A LATTICE-PATH REDUCTION FOR CANON PERMUTATIONS AVOIDING $\{12^j1,\ 21^j2\}$

KOREA SUPERINTELLIGENCE LABS

Abstract. A *canon permutation* is a k-regular word over $[n]$ in which, for every j, the subsequence of j-th copies of the letters is one and the same permutation. Laudone enumerated the canon permutations avoiding the symmetric pattern set $\mathcal{O}(12^j1) = \{12^j1, 21^j2\}$ in every regime except one, and asked what the count is in the remaining range $n \geq 3$, $4 \leq j < k < 2(j-1)$, printing seven terms of the smallest open case $(j,k) = (4,5)$. We give a column-merging reduction which turns the count, for every such (j,k), into an excursion count for a lattice path in an orthant of dimension $2j-k-1$; on the diagonal $k = 2j-3$, which contains the smallest open case, the model is a walk in the quarter plane with the three steps $(1,0)$, $(-s,s)$, $(0,-1)$, where $s = k-j+1$. The reduction makes the open cells computable in polynomial time, where direct enumeration is not: we extend the smallest open case from Laudone's seven terms to forty, and give the first terms of the nine further open cells with $j \leq 7$, for which no values have been published. We then prove a negative structural result: the smallest open case admits no P-recursive recurrence with rational polynomial coefficients in any of seven (order, degree) boxes — none of order ≤ 15 and degree ≤ 7, none of order ≤ 1 and degree ≤ 63, and no constant-coefficient recurrence of order ≤ 74 — so the open row of Laudone's table has no answer of the shape its other rows have. A Denisov–Wachtel computation on the quarter-plane model predicts an irrational polynomial-correction exponent, which leads us to conjecture that the generating function is not D-finite on the whole diagonal $k = 2j-3$. All finite statements below are formally verified in Lean 4.

1. Introduction

Let $[n] = \{1,\dots,n\}$ and let $\mathcal{S}_n^k$ be the set of *k-regular words* over $[n]$, i.e. the permutations of the multiset $\{1^k,\dots,n^k\}$. A word $\pi \in \mathcal{S}_n^k$ *contains* the pattern $\sigma = \sigma_1\cdots\sigma_\ell$ if some subsequence $\pi_{i_1}\cdots\pi_{i_\ell}$, $i_1 < \cdots < i_\ell$, has the same relative order as σ (equal letters matching equal letters), and *avoids* σ otherwise. Following Elizalde, $\pi \in \mathcal{S}_n^k$ is a *canon permutation* if there is a $\sigma \in \mathcal{S}_n$ such that for each $j \in [k]$ the subsequence of j-th copies of the letters, read left to right, is σ; these generalise the nonnesting multipermutations, which are the case $k = 2$ [8]. We write $\mathcal{C}_n^k(\Lambda)$ for the canon permutations avoiding every element of a set Λ of patterns and $c_n^k(\Lambda) = |\mathcal{C}_n^k(\Lambda)|$.

Laudone [1] studies classical pattern avoidance in canon permutations for arbitrary k. A set of patterns is *symmetric* if it is closed under relabelling the letters; the orbit $\mathcal{O}(\lambda) = \mathcal{S}_\ell \cdot \lambda$ of a pattern on ℓ letters always is, and $\mathcal{O}(12^j1) = \{12^j1,\ 21^j2\}$. Symmetric avoidance is governed by [1, Thm. 5.2]: if Λ is symmetric then

$$c_n^k(\Lambda) = n!\,\bigl|\mathcal{L}_n^k(\Lambda)\bigr|, \tag{1}$$

where $\mathcal{L}_n^k$ is the set of *k-regular lattice words* of length kn on $[n]$ — words in which every prefix contains at least as many i's as $(i+1)$'s — and $\mathcal{L}_n^k(\Lambda)$ those avoiding Λ. Every count in this paper is therefore a lattice-word count; the canon-permutation counts are $n!$ times ours. All numbered references to [1] in this paper are to the numbering of v1.

For the family $\mathcal{O}(12^j1)$, Laudone's answers are complete except in one range. Writing $C_n^{(k)}$ for the number of standard Young tableaux of rectangular shape (k^n) (the k-dimensional Catalan numbers [12]) and $T_N(a,b)$ for the solution of the (a,b)-tennis ball problem [6, 9], [1, Thms. 5.6, 5.7, 5.13 and

Prop. 5.9] proves

$$\left|\mathcal{L}_n^k(\mathcal{O}(12^j1))\right| = \begin{cases} \binom{2j-2}{j-1}^{n-1}, & k > 2(j-1), \\ T_{n-1}(k,\, j-1), & k = 2(j-1), \\ C_n^{(k)}, & j \geq k, \\ \binom{2j-2}{j-1} - \binom{2j-2}{2j-k-3}, & n = 2,\ 2 \leq j \leq k, \end{cases} \tag{2}$$

and leaves exactly one cell of the table open. Of the remaining range [1] writes that "[t]he first case seems complicated" and poses:

Question 1.1 ([1, Question 5.12])**.** For $n \geq 3$ and $4 \leq j < k < 2(j-1)$, what is $c_n^k(\mathcal{O}(12^j1))$? The smallest open case is $j = 4$, $k = 5$, where $|\mathcal{L}_n^5(\mathcal{O}(12^41))|$ begins (for $n \geq 1$)

$$1,\ 19,\ 795,\ 48875,\ 3780362,\ 340746821,\ 34264303289,\ \ldots$$

and $c_n^5(\mathcal{O}(12^41))$ is $n!$ times these values. It would also be interesting to answer this for $k = j+1$ in general.

The open cells are the pairs (j,k) with $j < k < 2j-2$; there are $j-3$ of them for each $j \geq 4$, ten in all with $j \leq 7$. Nothing beyond the seven printed values above appears in the literature: a search of the OEIS on 28 and 29 August 2026 returned nothing for the leading terms of any of the ten open cells with $j \leq 7$, against positive controls that fired (the entries A000108, A005791, A066357 and A381422 of [11]); and among the papers on canon permutations known to us only [1] treats lattice words or the family 12^j1 at all.

What this paper settles. *A reduction* (§3). Put

$$s := k - j + 1, \qquad m := 2j - k - 2, \qquad \text{so that } j = s + m + 1,\ k = 2s + m,$$

and note that the open range $j < k < 2(j-1)$ is exactly $s \geq 2$, $m \geq 1$. Laudone's characterisation of avoidance by two positional inequalities *merges columns* of the tableau: columns $1, \ldots, s$ of consecutive rows form a single increasing chain, and so do columns $j, \ldots, k$, leaving the m middle columns as separate chains. Counting the linear extensions of the resulting poset — posets attached to canon permutations are studied in [2] — turns $|\mathcal{L}_n^k(\mathcal{O}(12^j1))|$ into a lattice-path excursion count in an orthant of dimension $m+1 = 2j-k-1$ (Theorem 3.2). On the diagonal $k = 2j-3$, that is $m = 1$, which contains Laudone's smallest open case, the model is a walk in the quarter plane with the three steps $(1,0)$, $(-s,s)$, $(0,-1)$ (Corollary 3.3). The reduction is what makes the open cells computable: the count becomes a dynamic program with $O(n^{m+1})$ states, where direct enumeration is hopeless — at $n = 7$ the smallest open case already has $|\mathcal{L}_7^5| = 2.8 \times 10^{14}$ words to sift, of which 3.4×10^{10} avoid.

Values (§4). We extend the smallest open case from the seven printed terms to forty (Theorem 4.2), and give the first terms of the nine further open cells with $j \leq 7$, for none of which [1] prints a single value (Theorem 4.3). Every one of the four closed forms (2) is reproduced by the same machinery as a positive control (Proposition 4.4), and the open values are shown to lie strictly between the two neighbouring closed forms (Proposition 4.5), so no closed case of [1] extends into the open range.

No recurrence (§5). Every other cell of (2) is a closed form, hence P-recursive. The smallest open case is not, in a precise and finite sense: it satisfies no recurrence $\sum_{i \leq r} p_i(n) a_{n+i} = 0$ with $p_i \in \mathbb{Q}[t]$ of degree $\leq d$ for any of seven boxes (r,d) — in particular none of order ≤ 15 and degree ≤ 7, none of order ≤ 1 and degree ≤ 63, and no constant-coefficient recurrence of order ≤ 74 (Theorem 5.2). This is, as far as we know, the first negative structural information about Question 1.1.

Why (§6). The quarter-plane model of Corollary 3.3 has zero drift under the tilt forced by the step multiplicities, and correlation $-s/(s+1)$. The Denisov–Wachtel exponent for excursions in a cone [7] is then $-1 - \pi/\arccos\big(s/(s+1)\big)$, which by Niven's theorem [10] is irrational for every $s \geq 2$, while a P-recursive sequence has a rational exponent [5, 13]. This is the mechanism by which Bostan, Raschel and Salvy [4] proved non-D-finiteness for the quarter-plane excursion sequences with small steps, and it leads to Conjecture 6.1: on the whole diagonal $k = 2j-3$ the generating function is not D-finite. We stress that §6 is evidence, not proof.

The finite statements of Sections 2–5 are formally verified; §7 says exactly what that means and where the finite searches enter.

2. Preliminaries

2.1. **Lattice words and tableaux.** A word π on $[n]$ is a *lattice word* if in every prefix the number of occurrences of i is at least the number of occurrences of $i+1$, for every $1 \le i \le n-1$; it is k-regular if every letter occurs exactly k times. We write $\mathcal{L}_n^k$ for the k-regular lattice words of length kn on $[n]$. The map sending a standard Young tableau T of shape (k^n) to the word π_T with $(\pi_T)_i =$ the row of T containing the entry i is a bijection from those tableaux onto $\mathcal{L}_n^k$, so $|\mathcal{L}_n^k| = C_n^{(k)}$. For $\pi \in \mathcal{L}_n^k$ we write

$$\pi_{i,c} \;:=\; \text{the position of the } c\text{-th occurrence of the letter } i \text{ in } \pi,$$

which is the entry in row i and column c of the corresponding tableau; so $\pi_{i,1} < \cdots < \pi_{i,k}$ and $\pi_{i,c} < \pi_{i+1,c}$ throughout.

2.2. **The two rules.** Laudone's characterisation of avoidance is the tool everything here is built on.

Lemma 2.1 ([1, Lemma 5.4]). *Let $n \ge 1$, $k \ge 2$, $2 \le j \le k$, and let $\pi \in \mathcal{L}_n^k$. Then π avoids 12^j1 and 21^j2 if and only if for all $2 \le i \le n$,*

$$\pi_{i,j} > \pi_{i-1,k} \qquad \textit{and} \qquad \pi_{i,1} > \pi_{i-1,k-j+1}.$$

Read on tableaux, Lemma 2.1 says: no entry may be placed in row i until row $i-1$ has at least $k-j+1$ entries, and no entry may be placed in columns $j, \dots, k$ of row i until row $i-1$ is full.

Containment of the two patterns of $\mathcal{O}(12^j1)$ in a k-regular word admits a criterion that avoids inspecting position tuples, and it is what all the larger sweeps below use.

Lemma 2.2. *A word w contains 12^j1 or 21^j2 if and only if there is an ordered pair (a, b) of distinct letters such that at least j occurrences of b lie strictly between the first and the last occurrence of a in w.*

Proof. An occurrence of 12^j1 is a letter a, then j copies of some $b > a$, then a further copy of a; an occurrence of 21^j2 is the same with $b < a$. In either case the two copies of a may be taken to be the first and the last, which only widens the window available to the copies of b. Conversely such a configuration is an occurrence of 12^j1 if $b > a$ and of 21^j2 if $b < a$. □

The equivalence in Lemma 2.2 was also checked against the definition of classical containment — all increasing position tuples, compared for relative order — on every word of $\mathcal{L}_n^k$ for $(n, k) = (2, 5), (3, 3), (3, 4), (4, 3), (2, 6)$ and every $2 \le j \le k$ simultaneously, and Lemma 2.1 was checked on the same enumerations.

Proposition 2.3. *Lemma 2.1 holds on every k-regular lattice word of the cells $(n, k) = (2, 5)$, $(3, 3)$, $(3, 4)$, $(4, 3)$, $(2, 6)$, for every $2 \le j \le k$, with containment directly from the definition; and, with containment computed by the criterion of Lemma 2.2, on all $1\,662\,804$ standard Young tableaux of shape (5^4) at $j = 4$, of which exactly 48875 avoid $\mathcal{O}(12^41)$.*

Remark 2.4 (the price of the literal test). Checking containment from the definition at the cell $(n, k, j) = (3, 5, 4)$ means testing, for each of the 6006 words of $\mathcal{L}_3^5$ and each of the two patterns, all $\binom{15}{6} = 5005$ increasing 6-tuples of positions: 3.0×10^7 tests per pattern. Under compiled evaluation this ran for more than 31 minutes before being abandoned. That is why the definition-level checks above stop at the five listed cells — which cover $j = 4$ through $(n, k) = (3, 4)$ and $(2, 6)$ — while $(3, 5)$ and $(4, 5)$ use the criterion of Lemma 2.2 that those checks validate. A pruning matcher would make $(3, 5)$ affordable; that is a different algorithm, not a larger budget.

2.3. **The growth process.** Lemma 2.1, in its tableau form, describes a growth process on shapes, and counting its paths is a dynamic program.

Proposition 2.5. *Let $n \geq 1$, $k \geq 2$, $2 \leq j \leq k$. Then $|\mathcal{L}_n^k(\mathcal{O}(12^j1))|$ is the number of sequences $\varnothing = r^{(0)}, r^{(1)}, \dots, r^{(kn)} = (k^n)$ of partitions with at most n parts, each at most k, in which $r^{(t)}$ arises from $r^{(t-1)}$ by adding one cell in some row i, at column $c = r_i^{(t-1)} + 1$, subject to*

$$c \leq r_{i-1}^{(t-1)}, \qquad c = 1 \Rightarrow r_{i-1}^{(t-1)} \geq k-j+1, \qquad c \geq j \Rightarrow r_{i-1}^{(t-1)} = k \qquad (i \geq 2).$$

Proof. Reading the word π left to right builds the tableau one entry at a time, and the first condition is the requirement that the shape stay a partition, i.e. that π be a lattice word. The remaining two are the two inequalities of Lemma 2.1: the entry entering column 1 of row i is $\pi_{i,1}$, which must exceed $\pi_{i-1,k-j+1}$, i.e. row $i-1$ must already have $k-j+1$ entries; and the entry entering column $c \geq j$ of row i is at least $\pi_{i,j}$, which must exceed $\pi_{i-1,k}$, i.e. row $i-1$ must be full. □

Proposition 2.5 was checked against complete enumeration of $\mathcal{L}_n^k$ for $1 \leq n \leq 3$ and all $2 \leq j \leq k \leq 5$, and at $(n,k,j) = (4,5,4)$ against the sweep of Proposition 2.3.

3. The column-merging reduction

Throughout this section put

$$s = k-j+1, \qquad m = 2j-k-2, \qquad j = s+m+1, \qquad k = 2s+m,$$

and assume $s \geq 1$ and $m \geq 1$, i.e. $3 \leq j \leq k \leq 2j-3$. The open range of Question 1.1 is $s \geq 2$, $m \geq 1$.

Definition 3.1. For $s, m \geq 1$ and $n \geq 1$ let $W_{s,m}(n)$ be the number of words w over the alphabet $\{0, 1, \dots, m+1\}$ with

$$\#0 = \#(m+1) = sn, \qquad \#t = n \ \ (1 \leq t \leq m),$$

such that every prefix of w satisfies

$$\#0 \ \geq \ s \cdot \#1, \qquad \#1 \ \geq \ \#2 \ \geq \ \dots \ \geq \ \#m, \qquad s \cdot \#m \ \geq \ \#(m+1),$$

where $\#t$ denotes the number of occurrences of t in that prefix.

Such a word is a lattice path in the orthant $\mathbb{Z}_{\geq 0}^{m+1}$: track the $m+1$ slacks $\#0 - s\#1$, $\#1 - \#2$, $\dots$, $\#(m-1) - \#m$, $s\#m - \#(m+1)$. Each letter changes them by a fixed vector, the conditions say the path stays in the orthant, and the prescribed letter multiplicities say it starts and ends at the origin.

Theorem 3.2 (column merging)**.** *Let $s \geq 1$, $m \geq 1$, $j = s+m+1$ and $k = 2s+m$. Then for every $n \geq 1$,*

$$\left|\mathcal{L}_n^k(\mathcal{O}(12^j1))\right| \ = \ W_{s,m}(n).$$

Proof. Identify $\pi \in \mathcal{L}_n^k$ with its tableau and write (i,c) for the cell in row i, column c of the $n \times k$ rectangle. A word of $\mathcal{L}_n^k$ is a linear extension of the Young diagram poset of shape (k^n), and by Lemma 2.1 (using $k-j+1 = s$) the words avoiding $\mathcal{O}(12^j1)$ are exactly the linear extensions of the poset P on the rectangle generated by

$$\text{(R) } (i,c) < (i,c+1), \quad \text{(C) } (i,c) < (i+1,c), \quad \text{(A) } (i-1,k) < (i,j), \quad \text{(B) } (i-1,s) < (i,1).$$

Since $s \geq 1$ and $m \geq 1$ we have $1 \leq s < s+1 \leq j-1 < j \leq k$, so the three column blocks $\{1, \dots, s\}$, $\{s+1, \dots, j-1\}$ and $\{j, \dots, k\}$ are all nonempty, the middle one having $j-1-s = m$ columns. Define

$$\begin{aligned} X &: (1,1), \dots, (1,s), (2,1), \dots, (2,s), \ \dots, \ (n,1), \dots, (n,s) && \text{(length } sn), \\ Y^{(t)} &: (1,s+t), (2,s+t), \dots, (n,s+t) && \text{(length } n), \ 1 \leq t \leq m, \\ Z &: (1,j), \dots, (1,k), (2,j), \dots, (2,k), \ \dots, \ (n,j), \dots, (n,k) && \text{(length } sn). \end{aligned}$$

These $m+2$ sequences partition the rectangle. We claim that P is generated by

(i) X, Z and each $Y^{(t)}$ are chains in the orders listed;
(ii) $X_{si} < Y_i^{(1)}$ for $1 \le i \le n$;
(iii) $Y_i^{(t)} < Y_i^{(t+1)}$ for $1 \le t < m$, $1 \le i \le n$;
(iv) $Y_i^{(m)} < Z_{s(i-1)+1}$ for $1 \le i \le n$.

Each of (i)–(iv) holds in P: inside X the steps within a row are (R) and the step $(i,s) < (i+1,1)$ is (B); inside Z the steps within a row are (R) and the step $(i,k) < (i+1,j)$ is (A); each $Y^{(t)}$ is a column, so (C); and (ii), (iii), (iv) are the row relations $(i,s) < (i,s+1)$, $(i,s+t) < (i,s+t+1)$ and $(i,j-1) < (i,j)$. Conversely every generator of P follows from (i)–(iv). For (R), the relation $(i,c) < (i,c+1)$ is a step of the chain X if $c+1 \le s$, is (ii) if $c = s$, is (iii) if $s+1 \le c \le j-2$, is (iv) if $c = j-1$, and is a step of Z if $c \ge j$. For (C), the relation $(i,c) < (i+1,c)$ lies inside the chain X if $c \le s$, inside $Y^{(c-s)}$ if $s < c < j$, and inside Z if $c \ge j$. Finally (A) and (B) are consecutive steps of Z and of X. This proves the claim.

It remains to count the linear extensions of the poset generated by (i)–(iv). Record a linear extension by the word w of length kn over $\{0,\dots,m+1\}$ whose r-th letter names the chain contributing the r-th smallest element: 0 for X, t for $Y^{(t)}$, $m+1$ for Z. Because (i) makes each of the $m+2$ classes a chain, w determines the extension, and the letter multiplicities are exactly $\#0 = \#(m+1) = sn$ and $\#t = n$. A word with those multiplicities is a linear extension if and only if each of (ii)–(iv) is respected at the moment of placement. Placing the i-th letter 1 requires X_{si} to be already placed, i.e. $\#0 \ge si$ at that moment; over all prefixes this is $\#0 \ge s \cdot \#1$. Placing the i-th letter $t+1$ requires $\#t \ge i$, i.e. $\#t \ge \#(t+1)$. Placing the q-th letter $m+1$ requires $Y_i^{(m)}$ to be placed for $i = \lceil q/s \rceil$, the largest i with $s(i-1)+1 \le q$, i.e. $\#m \ge \lceil q/s \rceil$, which for integers is $s \cdot \#m \ge q = \#(m+1)$. These are exactly the conditions of Definition 3.1. □

Corollary 3.3 (the diagonal $k = 2j-3$ is a quarter-plane walk)**.** *Let $j \ge 3$, $k = 2j-3$ and $s = j-2$. Then $|\mathcal{L}_n^k(\mathcal{O}(12^j1))|$ is the number of excursions of length $(2s+1)n = kn$ — walks from the origin to the origin staying in $\mathbb{Z}^2_{\ge 0}$ — of the walk model with the three steps*

$$(1,0), \qquad (-s,s), \qquad (0,-1).$$

Proof. Here $m = 1$, so Theorem 3.2 applies with the alphabet $\{0,1,2\}$. Put $x = \#0 - s\,\#1$ and $y = s\,\#1 - \#2$. The letters 0, 1, 2 change (x,y) by $(1,0)$, $(-s,s)$ and $(0,-1)$ respectively, the two prefix conditions of Definition 3.1 are $x \ge 0$ and $y \ge 0$, and the prescribed multiplicities (sn,n,sn) put both ends at the origin. Conversely, if a walk with these steps has length $N = (2s+1)n$ and returns to the origin, its step multiplicities (c_0,c_1,c_2) satisfy $c_0 - sc_1 = 0$, $sc_1 - c_2 = 0$ and $c_0 + c_1 + c_2 = N$, whose unique solution is (sn,n,sn). So excursions of length kn correspond exactly to the words counted by $W_{s,1}(n)$. □

Remark 3.4 (what the reduction buys). The dynamic program implied by Definition 3.1 has one state per admissible vector of letter counts, of which there are $O(n^{m+2})$, and $m+2$ transitions from each; grouping the states by the length $\#0 + \cdots + \#(m+1)$ of the prefix, which is determined by the other coordinates, leaves $O(n^{m+1})$ of them alive at a time. So $|\mathcal{L}_n^k(\mathcal{O}(12^j1))|$ is computable in $O(n^{m+2})$ arithmetic operations on integers of $O(n)$ digits. Direct enumeration is not an option: at $(j,k) = (4,5)$ the objects to be enumerated at $n = 7$ already number $|\mathcal{L}_7^5| = C_7^{(5)} = 2.8 \times 10^{14}$, of which $a_7 = 3.4 \times 10^{10}$ avoid. This is why the tables of §4 exist at all, and it is also why the cells with large m stop early: at $(j,k) = (7,8)$ one has $m = 4$, and the state count $O(n^5)$ bites.

Remark 3.5 (a fixed point of the reduction). At $s = 1$, $m = 1$ one has $(j,k) = (3,3)$, which is the closed case $j \ge k$ of (2), and Corollary 3.3 degenerates to the classical tandem walk $\{(1,0),(-1,1),(0,-1)\}$, whose excursions of length $3n$ are counted by $C_n^{(3)}$. The two agree, as they must; we have checked the identity $W_{1,1}(n) = C_n^{(3)}$ numerically for $n \le 6$. This check is external to the formal development: the machine-checked instances of Theorem 3.2 recorded in §7 are the ten open cells with $j \le 7$, all of which have $s \ge 2$.

TABLE 1. $a_n = |\mathcal{L}_n^5(\mathcal{O}(12^4 1))|$ for $1 \le n \le 40$; the canon-permutation counts are $n!\,a_n$. The first seven entries are the values printed in Question 1.1.

n	a_n
1	1
2	19
3	795
4	48875
5	3780362
6	340746821
7	34264303289
8	3740554409595
9	435337019298913
10	53331481655131738
11	6813743109121574171
12	901591847135383270261
13	122893508809977360047025
14	17183510912889946651432237
15	2456363536880972170310498554
16	357996382651746732223099860627
17	53074616924559171393943321264039
18	7989091712217321519714999052929237
19	1219043518228934089734890704331114846
20	188306116639329069516297792294966547698
21	29412288511701979420790475931641452266701
22	4640626315282821003219555667207286303863415
23	738973023314982191279483148153365629990287822
24	118673236461302302561927712521164271602789455865
25	19206882859789029004342210216712297172993956438683
26	3130993458487160553934235205523426180116697578909187
27	513805214680538545441300486743287046279638819085288226
28	84839946871423684088502815898941201948359049185690369769
29	14089754893616364352456351120163638071308837061060531038999
30	2352565157485017365520420816306827756951816504417442230246404
31	394789015577619148589972856909326506798536394371452029841242182
32	66563730322366951304650521405490318932774012860084301204727213983
33	11272902931512179882995786959883675276204871819600472133776188760854
34	1917108130312105285707436582246580421201815349950949969445200306558005
35	327315833599623522571071517122141631193985604148313111904148765725877554
36	56092188835981520377929048995503108216388890104104222146709893396859304229
37	9646411299672509029206151263357109046548118142184495943432574276930689336398
38	1664474194182269094142104719662836230679137659248348618864195451753496532472012
39	288112625899639215745055888979929287384363754046358335227559937656888135364386948
40	50021073507665117355172470803712652018074771526303913354440523858947305824677497854

The reduction has been checked cell by cell against the growth process of Proposition 2.5 on all ten open cells with $j \le 7$; the ranges of n are recorded in §7.

4. Values

Write $a_n := |\mathcal{L}_n^5(\mathcal{O}(12^4 1))|$ for the smallest open case, $(j,k) = (4,5)$, i.e. $(s,m) = (2,1)$. By (1), $c_n^5(\mathcal{O}(12^4 1)) = n!\,a_n$.

Proposition 4.1. *The seven values printed in Question 1.1 are reproduced by the growth process of Proposition 2.5:*

$$(a_1, \dots, a_7) = (1,\ 19,\ 795,\ 48875,\ 3780362,\ 340746821,\ 34264303289).$$

The next theorem is the reduction at work: the terms $a_8, \dots, a_{40}$ are, to our knowledge, new.

Theorem 4.2. *For $1 \le n \le 40$, a_n is the value listed in Table 1.*

Theorem 4.3. *For each of the nine open cells (j,k) with $j \le 7$ other than $(4,5)$, the first five values of $|\mathcal{L}_n^k(\mathcal{O}(12^j 1))|$ are those listed in Table 2; the same computation determines, for each cell, the number of terms shown in the last column.*

The $n = 2$ column of Table 2 is a control that was not fitted: each entry agrees with the fourth line of (2), $\binom{2j-2}{j-1} - \binom{2j-2}{2j-k-3}$, which gives $70-8$, $70-1$, $252-45$, $252-10$, $252-1$, $924-220$, $924-66$, $924-12$, $924-1$ for the nine cells in the order listed, and $20-1 = 19 = a_2$ for $(4,5)$.

TABLE 2. The nine open cells with $j \le 7$ other than $(4,5)$. No value for any of these cells appears in [1]. The last column is the number of terms established here.

(j,k)	(s,m)	$\lvert\mathcal{L}_n^k(\mathcal{O}(12^j1))\rvert$, $n=1,\dots,5$	terms
$(5,6)$	$(2,2)$	1, 62, 12744, 4841100, 2685714295	16
$(5,7)$	$(3,1)$	1, 69, 12796, 3779301, 1466678055	20
$(6,7)$	$(2,3)$	1, 207, 215811, 528617911, 2217650499936	12
$(6,8)$	$(3,2)$	1, 242, 250986, 540396876, 1817387396032	14
$(6,9)$	$(4,1)$	1, 251, 197505, 263014139, 475806141651	18
$(7,8)$	$(2,4)$	1, 704, 3826143, 62666675556, 2078657193874495	10
$(7,9)$	$(3,3)$	1, 858, 4996916, 78927500713, 2328252921224636	11
$(7,10)$	$(4,2)$	1, 912, 4365721, 47540180434, 852883408206600	12
$(7,11)$	$(5,1)$	1, 923, 3022503, 17553783932, 142212268050976	14

The same machinery reproduces all four closed forms of (2).

Proposition 4.4. *The counts of Proposition 2.5 agree with* (2) *in the following ranges: with* $\binom{2j-2}{j-1}^{n-1}$ *for* $(j,k) = (2,3)$, $(2,4)$, $(3,5)$, $(3,6)$, $(4,7)$, $(4,8)$, $(5,9)$, $(5,10)$ *and* $1 \le n \le 4$*; with* $C_n^{(k)}$ *for* $(j,k) = (2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(3,2)$, $(4,3)$, $(5,4)$, $(6,5)$ *and* $1 \le n \le 4$*, where moreover* $C_n^{(5)} = 1, 42, 6006, 1662804, 701149020, 396499770810$ *for* $1 \le n \le 6$*; with the tennis-ball numbers* $T_{n-1}(k, j-1)$ *for* $j \in \{2,3,4,5\}$*,* $k = 2(j-1)$ *and* $1 \le n \le 5$*; and with* $\binom{2j-2}{j-1} - \binom{2j-2}{2j-k-3}$ *at* $n = 2$ *for all* $3 \le j \le 9$ *and* $j \le k \le 2j-3$*.*

Finally, the open values are pinned away from their neighbours on both sides, so that no closed case of (2) extends into the open range.

Proposition 4.5. (1) *For every open cell* (j,k) *with* $j \le 7$ *and every* $3 \le n \le 5$*,*

$$\binom{2j-2}{j-1}^{n-1} \;<\; \left|\mathcal{L}_n^k(\mathcal{O}(12^j1))\right| \;<\; C_n^{(k)}.$$

(2) *The tennis-ball formula does not extend below* $k = 2(j-1)$*: at* $(j,k) = (4,5)$ *it gives* $T_{n-1}(5,3) = 1, 10, 155, 2885$ *for* $1 \le n \le 4$*, while* $a_n = 1, 19, 795, 48875$*.*

(3) *The avoidance condition is neither vacuous nor unsatisfiable at* $(n,k,j) = (3,5,4)$*: of the* $|\mathcal{L}_3^5| = 6006$ *tableaux of shape* (5^3)*, exactly* 795 *satisfy the two rules of Lemma 2.1, and* 795 *is neither* 6006 *nor* $\binom{6}{3}^2 = 400$*.*

5. NO P-RECURSIVE RECURRENCE FOR THE SMALLEST OPEN CASE

Every closed form in (2) is P-recursive in n. We show that the smallest open case is not — in the finite, checkable sense of ruling out all recurrences inside a box of orders and degrees.

Definition 5.1. A sequence $(a_n)_{n\ge1}$ *satisfies a recurrence of order* $\le r$ *and degree* $\le d$ *on* $[1,N]$ if there are $p_0,\dots,p_r \in \mathbb{Q}[t]$ with $\deg p_i \le d$, not all zero, such that $\sum_{i=0}^r p_i(n)\, a_{n+i} = 0$ for $n = 1,\dots,N$. A P-recursive recurrence valid for all $n \ge 1$ is in particular one on $[1,N]$ for every N.

Theorem 5.2. *Let* $a_n = |\mathcal{L}_n^5(\mathcal{O}(12^41))|$ *and let* (r,d) *be any of*

$$(1,63),\quad (3,31),\quad (7,15),\quad (15,7),\quad (31,2),\quad (43,1),\quad (74,0).$$

Then a_n *satisfies no recurrence of order* $\le r$ *and degree* $\le d$ *on* $[1,(r+1)(d+1)]$*. In particular* a_n *satisfies no P-recursive recurrence of order* ≤ 15 *with coefficients of degree* ≤ 7*, none of order* ≤ 1 *with coefficients of degree* ≤ 63*, and no constant-coefficient linear recurrence of order* ≤ 74*.*

Proof. Fix (r,d) and set $C = (r+1)(d+1)$. Index a candidate coefficient vector by $x = i(d+1)+e$, $0 \le i \le r$, $0 \le e \le d$, so that c_x is the coefficient of $n^e a_{n+i}$ in $\sum_i p_i(n) a_{n+i}$. Requiring the recurrence at $n = 1,\dots,C$ is requiring $Mc = 0$ for the $C \times C$ matrix $M[u][x] = (u+1)^e a_{u+1+i}$, $0 \le u \le C-1$. A nonzero rational solution may be scaled by a common denominator and divided by the content to

TABLE 3. Measured versus predicted polynomial exponent on the diagonal $k = 2j-3$. Computed outside the formal development.

s	(j,k)	μ_s	exact terms used	Richardson	predicted α_s
1	$(3,3)$, closed	27	closed form $C_n^{(3)}$	-4 exactly	-4
2	$(4,5)$	3125/16	400	-4.73518396	-4.73523918
3	$(5,7)$	823543/729	180	-5.34657347	-5.34681581
4	$(6,9)$	387420489/65536	150	-5.88160127	-5.88203145

a *primitive* integer solution, whose coordinates are not all divisible by any given prime. So it suffices to exhibit, for one prime P, a matrix L over $\mathbb{Z}/P$ with $LM \equiv I \pmod P$: then any integer solution satisfies $c \equiv L(Mc) \equiv 0 \pmod P$, contradicting primitivity.

We take $P = 2^{31}-1$ and produce L by Gauss–Jordan elimination modulo P on the matrix M built from the first 150 terms of a_n, computed by the excursion model of Theorem 3.2. The verification of $LM \equiv I$ is a finite computation with $C^3 \le 128^3$ modular multiplications, carried out by compiled evaluation; *the deduction above from $LM \equiv I$ to the conclusion is not, and is checked by the proof kernel alone* (see §7). The seven boxes have $C = 128, 128, 128, 128, 96, 88, 75$ respectively. Finally, a recurrence of order $\le r'$ and degree $\le d'$ with $r' \le r$ and $d' \le d$ is one of order $\le r$ and degree $\le d$ with zero coefficients, so each box also rules out every box it contains. □

Remark 5.3 (larger boxes, outside the formal development). The seven boxes above were chosen after a wider search carried out off-line with the same prime and more terms. Every coefficient matrix tried there — at least eleven of them — had full column rank modulo P, among them $(r,d) = (239,0)$ with 240 unknowns and $(8,74)$ with 675 unknowns and 690 equations. For contrast, the Berlekamp–Massey algorithm applied to 70 terms returns the generic order 35, which is the "no recurrence" answer. None of this is part of the verified development; only the seven boxes of Theorem 5.2 are, and only they are claimed.

6. Asymptotics on the diagonal, and a conjecture

This section is evidence rather than proof, and nothing in it is part of the formal development.

Fix $j \ge 3$, let $k = 2j-3$ and $s = j-2$, and let $a_n^{(s)} = |\mathcal{L}_n^k(\mathcal{O}(12^j 1))|$; by Corollary 3.3 this counts excursions of length $(2s+1)n$ of the quarter-plane walk $\Sigma_s = \{(1,0), (-s,s), (0,-1)\}$. The step multiplicities are forced to (sn, n, sn), so the relevant tilt gives the three steps the probabilities $(s,1,s)/(2s+1)$. Under that tilt the drift is

$$\frac{s\,(1,0) + 1\,(-s,s) + s\,(0,-1)}{2s+1} = (0,0),$$

the exponential growth rate is the multinomial one, $\mu_s = (2s+1)^{2s+1}/s^{2s}$, and the covariance matrix is

$$\frac{1}{2s+1}\begin{pmatrix} s+s^2 & -s^2 \\ -s^2 & s+s^2 \end{pmatrix}, \qquad \text{of correlation} \quad \rho_s = -\frac{s}{s+1}.$$

For a zero-drift walk in a two-dimensional cone, under moment and aperiodicity hypotheses, the number of excursions of length N decays polynomially with exponent $-1-\pi/\theta$, where θ is the opening angle of the cone after the linear change of variables that makes the covariance the identity [7]; for the quarter-plane models with small steps this is the exponent tabulated in [3, 4]. Here $\theta_s = \arccos(-\rho_s) = \arccos\big(s/(s+1)\big)$. Note that Σ_s is not a small-step model for $s \ge 2$ — the step $(-s,s)$ leaves $\{0,\pm1\}^2$ — so the small-step results of [3, 4] do not themselves cover it, and it is [7] that is being invoked. This predicts

$$a_n^{(s)} \sim C\,\mu_s^n\, n^{\alpha_s}, \qquad \alpha_s = -1 - \frac{\pi}{\arccos\big(s/(s+1)\big)}. \tag{3}$$

Table 3 compares (3) against the exact terms by Richardson extrapolation of order 8 applied to $n\big(a_n^{(s)}/(\mu_s a_{n-1}^{(s)}) - 1\big)$.

The point of (3) is the arithmetic of the exponent. Since $\cos\theta_s = s/(s+1)$ is rational and lies outside $\{0, \pm\frac{1}{2}, \pm 1\}$ for every $s \geq 2$, Niven's theorem [10, Cor. 3.12] says that θ_s/π is irrational, hence so is π/θ_s and so is α_s. On the other hand a P-recursive sequence has an asymptotic expansion in which the polynomial exponent is rational [5, 13]. This is exactly the argument by which Bostan, Raschel and Salvy [4] established non-D-finiteness for the quarter-plane excursion sequences with small steps and infinite group, and it makes the following plausible. Note $s = 1$ is the escape hatch: there $\cos\theta_1 = \frac{1}{2}$, the exponent is the rational -4, and the count is indeed the closed form $C_n^{(3)}$.

Conjecture 6.1. For every $j \geq 4$ the generating function $\sum_{n\geq 1} |\mathcal{L}_n^{2j-3}(\mathcal{O}(12^j1))|\, x^n$ is not D-finite; equivalently, $|\mathcal{L}_n^{2j-3}(\mathcal{O}(12^j1))|$ satisfies no P-recursive recurrence. The same then holds for $c_n^{2j-3}(\mathcal{O}(12^j1))$, since multiplication by $n!$ preserves P-recursiveness in both directions.

Conjecture 6.1 is a structural answer to Question 1.1 on the whole diagonal $k = 2j-3$: the open row of Laudone's table would have no answer of the shape its other four rows have, and the smallest instance $j = 4$ is precisely his "smallest open case". Theorem 5.2 is the unconditional fragment of it that we can prove. Two ingredients are missing for a proof, both available in the literature but not assembled here: the Denisov–Wachtel local limit theorem for excursions has to be applied to the tilted step distribution of Σ_s with its hypotheses verified, and the passage from P-recursiveness to a rational exponent has to be quoted in a form that covers a sequence with a single dominant exponential scale. To be explicit about the status of (3): it is what the results of [7] predict for this model, and it is what Table 3 measures to four or five significant digits, but the hypotheses of that local limit theorem are not verified for Σ_s anywhere in this paper. Equation (3) is therefore a prediction, and Conjecture 6.1 rests on it as such.

7. Verification

Every *finite* statement of §§2–5 has been formally verified in Lean 4; the general statements — Lemma 2.2, Proposition 2.5, Theorem 3.2 and Corollary 3.3 — are proved on paper above, and what is machine-checked is their instances at the cells listed below. The development contains no `sorry`; it was compiled with the toolchain `leanprover/lean4:v4.32.0` against *Mathlib* at revision `81a5d257c8e410db227a6665ed08f64fea08e997` (tag `v4.32.0`), and a repository reference will be added at submission. What is delegated to compiled evaluation (Lean's `native_decide`) is exactly the finite searches, and they are these. Proposition 2.3 and Lemma 2.2 are one traversal of the complete enumerations of $\mathcal{L}_n^k$ for $(n,k) = (2,5), (3,3), (3,4), (4,3), (2,6)$, testing every $2 \leq j \leq k$ at once, with containment computed from the definition; and one traversal of all 1 662 804 tableaux of shape (5^4) at $j = 4$, which simultaneously confirms the enumeration is complete ($1\,662\,804 = C_4^{(5)}$), that 48875 words avoid, and that Lemma 2.1 holds on every one of them. Proposition 2.5 is checked against those enumerations for $1 \leq n \leq 3$, $2 \leq j \leq k \leq 5$, and at $(4,5,4)$. Theorem 3.2 is verified cell by cell against Proposition 2.5: at $(j,k) = (4,5)$ for $1 \leq n \leq 16$; at $(5,6)$ and $(5,7)$ for $n \leq 6$; at $(6,7)$, $(6,8)$, $(6,9)$ for $n \leq 5$; and at $(7,8)$, $(7,9)$, $(7,10)$, $(7,11)$ for $n \leq 4$ — the general statement of Theorem 3.2 is proved above and is not itself formalised. Proposition 4.1 and Theorems 4.2 and 4.3 are evaluations of the two dynamic programs, the first from the growth process of Proposition 2.5 and the others from the excursion model; the number of terms verified for each cell is the last column of Table 2, and 40 for $(4,5)$. Propositions 4.4 and 4.5 are evaluations over the ranges stated in them. In Theorem 5.2 the 150-term table is the excursion model of Theorem 3.2, verified against the growth process for $n \leq 16$ and against the seven values of Question 1.1 for $n \leq 7$; the certificate $LM \equiv I \bmod P$ is the only compiled evaluation in the refutation itself; the implication from it — that every integer coefficient vector annihilating the sequence on $[1, C]$ is divisible by P — contains no `native_decide`, and so depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. Independence of implementation was maintained throughout: the seven values of Question 1.1 were reproduced on the first run of a dynamic program written from Lemma 2.1 alone, before anything else here was written; the excursion model was implemented a second time, sharing no code, and agrees with the growth process on every cell tested; and the 400 terms of a_n computed off-line by two independent programs agree at every index, the first 40 of them being Theorem 4.2.

References

[1] R. Laudone, *Pattern avoidance in canon permutations*, arXiv:2608.21351 (v1, 21 August 2026).

[2] M. Beck and D. Deligeorgaki, *Canon permutation posets*, arXiv:2410.03245 (v2, 27 July 2025).

[3] M. Bousquet-Mélou and M. Mishna, *Walks with small steps in the quarter plane*, in: Algorithmic Probability and Combinatorics, Contemp. Math. **520**, Amer. Math. Soc., Providence, RI, 2010, 1–39.

[4] A. Bostan, K. Raschel and B. Salvy, *Non-D-finite excursions in the quarter plane*, J. Combin. Theory Ser. A **121** (2014), no. 1, 45–63.

[5] G. D. Birkhoff and W. J. Trjitzinsky, *Analytic theory of singular difference equations*, Acta Math. **60** (1933), 1–89.

[6] A. de Mier and M. Noy, *A solution to the tennis ball problem*, Theoret. Comput. Sci. **346** (2005), no. 2–3, 254–264.

[7] D. Denisov and V. Wachtel, *Random walks in cones*, Ann. Probab. **43** (2015), no. 3, 992–1044.

[8] S. Elizalde, *Descents on nonnesting multipermutations*, European J. Combin. **121** (2024), Paper No. 103846.

[9] D. Merlini, R. Sprugnoli and M. C. Verri, *The tennis ball problem*, J. Combin. Theory Ser. A **99** (2002), no. 2, 307–344.

[10] I. Niven, *Irrational Numbers*, Carus Mathematical Monographs **11**, Mathematical Association of America, 1956.

[11] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A000108, A005791, A066357, A381422.

[12] R. A. Sulanke, *Generalizing Narayana and Schröder numbers to higher dimensions*, Electron. J. Combin. **11** (2004), no. 1, #R54, 20 pp.

[13] J. Wimp and D. Zeilberger, *Resurrecting the asymptotics of linear recurrences*, J. Math. Anal. Appl. **111** (1985), no. 1, 162–176.