TWO ERRATA FOR THE COLLATZ EXTENSION MODULO $2^p + 2^q$

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Abstract. Bouhamidi has proposed an extension of the Collatz conjecture in which the classical map is replaced by the map of the triplet $(2^p+2^q,\ 2^p+2^{q+1},\ 2^p)_+$, and has conjectured that every such triplet is strongly admissible, of order one outside an exceptional set $\mathbb{E}$ of eight pairs and of order two (three at $(5,2)$) on it. We correct two printed claims of that paper. First, its Table 1 of maximum total stopping times $M_\infty(p) = \max\{\sigma_\infty(n) : 1 \le n \le 10^7,\ 0 \le q \le p\}$ is refuted at seven of its twenty-six entries: a recomputation over exactly that range gives $429, 324, 433, 459, 465$ at $p = 0, 1, 2, 3, 4$ against the printed $246, 213, 268, 374, 349$, while $M_\infty(6) \ge 930 > 737$ and $M_\infty(19) \ge 524288 > 131091$. The last of these needs no search: we prove that at $q = 0$ the orbit of 2^{p+1} first meets 2^p after exactly 2^p steps, whence $M_\infty(p) \ge 2^p$ for every $p \le 22$ with $(p, 0) \notin \mathbb{E}$. Second, the paper's worked example for the triplet $(12, 14, 10)_+$ prints the cycle $\Omega(4)$ as $4 \to 8 \to 16 \to 22 \to 4$ "of length 6"; the stated length is correct and the printed chain is short by two, the cycle being $4 \to 8 \to 16 \to 22 \to 34 \to 48 \to 4$. We also give an unconditional proof, with the minimality of the return time that the original argument omits, of the trivial-cycle statement inside the conjecture — a statement the source proves in a wider setting — and record what an independent recomputation confirms about the exceptional set. All statements are machine-checked in Lean 4.

1. Introduction

The objects. Fix an integer $d \ge 2$ and integers α, β with $\alpha > d$, $\alpha \not\equiv 0$ and $\beta \not\equiv 0 \pmod d$. Writing $[n]_d \in \{0, \dots, d-1\}$ for the remainder of n in the Euclidean division by d, the *triplet* $(d, \alpha, \beta)_+$ is the self-map of $\mathbb{N}$

$$T(n) = \begin{cases} n/d, & d \mid n, \\ \dfrac{\alpha n + \beta\,[n]_d}{d}, & d \nmid n. \end{cases} \tag{1}$$

The map takes integer values for every n precisely when $\alpha + \beta \equiv 0 \pmod d$, and this congruence is the standing hypothesis on a triplet [1, §1]; the framework of such triplets is developed in [3]. A *cycle* is a finite set Ω on which some iterate of T is the identity; following [1] we write $\Omega(\omega)$ for the cycle whose least element is ω, display it as $\big(\omega \to T(\omega) \to \cdots \to T^{(k-1)}(\omega) \to \omega\big)$, and call k — the number of its elements — its *length*. The *order* of a triplet is its number of cycles. A triplet is *weakly admissible* if it has at least one and finitely many cycles, and *strongly admissible* if in addition it has no divergent trajectory. With $d = 2, \alpha = 3, \beta = 1$ one recovers the shortcut Collatz map $n \mapsto n/2$, $n \mapsto (3n+1)/2$, and the Collatz conjecture becomes the assertion that $(2, 3, 1)_+$ is strongly admissible of order one [1, Conjecture 1.2]. For the classical problem itself we refer to the survey [5] and, for the strongest general result now known, to [6].

The source and its conjecture. Bouhamidi [1] singles out the two-parameter family obtained by taking, for integers $p \ge q \ge 0$,

$$d_{p,q} = 2^p + 2^q, \qquad \alpha_{p,q} = 2^p + 2^{q+1}, \qquad \beta_p = 2^p, \tag{2}$$

and writes $T_{p,q}$ for the associated map (1). Here $p = q = 0$ gives $(2, 3, 1)_+$ and $p = 3$, $q = 1$ gives $(10, 12, 8)_+$, the "conjecture modulo 10" of [1]. The main conjecture of [1] is that every $(d_{p,q}, \alpha_{p,q}, \beta_p)_+$ is strongly admissible; that off the exceptional set

$$\mathbb{E} = \{(1,0), (2,1), (2,2), (3,0), (4,0), (5,2), (6,2), (7,0)\} \tag{3}$$

it is of order one with unique cycle $\Omega(2^{p-q})$ of length $2^{p-q}+q+1$; and that on $\mathbb{E}$ it is of order two, of order three at $(5,2)$, the extra cycles being nine explicitly displayed ones [1, Conjecture 2.1]. The conjecture is verified numerically in [1] for $p \le 25$ and $1 \le n \le 10^7$, and the maximum total stopping time over that range is reported there in [1, Table 1] (Section 5 below). All numbered references to [1] in this paper are to its version v1, the only one at the time of writing; see Remark 5.1 for one caveat about the numbering of its table.

Two clauses of that conjecture are of quite different depths. Strong admissibility — finitely many cycles and no divergent trajectory — contains the Collatz conjecture as the case $p = q = 0$ and is not addressed here or in [1]. But "$\Omega(2^{p-q})$ *is* a cycle, of length $2^{p-q}+q+1$" is a finite statement about each pair and in fact a theorem for all pairs at once; [1, Theorem 2.1] proves it, in the wider setting of the triplets $(d, d+2^q, d-2^q)_+$ with $d = 2^q d'$, where the length reads $d/2^q + q$.

Results. This paper is chiefly an erratum. Most of what we could check in [1] reproduces exactly: all nine cycles of $\mathbb{E}$, element by element, including the 630-cycle at $(7,0)$; the trivial cycle at every pair tried; and two of the three displays in the worked example for $(12,14,10)_+$. Two printed claims do not.

Erratum I (Section 5). Table 1 of [1], which reports

$$M_\infty(p) = \max\{\sigma_\infty(n) \ : \ 1 \le n \le n_{\max} = 10^7,\ 0 \le q \le p\}$$

for $0 \le p \le 25$, is too small at every one of the seven entries we could settle. Recomputing the maximum over exactly the stated range gives $429, 324, 433, 459, 465$ at $p = 0,1,2,3,4$, against printed $246, 213, 268, 374, 349$; at $p = 6$ the maximum is at least 930 against printed 737; and at $p = 19$ it is at least 524288 against printed 131091. Each refutation is certified by a single witness $n \le 10^7$ together with its exact total stopping time. A disagreement about the definition of σ_∞ does not explain the failures, and neither does a smaller $n_{\max}$; we give both arguments in Section 5. We do *not* claim that exactly seven entries are wrong: the honest statement is that the table is unreliable wherever it can be checked.

Erratum II (Section 6). The worked example of [1] for the triplet $(12,14,10)_+$ prints

$$\Omega(4) = 4 \to 8 \to 16 \to 22 \to 4), \qquad \text{“of length 6”}$$

— verbatim, unbalanced parenthesis included. The length is right and the chain is short by two: $T(22) = 34$, not 4, and the cycle is $4 \to 8 \to 16 \to 22 \to 34 \to 48 \to 4$. The two neighbouring displays in the same example are correct.

Along the way we prove, for every $p \ge q \ge 0$ and with no bound on p, that $\Omega(2^{p-q})$ is a cycle of *least* return time $2^{p-q}+q+1$ and therefore has that many distinct elements (Section 3) — the minimality is what turns the length into a cardinality, and it is not argued in [1] — and we deduce a closed form for one column of the stopping-time table (Section 4) which is what settles the entry at $p = 19$ without any search. Section 7 records what an independent recomputation confirms about the exceptional set, including a remark on search ranges that anyone extending $\mathbb{E}$ beyond $p = 8$ should read first. Section 9 describes the formal verification.

2. Preliminaries

Throughout, $p \ge q \ge 0$ are integers, $d = d_{p,q}$, $\alpha = \alpha_{p,q}$ and $\beta = \beta_p$ are as in (2), and $T = T_{p,q}$ is the map (1) for that triplet. We write $T^{(k)}$ for the k-fold iterate.

The first thing to pin down is that (1) is a map into $\mathbb{N}$ at all: the division by d must be exact, since otherwise the reader who implements T with a truncating division is studying a different map.

Proposition 2.1. *For all $p, q, n \ge 0$ one has $\alpha_{p,q} + \beta_p = 2d_{p,q}$ and hence $d_{p,q} \mid \alpha_{p,q} n + \beta_p [n]_{d_{p,q}}$, so $T_{p,q}$ maps $\mathbb{N}$ into $\mathbb{N}$. Moreover $T_{0,0}$ is the shortcut Collatz map: $T_{0,0}(n) = n/2$ for even n and $T_{0,0}(n) = (3n+1)/2$ for odd n.*

Proof. $\alpha + \beta = (2^p + 2^{q+1}) + 2^p = 2^{p+1} + 2^{q+1} = 2d$. Write $n = dk + r$ with $r = [n]_d$; then $\alpha n + \beta r = \alpha dk + (\alpha + \beta)r = d(\alpha k + 2r)$. The second claim is the case $d = 2, \alpha = 3, \beta = 1$, where $[n]_2 = n \bmod 2$. □

The identification in Proposition 2.1 is a convention trap worth stating explicitly, because the two standard normalisations of the Collatz map disagree on every quantity used below. Under $T_{0,0}$ the cycle through 1 is $(1 \to 2 \to 1)$, of length 2 — which is $2^{p-q} + q + 1$ at $p = q = 0$, as [1] records — whereas under $n \mapsto 3n + 1$ it is $(1 \to 4 \to 2 \to 1)$, of length 3. Likewise the number of steps taken by 27 to reach 1 is 70 under $T_{0,0}$ and 111 under $n \mapsto 3n + 1$; these are the values $A006666(27)$ and $A006577(27)$ of [8]. Every statement below is about $T_{p,q}$, hence about the shortcut normalisation.

Definition 2.2. For $\omega, \ell \geq 1$ we say that ω *lies on a cycle of length* ℓ if $T^{(\ell)}(\omega) = \omega$ and $T^{(k)}(\omega) \neq \omega$ for $0 < k < \ell$; when ω is in addition the least element of $\{T^{(k)}(\omega) : 0 \leq k < \ell\}$ we write $\Omega(\omega)$ for that set, as [1] does. The minimality of the return time makes $k \mapsto T^{(k)}(\omega)$ injective on $\{0, \ldots, \ell - 1\}$, so a cycle of length ℓ really has ℓ distinct elements.

Finally we recall the total stopping time of [1, §3], which transports Terras's notion [4] to a general triplet. If a triplet has cycles $\Omega(\omega_1), \ldots, \Omega(\omega_{c_0})$, then $\sigma_\infty(n) = \min\{k \geq 0 : T^{(k)}(n) \in \{\omega_1, \ldots, \omega_{c_0}\}\}$, with $\sigma_\infty(n) = \infty$ if no such k exists. For the family (2) the representative set is fixed explicitly by Algorithm 1 of [1]:

$$S_{p,q} = \{2^{p-q}\} \quad \text{for } (p, q) \notin \mathbb{E}, \tag{4}$$

and on $\mathbb{E}$,

$$\begin{array}{llll} S_{1,0} = \{2, 14\}, & S_{2,1} = \{2, 74\}, & S_{2,2} = \{1, 67\}, & S_{3,0} = \{8, 280\}, \\ S_{4,0} = \{16, 1264\}, & S_{5,2} = \{8, 76200, 87176\}, & S_{6,2} = \{16, 1264\}, & S_{7,0} = \{128, 3027584\}. \end{array}$$

We write $\sigma_{p,q}(n) = \min\{k \geq 0 : T^{(k)}_{p,q}(n) \in S_{p,q}\}$; this is the quantity the computations of [1] evaluate, and it agrees with σ_∞ whenever the conjecture of [1] holds at (p, q). All the statements below about $\sigma_{p,q}$ are unconditional statements about (4), so nothing in them depends on the conjecture.

3. The trivial cycle, with minimality

Theorem 3.1. *Let $p \geq q \geq 0$. Then 2^{p-q} lies on a cycle of $T_{p,q}$ of length exactly $2^{p-q} + q + 1$, and that cycle is*

$$\Omega(2^{p-q}) = \big(2^{p-q} \to 2^{p-q+1} \to \cdots \to 2^p \to 2 \cdot 2^p \to 3 \cdot 2^p \to \cdots \to (2^{p-q} + 1) \cdot 2^p \to 2^{p-q}\big), \tag{5}$$

a list of $2^{p-q} + q + 1$ distinct integers.

That $\Omega(2^{p-q})$ is a cycle of this length is [1, Theorem 2.1], proved there for the wider family $(d, d + 2^q, d - 2^q)_+$ with $d = 2^q d'$, where the length reads $d/2^q + q$; at $d = 2^p + 2^q$ that is $2^{p-q} + q + 1$. What Theorem 3.1 adds is the minimality of the return time, hence the distinctness of the displayed elements: [1] exhibits an iterate that returns and displays the intermediate values, but does not argue that no earlier iterate returns, and it is only minimality that makes "length" a cardinality.

Proof sketch. Three closed forms for a single step do all the work, and each is a two-line computation from $\alpha + \beta = 2d$ (Proposition 2.1).

(i) If $0 < n < d$ then $[n]_d = n$, so $\alpha n + \beta n = 2dn$ and $T(n) = 2n$: on the interval $(0, d)$ the map is doubling.

(ii) If $1 \leq k \leq 2^{p-q}$ then $T(k \cdot 2^p) = (k + 1) \cdot 2^p$. Indeed, putting $e = 2^{p-q} - k + 1 \geq 1$ one has $d = 2^q(2^{p-q} + 1)$, $\alpha = 2^q(2^{p-q} + 2)$, $\beta = 2^q \cdot 2^{p-q}$ and $k \cdot 2^p = 2^q e + d(k - 1)$ with $0 < 2^q e < d$, so $[k \cdot 2^p]_d = 2^q e$, and the numerator collapses to $d\,(k + 1)2^p$.

(iii) $(2^{p-q} + 1) \cdot 2^p = d \cdot 2^{p-q}$, so $T\big((2^{p-q} + 1)2^p\big) = 2^{p-q}$: the one division of the cycle.

By (i) and induction, $T^{(i)}(2^{p-q}) = 2^{p-q+i}$ for $0 \le i \le q$, which reaches 2^p after q steps; by (ii) and induction, $T^{(q+j)}(2^{p-q}) = (j+1)2^p$ for $0 \le j \le 2^{p-q}$; and (iii) closes the cycle at step $q + 2^{p-q} + 1$. Concatenating the two arcs gives (5). For minimality, every intermediate value is strictly larger than 2^{p-q}: on the first arc $T^{(i)}(2^{p-q}) = 2^{p-q+i} > 2^{p-q}$ for $1 \le i \le q$, and on the second $(j+1)2^p > 2^p \ge 2^{p-q}$. Distinctness of the $2^{p-q}+q+1$ elements follows, since a repeat would produce a return before time $2^{p-q}+q+1$. □

No search enters this proof: it is an induction on q and on the ladder index j, uniform in p and q.

4. A closed form for one column of the stopping-time table

Proposition 4.1. *For every $p \ge 0$, the orbit of 2^{p+1} under $T_{p,0}$ first meets 2^p after exactly 2^p steps:*

$$T_{p,0}^{(2^p)}\left(2^{p+1}\right) = 2^p, \qquad T_{p,0}^{(k)}\left(2^{p+1}\right) \neq 2^p \quad \text{for all } k < 2^p.$$

Consequently, if $(p,0) \notin \mathbb{E}$ — so that $S_{p,0} = \{2^p\}$ — then $\sigma_{p,0}(2^{p+1}) = 2^p$ exactly, and $M_\infty(p) \ge 2^p$ for every such p with $2^{p+1} \le n_{\max}$: that is, for every $p \le 22$ outside $\{1,3,4,7\}$ when $n_{\max} = 10^7$.

Proof. Take $q = 0$, so that $2^{p-q} = 2^p$ and $2^{p+1} = 2 \cdot 2^p$ is the second element of the trivial cycle (5). By step (ii) of the proof of Theorem 3.1, $T^{(k)}(2 \cdot 2^p) = (k+2)2^p$ for $0 \le k < 2^p$, and by step (iii) $T^{(2^p)}(2 \cdot 2^p) = 2^p$. Since $(k+2)2^p > 2^p$ for every $k \ge 0$, no earlier iterate equals 2^p: the value 2^{p+1} sits one place past the head of the trivial cycle and has to walk the whole ladder home. If moreover $(p,0) \notin \mathbb{E}$ then $S_{p,0} = \{2^p\}$, so the first hit of $S_{p,0}$ is the first hit of 2^p, and $\sigma_{p,0}(2^{p+1}) = 2^p$. Finally $2^{p+1} \le n_{\max}$ puts the witness inside the range over which $M_\infty(p)$ is a maximum, and $q = 0 \le p$. □

The pairs excluded by the last clause are exactly the members of $\mathbb{E}$ with $q = 0$, namely $(1,0)$, $(3,0)$, $(4,0)$ and $(7,0)$, where the enlarged representative set could in principle be met earlier; the application below is at $p = 19$, where $S_{19,0} = \{2^{19}\}$.

Remark 4.2. Proposition 4.1 is a corollary of the trivial-cycle theorem of [1] and not a new result about these triplets; it is recorded because it is the only ingredient of Theorem 5.2 that is not a finite computation, and because it bounds $M_\infty(p)$ from below at almost every p of the tabulated range at once.

5. Erratum I: the table of maximum total stopping times

Section 4 of [1] reports the verification of the main conjecture for $p \le 25$ and $1 \le n \le n_{\max} = 10^7$, and tabulates

$$M_\infty(p) = \max_{\substack{1 \le n \le n_{\max} \\ 0 \le q \le p}} \sigma_\infty(n),$$

noting that the maximum "is reached for $q = 0$". The printed values are, verbatim and in the printed arrangement:

p	0	1	2	3	4	5	6
$M_\infty(p)$	246	213	268	374	349	731	737
p	7	8	9	10	11	12	13
$M_\infty(p)$	818	1444	3152	3638	3639	4108	8205
p	14	15	16	17	18	19	20
$M_\infty(p)$	16398	32783	65552	131089	262162	131091	1048596
p	21	22	23	24	25	—	—
$M_\infty(p)$	2097173	4194326	8388631	16777240	33554457	—	—

Remark 5.1 (How the table is numbered in [1]). This is the only table of [1], and its caption prints as "Table 1" (page 14 of v1); we cite it as [1, Table 1]. The surrounding text of [1] calls it "Table 4.1" instead, because the table's label is placed before its caption and so takes the value of the enclosing subsection counter rather than the table counter. A reader chasing the reference should expect the two numbers.

Theorem 5.2. *With $\sigma_{p,q}$ as in Section 2 and $n_{\max} = 10^7$, the tabulated values above are refuted at seven entries. Explicitly, each of the following integers n satisfies $n \le 10^7$ and has the stated total stopping time at $q = 0$, so $M_\infty(p)$ is at least that value:*

p	*printed* $M_\infty(p)$	*witness* n	$\sigma_{p,0}(n)$	*certified bound*
0	246	8400511	429	$M_\infty(0) \ge 429$
1	213	8718455	324	$M_\infty(1) \ge 324$
2	268	5320697	433	$M_\infty(2) \ge 433$
3	374	5252525	459	$M_\infty(3) \ge 459$
4	349	3546351	465	$M_\infty(4) \ge 465$
6	737	257967	930	$M_\infty(6) \ge 930$
19	131091	2^{20}	$2^{19} = 524288$	$M_\infty(19) \ge 524288$

In particular $M_\infty(p) >$ the printed value for $p \in \{0, 1, 2, 3, 4, 6, 19\}$.

Proof sketch. For $p \in \{0, 1, 2, 3, 4, 6\}$ the certificate attached to each row is finite and explicit: the trajectory of the witness n under $T_{p,0}$ is evaluated for σ steps, where σ is the value in the fourth column; one checks that $T^{(\sigma)}_{p,0}(n) \in S_{p,0}$ and that $T^{(k)}_{p,0}(n) \notin S_{p,0}$ for every $k \le \sigma - 1$. This is a single linear scan of length $\sigma \le 930$, carried out by evaluation, and it establishes $\sigma_{p,0}(n) = \sigma$. Since $n \le 10^7$ and $0 \le p$, the pair $(n, q) = (n, 0)$ competes in the maximum defining $M_\infty(p)$, so $M_\infty(p) \ge \sigma$. For $p = 19$ no computation is involved: $2^{20} = 1048576 \le 10^7$ and $\sigma_{19,0}(2^{20}) = 2^{19}$ by Proposition 4.1. □

Two things should be said precisely about what rests on exhaustive computation and what does not. *The refutations do not.* Each is a one-witness statement, and the witness is verified by a bounded scan as above. *The claim that* $429, 324, 433, 459, 465$ *are the true maxima* at $p = 0, \dots, 4$ does: those five values were produced by a memoised recomputation of $\max\{\sigma_{p,q}(n) : 1 \le n \le 10^7,\ 0 \le q \le p\}$ over exactly the range [1] states, in a C program written for the purpose, and that computation is not formally verified. Nothing in Theorem 5.2 depends on it. The entry at $p = 6$ is a lower bound only: the value 930 was found by scanning $n \le 3 \cdot 10^5$ and no larger search was run, so the true $M_\infty(6)$ may be larger still. The entry at $p = 5$ (printed 731) survives a search to $n \le 10^6$, whose maximum is 676, and was not pushed to 10^7; the entries at $p \in \{7, \dots, 18\} \cup \{20, \dots, 25\}$ were not tested at all. We therefore do not assert that exactly seven entries of the table are wrong. What the evidence supports is the weaker and more useful statement: *every entry of the table whose value we could settle is too small*, so the table should not be used as data.

Remark 5.3 (The failures are not a disagreement about σ_∞). The obvious alternative explanation for Theorem 5.2 is that [1] computes a different quantity — for instance a stopping time counting only the steps with $d \nmid n$. It does not. Every tabulated entry with $12 \le p \le 25$ except the one at $p = 19$ is exactly $2^p + p$: $4108 = 2^{12} + 12$, $8205 = 2^{13} + 13$, …, $33554457 = 2^{25} + 25$. By Proposition 4.1, 2^p is precisely the walk length forced on 2^{p+1} by the trivial cycle under the definition of $\sigma_{p,0}$ used here, so the high end of the table is computed with the same total stopping time as this paper. That is what makes the low-end failures a discrepancy in the computation rather than in the definitions. It also isolates $p = 19$: the printed 131091 is $2^{17} + 19$, so it has the additive term of the row $p = 19$ and the exponent of the row $p = 17$, and $2^{19} + 19 = 524307$ would fit the pattern. We record that reading as a plausible transcription slip only; the refutation at $p = 19$ is independent of it.

Remark 5.4 (The failures are not a smaller $n_{\max}$). Nor can the printed low-end values be maxima over some uniformly smaller range. Over $1 \le n \le 10^5$ the maxima are $221, 234, 251, 260, 238, 307, 461$ for $p = 0, \dots, 6$; already $213 < 234$ rules out any $n_{\max} \ge 10^5$ at $p = 1$. In the other direction, the printed 374 at $p = 3$ exceeds 369, the maximum over $1 \le n \le 10^6$ at that p, so no $n_{\max} \le 10^6$ can produce it. The printed entries are thus mutually inconsistent, not uniformly computed with a smaller bound. (These two auxiliary maxima are outputs of the same unverified C computation and are quoted as evidence about the table, not as theorems.)

Remark 5.5 (An external anchor). The row $p = 0$ can be checked against the classical literature, because $T_{0,0}$ is the shortcut Collatz map and $\sigma_{0,0}$ is therefore the number of halving steps to reach 1, the sequence $A006666$ of [8]. Our witness $n = 8400511$ has $A006666(8400511) = 429$, the value certified in Theorem 5.2, and $A006577(8400511) = 685 = 429 + 256$. It is moreover the classical record holder in the tabulated range: 8400511 is the largest term not exceeding 10^7 of $A006877$, the starting values that set successive records for the number of $3x + 1$ steps to reach 1 — the next term is 11200681 — and the corresponding record in $A006878$ is 685. The record for $A006577$ need not a priori sit at the same n as the record for $A006666$, which counts only the halving steps; our recomputation of $\max\{A006666(n) : 1 \le n \le 10^7\}$ nevertheless returns 429 at that same n.

Remark 5.6 (An unrelated internal inconsistency). While reading the same section of [1] we noticed that the verification range for the pair $(3, 1)$ is given twice and differently: the text first reports that the computation, using a descent criterion, was extended to $n \le 6.5 \times 10^9$, and states a few paragraphs later that the conjecture was verified for $1 \le n \le 6.5 \times 10^6$. Only one of the two can be meant. This is a typographical matter with no mathematical content, and we make no claim about which range was achieved.

6. Erratum II: the cycle $\Omega(4)$ of the triplet $(12, 14, 10)_+$

Immediately after its trivial-cycle theorem, [1] works out an example that is not a member of the family (2) — $12 = 2^3 + 2^2$ forces $p = 3$, $q = 2$ and hence $\alpha = 2^p + 2^{q+1} = 16$, not 14 — but of the wider family $(d, d + 2^q, d - 2^q)_+$ at $d = 12$, $q = 1$, namely the triplet $(12, 14, 10)_+$ with map

$$T(n) = \begin{cases} n/12, & 12 \mid n, \\ \big(14n + 10\,[n]_{12}\big)/12, & 12 \nmid n. \end{cases}$$

The example states that this triplet has the trivial cycle $\Omega(5)$ of length 7 and "two other cycles", displayed as $\Omega(4)$ of length 6 and $\Omega(1305)$ of length 17. The display of $\Omega(4)$ is printed as

$$\Omega(4) = 4 \to 8 \to 16 \to 22 \to 4), \qquad \text{"of length 6"},$$

reproduced here exactly, including the unmatched closing parenthesis.

Theorem 6.1. *For the triplet* $(12, 14, 10)_+$ *one has* $T(22) = 34 \ne 4$. *The cycle through* 4 *has length exactly* 6 *and is*

$$\Omega(4) = \big(4 \to 8 \to 16 \to 22 \to 34 \to 48 \to 4\big).$$

The stated length in [1] *is therefore correct and the printed chain omits the two elements* 34 *and* 48. *The neighbouring displays of the same example are correct:* $\Omega(5) = (5 \to 10 \to 20 \to 30 \to 40 \to 50 \to 60 \to 5)$ *has length* 7, *and the displayed* $\Omega(1305)$, $1305 \to 1530 \to 1790 \to 2090 \to 2440 \to 2850 \to 3330 \to 3890 \to 4540 \to 5300 \to 6190 \to 7230 \to 8440 \to 9850 \to 11500 \to 13420 \to 15660 \to 1305$, *has length* 17.

Proof. Every step is an evaluation of the displayed map. $[4]_{12} = 4$ and $(14\cdot4+10\cdot4)/12 = 96/12 = 8$; $[8]_{12} = 8$ and $(112 + 80)/12 = 16$; $[16]_{12} = 4$ and $(224 + 40)/12 = 264/12 = 22$; $[22]_{12} = 10$ and $(308 + 100)/12 = 408/12 = 34$; $[34]_{12} = 10$ and $(476 + 100)/12 = 576/12 = 48$; and $12 \mid 48$, so $T(48) = 4$. Hence the orbit of 4 is $4, 8, 16, 22, 34, 48$ and returns at time 6; the six values are distinct, so the return time is minimal and the length is 6. The two other displays are checked the same way, by evaluating 7 and 17 steps respectively. $\square$

The correction is small but it is the kind that propagates: the display is the only worked instance of the trivial-cycle theorem's "two other cycles" phenomenon given in [1] before the exceptional set $\mathbb{E}$ is introduced, and a reader recomputing it will not reproduce the printed chain. The follow-up paper [2] of the same author, which restates the main conjecture with the same $\mathbb{E}$ and the same nine cycles, does not repeat this example, so the display is not corrected there.

7. What reproduces: the exceptional set

For completeness we record what an independent recomputation confirms about the material of [1, Conjecture 2.1], since an erratum is only useful if it says what is *not* in error. The statements of this section are verifications of published values, not new results.

Proposition 7.1. *Each of the nine cycles displayed in* [1, Conjecture 2.1] *exists, with the printed starting value and the printed length, and in each case the printed starting value is the least element of the cycle, as the notation* $\Omega(\omega)$ *of* [1] *requires. The data are*

$$
\begin{array}{llll}
(p,q)=(1,0):\ \omega=14,\ \ell=9; & (2,1):\ 74,\ 7; & (2,2):\ 67,\ 6; & (3,0):\ 280,\ 21; \\
(4,0):\ 1264,\ 49; & (5,2):\ 76200,\ 70; & (5,2):\ 87176,\ 35; & (6,2):\ 1264,\ 69; \\
(7,0):\ 3027584,\ 630. & & &
\end{array}
$$

Moreover, wherever [1] *prints such a cycle element by element, its display agrees with the computed orbit at every entry, as do its displays of the trivial cycles for* $(3,1)$ *and* $(5,2)$*. The display at* $(7,0)$ *is the one printed with an ellipsis; its fourteen leading and nine trailing values are covered by Proposition 7.2.*

Proposition 7.2. *At* $(p,q)=(7,0)$ *the cycle* $\Omega(3027584)$ *has length* 630*; its least element is* 3027584 *and its greatest is* 390558336*. In particular none of its* 630 *elements is at most* $3\cdot 10^6$*. The fourteen leading and nine trailing values printed in* [1] *agree with the computed orbit.*

Proposition 7.3. *Every pair in* $\mathbb{E}$ *has order at least two, and* $(5,2)$ *has order at least three: for each* $(p,q)\in\mathbb{E}$ *the cycle* $\Omega(\omega_{p,q})$ *of Proposition 7.1 is disjoint from the trivial cycle* $\Omega(2^{p-q})$*, and at* $(5,2)$ *the cycles* $\Omega(76200)$ *and* $\Omega(87176)$ *are disjoint from each other and from* $\Omega(8)$*.*

Proof sketch for Propositions 7.1–7.3. For a claimed pair (ω,ℓ) one evaluates the orbit ω, $T(\omega)$, ..., $T^{(\ell)}(\omega)$ — at most 631 steps — and checks three Booleans: that $T^{(\ell)}(\omega)=\omega$; that $T^{(k)}(\omega)\neq\omega$ for $0<k<\ell$, which gives minimality and hence ℓ distinct elements; and that $T^{(k)}(\omega)\neq\tau$ for $k<\ell$, where τ is the representative of the cycle to be separated from, which gives disjointness. The extremes in Proposition 7.2 are the running minimum and maximum of the same single pass. Each check is one linear scan; the largest is the 630-cycle. □

We stress what Proposition 7.3 does *not* say. [1] conjectures order *exactly* two on $\mathbb{E}\setminus\{(5,2)\}$ and exactly three at $(5,2)$, and exactly one off $\mathbb{E}$. Exactness requires excluding all further cycles, which for $p=q=0$ is the Collatz conjecture; nothing here bears on it.

Remark 7.4 (A warning about search ranges). Proposition 7.2 has a practical consequence for anyone testing whether $\mathbb{E}$ is complete beyond $p=8$, which [1] asserts for $p\le 25$ and which nothing computed here tests. The natural probe — enumerate starting values $n\le M$ and follow each trajectory while it stays in $[1,M]$ — misses the cycle at $(7,0)$ for every $M<3027584$, and can only close it once $M\ge 390558336$. We ran such a probe at $M=3\cdot 10^6$: it recovered seven of the eight pairs of $\mathbb{E}$ and missed $(7,0)$, whose least element exceeds that bound by 0.92% and whose largest element exceeds it by a factor of 130. A flat bound is the wrong instrument here; the minima of the exceptional cycles grow with the triplet, so the search range must be indexed by (p,q) rather than fixed.

8. Bounded convergence, and controls

Proposition 8.1. *For each of the fifteen pairs listed below and the stated* N *and* f*: for every* n *with* $1\le n\le N$ *there is a* $k\le f$ *with* $T_{p,q}^{(k)}(n)\in S_{p,q}$*, with* $S_{p,q}$ *the representative set* (4) *of Algorithm 1 of* [1].

$$
\begin{array}{ll}
N=10^5: & (0,0)\ [f=300],\ (3,1)\ [f=300]; \\
N=5\cdot 10^4: & (2,0)\ [300],\ (4,1)\ [400],\ (5,0)\ [400],\ (6,0)\ [500],\ (8,0)\ [500]; \\
N=5\cdot 10^4: & (1,0)\ [400],\ (2,1)\ [300],\ (2,2)\ [300],\ (3,0)\ [400],\ (4,0)\ [400], \\
 & (5,2)\ [400],\ (6,2)\ [300],\ (7,0)\ [1000].
\end{array}
$$

These ranges are far below the $n_{\max} = 10^7$ of [1], and far below what dedicated software reaches for the classical map [7]; they prove nothing about strong admissibility, and their only merit is that each is a checked consequence of the definitions rather than a reported one. Each is decided by compiled evaluation of one Boolean — a scan of N trajectories with a fuel bound — and then transported to the quantified statement by a bridge lemma about the scanning function.

Proposition 8.2. *The following hold, and together they show that the statements above are neither vacuous nor off by one nor about the wrong map. (i) The two normalisations really differ: under* $n \mapsto 3n+1$ *the cycle through* 1 *has length* 3*, whereas under* $T_{0,0}$ *the value* 1 *does not lie on a cycle of length* 3*, its length there being* 2*; and the number of steps taken by* 27 *to reach* 1 *is* 70 *under* $T_{0,0}$ *against* 111 *under* $n \mapsto 3n+1$*. (ii) The periods above are exact: at* $(7,0)$ *one has* $T^{(629)}(3027584) \neq 3027584$ *and* $T^{(631)}(3027584) \neq 3027584$*, while at* $(3,1)$ *one has* $T^{(5)}(4) \neq 4$ *and* 7 *is not the least return time of* 4*. (iii) The hypothesis* $q \leq p$ *of Theorem 3.1 is load-bearing: for the triplet* $(3,5,1)_+$ *obtained by reading* (2) *at* $(p,q) = (0,1)$*, the head* 1 *of the putative cycle satisfies* $T^{(k)}(1) \neq 1$ *for every* k *with* $0 < k \leq 200$*, although that triplet does have a cycle, namely* $(4 \to 7 \to 12 \to 4)$ *of length* 3*; so what fails when* $q > p$ *is the starting point, not the length formula. (iv) The enlargement of* $S_{p,q}$ *on* $\mathbb{E}$ *is necessary: at* $(1,0)$ *the trajectory of* 14 *does not reach* 2 *within* 5000 *steps, and at* $(5,2)$ *the trajectory of* 87176 *reaches neither* 8 *nor* 76200 *within* 3000 *steps, so the sweeps of Proposition 8.1 would fail with the trivial representative alone. (v) A fuel bound that is too small is detected:* 27 *does not reach* 1 *in* 69 *steps under* $T_{0,0}$*. (vi) The remainder term of* (1) *is what makes the map integer-valued: at* $(3,1)$ *one has* $10 \nmid 12 \cdot 7$ *but* $10 \mid 12 \cdot 7 + 8 \cdot 7 = 140$*.*

9. Verification

Every statement of this paper has been formally verified in Lean 4 against *Mathlib*; the development contains no `sorry` and no `axiom` declaration of its own, and a repository reference will be added at submission. The division of labour is the one described above. The unconditional statements — Proposition 2.1, Theorem 3.1 with its distinctness clause, Proposition 4.1 and the entry at $p = 19$ of Theorem 5.2 — are proved by induction, uniformly in p and q, and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. The finite claims — the six explicit witnesses of Theorem 5.2, the whole of Theorem 6.1, Propositions 7.1–7.3, and the controls of Proposition 8.2 — are decided by evaluation inside Lean's kernel, with no appeal to compiled code; the two statements of Theorem 6.1 about $\Omega(4)$ depend on no axioms at all. Keeping the cycle predicates linear in the period (one pass computing the return check, the running minimum and the running maximum, with a bridge lemma to the quantified form) is what makes the 630-cycle and the 930-step stopping time affordable in the kernel. The only statements delegated to compiled evaluation are the fifteen bounded sweeps of Proposition 8.1, which accordingly rest on Lean's native-evaluation axiom in addition to the three named above. Outside the formal development stand exactly two computations, both flagged as such in Section 5: the exhaustive recomputation of $M_\infty(p)$ over $1 \leq n \leq 10^7$ for $p \leq 4$, and the auxiliary maxima at $n_{\max} \in \{10^5, 10^6, 3 \cdot 10^5\}$ used in Remark 5.4 and for the witness at $p = 6$. Every numerical value quoted here was first produced by an independent implementation in C or Python, and the formal statement was then written against it.

References

[1] A. Bouhamidi, *An Extension of the Collatz Conjecture modulo* $2^p + 2^q$, arXiv:2601.06208v1 [math.GM], submitted 8 January 2026. (Its LaTeX source carries the date 5 November 2025.) All numbered references to this work in the text — Conjecture 1.2, Theorem 2.1, Conjecture 2.1, Algorithm 1 and Table 1 — are to this version.

[2] A. Bouhamidi, *Weakly and Strongly Admissible Triplets for a Collatz-Type Map*, arXiv:2601.17573v1 [math.GM], submitted 24 January 2026.

[3] A. Bouhamidi, *Weak and strong admissible triplets associated to Collatz like map*, preprint, hal-04552296v1, deposited 19 April 2024.

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[7] D. Barina, *Convergence verification of the Collatz problem*, J. Supercomput. **77** (2021), no. 3, 2681–2688.

[8] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A006577, A006666, A006877 and A006878.