NEW LOWER BOUNDS FOR THE COLUMN NUMBER OF Δ-MODULAR MATRICES AT RANKS FOUR AND FIVE

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. The column number $s(\Delta, r)$ is the largest number of pairwise non-parallel columns of a rank-r integer matrix all of whose $r \times r$ minors are at most Δ in absolute value. Paat, Walsh and Xu recently determined $s(3, r)$ for sufficiently large r, and pointed at sporadic low-rank matrices as the obstruction to small ranks; the only such matrix known for $\Delta = 3$ had rank 3. We exhibit four integer matrices with more pairwise non-parallel columns than any published construction supplies:

$$s(3, 4) \geq 17, \qquad s(3, 5) \geq 25, \qquad s(4, 4) \geq 22, \qquad s(5, 4) \geq 24,$$

against the values 16, 23, 21, 22 derivable from the literature. Ranks 4 and 5 are therefore sporadic for $\Delta = 3$ as well, so any threshold in the phrase "sufficiently large" of the $\Delta = 3$ theorem is at least 6. The $\Delta = 4$ witness is one of Paat–Walsh–Xu's own extremal matrices $A(4, (1, 1, 1), 4)$ with three columns appended: that matrix, a member of the family they introduce in order to build exponentially many pairwise non-isomorphic extremal matroids, is not maximal, and of the nine partition matrices of rank 4 with $\Delta \leq 5$ exactly two are not. All four witnesses are maximal in a precise identity-normalised sense: no primitive column at all can be appended to any of them. The engine behind both the search and the maximality proofs is the identity $\langle \mathbf{v}_1 \times \ldots \times \mathbf{v}_{r-1}, \mathbf{c} \rangle = \det[\mathbf{v}_1 | \ldots | \mathbf{v}_{r-1} | \mathbf{c}]$, which replaces a recomputation of every $r \times r$ minor by one dot product per candidate column. Every statement below is machine-checked in Lean 4.

1. Introduction

A matrix $\mathbf{A} \in \mathbb{Z}^{r \times n}$ is **Δ-modular**, for $\Delta \in \mathbb{Z}_{\geq 1}$, if the determinant of every $\text{rank}(\mathbf{A}) \times \text{rank}(\mathbf{A})$ submatrix of $\mathbf{A}$ has absolute value at most Δ; two columns are **parallel** if they are linearly dependent. Integer programs with Δ-modular constraint matrices are solvable in polynomial time for $\Delta \leq 2$ [5, 2], and whether this persists for fixed $\Delta \geq 3$ is open; the structural question that accompanies it is how wide such a matrix can be.

Problem 1.1 ([10, Problem 1.1])**.** For $\Delta, r \in \mathbb{Z}_{\geq 1}$ determine the **column number**

$$s(\Delta, r) := \max\Big\{n : \ \mathbf{A} \in \mathbb{Z}^{r \times n} \text{ is rank-}r \text{ and } \Delta\text{-modular, with pairwise non-parallel columns}\Big\}.$$

Heller [4] proved $s(1, r) = \binom{r+1}{2}$. For $\Delta = 2$ the answer is $\binom{r+1}{2} + r - 1$ for large r (Oxley–Walsh [8]; Lee, Paat, Stallknecht and Xu [7], whose own analysis covers every $r \notin \{3, 5\}$, though [10] summarises it as $r \geq 6$ — see the footnote to the table in Section 4), with the two exceptions $s(2, 3) = 9$ and $s(2, 5) = 20$, one above the formula. Paat, Walsh and Xu [10] settled $\Delta = 3$:

Theorem 1.2 ([10, Theorem 1.2])**.** *If r is sufficiently large, then $s(3, r) = \binom{r+1}{2} + 2(r - 1)$.*

Their abstract records the state of the art frankly: "Exact values for the column number are only known for $r \leq 2$ or $\Delta \leq 2$." For general Δ and r the best known lower bound is

$$s(\Delta, r) \ \geq \ \binom{r+1}{2} + (\Delta - 1)(r - 1), \tag{1}$$

attained by a matrix of Lee et al. [7, Proposition 1], and, as [10, Theorem 1.3] shows for $\Delta \geq 2$ and $r \geq \Delta + 1$, by at least $N_\Delta + 1$ matrices with pairwise non-isomorphic vector matroids, N_Δ being the number of partitions of $\Delta - 1$. Upper bounds have moved faster than lower ones. In the count of Problem 1.1, $s(\Delta, r) \leq \binom{r+1}{2} + 80\Delta^7 r$ for fixed Δ and large r [9, Theorem 1.3]; the bounds

$\Delta^2\binom{r+1}{2}$ for all Δ, r [7, Theorem 2] and $O(r^4)\Delta$ [1, Theorem 1.2] are proved for the wider counts c and h recalled in Section 2 and descend to s through (2) and (3), which is how [10] quotes them. Kriepke and Schymura determine the *generic* column number $g(\Delta, 2)$ — the same count restricted to matrices with no vanishing 2×2 minor — for every $\Delta \geq 10^8$ and for $385 \leq \Delta \leq 1550$, and record there, for g, s and h alike, the same erratic behaviour at small parameters that concerns us here [6, Theorem 1.1 and §1]. Most recently Pribytkova and Gribanov improved the $O(r^4)\Delta$ bound to $h(\Delta, r) \in O(r^3)\Delta$ for every Δ, a bound previously available only for odd Δ [11].

Two obstructions to computing $s(\Delta, r)$ for $\Delta \geq 4$ are named in [10]. The first is that (1) is not always sharp: Averkov and Schymura [1] refuted the corresponding conjecture of Lee et al. for $\Delta \in \{4, 8, 16\}$, although — as we recall in Section 6 — their counterexamples refute the conjecture in the weaker count that allows the parallel pair $\mathbf{u}, 2\mathbf{u}$, and none of them is known to beat (1) in the count of Problem 1.1. The second obstruction is *sporadic low-rank matrices*: values of $s(\Delta, r)$ at small r that exceed the formula valid for large r. Three were known — $s(2, 3) = 9$, $s(2, 5) = 20$, and a rank-3, 3-modular matrix with $11 > 10$ pairwise non-parallel columns displayed in the introduction of [10] — and the authors write that "the number of these sporadic matrices will increase with Δ", that "computing assistance may be helpful (or possibly, required)" for $\Delta \geq 4$, and that they expect a sharp bound for $\Delta \in \{4, 5\}$ to be reachable "with the help of computing".

This paper supplies four such matrices, at ranks 4 and 5.

	(1)	best derivable from published work	here
$s(3, 4)$	16	16	**17**
$s(3, 5)$	23	23	**25**
$s(4, 4)$	19	21	**22**
$s(5, 4)$	22	22	**24**

The middle column is discussed in Remark 4.8; at $(4, 4)$ it is the count, in the sense of Problem 1.1, of the Averkov–Schymura pyramid, and we formalise that 21 as well (Proposition 4.5) so that the improvement is one explicit matrix against another. Two consequences are worth stating separately. First, since (1) with $\Delta = 3$ is the formula of Theorem 1.2, ranks 4 and 5 join rank 3 as exceptions for $\Delta = 3$, so "sufficiently large" there cannot mean anything below 6 (Corollary 4.3). Second, the $\Delta = 4$ witness is not a new construction at all: it is $A(4, (1, 1, 1), 4)$ of [10, Definition 4.1] — a member of the family $A(\Delta, \lambda, r)$ behind their Theorem 1.3 — with three further columns appended (Theorem 4.4). Attaining the bound (1) and being maximal are different properties, and exactly two of the nine rank-4 partition matrices with $\Delta \leq 5$ fail the second (Proposition 4.6).

All four matrices are *maximal*: no further column can be appended to any of them. That statement needs care, and Section 5 makes it precise. In brief, a Δ-modular matrix that possesses a unimodular $r \times r$ submatrix may be normalised so that its columns include $\mathbf{e}_1, \ldots, \mathbf{e}_r$, and then every column that could ever be appended lies, up to sign and a positive factor, in the finite set $\mathcal{P}(\Delta, r)$ of primitive vectors in $[-\Delta, \Delta]^r$; maximality is the assertion that no member of $\mathcal{P}(\Delta, r)$ can be appended. It is maximality, not maximality of the size: $s(3, 4) \geq 17$ and 17 is the largest size an identity-normalised matrix can have at $(3, 4)$ by an exhaustive search we report but do not formalise (Remark 5.4), while $s(3, 4) \leq 18$ follows from published enumeration data (Remark 5.5).

The tool that makes both the search and the maximality proofs affordable is elementary and is stated as Proposition 3.4: for $r - 1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1} \in \mathbb{Z}^r$ the generalised cross product $\mathbf{v}_1 \times \ldots \times \mathbf{v}_{r-1}$ is the linear functional $\mathbf{c} \mapsto \det[\mathbf{v}_1|\cdots|\mathbf{v}_{r-1}|\mathbf{c}]$. Testing whether a column may be appended to an n-column matrix therefore costs $\binom{n}{r-1}$ dot products against normals computed once, rather than $\binom{n+1}{r}$ determinants recomputed per candidate; at $(\Delta, r) = (3, 5)$ with 25 columns and 8161 candidates this is the difference between $5.4 \cdot 10^8$ determinants of size 5 and 12 650 of size 4 followed by dot products. This is what turns maximality from a computation one cannot afford into a proof.

Every statement in this paper is machine-checked in Lean 4 against Mathlib; what is delegated to compiled evaluation and what is proved by hand is set out in Section 7. No upper bound on any $s(\Delta, r)$ is claimed here.

2. Preliminaries

Throughout, $\mathbf{e}_1, \ldots, \mathbf{e}_r$ is the standard basis of $\mathbb{Z}^r$ and $[d] = \{1, \ldots, d\}$. For $\mathbf{A} \in \mathbb{Z}^{r\times n}$ and $k \leq r$ we write $\Delta_k(\mathbf{A})$ for the largest absolute value of a $k \times k$ minor of $\mathbf{A}$. We identify a matrix with its list of columns; a *submatrix* $\mathbf{A}[R, E]$ is indexed by a set R of rows and a set E of columns, so ranging over r-element sub-lists of the column list is exactly ranging over the $r \times r$ submatrices. A matrix of rank r is Δ-modular in the sense of [10] precisely when $\Delta_r(\mathbf{A}) \leq \Delta$, and it has rank r precisely when $\Delta_r(\mathbf{A}) \neq 0$; we use both characterisations without further comment. A vector $\mathbf{v} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$ is **primitive** if the gcd of its entries is 1.

Three column counts. Besides $s(\Delta, r)$ of Problem 1.1 the literature uses two relatives, and keeping them apart is essential to reading the table in Section 1. Following [1], let

$$h(\Delta, m) := \max\bigl\{n : \mathbf{A} \in \mathbb{Z}^{m\times n} \text{ has pairwise distinct columns and } \Delta_m(\mathbf{A}) = \Delta\bigr\},$$

and let $c(\Delta, m)$ be the maximum number of columns of an integer matrix $\mathbf{A}$ with m rows, no zero column, $\Delta_m(\mathbf{A}) \leq \Delta$, and no two columns equal up to sign. (Averkov and Schymura reserve "Δ-modular" for $\Delta_m(\mathbf{A}) = \Delta$ and say "Δ-submodular" for $\Delta_m(\mathbf{A}) \leq \Delta$, noting in a footnote that [3, 7] — and hence [10] and this paper — use the first word for the second meaning.) They record the identity

$$c(\Delta, m) = \tfrac{1}{2}\bigl(\max\{h(1, m), \ldots, h(\Delta, m)\} - 1\bigr). \tag{2}$$

Equality up to sign is weaker than non-parallelism — it permits the pair $\mathbf{u}$, $2\mathbf{u}$ — so

$$s(\Delta, r) \;\leq\; c(\Delta, r). \tag{3}$$

Conversely, dividing a column by the gcd of its entries only shrinks minors and preserves both rank and the parallelism relation, so in Problem 1.1 all columns may be assumed primitive; for primitive columns "distinct up to sign" and "non-parallel" agree, which is why $s(\Delta, r)$ is the quantity Lee et al. write $c^{\mathrm{p}}(\Delta, r)$ — the identification is the one [10] makes when it restates their exceptional values $c^{\mathrm{p}}(2, r) = c(1, r) + r$, $r \in \{3, 5\}$, directly as statements about s. Finally $s(\Delta, r)$ is non-decreasing in Δ, so a Δ'-modular witness with $\Delta' \leq \Delta$ bounds $s(\Delta, r)$ from below.

The conjecture that drives the subject is due to Lee et al.: for all Δ, m,

$$h(\Delta, m) = m^2 + m + 1 + 2m(\Delta - 1), \qquad \text{equivalently} \qquad c(\Delta, m) = \binom{m+1}{2} + (\Delta - 1)m, \tag{4}$$

whose right-hand side exceeds the lower bound (1) by $\Delta - 1$. It is a theorem for $\Delta \leq 2$ or $m \leq 2$ [7] and false for $\Delta \in \{4, 8, 16\}$ [1].

The matrices of the literature. All the constructions we compare against are generated from their definitions rather than copied from tables. Heller's extremal unimodular matrix is $H_r := [\,\mathbf{e}_i\ (i \in [r]) \mid \mathbf{e}_i - \mathbf{e}_j\ (i < j)\,]$, with $\binom{r+1}{2}$ columns and $\Delta_r = 1$. The matrix behind (1) is

$$A(\Delta, r) := \bigl[\, H_r \;\bigm|\; k\mathbf{e}_1 - \mathbf{e}_i \quad (i \in \{2, \ldots, r\},\ k \in \{2, \ldots, \Delta\})\,\bigr]$$

[10, Definition 4.2]; [7, Proposition 1] is the same construction in the c count, where the further columns $k\mathbf{e}_1$, $2 \leq k \leq \Delta$, are admissible. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of $\Delta - 1$ and $r \geq m + 1$, [10, Definition 4.1] sets

$$A(\Delta, \lambda, r) := \Bigl[\, H_r \;\Bigm|\; k\mathbf{e}_1 + \mathbf{e}_{i+1}, \quad k\mathbf{e}_1 + \mathbf{e}_{i+1} - \mathbf{e}_j \quad \bigl(i \in [m],\ k \in [\lambda_i],\ j \in [r] \setminus \{1, i+1\}\bigr)\Bigr],$$

again with $\binom{r+1}{2} + (\Delta - 1)(r - 1)$ columns; each is Δ-modular by [10, Proposition 4.3], and for $r \geq \Delta + 1$ their vector matroids are pairwise non-isomorphic [10, Proposition 4.4], which is [10, Theorem 1.3]. Finally, writing $D(S) = S - S$ and $\mathrm{pyr}(S) = (S \times \{0\}) \cup \{\mathbf{e}_{m+1}\}$, Averkov and

Schymura's counterexamples to (4) are the difference sets of $C_m^\ell := \operatorname{pyr}^{m-\ell}(H_\ell^{\text{set}})$, where $H_\ell^{\text{set}} = D(\{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_\ell\})$; their Lemma 5.4 gives $|D(\operatorname{pyr} S)| = |D(S)| + 2|S|$ with $\Delta(D(\operatorname{pyr} S)) = \Delta(D(S))$ for every finite non-empty $S \subseteq \mathbb{R}^m$.

3. The computational model and the cross-product test

Determinants of column lists. The computations below are carried out on lists of integer vectors, with a Laplace expansion along the first row that recurses on the matrix size, and with an explicit enumeration of the r-element sub-lists of the column list. Neither is a private notion: both agree with the standard ones, at every rank.

Proposition 3.1. *For every r and every list of columns, the recursive Laplace expansion used here equals the determinant of the corresponding $r \times r$ matrix, and the enumeration of r-element sub-lists is exhaustive: it lists exactly the sublists of length r. Consequently "$\Delta_r(\mathbf{A}) \leq \Delta$" as computed is the quantifier over all $r \times r$ submatrices of $\mathbf{A}$, and the vanishing of all 2×2 minors of a pair of columns is equivalent to their linear dependence.*

Proof sketch. Deleting the entry in position i of a list shifts index k to k or $k+1$ according as $k < i$ or not, which is exactly the order-preserving injection that skips i — the index map of the submatrix in the cofactor expansion. The recursive argument of the expansion is therefore literally the submatrix appearing in the expansion of det along row 0, and induction on the size closes the identification. Sub-list enumeration is proved equal to the standard one by a double induction. For the last sentence: if $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$ with $(a, b) \neq (0, 0)$ then all 2×2 minors $u_i v_j - u_j v_i$ vanish; conversely, if they all vanish, pick k with $u_k \neq 0$ and take $(a, b) = (v_k, -u_k)$, or $(1, 0)$ if $\mathbf{u} = \mathbf{0}$. All of this is proved in the proof assistant without appeal to compiled evaluation. □

Identity normalisation.

Definition 3.2. For $\Delta, r \geq 1$ let

$$\mathcal{P}(\Delta, r) := \big\{\mathbf{v} \in \mathbb{Z}^r : \ \mathbf{v} \neq \mathbf{0}, \ \|\mathbf{v}\|_\infty \leq \Delta, \ \mathbf{v} \text{ primitive}, \text{ the first nonzero entry of } \mathbf{v} \text{ is positive}\big\},$$

one representative of each parallel class of primitive vectors with entries bounded by Δ. Call $\mathbf{A}$ **identity-normalised** if its columns include $\mathbf{e}_1, \dots, \mathbf{e}_r$.

Lemma 3.3. *Let $\mathbf{A} \in \mathbb{Z}^{r \times n}$ be identity-normalised and Δ-modular, and let $\mathbf{b} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$ be such that $[\mathbf{A}|\mathbf{b}]$ is Δ-modular with pairwise non-parallel columns. Then there is $\mathbf{c} \in \mathcal{P}(\Delta, r)$ parallel to $\mathbf{b}$, and $[\mathbf{A}|\mathbf{c}]$ is again Δ-modular with pairwise non-parallel columns.*

Proof. For each i the submatrix of $[\mathbf{A}|\mathbf{b}]$ on the columns $\{\mathbf{e}_j : j \neq i\} \cup \{\mathbf{b}\}$ has determinant $\pm b_i$, so $\|\mathbf{b}\|_\infty \leq \Delta$. Let $g = \gcd(\mathbf{b})$ and let $\mathbf{c}$ be $\pm\mathbf{b}/g$ with the sign making the first nonzero entry positive; then $\mathbf{c} \in \mathcal{P}(\Delta, r)$. Every $r \times r$ minor of $[\mathbf{A}|\mathbf{c}]$ either does not involve $\mathbf{c}$, and is a minor of $\mathbf{A}$, or equals $\pm 1/g$ times the corresponding minor of $[\mathbf{A}|\mathbf{b}]$; in both cases it is at most Δ in absolute value. Since $\mathbf{c}$ and $\mathbf{b}$ are parallel, $\mathbf{c}$ is non-parallel to a column of $\mathbf{A}$ exactly when $\mathbf{b}$ is. □

Lemma 3.3 is elementary and is not part of the formal development, which quantifies over $\mathcal{P}(\Delta, r)$ directly; it is what licenses reading the theorems of Section 5 as "no column whatsoever can be appended". The set $\mathcal{P}(\Delta, r)$ is small enough to enumerate: $|\mathcal{P}(3, 4)| = 1120$, $|\mathcal{P}(4, 4)| = 2928$, $|\mathcal{P}(5, 4)| = 6928$, $|\mathcal{P}(2, 5)| = 1441$, $|\mathcal{P}(3, 5)| = 8161$.

The cross product.

Proposition 3.4. *Let $n \geq 0$ and let $T = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be n vectors in $\mathbb{Z}^{n+1}$. Define $\mathbf{v}_1 \times \dots \times \mathbf{v}_n \in \mathbb{Z}^{n+1}$ by*

$$(\mathbf{v}_1 \times \dots \times \mathbf{v}_n)_i = (-1)^{i+n} \det\big(T \textit{ with row } i \textit{ deleted}\big), \qquad i = 0, \dots, n.$$

Then for every $\mathbf{c} \in \mathbb{Z}^{n+1}$,

$$\big\langle\, \mathbf{v}_1 \times \dots \times \mathbf{v}_n, \ \mathbf{c} \,\big\rangle \ = \ \det\big[\mathbf{v}_1 | \cdots | \mathbf{v}_n | \mathbf{c}\big].$$

Proof sketch. Laplace expansion along the last column, that is, cofactor expansion at the column index n, whose index map is the inclusion $\{0, \dots, n-1\} \hookrightarrow \{0, \dots, n\}$; the row-deletion identification is the one used in Proposition 3.1, applied to the row index of the cofactor instead of to the recursive argument. No compiled evaluation is involved. $\square$

Proposition 3.5. *Let $\mathbf{A}$ be a Δ-modular matrix with n pairwise non-parallel columns of length r, and let N be the set of the $\binom{n}{r-1}$ cross products of $(r-1)$-element sub-lists of the columns of $\mathbf{A}$. If every $\mathbf{c} \in \mathcal{P}(\Delta, r)$ either is parallel to a column of $\mathbf{A}$ or satisfies $|\langle \mathbf{v}, \mathbf{c} \rangle| > \Delta$ for some $\mathbf{v} \in N$, then no $\mathbf{c} \in \mathcal{P}(\Delta, r)$ makes $[\mathbf{A}|\mathbf{c}]$ Δ-modular with pairwise non-parallel columns.*

Proof sketch. Immediate from Proposition 3.4: the $r \times r$ minors of $[\mathbf{A}|\mathbf{c}]$ that are not minors of $\mathbf{A}$ are exactly the numbers $\langle \mathbf{v}, \mathbf{c} \rangle$, $\mathbf{v} \in N$. The parallel branch uses the equivalence of Proposition 3.1 between the vanishing of all 2×2 minors and linear dependence, in the direction that turns a rejected candidate into a contradiction with pairwise non-parallelism. $\square$

The gain is one of arithmetic scale rather than of principle. Deciding maximality by the definition costs $|\mathcal{P}(\Delta, r)| \cdot \binom{n+1}{r}$ determinants of size r; through Proposition 3.5 it costs $\binom{n}{r-1}$ determinants of size $r-1$, computed once, and at most $|\mathcal{P}(\Delta, r)| \cdot \binom{n}{r-1}$ dot products of length r. At $(\Delta, r) = (2, 5)$ with the 20-column witness of Lee et al. that is $1441 \cdot \binom{21}{5} = 2.9 \cdot 10^7$ five-by-five determinants against 4845 four-by-four ones; at $(3, 5)$ with 25 columns, $8161 \cdot \binom{26}{5} = 5.4 \cdot 10^8$ against 12 650.

4. Four lower bounds

In each theorem of this section the assertions about one displayed matrix — its number of columns, the exact value of Δ_r, and pairwise non-parallelism — are finite computations, carried out by compiled evaluation inside the proof assistant and re-computed independently outside it (Section 7). We state once what they are: for a matrix with n columns of length r, computing Δ_r means evaluating all $\binom{n}{r}$ determinants of size r, and pairwise non-parallelism means checking, for each of the $\binom{n}{2}$ pairs of columns, that at least one of the r^2 products $u_i v_j - u_j v_i$ is nonzero. Nothing else in the theorems is a computation: $\Delta_r(\mathbf{A}) \neq 0$ gives the rank, and the numerical comparison with (1) is arithmetic.

Theorem 4.1. *The matrix*

$$M_{3,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 & 3 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{Z}^{4 \times 17} \tag{5}$$

has $\Delta_4(M_{3,4}) = 3$ and pairwise non-parallel columns. Hence

$$s(3,4) \ \geq \ 17 \ > \ 16 = \binom{5}{2} + 2 \cdot 3,$$

and $A(3,4)$, $A(3,(2),4)$ and $A(3,(1,1),4)$, which have 16 columns each, are not extremal at $(\Delta, r) = (3,4)$.

Proof sketch. $\binom{17}{4} = 2380$ determinants of size 4 give $\Delta_4 = 3$, and $\binom{17}{2} = 136$ pairs give non-parallelism; $\Delta_4 \neq 0$ gives rank 4, so $M_{3,4}$ is admissible in Problem 1.1. The column count of $A(3,4)$ is computed from the definition in Section 2, not quoted. $\square$

Theorem 4.2. *The matrix*

$$M_{3,5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{Z}^{5 \times 25} \tag{6}$$

has $\Delta_5(M_{3,5}) = 3$ *and pairwise non-parallel columns. Hence*

$$s(3,5) \;\geq\; 25 \;>\; 23 = \binom{6}{2} + 2 \cdot 4.$$

Proof sketch. As for Theorem 4.1, with $\binom{25}{5} = 53\,130$ determinants of size 5 and $\binom{25}{2} = 300$ pairs. □

Corollary 4.3. *Let* r_0 *be such that* $s(3,r) = \binom{r+1}{2} + 2(r-1)$ *for all* $r \geq r_0$*, as provided by Theorem 1.2. Then* $r_0 \geq 6$.

Proof. The formula gives 10, 16, 23 at $r = 3,4,5$, while $s(3,3) \geq 11$ [10, §1], $s(3,4) \geq 17$ and $s(3,5) \geq 25$. □

The next theorem concerns a matrix that is not ours.

Theorem 4.4. *Let* $A(4,(1,1,1),4) \in \mathbb{Z}^{4\times 19}$ *be the matrix of* [10, Definition 4.1] *for the partition* $\lambda = (1,1,1)$ *of* $\Delta - 1 = 3$*, and put*

$$M_{4,4} := \left[\, A(4,(1,1,1),4) \;\middle|\; \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4,\quad \mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_4,\quad \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \,\right] \in \mathbb{Z}^{4\times 22}.$$

Then $\Delta_4(M_{4,4}) = 4$ *and the columns of* $M_{4,4}$ *are pairwise non-parallel, so*

$$s(4,4) \;\geq\; 22 \;>\; 21 \;>\; 19 = \binom{5}{2} + 3 \cdot 3.$$

Moreover the three appended columns are exactly the members of $\mathcal{P}(4,4)$ *that can be appended to* $A(4,(1,1,1),4)$.

Proof sketch. The first nineteen columns of $M_{4,4}$ are generated from the definition of $A(\Delta,\lambda,r)$ recalled in Section 2 and are not transcribed; that they agree with the first nineteen columns of the displayed matrix is a definitional identity in the formal development. Then $\binom{22}{4} = 7315$ determinants give $\Delta_4 = 4$, and $\binom{22}{2} = 231$ pairs give non-parallelism. The last sentence is one run of the test of Proposition 3.5 on $A(4,(1,1,1),4)$: over all 2928 members of $\mathcal{P}(4,4)$, filtered against the $\binom{19}{3} = 969$ cross products of its columns, exactly three survive. The comparison with 21 is Proposition 4.5. □

Proposition 4.5. *Let* $S = D(\mathrm{pyr}^2(H_2^{\mathrm{set}})) \subseteq \mathbb{Z}^4$ *be the point set of* [1, Theorem 1.3] *at* $(\Delta, m) = (4,4)$*, and let* $\mathbf{A}$ *consist of one representative of each parallel class of nonzero points of* S*. Then* $\mathbf{A}$ *has* 21 *columns,* $\Delta_4(\mathbf{A}) = 4$*, and its columns are pairwise non-parallel; hence* $s(4,4) \geq 21$*. In the weaker count that identifies columns only up to sign the same set gives* 24 *columns, which is Averkov and Schymura's refutation* $c(4,4) \geq 24 > 22$ *of* (4).

Proof sketch. The point set is built from the definitions of D and pyr recalled in Section 2, and the two representative selections are then computed; the verification of Δ_4 and of the two pairwise conditions is as above. □

Proposition 4.6. *Among the nine matrices* $A(\Delta,\lambda,4)$ *with* $\lambda \vdash \Delta - 1$ *and* $\Delta \in \{3,4,5\}$*, exactly two admit an extension by a member of* $\mathcal{P}(\Delta,4)$*:* $\lambda = (1,1,1)$ *at* $\Delta = 4$ *admits the three columns of Theorem 4.4, and* $\lambda = (2,1)$ *at* $\Delta = 4$ *admits exactly one, namely* $\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$*. The remaining seven admit none, and neither does* $A(\Delta,4)$ *for* $\Delta \in \{4,5\}$*. In particular* $A(4,(2,1),4)$ *together with its extra column gives a second, independent witness of* $s(4,4) \geq 20$.

Proof sketch. Eleven runs of the test of Proposition 3.5, one per matrix, each over the whole of $\mathcal{P}(\Delta,4)$. Failure of maximality is thus a property of the partition, not of the construction and not an artefact of $\Delta = 4$: at $\Delta = 5$ all four partitions $(4),(3,1),(2,2),(2,1,1)$ give maximal matrices. □

Theorem 4.7. *The matrix*

$$M_{5,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 & 0 & -2 & -1 & -2 & 0 & 1 & 0 & 1 & 1 & -2 & -1 & -1 & 2 & -3 & -2 & -3 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & -1 & -1 & -1 & -2 & 0 & 0 & -1 & 1 & 1 & -2 & -2 & -2 & 1 & -2 & -1 & -3 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 & 1 & 3 & 1 & 0 & 1 & -1 & 0 & 2 & 2 & 2 & -2 & 3 & 3 & 4 & 1 & -1 & 2 \end{bmatrix} \in \mathbb{Z}^{4\times 24} \tag{7}$$

has $\Delta_4(M_{5,4}) = 5$ *and pairwise non-parallel columns. Hence*

$$s(5,4) \ \geq \ 24 \ > \ 22 = \binom{5}{2} + 4 \cdot 3.$$

Proof sketch. $\binom{24}{4} = 10\,626$ determinants of size 4 and $\binom{24}{2} = 276$ pairs. □

Remark 4.8. The column "best derivable from published work" in Section 1 is the maximum of three quantities: (1); the same bound at a smaller Δ, by monotonicity; and the number of parallel classes in the Averkov–Schymura pyramids $D(\mathrm{pyr}^{m-\ell}(H_\ell^{\mathrm{set}}))$, which we computed to be 13, 21, 34, 30, 48, 76 at $(\ell, m) = (2,3), (2,4), (3,4), (2,5), (3,5), (4,5)$ against the counts 16, 24, 40, 33, 54, 86 up to sign. The pyramids fire only at $\Delta = 2^\ell$, so they contribute nothing at $\Delta = 3$; at $\Delta = 4$ and $r = 4$ they give 21, and the $\Delta = 4$ instance is 5-modular as well but still gives only $21 < 22$. Averkov and Schymura publish no classification in dimension 5 at all.

The three quantities are computations; that they exhaust the literature is not. The search behind that claim covered [1, 7, 9, 6, 10, 11], each read in its own LaTeX source or published PDF rather than through a third paper's summary, together with the classification data of [1]; the arXiv full-text corpus and the arXiv API; OpenAlex; Semantic Scholar; and the OEIS. It did *not* cover MathSciNet, zbMATH by subject, or venues that publish outside those indexes. The middle column should therefore be read as *the best we could derive from the published work we found*, and a construction we missed that gives more than 16, 23, 21 or 22 in these four cells would displace it. No theorem of this paper depends on it: the four lower bounds of Section 4, the maximality statements of Section 5 and Corollary 4.3 are all statements about explicit matrices and would be unaffected.

How the matrices were found. The four witnesses were produced by a randomised search over identity-normalised column sets, implemented in C, and the theorems above do not depend on it: each theorem is a finite verification of a displayed matrix. We describe the search because its control cells are the reason we trust the cells that are open. The search starts from $\mathbf{e}_1, \ldots, \mathbf{e}_r$ — which also forces rank r — and repeatedly adds a member of $\mathcal{P}(\Delta, r)$ that keeps $\Delta_r \leq \Delta$, using Proposition 3.4 as the incremental test, accepting moves that leave the size unchanged as well as moves that increase it, and restarting; each of eight cells was run for 60 000 rounds within 1500 seconds.

Two failures of earlier versions are worth recording, both caught by control cells rather than by inspection. Without the identity seed the search is unsound for Problem 1.1: a rank-deficient set has all $r \times r$ minors zero, hence is Δ-modular for every Δ, and the first version returned 13 columns at $(1,4)$ — all with two equal rows — where Heller's theorem gives 10. And a search accepting only strict improvements never reaches the known sporadic value $s(2,5) = 20$: 20 000 restarts stalled at 19, while accepting ties finds 20 in the same budget. After both fixes the search reproduces every value that the literature pins down at these ranks:

(Δ, r)	(1)	search	literature
$(1,3)$	6	6	$s(1,3) = 6$ [4]
$(1,4)$	10	10	$s(1,4) = 10$ [4]
$(1,5)$	15	15	$s(1,5) = 15$ [4]
$(2,3)$	8	**9**	$s(2,3) = 9$ [7]
$(2,4)$	13	13	$s(2,4) = 13$ [7]†
$(2,5)$	19	**20**	$s(2,5) = 20$ [7]
$(3,3)$	10	**11**	$s(3,3) \geq 11$ [10, §1]
$(3,4)$	16	**17**	open
$(3,5)$	23	**25**	open
$(4,4)$	19	**22**	$s(4,4) \geq 21$ (Proposition 4.5)
$(5,4)$	22	**24**	open

† Two published readings of the $(2,4)$ cell differ, and it is worth saying which one the control is against. Lee et al. assert, in the introduction of [7] (arXiv v5), "*Our analysis shows* $c^{\mathrm{p}}(2,m) = c(1,m) + m$ *for* $m \in \{3,5\}$ *and* $c^{\mathrm{p}}(2,m) = c(1,m) + m - 1$ *otherwise*", pointing for the lower half at their tight examples and for the upper half at the three case bounds (23), (24), (25) of their Section 4. Since $c^{\mathrm{p}} = s$ and $c(1,4) = 10$, that is $s(2,4) = 13$, and at $m = 4$ those three bounds read 13, 13, 12 while the remaining subcase of that proof gives 13 as well. Kriepke and Schymura state the same conclusion directly in the notation used here: $s(2,r) = \binom{r+1}{2} + r$ for $r \in \{3,5\}$ and $\binom{r+1}{2} + r - 1$ for $r \in \mathbb{Z}_{>0} \setminus \{3,5\}$, which they attribute to [7, §4] together with [8]; see [6, §1]. Paat, Walsh and Xu summarise the same result more cautiously — "*for* $r \geq 6$*, they improve to* $s(2,r) = \binom{r+1}{2} + r - 1$" [10, §1] — which is silent at $r = 4$ rather than in conflict with it. The control is against the first reading. Under the second, the cell asserts only $s(2,4) \geq 13$, which is (1), and the control degrades from an equality match to an inequality match; nothing in this paper depends on which reading is right.

Seven of the eleven cells are controls, and all seven agree, including both known sporadic values and the rank-3, $\Delta = 3$ matrix displayed in [10]. We do not claim the four open cells as exact values: the search is a heuristic, and only its output matrices, not its verdicts, enter the theorems. Values of $s(\Delta, 3)$ that the same search produces for $\Delta \in \{5,6,7\}$ are deliberately not reported as results, because Averkov and Schymura's classification in dimension 3 is published up to $\Delta = 7$ (and its extremal part to $\Delta = 12$), so a maximum-subset computation inside their data is an available route to those cells that we have not carried out.

5. Maximality

Definition 5.1. A Δ-modular matrix $\mathbf{A}$ with pairwise non-parallel columns of length r is **maximal among identity-normalised extensions** if for every $\mathbf{c} \in \mathcal{P}(\Delta, r)$ the matrix $[\mathbf{A}|\mathbf{c}]$ fails to be Δ-modular with pairwise non-parallel columns.

By Lemma 3.3, for an identity-normalised $\mathbf{A}$ this says exactly that no integer column at all can be appended to $\mathbf{A}$ keeping it Δ-modular with pairwise non-parallel columns. It does *not* say that $\mathbf{A}$ has the largest possible number of columns: it is maximality under inclusion, not maximality of size.

Theorem 5.2. *Each of the following is maximal among identity-normalised extensions: the* 17*-column matrix* $M_{3,4}$ *of* (5) *at* $\Delta = 3$*; the* 25*-column matrix* $M_{3,5}$ *of* (6) *at* $\Delta = 3$*; the* 22*-column matrix* $M_{4,4}$ *of Theorem 4.4 at* $\Delta = 4$*; the* 24*-column matrix* $M_{5,4}$ *of* (7) *at* $\Delta = 5$*; and the* 20*-column matrix of Lee et al. witnessing* $s(2,5) = 20$ *at* $\Delta = 2$.

Proof sketch. Five runs of the test of Proposition 3.5, each exhausting the whole candidate set: $1120 \times \binom{17}{3}$, $8161 \times \binom{25}{4}$, $2928 \times \binom{22}{3}$, $6928 \times \binom{24}{3}$ and $1441 \times \binom{20}{4}$ dot products respectively, on top of 680, 12 650, 1540, 2024 and 4845 cross products. In each case no candidate survives. The reduction of this Boolean computation to a statement about determinants and linear dependence is Propositions 3.1, 3.4 and 3.5, and is proved without compiled evaluation. □

Proposition 5.3. *The rank-*3*,* 3*-modular matrix with* 11 *columns displayed in* [10, §1]*, a* 9*-column witness of* $s(2,3) = 9$*, and Heller's matrix* H_3 *at* $\Delta = 1$ *are all maximal among identity-normalised extensions.*

Proof sketch. Here the candidate sets are small enough ($|\mathcal{P}(3,3)| = 145$, $|\mathcal{P}(2,3)| = 49$, $|\mathcal{P}(1,3)| = 13$) that the definition can be used directly, recomputing every 3×3 minor of the extended matrix per candidate. □

Remark 5.4. Maximality under inclusion leaves open whether a different matrix does better. Let $s_I(\Delta, r)$ denote the maximum of n over Δ-modular $\mathbf{A} \in \mathbb{Z}^{r\times n}$ of rank r with pairwise non-parallel columns that contain $\mathbf{e}_1, \dots, \mathbf{e}_r$. If $\mathbf{A}$ has a unimodular $r \times r$ submatrix $\mathbf{B}$ then $\mathbf{B}^{-1}\mathbf{A}$ is integral, has the same $r \times r$ minors up to the global sign $\det \mathbf{B}$, has the same parallelism relation, and contains the identity; so $s_I(\Delta, r)$ is the maximum of n over those $\mathbf{A}$ possessing a unimodular $r \times r$ submatrix, and $s_I \le s$. An exhaustive depth-first search over $\mathcal{P}(3,4)$ seeded with $\mathbf{e}_1, \dots, \mathbf{e}_4$, run in C and bounded below by the 17 of Theorem 4.1, terminated without finding 18 columns, so $s_I(3,4) = 17$. The search is deterministic — it enumerates subsets of $\mathcal{P}(3,4)$ containing $\mathbf{e}_1, \dots, \mathbf{e}_4$ in a fixed order, pruning by the cross-product test of Proposition 3.5 and by the bound $n_{\text{chosen}} + n_{\text{remaining}} \le 17$ — and two independent runs of the same program, on different days and different machine loads, both report the same 11 577 318 nodes; the wall times were 230 and 128 seconds. The node count, not the time, is the reproducible figure. We report this as a search record and not as a theorem: it is not formalised, and the same search did not terminate within a 25-minute budget at $(3,5)$, $(4,4)$ or $(5,4)$. Nothing else in this paper depends on it.

Remark 5.5. In the opposite direction, (2) and (3) give

$$s(3,4) \;\le\; c(3,4) = \tfrac{1}{2}\big(\max\{h(1,4), h(2,4), h(3,4)\} - 1\big).$$

The values $h(1,4) = 21$ and $h(2,4) = 29$ are in [1, Table 1]. The cell $(m,\Delta) = (4,3)$ of that table is blank, so $h(3,4)$ has to be taken from the classification data published with [1] instead, and it is worth being exact about what that involves. Averkov and Schymura prove

$$h(\Delta, m) = \max\big\{|P \cap \mathbb{Z}^m| :\ P \subseteq \mathbb{R}^m \text{ an } o\text{-symmetric lattice polytope with } \Delta(P) = \Delta\big\}$$

[1, eq. (8)], and their Problem 5.2 asks for the polytopes attaining that maximum; the files named `…_extremal.txt` in their repository are the announced solutions of that problem, one file per pair (m, Δ), each listing vertex sets. *What we checked:* their dimension-4, $\Delta = 3$ extremal file lists three polytopes; we enumerated the lattice points of each — facets from the 4-subsets of the vertex list, then a sweep of the bounding box — and found that all three are o-symmetric, satisfy $\Delta(P) = 3$, and contain exactly 37 lattice points. The same computation on their dimension-4 extremal files at $\Delta = 1$ and $\Delta = 2$ returns 21 and 29, which are the $h(1,4)$ and $h(2,4)$ entries of their Table 1; that is the control. *What we assume:* that the entries of the $\Delta = 3$ file really are extremal, that is, that no o-symmetric lattice 4-polytope with $\Delta(P) = 3$ has more than 37 lattice points. Completeness of the list is not needed — one correct maximiser determines h — but this correctness is. Granting it, $h(3,4) = 37$ and $c(3,4) = 18$, and with Theorem 4.1,

$$17 \;\le\; s(3,4) \;\le\; 18.$$

The upper half is thus not a theorem of this paper: it rests on somebody else's computation and is not formalised.

Remark 5.6. The published classification data of [1] is a genuine alternative route to lower bounds — one takes a maximum subset with $\Delta_r \le \Delta$ inside the lattice points of each listed polytope — and it does not reach the matrices of Section 4. In dimension 4 at $\Delta = 3$ the extremal file is the only one published, and the largest number of parallel classes with $\Delta_4 \le 3$ inside any of its three polytopes is 16, one short of 17. The same route applied in dimension 3 at $\Delta = 3$ returns 10, while 11 is known from the matrix displayed in [10]: it under-reports, and it under-reports exactly on the known sporadic example. The reason is visible on $M_{3,4}$: the polytope $\mathrm{conv}(\pm M_{3,4} \cup \{\mathbf{0}\})$ has 35 lattice points and $\Delta_4 = 3$, so it is an o-symmetric $\Delta = 3$ lattice polytope, but $35 < 37$, so it is not extremal and does not appear in that file. In dimension 5 nothing at all is published. Every number in this remark is a computation we ran on their published files, not a statement of theirs.

6. The published values, re-derived

For the comparisons above to mean anything, the values they are compared against have to be present in the same form. They are: every construction of Section 2 is generated from its definition and its invariants computed, so that "one column more" is one machine-checked statement against another. We record what is there.

Proposition 6.1. *Heller's matrices H_r have $\binom{r+1}{2}$ pairwise non-parallel columns and $\Delta_r = 1$, giving $s(1,r) \geq \binom{r+1}{2}$ for $r = 2,3,4,5$. The matrices $A(\Delta, r)$ have $\binom{r+1}{2}+(\Delta-1)(r-1)$ pairwise non-parallel columns and $\Delta_r = \Delta$, giving (1) for $\Delta \in \{2,3,4,5\}$ and $r \in \{3,4\}$ and for $(\Delta, r) \in \{(2,5),(3,5)\}$. The nine matrices $A(\Delta, \lambda, 4)$ with $\lambda \vdash \Delta - 1$, $\Delta \in \{3,4,5\}$, and $A(3,(2),5)$ likewise attain (1).*

Proposition 6.2. *The three sporadic values known before this work are witnessed: the rank-3, 3-modular matrix displayed in [10, §1] has 11 pairwise non-parallel columns and $\Delta_3 = 3$, so $s(3,3) \geq 11 > 10$; a 9-column rank-3 matrix with $\Delta_3 = 2$ gives $s(2,3) \geq 9 > 8$, a value neither [7] nor [10] exhibits a matrix for, so this witness is one found by an exhaustive identity-normalised search; and the 5×20 matrix displayed by Lee et al. under* Tight examples *gives $s(2,5) \geq 20 > 19$, in both the pairwise non-parallel and the up-to-sign count. For the last, the matrix is also rebuilt from the authors' own second description of it — the vertex-edge incidence matrix of the directed complete graph on 5 vertices minus the column $\mathbf{e}_1 - \mathbf{e}_2$, together with the identity and six further columns — and the two constructions are checked to give the same certificate.*

The other count, and why it is not this one. The counterexamples of [1] to (4) are present as well, in both counts, and the comparison is instructive.

Proposition 6.3. *$c(4,3) \geq 16 > 15$, $c(4,4) \geq 24 > 22$ and $c(8,4) \geq 40 > 38$, from $D(\mathrm{pyr}^{m-\ell}(H_\ell^{\mathrm{set}}))$ built from the definitions; the intermediate counts $|D(C_3^2)| = 33$, $|D(C_4^2)| = 49$, $|D(C_4^3)| = 81$ reproduce [1, Table 1] and [1, Theorem 1.3] at those parameters. A planar base set found by search gives $c(8,4) \geq 39 > 38$ independently, where [1] use a three-dimensional one.*

Proposition 6.4. *There is an explicit $\mathbf{A} \in \mathbb{Z}^{4\times 43}$ with no zero column, no two columns equal up to sign, and $\Delta_4(\mathbf{A}) = 9$; since (4) predicts $c(9,4) = 42$, the conjecture of Lee et al. fails at $\Delta = 9$, outside the set $\{4,8,16\}$ recorded in the literature.*

Proof sketch. The pyramid lemma [1, Lemma 5.4] is stated there for an arbitrary finite non-empty set S, and the counting identity uses only $|S|$ and $|D(S)|$; so the criterion "$\Delta(S-S) \leq \Delta$ and $|S| > \Delta + \ell$ refutes (4) at Δ" is theirs with the base set left general. Taking S a planar 12-point set with $\Delta(S-S) = 9$ gives $12 > 9+2$, and the smallest instance $m = 4$ has 87 points, hence 43 columns up to sign. This is a routine consequence of published work rather than a new result: the required base set is already contained in the classification data published with [1], whose list of o-symmetric lattice polygons with $\Delta = 9$ contains a hexagon whose 37 lattice points are $S - S$ for such an S, and one maximum-clique computation on that file exposes it. The same computation reproduces $\Delta = 4$ and $\Delta = 8$, which is the control. The criterion also fires at $\Delta = 12$, where we have not built the matrix. ☐

Proposition 6.5. *In the count of Problem 1.1 the same two matrices give only $s(4,3) \geq 13$ and $s(9,4) \geq 37$, against the conjectured values 15 and 42.*

Proposition 6.5 is the reason the middle column of the table in Section 1 is what it is: no known counterexample to (4) refutes (1), and the pyramids improve on (1) in the count of Problem 1.1 only at $(4,4)$, where they give 21.

Negative controls. A development of this kind can be satisfied by a misplaced quantifier, so each layer carries refutations as well as proofs.

Proposition 6.6. *The following all hold, and are proved in the same language as the theorems above: $M_{3,4}$ is not 2-modular and $M_{4,4}$ is not 3-modular, by exhibited 4×4 submatrices of determinant -3 and -4; $A(3,4)$ has $16 < 17$ columns; a matrix of four 4-vectors of rank 3 fails the rank hypothesis, so that hypothesis is not vacuous; appending $-\mathbf{e}_1$ to $M_{3,4}$ leaves $\Delta_4 = 3$ but destroys non-parallelism, so Δ-modularity alone does not force the columns apart; appending $2\mathbf{e}_1$ to $M_{3,5}$, which the up-to-sign count would allow, pushes a 5×5 minor to 6; the cross product of Proposition 3.4 takes the value 2 on a concrete instance rather than 0, changes sign when two of its arguments are swapped, and vanishes identically on a dependent argument; and the extension test is not vacuously empty — it returns 205 candidates for H_4 at $\Delta = 3$, while returning none for H_4 at $\Delta = 1$ and none for $A(2,4)$ at $\Delta = 2$, the two cells where $s(1,4) = 10$ and $s(2,4) = 13$ are the published values.*

7. Verification

Every statement in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development contains no `sorry`, and the repository reference is to be added at submission. The reasoning layer is proved by hand and depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: this covers Proposition 3.1 (the list-based Laplace expansion is det at every rank, and the sub-list enumeration is the set of $r \times r$ submatrices), Proposition 3.4 (the cross-product identity, at every rank), and Proposition 3.5 together with both directions of the equivalence between vanishing 2×2 minors and linear dependence, which is what allows a rejected candidate to contradict pairwise non-parallelism. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches over explicit integer data: for each displayed matrix, the number of columns, the value of Δ_r and pairwise non-parallelism (Theorems 4.1, 4.2, 4.4, 4.7 and Propositions 4.5, 6.1, 6.2, 6.3, 6.4, 6.5); the eleven extension tests of Proposition 4.6; the five of Theorem 5.2 and the three of Proposition 5.3, together with the width checks on the candidate sets $\mathcal{P}(\Delta, r)$; and the controls of Proposition 6.6. The whole maximality layer — five candidate-set exhaustions, eight width checks and seven controls — evaluates in 32 seconds and 6.9 GB of memory, against an `import Mathlib` baseline of 6.5 GB on the same machine; the witness file costs 1 minute 51 seconds. Independently of the proof assistant, every number quoted for a displayed matrix was recomputed outside it: the minors of the four witnesses by three different determinant routines (fraction-free Bareiss, cofactor expansion, and Gaussian elimination over $\mathbb{Q}$) over all $\binom{n}{r}$ subsets, and the extension tests by a separate implementation, with the same verdicts, including the three columns of Theorem 4.4 and the count 205 of Proposition 6.6.

8. Questions

Question 8.1. Is $s(3,4) = 17$ or 18? Remark 5.5 boxes it, and Remark 5.4 settles the identity-normalised half; what is missing is a matrix with no unimodular 4×4 submatrix, or a proof that none helps.

Question 8.2. How far does the sporadic behaviour at $\Delta = 3$ extend? Corollary 4.3 gives $r_0 \geq 6$ for the threshold of Theorem 1.2; is $s(3,6) > \binom{7}{2} + 2\cdot 5 = 26$?

Question 8.3. Theorem 4.4 shows that attaining (1) and being maximal are different. Which of the matrices $A(\Delta, \lambda, r)$ of [10, Definition 4.1] — the family behind their Theorem 1.3 — are maximal? At rank 4 the answer is: all but $\lambda = (1,1,1)$ and $\lambda = (2,1)$ at $\Delta = 4$.

Question 8.4. The criterion in the proof of Proposition 6.4 fires at $\Delta = 4, 8, 9, 12$ among $\Delta \le 13$ in base dimension 2, and on the threshold, without firing, at the primes $\Delta = 5, 7$. Is there a full-dimensional $S \subseteq \mathbb{Z}^\ell$ with $\Delta(S - S) \le 5$ and $|S| > 5 + \ell$? For $\ell = 2, 3$ there is none inside the normalised box; $\ell = 4$ is open.

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