter a is reserved for the step exponent. Let $\alpha_{\max}$ be the largest exponent with $p^{\alpha_{\max}} \le d$; no modulus is divisible by $p^{\alpha_{\max}+1}$, as moduli are at most d. Now descend: at stage $\alpha = \alpha_{\max}, \alpha_{\max} - 1, \dots, a$ the system has, by the earlier stages, no modulus divisible by $p^{\alpha+1}$, so every modulus divisible by p^α has v_p exactly α and hence lies among the fewer than p elements counted by $E_{p,\alpha}(A)$; Lemma 2.4 deletes them and the covering survives. After the stage $\alpha = a$ no multiple of p^a remains. □

The crucial point is where the count in Definition 5.1 is taken: inside A, which shrinks, rather than inside $[c, d]$, which does not. The smallest example is the cell that Proposition 2.6 misses at $k = 10$ and this section recovers.

Example 5.3. Take $[c, d] = [15, 150]$. The interval contains $\nu_{11}(15, 150) = \lfloor 150/11 \rfloor - \lfloor 14/11 \rfloor = 12$ multiples of 11, and $12 > 11$, so no test taken over the interval can discard the prime 11. But $|\mathcal{A}(15, 150)| = 49$ and only ten of the twelve multiples survive: $121 = 11^2$ goes because $[15, 150]$ holds a single multiple of 121, and $143 = 11 \cdot 13$ goes with the prime 13. Since $10 < 11$ and no surviving modulus is divisible by 121, the step $(11, 1)$ is licensed, and it deletes all ten. The reciprocal sum falls from $S(15, 150) = 27493/25200 = 1.0909920\ldots$ to $3269/3600 = 0.9080555\ldots < 1$.

5.2. **The certificate.** A step list plus a reciprocal-sum test is a proof of $\mathcal{N}(c, d)$, and we make the test integral so that no rational arithmetic enters the check. For $D \geq 1$ and a finite set A of positive integers, $1/n \leq (\lfloor D/n \rfloor + 1)/D$ gives

$$\sum_{n \in A} \left\lfloor \frac{D}{n} \right\rfloor + |A| \;<\; D \qquad \Longrightarrow \qquad \sum_{n \in A} \frac{1}{n} < 1. \tag{2}$$

Definition 5.4. A *certificate* for $[c, d]$ is a pair (D, σ) with $D \geq 1$ and σ a finite list of steps such that each step of σ is licensed for the set the previous steps leave, starting from $\mathcal{A}(c, d)$, and such that the final set A_σ satisfies the left side of (2).

Corollary 5.5. *If $c \geq 1$ and $[c, d]$ has a certificate, then $\mathcal{N}(c, d)$.*

Proof. Theorem 5.2 gives (1) for A_σ; (2) and Theorem 2.2 applied to the sub-system inside A_σ give the contradiction. □

Every certificate in this paper uses $D = 10^9$ and a step list of length one. The margin is comfortable: the tightest cell, $m = 145$ at $k = 13$, clears the left side of (2) by 1 890 152.

5.3. **Two cells of Dalton and Jones, recovered uniformly.**

Theorem 5.6. *$\mathcal{N}(15, 150)$ and $\mathcal{N}(33, 330)$ hold, each by a certificate with the single step $(11, 1)$ and $(17, 1)$ respectively.*

These are two of the twenty-three cells that [2] treats individually, so the results are known; the interest is that one uniform certificate replaces two bespoke arguments, and that the arithmetic can be compared line by line with the published one.

Theorem 5.7 (Published values, recomputed)**.**

$$S(15, 150) = \frac{27493}{25200}, \qquad \sum_{n \in A_{(11,1)}(15,150)} \frac{1}{n} = \frac{3269}{3600},$$

$$S(33, 330) = \frac{26466589}{25945920}, \qquad \sum_{n \in A_{(17,1)}(33,330)} \frac{1}{n} = \frac{3249751}{3706560},$$

where $A_{(p,a)}(c, d)$ denotes $\mathcal{A}(c, d)$ after the step (p, a). Moreover $\frac{3249751}{3706560} + \frac{1}{720720} = \frac{22748293}{25945920} = 0.8767580\ldots$

The four rationals are exactly the numbers [2] prints for these two cells: 1.0909921, 0.908056, 1.0200675 and 0.876758. The last needs a word. For $m = 33$ the argument in [2] counts the permissible multiples of 17 in $[33, 330]$ and replaces the corresponding congruences by a single congruence modulo 720720, which is why the printed value is our 3249751/3706560 plus 1/720720. Here that replacement is unnecessary: the surviving multiples of 17 are the sixteen moduli

$$34, 51, 68, 85, 102, 119, 136, 153, 170, 187, 204, 221, 238, 255, 272, 306,$$

and $16 < 17$, so Lemma 2.4 deletes them outright.

Remark 5.8 (Provenance of the method). Recounting inside the survivors and reducing again is the move of [2, §3] itself. Its subsection on reduction handles $m \in \{6, 16, 20, 33, 42, 43, 45\}$ by counting permissible multiples of a prime power inside the surviving set, and for $m = 6$ and $m = 16$ it does this twice in a row, recounting after the first reduction. What Theorem 5.2 and Corollary 5.5 add is a uniform, checkable form that can be run at any (c, d), which is what makes the sweep of §6 possible. We claim no novelty for the idea.

Remark 5.9 (Two slips in the source, neither affecting its theorem). Two small things surfaced from the line-by-line comparison of Theorem 5.7 with [2]. First, the sentence treating $m = 33$ says there are 17 permissible multiples of 17 in $[33, 330]$ and then lists sixteen of them, one factor being printed as $18 \cdot 13$ where $18 \cdot 17$ is meant; the count is 16, as above, which if anything makes the argument shorter. Second, for $m = 15$ the source routes through its reduction modulo a prime and a pigeonhole on ten leftover moduli, while printing in that very argument the sum 0.908056 — our $3269/3600$ — for the moduli not divisible by 11; since those ten moduli are exactly the surviving multiples of 11 and $10 < 11$, the discard lemma already finishes the cell at that line. Neither observation affects the correctness of $\mathcal{N}(m, 10m)$.

6. Cells beyond $k = 10$

Where the tables of [2] stop, the certificate lands new values.

Theorem 6.1. *$\mathcal{N}(m, km)$ holds for each of the following twenty pairs (k, m), in every case by a certificate with a single licensed step:*

k	m	$d = km$	*step*
11	13	143	$(11, 1)$
11	30	330	$(17, 1)$
11	85	935	$(29, 1)$
12	12	144	$(11, 1)$
12	28	336	$(17, 1)$
12	78, 79	936, 948	$(29, 1)$
12	127, 128, 129	1524, 1536, 1548	$(37, 1)$
12	154, 155, 156	1848, 1860, 1872	$(41, 1)$
12	169, 170, 171, 173, 174, 175	2028, …, 2100	$(43, 1)$
13	145	1885	$(41, 1)$

None of these m appears in the corresponding list of Theorem 4.4: the one-pass filter leaves every one of them open, with $S(m, km)$ between 1.0015 and 1.1558, and the single step brings the sum down to between 0.8825 and 0.9982. At $k = 13$ the cell $m = 145$ is the first that any method in this paper settles; Theorem 4.4 records that the one-pass filter settles none with $m \le 80$ there.

Proof sketch. Each line is one instance of Corollary 5.5 with $D = 10^9$ and the step shown. What is computed is: the set $\mathcal{A}(m, km)$ from Definition 2.5; the valuation counts $E_{p,\alpha}$ over that set for every α with $p^\alpha \le d$, which is the licensing condition of Definition 5.1; and the integer inequality (2) for the survivors. These are exhaustive computations over finite sets of size at most d, carried out by compiled evaluation inside the proof assistant, one evaluation per value of k; the individual cell statements are then decided against that evaluation. Everything above the computed Booleans — Lemma 2.4, Proposition 2.6, Theorem 5.2, (2) and Corollary 5.5 — is proved, not computed. □

Remark 6.2 (The search that produced the table). The pairs in Theorem 6.1 were found by an independent implementation, in Python with exact rational arithmetic, which for each cell iterates licensed steps to a fixpoint and then tests the reciprocal sum. The ranges swept were $3 \le m \le 260$ for $k = 11$ and $k = 12$, and $3 \le m \le 220$ for $k = 13$, $k = 14$ and $k = 15$. Within those ranges the sweep returned exactly the cells listed, and *nothing at all* for $k = 14$ and $k = 15$. The sweep itself is not machine-checked — only its output is, as Theorem 6.1 — so the exhaustiveness claim in this

remark should be read as a measurement rather than as a theorem. It cost about 18 minutes on one core.

Remark 6.3 (What is not claimed). Theorem 6.1 settles sporadic cells. It is not an answer to Question 1.1 or Question 1.2, which ask for all $n \geq 4$ at $k = 12$ and all $n \geq 5$ at $k = 15$. At $k = 12$ the smallest cell we settle is $m = 12$; at $k = 15$ we settle none. What the theorem does show is that the interval $[m, km]$ can be closed for some m at $k = 11$, 12 and 13, which as far as we could determine had not been recorded for any $k > 10$.

Remark 6.4 (Search of the literature). We searched a local mirror of the arXiv complete through August 2026 (317 shards) for the terms *Krukenberg*, *Trifonov*, *covering system*, *distinct moduli*, *minimum modulus*, *incongruent moduli* and *distinct covering*; the 150 covering-systems papers this returned were examined individually. Eleven of the intervals appearing in Theorem 6.1 were then searched for literally: ten occur nowhere in the mirror, and the eleventh, $[12, 144]$, occurs three times, in two character-table computations and a Latin-square paper. A pattern search for intervals of the form $[n, kn]$ with $11 \leq k \leq 19$ over all mathematics shards returns 21 locations in 11 papers, of which the only covering-systems paper is [2] itself, and there only in Questions 1.1 and 1.2. As of August 2026 Semantic Scholar and OpenAlex both report no citations of [2] at all, and a search of the mirror for its identifier returns only its own header; the two papers citing [6] contain no interval statement. No OEIS sequence records the least k for which $[n, kn]$ carries a distinct covering system. A negative search is not a proof of novelty, and these are computations that anyone with the method of [2] could run; the reason to think they are new is that the $k > 10$ case appears not to have been treated at all.

Remark 6.5 (An observation, not a theorem). In all twenty-two cells settled by a certificate here — the twenty of Theorem 6.1 and the two of Theorem 5.6 — the prime dropped is the largest p with p^2 slightly below d: the ratio d/p^2 lies between 1.09 and 1.24. We have no proof that a useful step must have this shape.

7. Where the reduction provably stops

A certificate search that finds nothing has two possible meanings: the search was not clever enough, or no certificate exists. The following separates them.

Definition 7.1. A finite set A is *stuck* for d if every licensed step for A deletes nothing from A.

Checking stuckness is finite: a step (p, a) with $p^a > d$ deletes nothing from a set of integers bounded by d, so only the primes $p \leq d$ and the exponents a with $p^a \leq d$ need be examined.

Theorem 7.2. *If A is stuck for d and all its elements are positive and at most d, then every licensed step list leaves A unchanged. Consequently, if in addition $A = \mathcal{A}(c, d)$ and $S(c, d) \geq 1$, then $[c, d]$ has no certificate: for every D and every step list, the check of Definition 5.4 fails.*

Proof sketch. Induct along the step list, using stuckness for the steps within range and the remark above for those outside it. For the consequence, a passing check would give $\sum_{n \in \mathcal{A}(c,d)} 1/n < 1$ by (2), contradicting $S(c, d) \geq 1$. □

Theorem 7.3. *The intervals $[3, 36]$, $[4, 60]$, $[6, 167]$ and $[60, 600]$ have no certificate.*

The four instances are chosen to bound the method from both sides.

Above. $[3, 36]$ and $[4, 60]$ carry distinct covering systems by Theorems 3.1 and 3.2, so $\mathcal{N}(3, 36)$ and $\mathcal{N}(4, 60)$ are false. Theorem 7.3 says the certificate machinery provably never claims them — not that a search failed to claim them.

Below. $\mathcal{N}(6, 167)$ would give Klein's conjecture that a distinct covering system with smallest modulus 6 has largest modulus at least 168. That cell is stuck at its 60 admissible moduli, whose reciprocals sum to $2.0011\ldots$, so it is out of reach here and needs a strictly stronger filter; it is also

out of reach of the search of §3, since $\operatorname{lcm} \mathcal{A}(6, 167) = 1663200$. Likewise $[60, 600]$, the largest of the five cells of Remark 4.3, stays open to this method, so the gap in [2] is not repaired here.

Remark 7.4 (Non-vacuity). The licensing condition of Definition 5.1 is a real constraint: at $[15, 150]$ the step $(7, 1)$ is *refused*, because thirteen multiples of 7 survive the one-pass filter and $13 \geq 7$. And the recount is doing work the one-pass filter cannot do: $S(15, 150) \geq 1$ and $S(33, 330) \geq 1$, so Proposition 2.6 closes neither of the two cells of Theorem 5.6.

Remark 7.5 (An extension that fails). The obvious strengthening is to use the replacement form of Krukenberg's lemma: when exactly p congruences have modulus divisible by p^a, replace them by a single congruence modulo $p^{a-1} \operatorname{lcm}(n_1, \ldots, n_p)$ instead of discarding them. A greedy implementation of that rule, added to the step list above, appears to gain about twenty cells per k — and is unsound as stated. Its output includes a claim of $\mathcal{N}(3, 39)$, which is false by Theorem 3.1. The reason is that a replacement introduces a *new* modulus, which may lie outside $[c, d]$ and may duplicate one already present; the later counts must include it, and recounting only over the shrinking subset of $[c, d]$ undercounts, so the test "fewer than p" fires when it should not. Nothing in the present development uses replacement — every step here only discards, and the invariant (1) tracks the covering system's own moduli inside A, so the failure cannot arise. A correct replacement layer would have to carry the new moduli as a separate multiset; the numbers above suggest it is worth building.

8. Verification

Every statement in this paper has been formally verified in Lean 4 [8] against *Mathlib* [9], and the development contains no `sorry`. The reasoning layer is proved outright, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: Theorem 2.2 and its corollaries, Proposition 2.3, Lemma 2.4 and Proposition 2.6, the soundness bridge behind Proposition 3.3, Theorem 5.2 with Corollary 5.5, the implication (2), and Theorem 7.2. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite data: one Boolean per table in Theorems 4.1 and 4.4 (an exhaustive scan of m over the stated range, building $\mathcal{A}(m, km)$ from Definition 2.5 and comparing an exact rational sum with 1), the exact values of Theorems 4.2 and 5.7, one Boolean per value of k for the certificates of Theorems 5.6 and 6.1, the stuckness and one-pass-sum checks behind Theorem 7.3, one evaluation over a single period for each of Theorems 3.1 and 3.2, and, for Theorem 3.4, the replay of four LRAT refutations produced by `cadical`, where the refuted formula is rebuilt from the definitions inside Lean so that the certificate cannot be checked against a different formula from the one that was solved. The one computation that is *not* machine-checked is the sweep of Remark 6.2, which establishes only that no further cell was found in the ranges stated there; every value it produced was then re-derived inside Lean, and every published number quoted in Theorem 5.7 was recomputed from the definitions rather than transcribed. Repository reference to be added at submission.

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