PERIODS OF THE REVERSIBLE SECOND-ORDER CELLULAR AUTOMATON RESOCA 115

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. RESOCA 115 is the reversible second-order cellular automaton

$$x_i^{t+1} \;=\; \delta_{115}(x_{i-1}^t, x_i^t, x_{i+1}^t) \oplus x_i^{t-1}$$

built from Wolfram's elementary rule 115 on the bi-infinite line. Formenti and Kamilya computed the periods of three families of finite configurations, leaving the family generated by $1(01)^k$ open with two conjectured closed forms, and asked which multiples of 3 occur as periods and whether the set of periods is closed under squaring. We prove a splitting theorem: at every even k the configuration $[\underline{u}; 1(01)^k]$ falls apart, across a single cell pinned to the background for all time, into two autonomous halves, so its period is the least common multiple of theirs. This turns a search of length $\Theta(4^k)$ into one of length $\Theta(2^k)$ and identifies the two factors the conjectured closed form is built from. We also solve the column structure of the automaton, deduce a parity conservation law for its nonlinear "kicks", and answer both questions negatively: no finite configuration has minimal period 9, 18, 21 or 24; no configuration whatsoever has minimal period 9; the period spectrum below 13 is exactly $\{1, 3, 6, 12\}$; and 3 is a period while 9 is not. All statements are machine-checked in Lean 4.

1. INTRODUCTION

A *second-order* cellular automaton computes the next row of its space-time diagram from the two preceding rows, by adding the value two rows up modulo 2. The construction is the standard device for making an arbitrary cellular automaton reversible [5]: it turns any local rule δ into a bijective global map. Formenti and Kamilya [2] isolated the elementary case and, in [1], studied one rule of it in detail.

Write δ_{115} for the elementary rule with Wolfram number 115 and let

$$x_i^{t+1} \;=\; \delta_{115}\big(x_{i-1}^t, x_i^t, x_{i+1}^t\big) \oplus x_i^{t-1}, \qquad i \in \mathbb{Z},\; t \geq 1. \tag{1}$$

The state of the system is the pair $[b; c]$ consisting of the memory row and the current row, both configurations of the bi-infinite line $\mathbb{Z}$; the global map f sends $[b; c]$ to $[c\,; b \oplus \delta{\cdot}c]$ and is a bijection. This is *RESOCA 115.* The period of a state is the least $p \geq 1$ with $f^p[b; c] = [b; c]$, and it is a period of the whole state, not of a single cell and not a spatial period. There is no ring and no boundary condition anywhere in what follows.

The single fact that gives the system its character is that $\delta_{115}(0, 0, 0) = 1$, so the all-zero state is not quiescent. From $[\underline{0}; \underline{0}]$ every column of the space-time diagram reads $001001001\ldots$; [1] calls this the *background*, and its temporal period 3 is the reason ([1, Lemma 1]) that every period of a finite configuration is a multiple of 3.

Section 6 of [1] studies three families of finite configurations, all with memory row $\underline{u}$ — a single 1 at the origin — and current row a word of odd length $2k + 1$ laid on the cells $-k, \ldots, k$. For the words $10^{2k-1}1$ and 1^{2k+1} the period is a closed function of k with finitely many exceptions ([1, Propositions 4 and 5]). For the third family, the words $1(01)^k$, the authors write that they "have no clue for a closed formula" and record two conjectures:

Conjecture 1.1 ([1, Conjecture 2])**.** $P(2i + 1) = 4^{2i+3} - 1$ for all $i \in \mathbb{N}$.

Conjecture 1.2 ([1, Conjecture 3])**.** $P(2i) = \big(2 \cdot 4^{2i+2} - 7 \cdot 4^{i+1} + 5\big)/3$ for all $i \in \mathbb{N}$.

Section 7 of [1] lists the periods its program found — $3, 6, 9^*, 12, 15, 27, 30, 54, \ldots$ — and poses two questions. The first asks for omniperiodicity: writing H for the set of k such that $3k$ is the period of some finite configuration, is $H = \mathbb{N}$? The authors single out 18 as the first open case and remark that settling it by their method "would require the analysis of graphs with 2^{72} vertices and 2^{74} edges". The second question asks whether the set of periods is closed under squaring, motivated by the observation that $3 \mid p$ implies $3 \mid p^2$; the authors say they cannot answer it.

Results. This paper settles both questions, in the negative, and proves the structural theorem that the two conjectures are really about.

Splitting (Section 4). At every even k the cell -1 of $[\underline{u}; 1(01)^k]$ carries the background value at every time of its orbit, and the configuration is the merge of two autonomous halves across it; its period is therefore the least common multiple of theirs, exactly. The halves are two new one-parameter families E and L with observed periods $4^{m+2} - 1$ and $2^{2m+3} - 5$, and Conjecture 1.2 is precisely the assertion that those two closed forms hold. The reduction is a square root: it computes $P(20) = 11\,728\,114\,242\,903$ from $4\,194\,303 + 8\,388\,603$ steps instead of 1.2×10^{13}.

Columns (Section 3). Every column of every RESOCA 115 space-time diagram is one of exactly four sequences, and changes type only where the current row reads 110. The four types form a star, from which a conservation law follows with no hypotheses whatsoever: over any return time, every cell is kicked an even number of times. Section 5 reduces both conjectures, at *every* k, to a statement about a Poincaré section that we verify at sixteen widths and prove at none.

Which periods occur (Section 6). No finite configuration has minimal period 9, 18, 21 or 24, so $H \neq \mathbb{N}$. With finiteness dropped, no configuration of $\mathbb{Z}$ at all has minimal period 9 — which contradicts the assertion in [1] that non-finite period-9 configurations exist — and the period spectrum below 13 is exactly $\{1, 3, 6, 12\}$. Since 3 is a period and 9 is not, the answer to the squaring question is no under either reading.

Along the way we supply what appears to be the first proof on record of [1, Propositions 4 and 5] for every k (Theorem 2.5) — the source asserts both without printed proof — by an insertion-commutation induction that uses no computer search.

Every statement below has been formally verified; Section 7 says precisely what that means and where the finite searches enter.

2. Preliminaries

2.1. **The automaton.** A *configuration* is a map $x\colon \mathbb{Z} \to \{0, 1\}$, and a *state* is a pair $[b; c]$ of configurations. The rule δ_{115} is given by the table

abc	111	110	101	100	011	010	001	000
$\delta_{115}(a,b,c)$	0	1	1	1	0	0	1	1

whose output column, read as a binary numeral, is 115. Two closed forms for it are used throughout:

$$\delta_{115}(a, b, c) \;=\; \neg b \vee (a \wedge \neg c) \;=\; 1 \oplus b \oplus (a \wedge b \wedge \neg c). \tag{2}$$

The global map is

$$f[b; c] \;=\; \bigl[\, c\,;\; i \mapsto b_i \oplus \delta_{115}(c_{i-1}, c_i, c_{i+1}) \,\bigr],$$

which is (1) written on states. Given a state $s = [b; c]$ we write y^t for the t-th row of its space-time diagram, so $y^0 = b$, $y^1 = c$ and y^t is the current row of $f^{t-1}(s)$.

A state s is *periodic* if $f^p(s) = s$ for some $p \geq 1$, and its *period* is the least such p; we write $\mathrm{per}(s) = p$. A state is *finite* if only finitely many cells of its two rows carry 1. Following [1, §3] we say that *p is a period for RESOCA 115* if some finite state has period p, and we call *p realised over all configurations* if some state, finite or not, has period p.

The *background* is the orbit of $[\underline{0}; \underline{0}]$. Because $\delta_{115}(0, 0, 0) = 1$ and $\delta_{115}(1, 1, 1) = 0$, its rows are the constant configurations $\beta(t) = [\, t \equiv 1 \bmod 3\,]$; every column of its space-time diagram is $001001001\ldots$, of temporal period 3. A cell of a state is said to *deviate* at time t if $y_i^t \neq \beta(t)$; the *deviation interval* of the families below is the smallest interval outside which no cell ever deviates, and its length is the *deviation width* w.

2.2. **The families.** For a word w of length $2k+1$ let $\mathrm{cfg}_k(w)$ be the configuration with $\mathrm{cfg}_k(w)_{-k+j} = w_j$ for $0 \le j \le 2k$ and 0 elsewhere. The states of [1, §6] are

$$S_k(w) \;=\; [\,\underline{u}\;;\;\mathrm{cfg}_k(w)\,], \qquad \underline{u}_i = [\,i = 0\,],$$

for the three word families $A_k = 10^{2k-1}1$ $(k \ge 1)$, $B_k = 1^{2k+1}$ and $C_k = 1(01)^k$. Two further families, which will turn out to be the halves of the even members of the third, are introduced in Section 4.

2.3. **From the bi-infinite line to a finite search.** Every period claim below is a claim about a bi-infinite state, so it cannot be decided directly. The source proves containment lemmas ([1, §4]) bounding the perturbation inside $[\min \vec{c} - 1, \max \vec{c}]$, which would license a finite window; we use instead a transfer that *checks* containment, per instance.

A *window state* over $[\ell, \ell + n - 1]$ consists of two words $P, C \in \{0,1\}^n$ and two background values β_p, β_c; it denotes the state whose memory and current rows are P and C on that interval and constantly β_p, β_c outside, and it is stepped by (1) inside the interval and by the uniform rule outside. Call it *boundary consistent* if

$$\delta_{115}(\beta_c, \beta_c, C_0) = \delta_{115}(\beta_c, \beta_c, \beta_c) \quad \text{and} \quad \delta_{115}(C_{n-1}, \beta_c, \beta_c) = \delta_{115}(\beta_c, \beta_c, \beta_c), \tag{3}$$

i.e. if the two cells immediately outside the interval still update to the outside value.

Proposition 2.1 (Window transfer)**.** *Let s be a window state over $[\ell, \ell + n - 1]$ and let $\hat{s}$ be the bi-infinite state it denotes. If s is boundary consistent and the window orbit of s returns to s for the first time at step p, then* $\mathrm{per}(\hat{s}) = p$.

Proof sketch. One window step agrees with one application of f exactly when (3) holds, since (3) says precisely that the two cells flanking the window keep the uniform value. The simulation carries (3) as a monotone flag, conjoining it at every step; the flag never recovers once lost, so a closing orbit with the flag still true forces (3) at every intermediate time. Agreement on $[0, p]$ then transfers both $f^p(\hat{s}) = \hat{s}$ and the minimality of p. □

For rule 115 the condition (3) simplifies sharply. Its right conjunct is unconditionally true, because $\delta_{115}(a, \beta, \beta)$ does not depend on a; its left conjunct says *the leftmost window cell must be* 1 *whenever the background is* 1. This is the left–right asymmetry that the containment lemmas of [1, §4] exhibit — activity spills one cell to the left of the word and none to the right — recovered as a by-product of the transfer rather than assumed.

Remark 2.2. Four checks show that Proposition 2.1 is not vacuous and that its hypotheses are load-bearing. On the window $[-1, 1]$ for the word 101 — one cell of margin too few — boundary consistency fails after two steps and no return is found within 200 steps, so a too-narrow window certifies nothing, and the width at which it starts to work is exactly $[\min \vec{c} - 1, \max \vec{c}]$, re-derived. The margin-one window proves the same bi-infinite statement as the margin-three window used elsewhere. Minimality is doing work: $126 = 2 \cdot 63$ satisfies $f^{126}(S_1(101)) = S_1(101)$ and is nevertheless not the period. And the closed forms of [1] are not over-read: $2^{2k+4} - 1$ fails at $k = 3$ for the first family, the constant 6 fails at $k = 0$ for the second, and $4^{k+2} - 1$ fails at $k = 2$ for the third, where the period is 135 and not 255.

2.4. **Reversibility.**

Proposition 2.3. *f is injective, with $f^{-1}[c; d] = [\,d \oplus \delta{\cdot}c\;;\;c\,]$. Consequently two states on the same orbit have the same period.*

Proof. The current row of $f[b; c]$ is c and its memory row is $b \oplus \delta{\cdot}c$, so b is recovered by adding the same value back. Transport of the period along the orbit is then formal. □

Reversibility is asserted in [1, §2]; we record the proof because the transport statement is used in Section 5. One consequence is worth isolating. Let $Z_k = [\underline{0}; \mathrm{cfg}_k(C_k)]$ be the source's third family with its central memory 1 deleted. For $k = 1, 3, 5, 7, 9$ one has $f^{2^{k+2}+1}(Z_k) = S_k(C_k)$, so Z_k inherits its period; whereas at $k = 2$ the two states lie on *different* orbits at every time, since $255 \neq 135$ — an impossibility no simulation could establish.

2.5. **Certified values.** Proposition 2.1 makes each period claim a finite orbit search. Carrying these out reproduces the published data and extends it.

Proposition 2.4. *For $1 \leq k \leq 12$ the period of $S_k(A_k)$ is $63, 255$ at $k = 1, 2$ and 15 for $3 \leq k \leq 12$; for $0 \leq k \leq 12$ the period of $S_k(B_k)$ is $3, 12$ at $k = 0, 1$ and 6 for $2 \leq k \leq 12$; and $\mathrm{per}([\underline{0}; \underline{0}]) = 3$.*

These twenty-five instances agree with [1, Propositions 4 and 5]. The source asserts both propositions for every k but prints no proof of either, so the following theorem is, to our knowledge, the first proof on record; the statements themselves are of course not new.

Theorem 2.5. *For every $k \geq 1$ the period of $S_k(A_k)$ is $2^{2k+4} - 1$ at $k = 1, 2$ and 15 for $k \geq 3$; for every $k \geq 0$ the period of $S_k(B_k)$ is 3 at $k = 0$, 12 at $k = 1$, and 6 for $k \geq 2$.*

Proof sketch. Passing from parameter k to $k+1$ inserts one cell into each of two fixed locally constant stretches of the deviation picture. Since δ_{115} cannot see its own left argument when its other two arguments agree, such an insertion commutes with f as long as the current row stays constant across both insertion points; this flatness is an invariant of the orbit, so the commutation transports the exact period — minimality included — from k to $k + 1$. The base cases $k = 4$ (family A) and $k = 3$ (family B), together with the small parameters below them, are finite window-orbit checks carried out by the kernel by direct evaluation. Unlike the certificates behind Proposition 2.4, this induction uses no native evaluation at any point. $\square$

Proposition 2.6. *For $0 \leq k \leq 9$ the period $P(k)$ of $S_k(C_k)$ is*

$$3,\ 63,\ 135,\ 1023,\ 2583,\ 16383,\ 43095,\ 262143,\ 696663,\ 4194303.$$

The printed list of realised periods in [1] stops at $2583 = P(4)$, so $k = 5, \ldots, 9$ lie past the published data; each agrees with Conjecture 1.1 or 1.2, and since [1] says its program found "many others", these are an extension of its computation rather than a new question.

3. The columns of a space-time diagram

Read the space-time diagram of a state down a fixed cell i: the *column* at i is the sequence $t \mapsto y_i^t$. By the second form in (2), (1) becomes

$$y_i^{t+1} \;=\; 1 \oplus y_i^t \oplus y_i^{t-1} \;\oplus\; \kappa_i^t, \qquad \kappa_i^t = \left[\, y_{i-1}^t y_i^t y_{i+1}^t = 110 \,\right]. \tag{4}$$

The linear part of (4) is the *same* recurrence at every cell, it does not see the neighbours at all, and the indicator κ — the *kick* — is the entire nonlinearity of the automaton.

Theorem 3.1. *The recurrence $z^{t+1} = 1 \oplus z^t \oplus z^{t-1}$ over $\mathbb{F}_2$ has period 3 and exactly four solutions: the constant sequence 1, and the three cyclic shifts of $(001)^\infty$. Consequently a kick-free stretch of any column of any RESOCA 115 space-time diagram is one of these four sequences, and the column changes type only at a time when the current row reads 110 at that cell.*

Proof. On pairs (z^{t-1}, z^t) the recurrence is the map $(a, b) \mapsto (b, a \oplus \neg b)$, whose cube is the identity; the three-element orbit through $(0, 1), (1, 0), (0, 0)$ and the fixed point $(1, 1)$ exhaust the four pairs. Reading off the second coordinate gives the constant 1 and, for each of the other three pairs, the sequence carrying 1 exactly at the times in one residue class mod 3. The statement about kicks is (4). $\square$

[1, Lemma 1] — that every period of a finite configuration is a multiple of 3 — is Theorem 3.1 localised to one cell. The classification also has a consequence with no hypotheses at all.

Theorem 3.2 (The star and the parity law)**.** *Let s be any state and i any cell. Write $\mathbf{1}_i(s) = [\,b_i = c_i = 1\,]$ for the indicator that the column at i is the constant-1 column. Then*

$$\mathbf{1}_i(f(s)) \;=\; \mathbf{1}_i(s) \oplus \kappa_i(s),$$

exactly. Moreover a kick exchanges the constant-1 column with the shift of $(001)^\infty$ that carries a 1 at the kick's own time, and never one shift with another: the four column types form a star with the constant-1 column at its centre. Hence for every state s, every cell i and every p with $f^p(s) = s$, the number of times i is kicked in the first p steps is even.

Proof. A kick at i requires $y_i^t = 1$, so the pair (y_i^{t-1}, y_i^t) before the kick is $(1,1)$ or $(0,1)$; by (4) the pair after the kick is $(1, \neg y_i^{t-1})$, which is $(1,0)$ in the first case and $(1,1)$ in the second. Both possibilities are one turn of the column clock apart, which is the star. In particular the kick toggles $\mathbf{1}_i$, and never changes it otherwise, which is the displayed identity. If $f^p(s) = s$ then $\mathbf{1}_i$ returns to its initial value, so the toggles cancel in pairs. □

No finiteness and no confinement is assumed in Theorem 3.2; it holds for every configuration of $\mathbb{Z}$ and every return time, and it is the consistency check on every kick count computed in Section 5. The columns also settle a natural structural guess. The periods $4^{m+2} - 1 = 2^{2m+4} - 1$ of the family E of Section 4 are exactly the maximal periods of a linear feedback shift register of degree $2m + 4$, which invites the hypothesis that the dynamics on such an orbit is affine over $\mathbb{F}_2$ and the period is the order of a matrix. It is not.

Proposition 3.3. *There is no $\mathbb{F}_2$-affine map F on states with $F(f^{3n}(E_1)) = f^{3n+3}(E_1)$ for $n = 0, 1, 3, 4$. In particular $2^{2m+4} - 1$ is not the order of a matrix over $\mathbb{F}_2$ acting on these configurations, and the LFSR route to Conjectures 1.1 and 1.2 is closed.*

Proof. An $\mathbb{F}_2$-affine map preserves the relation $a \oplus b \oplus c \oplus d = 0$ on any four points. The four states of the orbit of E_1 at times $0, 3, 9, 12$ satisfy that relation and are pairwise distinct, while their images at times $3, 6, 12, 15$ do not. The verification is a finite computation on the sixteen-step prefix of one orbit, on which the window and the bi-infinite dynamics provably agree. □

Finally, the kicks on the orbits studied below are extremely sparse: at every single time at most one cell is kicked. This has a local explanation.

Proposition 3.4. *Let y be a row and $[c, h]$ an interval. If y contains no factor* 011 *inside $[c, h]$, then it contains at most one factor* 110 *with its centre in $[c - 1, h]$, hence at most one kick there. The rows occurring on the orbits of E_m and L_m do avoid* 011 *on their deviation intervals, for the whole period; this is verified for $m \le 5$.*

Proof. Suppose y is kicked at $i < j$. Then $y_{i+1} = 0$ and $y_{j-1} = y_j = 1$. Walk left from $j - 1$ to the last position carrying 0; the three cells starting there read 011, and they lie inside $[c, h]$. □

The restriction to the deviation interval is essential: at a time when the background is 1 the bi-infinite row is 1 far away on both sides, so it always contains 011 outside that interval, and the unrestricted hypothesis would be vacuous. Widening the reading window by a single cell already breaks the property at time 3 for E_1, and dropping the hypothesis breaks the conclusion — the row 110110 carries two kicks. We do not prove that the row language is forward invariant; each instance is one orbit walk.

4. Splitting at even k

4.1. **Two auxiliary families.** Write alt_m for the configuration carrying the word $1(01)^m$ on the cells $0, 1, \ldots, 2m$ and 0 elsewhere, and set

$$E_m \;=\; [\,\underline{0}\,;\, \mathrm{alt}_m\,], \qquad L_m \;=\; [\,\underline{u}\,;\, \mathrm{alt}_m\,].$$

So E_m and L_m carry the same current row, and L_m has a single memory 1 at the *left end* of that word — not at its centre, as the states $S_k(w)$ of [1] do.

Theorem 4.1. $\mathrm{per}(E_m) = 4^{m+2} - 1$ *for* $0 \le m \le 9$, *and* $\mathrm{per}(L_m) = 2^{2m+3} - 5$ *for* $0 \le m \le 10$.

Proof. Each instance is one application of Proposition 2.1: the orbit search on the window $[-3, 2m]$ for E_m, and on the window $[0, 2m+3]$ for L_m, closes for the first time at the stated value, with boundary consistency intact. The two searches together run $\sum_{m\le 9}(4^{m+2}-1) + \sum_{m\le 10}(2^{2m+3}-5)$ window steps. The window for L_m is chosen with its left edge at the word's own left end, so that the left conjunct of (3) says exactly that the cell immediately left of the word stays at the background; the same search therefore proves the wall condition used below. □

The two closed forms are *not* proved for all m: Theorem 4.1 is twenty-one instances. The memory bit of L_m is load-bearing — with it deleted, boundary consistency is lost after two steps and no orbit closes within 2000. The values are of independent interest: the periods of $L_0, L_1, \dots$ are 3, 27, 123, 507, 2043, 8187, …, of which $27, 123, 507, 2043$ appear verbatim in the list of realised periods printed in [1, §7], for the reason given after Proposition 4.6. Dividing by 3 gives $1, 9, 41, 169, 681, 2729$, which is [6, A083584].

4.2. **The merge and the wall.** The merge construction of [1, Proposition 3] glues two finite configurations across a gap of *five* background cells. For rule 115 one cell suffices, and the reason is the first form in (2): a cell sitting at the background value is blind to its left neighbour as soon as the cell to its right is 1 whenever the background is 1. That proviso is literally the left conjunct of (3), so it is certified for free by the search of Proposition 2.1.

Lemma 4.2 (One-cell merge)**.** *Let u and v be window states on the same background, with v boundary consistent, and let $u * v$ be the window state obtained by concatenating them with a single cell carrying the background value in between. Then $(u * v)' = u' * v'$, one step at a time. Consequently, if the orbits of u and v close first at a and b, then the orbit of $u * v$ closes first at* $\mathrm{lcm}(a, b)$.

The hypothesis is not removable: there are window states u, v on the same background with v not boundary consistent for which one step of the merge and the merge of one step differ both in the junction cell and in the flag.

Theorem 4.3 (Splitting)**.** *For every $i \ge 0$,*

$$S_{2i+2}\big(C_{2i+2}\big) \;=\; E_i * L_{i+1},$$

the junction being the cell -1. Moreover the cell -1 carries the background value at every time of the orbit, and therefore

$$\mathrm{per}\big(S_{2i+2}(C_{2i+2})\big) \;=\; \mathrm{lcm}\big(\mathrm{per}(E_i), \mathrm{per}(L_{i+1})\big).$$

Proof sketch. The word $C_k = 1(01)^k$ laid on $[-k, k]$ puts a 1 exactly at the cells of $[-k, k]$ congruent to k modulo 2. For even $k = 2i + 2$ these are the even cells, so the cell -1 carries 0, the background value at time 0. To its left the current row carries 1 at $-k, -k+2, \dots, -2$ and the memory row is identically 0, which is a translate of E_i; to its right it carries 1 at $0, 2, \dots, k$ with a single memory 1 at the cell 0, which is a translate of L_{i+1}. This is an identity between words, proved from the entry-by-entry description of $1(01)^m$, and it holds for every i rather than instance by instance. Lemma 4.2 applies because the search establishing $\mathrm{per}(L_{i+1})$ certifies the left conjunct of (3) for the right half; the wall statement and the lcm then follow. At odd k the argument fails at the first step and provably so: the cell -1 then belongs to the word and already carries 1 at time 0. □

Lemma 4.4. $\gcd\big(4^{i+2} - 1,\ 2^{2i+5} - 5\big) = 3$ *for every* $i \ge 0$.

Proof. With $x = 4^{i+1}$ the two numbers are $4x - 1$ and $8x - 5$, and $(8x - 5) - 2(4x - 1) = -3$, so the gcd divides 3; and $3 \mid 4x - 1$ because $4x \equiv 1 \bmod 3$. □

Theorem 4.5. *For every even k with $2 \le k \le 20$ the period of $S_k(C_k)$ is*

k	$P(k)$	k	$P(k)$	k	$P(k)$
10	11 175 255	14	2 863 158 615	18	733 005 305 175
12	178 918 743	16	45 812 372 823	20	11 728 114 242 903

together with the values $135, 2583, 43095, 696663$ *at* $k = 2, 4, 6, 8$ *already in Proposition 2.6. Each of these confirms Conjecture 1.2 at that* k.

Proof. Combine Theorems 4.3 and 4.1: at $k = 2i+2$ the period is $\mathrm{lcm}(4^{i+2}-1,\ 2^{2i+5}-5)$, which by Lemma 4.4 equals the product of the two halves divided by 3. A comparison with the closed form of Conjecture 1.2 is then Proposition 4.6 below. $\square$

The saving is a square root: a direct search at $k = 20$ needs 1.2×10^{13} window steps, the two half-searches $4\,194\,303 + 8\,388\,603$. The values $P(18)$ and $P(20)$ had not been computed directly by any method — they were available only as predictions from the halves, on the assumption that the splitting holds; here that assumption is a theorem. All these values are covered by Conjecture 1.2 and by the remark in [1] that its program found "many others", so they are recorded as evidence, not claimed as new.

4.3. **Why the conjectured closed form has the shape it has.**

Proposition 4.6. *The numerator of Conjecture 1.2 factors:* $2 \cdot 4^{2i+2} - 7 \cdot 4^{i+1} + 5 = (2 \cdot 4^{i+1} - 5)(4^{i+1} - 1)$, *so*

$$P(2i) \;=\; \big(2 \cdot 4^{i+1} - 5\big) \cdot \frac{4^{i+1}-1}{3},$$

the second factor being an integer.

Proof. With $x = 4^{i+1}$ the numerator is $2x^2 - 7x + 5 = (2x-5)(x-1)$, and $3 \mid 4^n - 1$ for all n. $\square$

The two factors are exactly the two halves of Theorem 4.3: the left factor $2 \cdot 4^{i+1} - 5 = 2^{2i+3} - 5$ is $\mathrm{per}(L_i)$, and the right factor $(4^{i+1}-1)/3$ is $\mathrm{per}(E_{i-1})/3$, the base-4 repunit sequence $1, 5, 21, 85, 341, 1365$ ([6, A002450]). This is why $27, 123, 507, 2043$ occur in the source's list of realised periods: they *are* periods, of the right-hand halves of this family's even members. It is also why Conjecture 1.2 carries a division by three: Lemma 4.4.

Corollary 4.7. *Conjecture 1.2 is equivalent to the conjunction of the two closed forms* $\mathrm{per}(E_m) = 4^{m+2} - 1$ *and* $\mathrm{per}(L_m) = 2^{2m+3} - 5$, *for all* m.

5. A Poincaré section

Section 4 reduces Conjecture 1.2 to two maximal-cycle-length claims for a nonlinear reversible map. This section converts such a claim into arithmetic.

Let $[c, c+w-1]$ be the deviation interval of E_m (so $c = -1$, $w = 2m+2$) or of L_m (so $c = 0$, $w = 2m+1$). Call a state on the orbit a *section point* if its *current row* deviates from the background at both ends of that interval, and let Σ be the set of section points and R the multiset of return times between consecutive section points along the orbit.

Lemma 5.1. *If every entry of R lies in* $\{1, 2, 3, 6\}$, *then*

$$\sum R \;+\; 2\,\#\{r \in R : r = 1\} \;+\; \#\{r \in R : r = 2\} \;=\; 3\,|R| \;+\; 3\,\#\{r \in R : r = 6\}.$$

In particular, if exactly one return time is 1, *exactly one is* 2, *and* b *of them are* 6, *then* $\sum R = 3|R| + 3b - 3$.

Proof. Each of the four admissible values contributes exactly 3 to each side. $\square$

Since $\sum R$ is the period, this reduces a maximal-cycle-length claim to a count. The count is the following statement, which we verify but do not prove.

Theorem 5.2. *For $w = 1, 2, \dots, 16$, for the family E at even w and L at odd w:*

$$|\Sigma| \;=\; 2^w, \qquad R \subseteq \{1, 2, 3, 6\}$$

with multiplicities 1, 1, $2^w - 2 - b$, b *where* $3b = 2^w + 2(-1)^w$*; hence, by Lemma 5.1,*

$$\mathrm{per} \;=\; 2^{w+2} - 3 + 2(-1)^w,$$

which is $4^{m+2} - 1$ *for* E_m *and* $2^{2m+3} - 5$ *for* L_m*. Moreover, at* every *w and unconditionally: the deviation at the left end c vanishes at every time $\equiv 1 \pmod 3$, and the times $3n+2$ and $3n+3$ carry the same value there — so the left-end deviation times are a union of consecutive pairs $\{3n+2, 3n+3\}$. Consequently the section never meets background phase* 1*, and the return times* 4*,* 5 *and* 7 *cannot occur.*

Proof of the unconditional half. Let c be the left end of the deviation interval, so that the cells $c-1$ and $c-2$ agree with the background at every time — which is what (3) certifies for these windows. Write $d(t)$ for the deviation at c. At a time with background 0 the cell c cannot be kicked, since its left neighbour does not deviate, so (4) gives $d(t+2) = d(t) \oplus d(t+1)$. At a time with background 1 the kick at $c-1$ equals $d(t)$, so confinement of $c-1$ forces $d(t) = 0$. Feeding in the background clock $0, 1, 0, 0, 1, 0, \dots$ yields $d(3n+1) = 0$ and $d(3n+3) = d(3n+2)$. A section point requires the current row to deviate at c, so it cannot occur at background phase 1; and every return time is therefore 1 (inside a pair) or $\equiv 0, 2 \pmod 3$, excluding 4, 5 and 7. □

What is verified. The counting half is one finite orbit walk per width: start at a section point, step the window until the current row deviates at both ends again, record the gap. The walks run to $w = 16$ for E and $w = 15$ for L, some 6×10^5 window steps in all. For E the starting state is not itself a section point — its current row does not deviate at the cell -1 — and the first section visit is three steps later, so the recorded walk splits one return of time 6 into $3+3$ and reports $2^w + 1$ gaps; the sum, which is all the period needs, is unchanged, and the undisplaced counts are recorded separately for even $w \le 14$. □

Both ends of the interval are needed in the definition of Σ. Asking only for the left end gives return times outside $\{1, 2, 3, 6\}$; asking only for the right end keeps them inside but gives the wrong number of points; using the union of the two rows instead of the current row alone gives $3 \cdot 2^w - 3$ points and return times $\{1, 4\}$. Moving b by one in either direction is refuted, and yields a period that is refuted in turn.

Theorem 5.3. *Assume the counting half of Theorem 5.2 at the widths $2i+2$ and $2i+3$. Then Conjecture 1.2 holds at $k = 2i+2$, with no further computation. Assume it at the width $2k+2$, together with the existence of an N with $f^N(Z_k) = S_k(C_k)$; then Conjecture 1.1 holds at that odd k.*

Proof. Lemma 5.1 turns the assumed counts into $\mathrm{per}(E_i) = 4^{i+2} - 1$ and $\mathrm{per}(L_{i+1}) = 2^{2i+5} - 5$; Theorem 4.3 and Lemma 4.4 then give $\mathrm{lcm} = (4^{i+2} - 1)(2^{2i+5} - 5)/3$, which is the value of Conjecture 1.2 by Proposition 4.6. For odd k the state $S_k(C_k)$ is not a merge, but it is on the orbit of Z_k, which is a translate of E_k; periods are invariant under translation and, by Proposition 2.3, along orbits. □

Theorem 5.3 does *not* settle Conjectures 1.1 and 1.2: its hypothesis is verified at sixteen widths and proved at none, and the shift hypothesis in the odd case is likewise one finite check per k, stated explicitly rather than hidden.

5.1. **The kick counts.** The section is not the only reduction of the two closed forms; a second one, in the language of Section 3, meets it at a single number.

Theorem 5.4. *Over one period, the cell at distance d from the left end c of the deviation interval is kicked exactly 2^{w+1-d} times for $1 \le d \le w-1$, and the cell c itself exactly $\big(2^{w+1} + 4(-1)^w\big)/3$ times; no cell outside the deviation interval is ever kicked. This is verified for the family E at $w = 2, 4, \dots, 12$ and for L at $w = 1, 3, \dots, 11$.*

For instance E_3 has $w = 8$ and period 1023, with kick counts $0, 0, 172, 256, 128, 64, 32, 16, 8, 4$ on the cells $-3, \ldots, 6$. The counts halve exactly, cell by cell: the orbit is a counter with a carry propagating rightwards. They are even, as Theorem 3.2 independently requires, and they vanish outside the deviation interval, which re-derives the containment that (3) certifies; moving one entry up or down by one is refuted, so the law is pinned and not merely consistent.

Theorem 5.4 is equivalent to the closed forms, since the kick counts and the idle times sum to $2^{w+2} - 3 + 2(-1)^w$. Its local cause, verified at $w = 4, \ldots, 8$ in both parities but not formalised, is an interleaving law: between two consecutive kicks of the cell at distance $d+1$ there are exactly 0 or 4 kicks of the cell at distance d, equally often, which forces the halving. At the left end the alternative is 0 or 2, and the number of gaps containing 2 is $(2^w + 2(-1)^w)/3$ — term for term the multiplicity b of Theorem 5.2. The section route and the counter route therefore bottom out on one number, reached two different ways, and that number is what Conjectures 1.1 and 1.2 are about.

6. Which periods occur

6.1. **Columns as a transfer system.** Suppose $f^p(s) = s$. Then $y^{t+p} = y^t$ for all t, so each cell i carries a *column* $v_i \in \{0,1\}^{\mathbb{Z}/p}$ with $v_i[t] = y_i^t$. Reading (1) at the cell i and using the first form of (2) splits the recurrence in two. If $v_i[t] = 0$ then $\delta_{115}(*, 0, *) = 1$ regardless of the neighbours, so $v_i[t+1] \neq v_i[t-1]$ — a condition on v_i *alone*; call such a column *admissible*. If $v_i[t] = 1$ then $v_{i-1}[t] \wedge \neg v_{i+1}[t] = v_i[t+1] \oplus v_i[t-1]$, a condition on the triple (v_{i-1}, v_i, v_{i+1}). The first clause is what makes the computation feasible: admissible columns of length n are counted by a four-state transfer matrix, their number satisfies $a(n) = a(n-1) + a(n-3)$ and is [6, A001609], giving $4, 10, 31, 98, 309, 973, 3063, 9642$ at $p = 3, 6, \ldots, 24$. So the graph relevant to $p = 18$ has 973 vertices at the column level, not the 2^{72} that a search over rows would require.

Recoding a p-periodic space-time diagram by cell histories and searching for a path in the resulting finite graph is the time-direction de Bruijn search of Eppstein [3]; we claim no novelty for it. What is specific to rule 115 is the admissibility pre-filter, which cuts the vertex set from 2^p to $a(p)$ before any graph is built, and the anchoring described next.

Proposition 6.1. *The Boolean predicates used for admissibility and for the local triple condition hold if and only if the corresponding statement about the rule does: the encoding is an equivalence, not an approximation. Furthermore, if s is finite and $f^p(s) = s$, then the bi-infinite sequence of pairs (v_{i-1}, v_i) is a closed walk at the background pair in the column graph; and if some proper divisor d of p is a temporal period of every column occurring on such a walk, then $f^d(s) = s$, so p is not the minimal period.*

Proof sketch. The equivalence is a direct computation on masks. For the anchoring, note that a period claim comes with a period: the trivial light-cone bound says row t agrees with the background outside $[-(N+t), N+t]$, and the column v_i reads only the rows $t < p$, so v_i is the background column for $|i| > N + p$. A p-dependent bound suffices, and no containment lemma from [1, §4] is needed. The certificate has three parts, each a quantifier-free check over indices: a set F closed under forward edges and containing the background pair, a set B closed under backward edges and containing it, and the assertion that every column named by a pair in $F \cap B$ has period d. The breadth-first searches that produce F and B are not part of the trusted computation. □

6.2. **Omniperiodicity fails.**

Theorem 6.2. *No finite configuration of RESOCA 115 has minimal period* 9*,* 18*,* 21 *or* 24*. Consequently $H = \{k : 3k$ is the period of some finite configuration$\}$ is not $\mathbb{N}$: $6 \notin H$.*

Proof. Apply Proposition 6.1 with witnessing divisors $d = 3, 6, 3, 12$ respectively. The certificate is checked on $K = 973$ admissible columns for $p = 18$, $K = 3063$ for $p = 21$ and $K = 9642$ for $p = 24$; it costs $(|F| + |B|) \cdot K$ edge tests, which is 2.7×10^7, 9.3×10^7 and 3.0×10^9 respectively. □

Three controls make this meaningful. The machinery does not over-fire: at the realised periods $3, 6, 12, 15$ *no* proper divisor admits a certificate. The backward anchor is load-bearing: forward reachability from the background pair alone refutes nothing — at $p = 18$ all 973 admissible columns are forward-reachable and 936 of them have minimal period 18, and they die only on the requirement to come back, so a one-sided certificate would be an unsound narrowing. And the machinery finds the walks it should: explicit closed walks at the background pair with column-period lcm exactly p exist for $p = 3, 6, 12, 15$, and rows 0 and 1 of each — [0011100; 0010100] at $p = 6$, [00111000; 00101100] at $p = 12$, [001000; 001100] at $p = 15$ — are rebuilt as finite states whose periods are re-certified by the independent route of Proposition 2.1.

The question answered here is the classical omniperiodicity question, whose best-known instance — Conway's Life — has the opposite answer [4]. For $p = 18, 21, 24$ the source makes no claim at all, so those three settle cases it left untouched; $p = 9$ is different, and is taken up next.

6.3. **Dropping finiteness.** The anchor of Proposition 6.1 is available only for finite configurations. It can be replaced by a soft graph-theoretic argument that has nothing to do with rule 115.

Proposition 6.3. *Let X_0 be the set of pairs of admissible columns of length p and let X_{n+1} consist of those elements of X_n having both a predecessor and a successor in X_n. Then every bi-infinite walk in the column graph has all its pairs in X_n, for every n. Consequently, if every column occurring in X_n has temporal period d, then every state s with $f^p(s) = s$ satisfies $f^d(s) = s$ — with no finiteness hypothesis and no anchor.*

Proof. In a finite graph a vertex visited twice by a walk lies on a cycle, so all but finitely many vertices of a bi-infinite walk lie on cycles, and in any case every visited vertex has a visited predecessor and a visited successor. The set of visited pairs is therefore contained in the largest set closed under that property, which is the fixed point of the trimming operator. The induction on n is one line: the successor comes from the walk at $i + 1$ and the predecessor from the walk at $i - 1$. □

Theorem 6.4. *No configuration of $\mathbb{Z}$ at all — finite or not — has minimal period 9: every state with $f^9(s) = s$ already satisfies $f^3(s) = s$. Equivalently, in a 9-periodic space-time diagram every single cell's history is either constantly 1 or one of the three phases of 001001....*

Proof. There are 31 admissible columns at $p = 9$, so 961 pairs. Trimming as in Proposition 6.3 gives $961 \to 97 \to 34 \to 16 \to 16$; the sixteen survivors use only four columns, namely the three phases of the background column and the all-1 column. All four have temporal period dividing 3, so Proposition 6.3 applies with $d = 3$. □

The third trimming round is needed: after two rounds 34 pairs remain, nine of whose columns still have minimal period 9. The trim does not over-fire: at the realised period 6 the same three rounds leave period-6 columns standing, so no divisor certificate exists there.

Remark 6.5. Theorem 6.4 contradicts [1, §7], which lists 9 among the realised periods with an asterisk and states that "we were only able to construct non-finite configurations exhibiting this period", emphasising that every other period in the list "arises from a finite configuration, with the notable exception of period 9". No such configuration exists; the source prints no witness and attributes the technique behind the period-9 construction to a private communication (footnote to §7). Two readings that might have rescued the claim were checked numerically and do not: no spatially periodic configuration on a ring of size ≤ 14 has a cycle of length 9 (the same computation reproduces 32 of the 33 periods printed in [1], all but 9), and no ring state of size ≤ 11 satisfies $f^9(x) = \sigma^k(x)$ for any shift σ^k. What remains open is only that "non-finite configuration" might mean something in [1] other than a configuration of $\{0, 1\}^{\mathbb{Z}}$; its preliminaries define a configuration as a function $\mathbb{Z} \to \{0, 1\}$, so this seems unlikely.

Theorem 6.6. *For $1 \leq p \leq 12$, some configuration of $\mathbb{Z}$ has minimal period p if and only if $p \in \{1, 3, 6, 12\}$.*

Proof. Positively, 1 is the period of the all-1 fixed point and $3, 6, 12$ are realised by finite configurations. Negatively, at $p = 4, 5, 7, 8, 10, 11$ four trimming rounds collapse the graph to the single vertex consisting of the all-1 column paired with itself, so Proposition 6.3 applies with $d = 1$ and the only p-periodic state is the all-1 fixed point, whose period is 1; $p = 2$ is Proposition 6.7 below and $p = 9$ is Theorem 6.4. □

Read against [1, Lemma 1], which says that every period of a *finite* configuration is a multiple of 3: over all configurations the only extra value below 13 is 1, and the one multiple of 3 in range that Lemma 1 permits, namely 9, does not occur. The gap at 9 is not an artefact of the finiteness hypothesis.

Proposition 6.7. *The all-*1 *state is the unique fixed point of f. No finite configuration has period* 1*, and no configuration at all has period* 2*. In particular the finiteness hypothesis of* [1, Lemma 1] *is load-bearing: the all-*1 *state has period* 1*, which is not a multiple of* 3*, and it is not finite.*

Proof. $f(s) = s$ forces $b = c$ and $\delta_{115}(c_{i-1}, c_i, c_{i+1}) = 0$ for all i; since $\delta_{115}(a, 0, c) = 1$ this forces $c \equiv 1$, and $\delta_{115}(1, 1, 1) = 0$ confirms it. If $f^2(s) = s$ then $b = c$ again, so s is that fixed point, whose period is 1. □

6.4. **Squaring.**

Theorem 6.8. *The set of periods of RESOCA 115 is not closed under squaring:* 3 *is a period and* $9 = 3^2$ *is not. This holds both under the definition of* [1, §3]*, which requires a finite witness, and with finiteness dropped. Moreover* 3 *is the smallest value at which the question has content, since no finite configuration has period* 1 *and nothing at all has period* 2*.*

Proof. 3 is the period of the all-zero state, which is finite. For the finite reading, 9 is excluded by Theorem 6.2; for the unrestricted reading, by Theorem 6.4. The last sentence is Proposition 6.7. □

Proposition 6.9. *The set of periods is closed under* lcm *(*[1, Proposition 1]*) but not under products:* 3 *and* 6 *are periods and* $18 = 3 \cdot 6$ *is not. It is not closed under multiplication by* 2*,* 3*,* 4*,* 7 *or* 8 *either: each of* $9 = 3 \cdot 3$*,* $18 = 3 \cdot 6$*,* $21 = 7 \cdot 3$ *and* $24 = 2 \cdot 12 = 4 \cdot 6 = 8 \cdot 3$ *is a dilation of a realised period and is itself unrealised.*

This is exactly what defeats the motivation offered for the squaring question. Closure under lcm is a merge statement — two configurations placed side by side on the same background, as in Lemma 4.2 — and merging cannot manufacture a period that the pieces do not already have as an lcm. Squaring and products have no construction behind them at all.

7. Verification

Every statement in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*, in the directory `LeanProblemSpec/CAPeriods/` of the repository `github.com/mazigle/lean-problem-spec`; the development is free of `sorry`. The reduction from the bi-infinite line to a finite object is kernel-checked throughout: Proposition 2.1; the merge, splitting, wall and gcd statements of Section 4; the column classification, star and parity law of Section 3; the counting lemma, the left-end analysis and both conditional theorems of Section 5; and the encoding equivalence, the trimming induction and the two divisor theorems of Section 6 all depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, while (2) and the parity identity of Theorem 3.2 depend on none at all. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: one Boolean per period instance in Propositions 2.4, 2.6 and Theorems 4.1, 4.5; one per width in Theorems 5.2 and 5.4; one per period in Theorems 6.2, 6.4 and 6.6 — in the case of Theorem 6.4 a single Boolean, that the sixteen pairs surviving three trimming rounds at $p = 9$ use only four columns. Every numerical value quoted was first produced by an independent implementation in C or Python and the formal proof was then written against it: the splitting identity was checked as a literal state equality at every even $k \leq 42$ before it was proved, and the certificates of Theorem 6.2 were reproduced by a separate C implementation with the same verdicts and witnessing divisors.

References

[1] E. Formenti and S. Kamilya, *The Curious Case of Reversible Elementary Second Order Cellular Automaton 115*, GASCom 2026, EPTCS 445 (2026) 95–103, doi:10.4204/EPTCS.445.12, arXiv:2606.13159.

[2] E. Formenti and S. Kamilya, *On Elementary Second Order Cellular Automata*, in: Cellular Automata Technology (ASCAT 2024), Communications in Computer and Information Science 2021, Springer, 2024, pp. 204–217, doi:10.1007/978-3-031-56943-2_16.

[3] D. Eppstein, *Searching for Spaceships*, in: More Games of No Chance, MSRI Publications 42, Cambridge University Press, 2002, pp. 433–453; arXiv:cs/0004003.

[4] N. Brown, C. Cheng, T. Jacobi, M. Karpovich, M. Merzenich, D. Raucci and M. Riley, *Conway's Game of Life is Omniperiodic*, arXiv:2312.02799 (2023).

[5] S. Wolfram, *A New Kind of Science*, Wolfram Media, 2002, p. 1018 (second-order cellular automata).

[6] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A001609, A002450, A083584.