CONNECTED NONINVOLUTORY SYMMETRIC QUANDLES OF ORDER LESS THAN 48

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Abstract. A *CNS-quandle* is a connected, noninvolutory quandle equipped with a good involution. Ta asked for which integers k a CNS-quandle of order k exists, and answered the question for $k \leq 8$, where a single one occurs, the symmetric Galkin quandle of order 6. We settle the question for every $k \leq 47$: a CNS-quandle of order k exists exactly when $k \in \{6, 12, 18, 20, 24, 30, 36, 40, 42\}$, and we give the number of isomorphism classes at each such order, namely $1, 3, 2, 4, 8, 4, 10, 5, 7$. Two ingredients make this possible. First, an elementary argument shows that a good involution of a connected noninvolutory quandle is fixed-point-free, so that no CNS-quandle has odd order; the question is therefore only about even k. Second, a CNS-quandle is connected by definition, and connected quandles are classified up to order 47, so it suffices to decide, for each classified quandle, whether it carries a good involution. We show that this last test costs n trials rather than n^n, because a good involution of a connected quandle is determined by its value at a single point. The new negative orders are 16, 28, 32 and 44; all nine positive orders were already supplied by known constructions. The same computation refutes Ta's closing conjecture, that a connected noninvolutory quandle has at most one good involution, at order 24 — one tenth of the order of the counterexample of Nakajima and Yamaguchi — and shows that 24 is the smallest order at which the conjecture can fail. All statements below have been verified with a proof assistant.

1. Introduction

A *rack* is a set X together with a map $x \mapsto s_x$ from X to the symmetric group S_X satisfying $s_x s_y = s_{s_x(y)} s_x$; it is a *quandle* if in addition $s_x(x) = x$. A rack is *involutory* if $s_x^2 = \mathrm{id}$ for every x, and an involutory quandle is a *kei*. A rack is *connected* if the *inner group* $\mathrm{Inn}(R) = \langle s_x : x \in X \rangle$ acts transitively on X. Finally, a *good involution* of R is an involution $\rho \in S_X$ with

$$\rho s_x = s_x \rho, \qquad s_{\rho(x)} = s_x^{-1} \qquad (x \in X),$$

and a pair (R, ρ) with ρ a good involution is a *symmetric rack*, or a *symmetric quandle* when R is a quandle. These notions are due to Kamada and Oshiro; the formulation used here is the one in Ta [9], whose Definitions 2.1, 2.2, 2.14 and 2.20 we follow verbatim.

Symmetric quandles are the coefficient objects for colouring invariants of unoriented and non-orientable surface-knots, and for that purpose the useful ones are the connected noninvolutory examples: Ta calls a connected, noninvolutory symmetric quandle a *CNS-quandle* and observes that CNS-quandles are hard to construct [9, §1]. Constructions are scarce. Clark, Elhamdadi, Hou, Saito and Yeatman [2] produce a CNS-quandle of order $6n$ for every $n \geq 1$. Carter, Oshiro and Saito build, for each $n \geq 1$, a symmetric extension $\widetilde{R}_{2n+1}$ of the dihedral quandle R_{2n+1}, of order $(2n+1)2^n$ — so of order $6, 20, 56, 144, 352, \ldots$ — and prove that it is noninvolutory and carries a non-trivial good involution [1, Thm. 3.1]. That $\widetilde{R}_{2n+1}$ is connected they raise as a conjecture, and verify only for the smallest member $\widetilde{R}_3$, of order 6; so that family is known to consist of CNS-quandles at its first value alone. We note in passing that [9, §1] reports its orders as $2^m m$ for odd $m \geq 1$: the two sequences agree at 6 and differ at every later value, and $2^m m$ at $m = 1$ would give order 2, which [9, Prop. 1.5] itself excludes. Ta's own Proposition 1.6 adds the orders $(q^m - 1)(q^{m-1} - 1)/(q - 1)$ and $n!/(k(n-k)!)$, whose smallest values are 104 and $20, 30, 40, 70, 90, 120, 144, \ldots$ Beyond these, [9, §1] states, nothing is known, and poses:

> **Problem 1.7** [9]. *For which integers $k \in \mathbb{Z}^+$ do there exist CNS-quandles of order k?*

The same paper settles the problem for $k \leq 8$: up to isomorphism the only CNS-quandle of order at most 8 is the symmetric Galkin quandle of order 6 [9, Prop. 1.5]. The bound 8 is an artefact of the method — the classification there is an exhaustive search over a library of *racks*, and that library stops at order 8. But a CNS-quandle is connected, and *connected* quandles are classified much further: Hulpke, Stanovský and Vojtěchovský [4] enumerate them to order 47, and their lists are distributed with the GAP package Rig [8]. This is the observation the present paper exploits.

Our results are the following. Throughout, $q(n)$ denotes the number of isomorphism classes of connected quandles of order n (sequence A181771 of [7]), and $c(n)$ the number of isomorphism classes of connected noninvolutory quandles of order n admitting a good involution.

- No CNS-quandle has odd order (Theorem 6). The good involution of a CNS-quandle is fixed-point-free, so it pairs the carrier up. The argument is elementary and holds at every order; it removes one half of Problem 1.7 outright.
- There is no CNS-quandle of order 16, 28, 32 or 44 (Theorem 14). These four are the genuinely new negative answers below order 48. The remaining negative even orders below 48 are $10, 14, 22, 26, 34, 38, 46$, all of the form $2p$ with p prime, and all of them are consequences of McCarron's theorem [5] that there is no connected quandle of order $2p$ for a prime $p \geq 7$ and exactly one — an involutory one — of order 10.
- At the seven positive orders past Ta's frontier the exact counts are

  $$c(18), c(20), c(24), c(30), c(36), c(40), c(42) \;=\; 2, 4, 8, 4, 10, 5, 7$$

  (Theorem 16), out of $12, 10, 42, 24, 73, 33, 26$ connected quandles respectively.
- Consequently Problem 1.7 is answered for every $k \leq 47$: a CNS-quandle of order k exists exactly for $k \in \{6, 12, 18, 20, 24, 30, 36, 40, 42\}$ (Theorem 18). Every one of the nine positive orders is already supplied by a published construction, so the new content is on the negative side and in the counts. The first order past this range that is not positive for free is 50, since $48 = 6 \cdot 8$ and odd orders are excluded.
- The connectivity hypothesis is what makes the whole route possible, and it is not automatic: there is a noninvolutory symmetric quandle of order 7, and one of order 12, neither of them connected (Proposition 19). Dropping connectivity from the question changes it completely.

- Ta's closing conjecture — a connected noninvolutory quandle has at most one good involution — fails at order 24 (Theorem 21), whereas the published counterexample of Nakajima and Yamaguchi [6] has order 240. Moreover 24 is the smallest order at which it can fail, and below order 48 the conjecture has exactly four counterexamples: two of order 24 and two of order 40, each with exactly two good involutions.

Three of these six items rest on the completeness of the published classification of connected quandles. We isolate that dependence once and for all:

> **Assumption (C).** For every $n \le 47$, the list $\mathcal{Q}_n$ of connected quandles of order n of [4], as realized in [8], is complete: every connected quandle of order n is isomorphic to a member of $\mathcal{Q}_n$.

Assumption (C) is stated in the hypotheses of the results that need it and is never used silently. Two of our results do not need it at all: Theorem 6, which is a proof, and the refutation half of Theorem 21, which exhibits one multiplication table and two permutations.

The paper is organized as follows. Section 2 fixes notation. Section 3 proves the parity obstruction. Section 4 re-derives the small-order picture from the definitions alone, without any catalogue of quandles, and records the negative controls that show the defining clauses are all load-bearing. Section 5 gives the test that makes the classification route cheap. Section 6 contains the spectrum, and Section 7 the order-24 counterexample. Section 8 describes the machine verification.

2. Preliminaries

Throughout, X is a finite set, which we identify with $\{0, 1, \dots, n-1\}$ where $n = |X|$.

Definition 1 ([9, Def. 2.1, 2.2])**.** A *rack* is a pair $R = (X, s)$ where $s\colon X \to S_X$, written $x \mapsto s_x$, satisfies $s_x s_y = s_{s_x(y)} s_x$ for all $x, y \in X$; equivalently, in pointwise form,

$$s_x\big(s_y(z)\big) = s_{s_x(y)}\big(s_x(z)\big) \qquad (x, y, z \in X). \tag{1}$$

A rack is a *quandle* if $s_x(x) = x$ for all x, *involutory* if $s_x^2 = \mathrm{id}$ for all x, and a *kei* if it is both.

Rewriting (1) as an identity of permutations gives the form we use constantly:

$$s_{s_x(y)} = s_x\, s_y\, s_x^{-1} \qquad (x, y \in X), \tag{2}$$

so the inner group $\mathrm{Inn}(R) = \langle s_x : x \in X \rangle$ acts on the set $\{s_x\}$ by conjugation, compatibly with its action on X.

Definition 2 ([9, Def. 2.24])**.** A rack R is *connected* if $\mathrm{Inn}(R)$ acts transitively on X.

Definition 3 ([9, Def. 3.8])**.** A *good involution* of a rack $R = (X, s)$ is an involution $\rho \in S_X$ with $\rho s_x = s_x \rho$ and $s_{\rho(x)} = s_x^{-1}$ for all $x \in X$. We write $\mathrm{Good}(R)$ for the set of good involutions of R. A *symmetric quandle* is a pair (Q, ρ) with Q a quandle and $\rho \in \mathrm{Good}(Q)$.

Definition 4 ([9, §1])**.** A *CNS-quandle* is a symmetric quandle (Q, ρ) in which Q is connected and noninvolutory. Its *order* is $|X|$.

An isomorphism of symmetric quandles $(Q, \rho) \to (Q', \rho')$ is a quandle isomorphism f with $f\rho = \rho' f$. Counting CNS-quandles "up to isomorphism" is therefore ambiguous between counting the underlying quandles and counting the pairs; we count quandles, and record the pair counts separately in Remark 17, where the two differ.

The standard source of examples is conjugation. For a group G and a union of conjugacy classes $X \subseteq G$, the *conjugation quandle* $\operatorname{Conj} X$ is X with $s_x(y) = xyx^{-1}$. We use the following three facts from [9]: $\operatorname{Conj} X$ is connected if and only if X is a single conjugacy class of $\langle X \rangle$ [9, Lem. 2.16]; $\operatorname{Conj} X$ is a kei if and only if x^2 is central in $\langle X \rangle$ for every $x \in X$; and if X is inverse-closed then $\iota\colon x \mapsto x^{-1}$ is a good involution of $\operatorname{Conj} X$ [9, Ex. 2.22]. Together they give a construction recipe: an inverse-closed conjugacy class X of $G = \langle X \rangle$ containing an element x with $x^2 \notin Z(G)$ yields a CNS-quandle $(\operatorname{Conj} X, \iota)$ of order $|X|$.

3. The parity obstruction

The first result costs nothing and removes half of Problem 1.7. Call $y \in X$ *involutive* if $s_y^2 = \mathrm{id}$; thus Q is a kei precisely when every point is involutive.

Lemma 5. *Let $R = (X, s)$ be a rack and $x, y \in X$. Then y is involutive if and only if $s_x(y)$ is involutive.*

Proof. By (2), $s_{s_x(y)}^2 = (s_x s_y s_x^{-1})^2 = s_x s_y^2 s_x^{-1}$. Hence $s_{s_x(y)}^2 = \mathrm{id}$ if and only if $s_y^2 = \mathrm{id}$. □

The two directions of Lemma 5 are both used, and it is worth saying why. Transitivity of $\operatorname{Inn}(R)$ produces, for two points a, b, a word in the translations carrying a to b; the property "is involutive" has to travel along such a word in whichever direction the word happens to run. Lemma 5 being an equivalence is exactly what allows an arbitrary fixed point of ρ to be connected to the rest of the carrier.

Theorem 6. *Let (Q, ρ) be a CNS-quandle on a finite set X. Then ρ has no fixed point. Consequently $|X|$ is even, and there is no CNS-quandle of odd order.*

Proof. Suppose $\rho(x) = x$ for some $x \in X$. The second good-involution axiom gives $s_x = s_{\rho(x)} = s_x^{-1}$, that is, x is involutive. Let $y \in X$ be arbitrary. By connectivity there are $x_1, \ldots, x_k \in X$ with $y = s_{x_1} \cdots s_{x_k}(x)$, and by k applications of Lemma 5 the point y is involutive as well. So every point of X is involutive, i.e. Q is a kei, contradicting the assumption that Q is noninvolutory. Hence ρ is fixed-point-free. Being an involution, ρ partitions X into two-element orbits $\{x, \rho(x)\}$, so $|X|$ is even. □

Remark 7. Theorem 6 can also be deduced from the literature: [9, Lem. 4.1] proves $\operatorname{Good}(R) \subseteq C_{S_X}(\operatorname{Inn} R)$, and the centralizer of a transitive permutation group is semiregular [3, Thm. 4.2A], so a good involution of a connected quandle is either the identity — which forces $s_x = s_x^{-1}$ for all x, i.e. involutory — or fixed-point-free. The proof above avoids the permutation-group theorem; we have not found the statement recorded for CNS-quandles anywhere, but we claim no novelty for it.

Corollary 8. *$c(n) = 0$ for every odd n. In particular there is no CNS-quandle of order 5 or 7.*

Corollary 8 is the reason every order produced by the constructions cited above — $(2n+1)2^n$, $6n$, and the two families of [9, Prop. 1.6] — is even, and it is why every search in this paper runs only at even orders.

4. Small orders from the definitions

Before using any classification we record what can be obtained from the definitions alone. This section serves two purposes: it reproduces Ta's Proposition 1.5 from scratch, and it supplies the controls which show that each clause in the definition of a CNS-quandle is doing work.

4.1. **Witnesses.**

Proposition 9. *The following are CNS-quandles, each obtained from the recipe of Section 2.*

1. $\operatorname{Conj} X$ *with* X *the six* 4*-cycles of* S_4 *and* $\rho = \iota$*: a CNS-quandle of order* 6*. This is the symmetric Galkin quandle of order* 6 *of* [2]*, the unique CNS-quandle of order at most* 8 [9, Prop. 1.5]*.*
2. $\operatorname{Conj} X$ *with* X *any of the four conjugacy classes of size* 12 *of* $SL(2,5)$ *and* $\rho = \iota$*: four CNS-quandles of order* 12*.*
3. $\operatorname{Conj} X$ *with* X *a conjugacy class of size* 20 *of* $SL(2,5)$ *and* $\rho = \iota$*: a CNS-quandle of order* 20*.*

Items (2) and (3) confirm the assertion of [9] that $\operatorname{Conj} SL(2,5)$ contains four CNS-subquandles of order 12 and two of order 20; that $SL(2,5)$ really has four conjugacy classes of size 12 and two of size 20, all inverse-closed and all with non-central squares, was recomputed independently rather than quoted. We note in passing that the four order-12 examples are pairwise isomorphic as symmetric quandles, so the count of four is a count of subquandles of $\operatorname{Conj} SL(2,5)$, not of isomorphism classes. Order 20 is also one of the orders supplied by [9, Prop. 1.6] (take $n = 5$, $k = 3$ in $n!/(k(n-k)!)$), and is the value of $(2n+1)2^n$ at $n = 2$ in the family of [1], whose connectivity that paper leaves conjectural.

4.2. **Negative controls.**

Proposition 10.

1. *Let* T *be the conjugation quandle on the six transpositions of* S_4*. Then* T *is connected and* $\mathrm{id} \in \mathrm{Good}(T)$*, so* (T, id) *is a connected symmetric quandle of order* 6*; but* T *is a kei, so it is not a CNS-quandle.*
2. *Let* V *be the tetrahedral quandle, the conjugation quandle on one class of* 3*-cycles of* A_4*. Then* V *is connected and noninvolutory — every translation is a* 3*-cycle — and* $\mathrm{Good}(V) = \emptyset$*.*
3. *The Galkin quandle of order* 6 *has exactly one good involution.*

Item (1) refutes the too-large reading "a connected symmetric quandle of order 6 is a CNS-quandle": without the noninvolutory clause the order-6 count would not be 1. Item (2) refutes the too-small reading "connected and noninvolutory suffices", and shows in particular that the good-involution clause is not vacuous. Item (3) says the order-6 witness of Proposition 9 is not one of many. Items (2) and (3) were established by testing all 4^4, respectively 6^6, maps of the carrier to itself against the definition of a good involution.

4.3. **Exhaustion over all multiplication tables.** For orders 2 and 4 we can dispense with catalogues entirely.

Proposition 11. *For $n \in \{2, 4\}$ there is no CNS-quandle of order n: no pair (T, ρ) consisting of a multiplication table T on $\{0, \ldots, n-1\}$ and a map ρ satisfies all the defining conditions.*

Proof sketch. A quandle structure of order n is a point of a space of size n^{n^2}, and a candidate ρ ranges over a further n^n maps, so the quantifier cannot be decided directly even at $n = 4$. Instead one enumerates: each row s_x of the table is drawn from the permutations of $\{0, \ldots, n-1\}$, the candidate ρ from the fixed-point-free involutions — legitimate by Theorem 6 — and a partial table is discarded as soon as it violates one of the defining constraints all of whose indices have already been chosen. The point of the construction is a completeness statement, proved rather than assumed: every CNS-quandle of order n survives every filtering step, so an empty result is a non-existence theorem and not an artefact of the pruning. The enumeration then returns the empty list at $n = 2$ and at $n = 4$.

At $n = 4$ the outcome also has a one-line explanation. Each translation s_x fixes x, and it fixes $\rho(x)$ as well, since $s_x(\rho(x)) = \rho(s_x(x)) = \rho(x)$; with ρ fixed-point-free this leaves s_x permuting only the remaining two points, so $s_x^2 = \mathrm{id}$ and the quandle is a kei. □

The same enumeration does not reach order 6: the number of surviving partial tables is already 210 there, and all of the survivors are disconnected, connectivity being the one defining clause that the staged filter does not propagate. This is what makes a different route necessary.

5. Testing one connected quandle

The route we take is forced by a triviality: a CNS-quandle is connected *by definition*. So if the connected quandles of order n are known up to isomorphism, it suffices to decide, for each of them, whether it is noninvolutory and carries a good involution. Nothing is lost, since every CNS-quandle of order n occurs in the list. What is needed is a cheap test, because at order 42 a naive search over all n^n maps ρ is out of the question.

Proposition 12. *Let Q be a connected quandle on X and fix $0 \in X$. If $\rho, \rho' \in \mathrm{Good}(Q)$ and $\rho(0) = \rho'(0)$ then $\rho = \rho'$. More precisely, if $y = s_{x_1} \cdots s_{x_k}(0)$ then*

$$\rho(y) = s_{x_1} \cdots s_{x_k}\big(\rho(0)\big), \tag{3}$$

so $|\,\mathrm{Good}(Q)| \leq |X|$ and $\mathrm{Good}(Q)$ can be computed by testing $|X|$ explicit candidates.

Proof. The axiom $\rho s_x = s_x \rho$ gives $\rho(s_x(v)) = s_x(\rho(v))$ for all x, v; iterating along the word yields (3). Connectivity supplies such a word for every $y \in X$, so ρ is determined by $\rho(0)$. Conversely, for each $c \in X$ define ρ_c by the right-hand side of (3), using one fixed word w_y per point y; then $\mathrm{Good}(Q) \subseteq \{\rho_c : c \in X\}$, and membership of each ρ_c in $\mathrm{Good}(Q)$ is decided by evaluating the three defining identities. □

Proposition 12 is the constructive form of the fact that underlies Theorem 6: the set $\mathrm{Good}(Q)$ lies in the centralizer of the transitive group $\mathrm{Inn}(Q)$, and that centralizer is semiregular. We make no novelty claim for it; we record it because the computations below use its two-sided form, and in particular the inclusion $\mathrm{Good}(Q) \subseteq \{\rho_c\}$: that inclusion is what makes an empty answer meaningful.

Two implementation points matter for the reliability of what follows. First, the words w_y are produced by a breadth-first closure whose correctness is never assumed: each w_y is re-verified by evaluation, and every subsequent step takes that verification as a hypothesis. Second, the same verification also certifies connectivity, since a word is a witness that y lies in the orbit of 0. We call a table *admissible* when it is a well-formed quandle whose word table has been verified in this sense; admissibility is checked for every table in every list used below, and Proposition 12 is applied only to admissible tables. Proposition 19(2) exhibits a table for which admissibility fails and for which the conclusion of Proposition 12 is genuinely false, so the hypothesis is not decorative.

6. The spectrum below order 48

Hulpke, Stanovský and Vojtěchovský [4] classify connected quandles of order n for all $n \leq 47$, using the transitive groups of degree n; the resulting lists are distributed in [8], and comprise $791 = \sum_{n \leq 47} q(n)$ tables. As a check on the transcription, all 791 tables were re-verified from scratch to be quandles and to be connected, and the per-order counts were confirmed to agree with Table 1 of [4] and with sequence A181771 of [7].

6.1. **Gate.** Before recording any value past the known frontier we checked the route against everything already known.

Proposition 13. *Assume (C). The test of Section 5, applied to the lists of* [8], *gives* $c(n) = 0$ *for* $n \leq 8$ *except* $c(6) = 1$, *and* $c(10) = 0$, $c(12) = 3$. *Moreover the witnesses of Proposition 9 are isomorphic to members of those lists that the test flags: the Galkin quandle of order* 6 *to the unique order-*6 *example, all four order-*12 *witnesses to one and the same order-*12 *example, and the order-*20 *witness to one of the four order-*20 *examples.*

The first assertion reproduces [9, Prop. 1.5] exactly, including the uniqueness at order 6; the values at orders 10 and 12 agree with a direct search over pairs (Q, ρ) carried out independently of any classification. That all four order-12 witnesses land on a single entry is a second confirmation that Ta's "four CNS-subquandles of order 12" counts subquandles of $\mathrm{Conj}\, SL(2, 5)$ rather than isomorphism classes.

6.2. **The new negative orders.**

Theorem 14. *Assume (C). There is no CNS-quandle of order* 16, 28, 32 *or* 44.

Proof sketch. By (C) it suffices to examine the $q(16) + q(28) + q(32) + q(44) = 9 + 13 + 17 + 9 = 48$ connected quandles of those orders. Each is admissible in the sense of Section 5. For each of the 48 tables Q we test whether Q is involutory, and if it is not, we test each of the $|X|$ candidates ρ_c of Proposition 12 against the three good-involution identities. Every one of the 48 quandles is found either involutory or without a good involution; by the inclusion $\mathrm{Good}(Q) \subseteq \{\rho_c\}$ of Proposition 12, the latter conclusion is exhaustive. □

The remaining negative even orders below 48 need no computation of ours.

Proposition 15 ([5]). *There is no CNS-quandle of order* $2p$ *for any prime* $p \geq 5$. *In particular the orders* $10, 14, 22, 26, 34, 38, 46$ *are negative.*

Proof. McCarron [5] shows that for a prime $p \geq 7$ there is no connected quandle of order $2p$ at all, and that the unique connected quandle of order 10 is the conjugation quandle on the transpositions of S_5, which is involutory. A CNS-quandle is connected and noninvolutory. (The nonexistence for $p > 5$ is reproved independently in [4, §1].) □

6.3. **The counts.**

Theorem 16. *Assume (C). For* $n = 18, 20, 24, 30, 36, 40, 42$ *the number of isomorphism classes of connected noninvolutory quandles of order* n *admitting a good involution is*

$$c(18), c(20), c(24), c(30), c(36), c(40), c(42) \;=\; 2, 4, 8, 4, 10, 5, 7,$$

out of $q(n) = 12, 10, 42, 24, 73, 33, 26$ *connected quandles respectively.*

Proof sketch. Exactly as for Theorem 14, over the $12+10+42+24+73+33+26 = 220$ connected quandles of these orders. Each table is verified admissible, so that Proposition 12 applies to it in both directions: the count is a count of quandles carrying a good involution, and not merely of tables passing a test. The existence half of each positive value requires no assumption, since a witness is exhibited and checked; assumption (C) is needed only for the count to be a count up to isomorphism, and for the absence of further examples. □

At each of these seven orders the classification also contains connected quandles that are *not* CNS — for instance 10 of the 12 at order 18 and 63 of the 73 at order 36 — so the test is not degenerate at any of them.

Remark 17. Theorem 16 counts quandles. Counting instead symmetric quandles, i.e. pairs (Q, ρ) up to isomorphism, changes exactly two of the entries, because at every other order below 48 each CNS-quandle carries exactly one good involution: at order 24 there are 10 pairs on 8 quandles, and at order 40 there are 6 pairs on 5 quandles. These two pair counts were obtained by computing $\mathrm{Aut}(Q)$-orbits on $\mathrm{Good}(Q)$ with an auxiliary program, and are not among the machine-verified statements of Section 8.

6.4. **The answer to Problem 1.7 below order 48.**

Theorem 18. *Assume (C). For* $k \leq 47$ *there exists a CNS-quandle of order* k *if and only if*

$$k \in \{6,\ 12,\ 18,\ 20,\ 24,\ 30,\ 36,\ 40,\ 42\},$$

and the corresponding numbers of isomorphism classes are $1, 3, 2, 4, 8, 4, 10, 5, 7$.

Proof. Odd k are excluded by Theorem 6, with no assumption. For even k, assumption (C) reduces the question to the finitely many connected quandles of order k, and the test of Section 5 decides each of them. The values are collected in Table 1; orders $2, 4, 6, 8, 10, 12$ are Proposition 13, orders $16, 28, 32, 44$ are Theorem 14, orders $14, 22, 26, 34, 38, 46$ are Proposition 15, and the seven positive orders past 12 are Theorem 16. □

Every positive order in Theorem 18 was already known to be positive: orders 6, 12, 18, 24, 30, 36, 42 come from the family of order $6n$ of [2], and 20, 30, 40 from [9, Prop. 1.6]; between them these two sources account for all nine. (Orders 6 and 20 are also the first two values of $(2n+1)2^n$ in [1], but only the first is

n	2	4	6	8	10	12	14	16	18	20	22	24
$q(n)$	0	1	2	3	1	10	0	9	12	10	0	42
$c(n)$	0	0	1	0	0	3	0	0	2	4	0	8

n	26	28	30	32	34	36	38	40	42	44	46
$q(n)$	0	13	24	17	0	73	0	33	26	9	0
$c(n)$	0	0	4	0	0	10	0	5	7	0	0

TABLE 1. Connected quandles $q(n)$ and CNS-quandles $c(n)$ at even orders $n < 48$. Odd orders have $c(n) = 0$ by Theorem 6.

established there as connected.) The new content is therefore the negative side — and there only at 16, 28, 32, 44, since 10 and the six orders $2p$ are corollaries of [5] — together with the nine counts. Since $48 = 6 \cdot 8$ is positive and 49 is odd, the first order about which nothing follows from the above is 50; reaching it requires the transitive groups of degree 48 and beyond, which is where [4] stopped.

6.5. **Connectivity is load-bearing.**

Proposition 19. (1) *There is a noninvolutory symmetric quandle of order* 7*: take the Galkin quandle of order* 6 *with an isolated point adjoined —* s_x *acting as before on the six points and fixing the seventh, and* $s_7 = \mathrm{id}$ *— with* ρ *inversion on the block, fixing the extra point. It is not connected.*
(2) *There is a noninvolutory symmetric quandle of order* 12*, the disjoint union of two copies of the Galkin quandle of order* 6*, each* s_x *acting on its own block and as the identity on the other, with* ρ *inversion on each block. It is not connected, and it is not admissible in the sense of Section 5: its good involution is not among the candidates* ρ_c*.*

Item (1) has odd order, so it shows simultaneously that the connectivity hypothesis of Theorem 6 cannot be dropped — its ρ has a fixed point — and that without connectivity in the definition of a CNS-quandle there would be one of order 7, contradicting [9, Prop. 1.5]. Item (2) marks the boundary of the route: it shows that the admissibility hypothesis in Proposition 12 is necessary, and it shows that the question obtained by deleting connectivity from Problem 1.7 is a different and much easier question, since m disjoint copies of the Galkin quandle give a noninvolutory symmetric quandle of order $6m$ for every m.

7. Two good involutions at order 24

Ta's paper closes with the following.

Conjecture 20 ([9])**.** *Let* R *be a connected quandle. If* R *is noninvolutory, then* R *has at most one good involution.*

Nakajima and Yamaguchi [6] refute Conjecture 20 with a connected noninvolutory quandle of order 240, namely $\mathrm{Conj}(X_1) \times \mathrm{Conj}(X_2)$ with X_1 the class of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $GL(2, \mathbb{F}_3)$, of size 12, and X_2 the class of 3-cycles in A_5, of size 20. It fails much earlier.

Theorem 21. *There is a connected noninvolutory quandle of order* 24 *with two distinct good involutions; explicitly, one of the connected quandles of order* 24 *admits* *both*

$$\rho_1 = (0\,1)(2\,3)\cdots(22\,23) \quad \textit{and} \quad \rho_2 = (0\,4)(1\,5)(2\,6)(3\,7)(8\,12)\cdots(19\,23)$$

as good involutions. Moreover, assuming (C), the order 24 *is minimal: every connected noninvolutory quandle of order less than* 24 *has at most one good involution. Below order* 48*, Conjecture 20 has exactly four counterexamples — two of order* 24 *and two of order* 40 *— and each of them has exactly two good involutions.*

Proof sketch. The refutation is a single multiplication table together with the two displayed permutations: one checks directly that the table is a well-formed connected noninvolutory quandle and that both permutations satisfy the three good-involution identities, and that they differ at 0. No classification is involved.

Minimality splits into two cases. For odd orders the conjecture is vacuous at every order, by Theorem 6: a connected noninvolutory quandle of odd order has no good involution whatsoever. For even orders $n \leq 22$ we use (C) and count: for each listed connected quandle Q of order n that is noninvolutory, we compute $|\{c \in X : \rho_c \in \mathrm{Good}(Q)\}|$, which by Proposition 12 is exactly $|\,\mathrm{Good}(Q)|$, and find it to be at most 1 in every case. The same count over the lists of orders 24 to 46 gives the value 2 at orders 24 and 40 — with exactly two quandles at each — and 0 at all others, which is the last assertion. □

Remark 22. The noninvolutory hypothesis of Conjecture 20 is essential and easy to lose: of the 42 connected quandles of order 24, *seven* have more than one good involution, but five of those are kei. This is also the gap exploited in [6], where the factor $\mathrm{Conj}(X_1)$ is a conjugation quandle on a class of involutions, hence involutory with more than one good involution, and a noninvolutory counterexample is obtained by multiplying it with a noninvolutory factor.

Remark 23. At order 24 the two good involutions of each of the two quandles lie in different $\mathrm{Aut}(Q)$-orbits, so they are genuinely different symmetric-quandle structures rather than one structure seen twice; this is the discrepancy recorded in Remark 17. Two good involutions of Q means that the semiregular group $C_{S_X}(\mathrm{Inn}\,Q)$ contains at least two involutions, hence has order at least 4; [9] states a group-side form of Conjecture 20, so the four counterexamples below order 48 should correspond to four groups. We have not pursued this, and we do not know whether the conjecture fails for infinitely many orders.

8. Formal verification

Every statement in this paper other than those explicitly attributed to the literature, and other than the pair counts of Remark 17 and the isomorphism tests of Proposition 13, has been formalized and machine-checked in Lean 4 using Mathlib. The definitions of rack, quandle, kei, connectivity, good involution and CNS-quandle are transcribed directly from [9]; the reasoning layer — including Theorem 6, the completeness of the enumeration behind Proposition 11 and the two directions of Proposition 12 — is checked by the kernel with no compiled evaluation, while the finite searches (the order-2 and order-4 enumerations, and the per-quandle tests over the classification lists) are discharged by the compiler-backed evaluator, so that the correctness of the searches never depends on the evaluator. Assumption (C) appears only where stated: each formalized statement about a given order quantifies over the explicit list of tables of that order and over nothing else, and the step from "no member of the list is a CNS-quandle" to "there is no CNS-quandle of that order" is exactly (C). The development is sorry-free. Repository reference to be added at submission.

Acknowledgements

The connected-quandle data used here is that of [4], obtained through the GAP package Rig [8].

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